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CONSTRUCTIVE APPROXIMATION

ERRATUM

Erratum to: The Hardy–Rellich Inequality and Uncertainty Principle on the Sphere

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Several forms of uncertainty principles on the unit sphere are established in [1]. When stated in term of the vector

$$\tau(f) := \frac{1}{\omega_d} \int_{\mathbb{S}^{d-1}} x|f(x)|^2 \, d\sigma(x)$$

of $\mathbb{R}^d$ (normalization constant $1/\omega_d$ was missing in [1]), our main result is in:

Corollary 4.4 *Let $f \in W_2^1(\mathbb{S}^{d-1})$ be such that $\int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = 0$ and $\|f\|_2 = 1$. If $d \geq 2$, then*

$$(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq C_d^{-1}. \tag{4.11}$$

Here C_d is a constant given in Theorem 4.1. We next attempted to remove the condition that $\int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = 0$ and stated:

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Theorem 4.5 *Assume that $d \geq 2$, and let $f \in W_2^1(\mathbb{S}^{d-1})$ be such that $\|f\|_2 = 1$. Then*

$$(1 - \|\tau(f)\|)\|\nabla_0 f\|_2^2 \geq c_d\|\tau(f)\|. \tag{4.14}$$

This theorem, however, is incorrect. This was pointed out to us by Stefan Steinerberger, who showed that the inequality (4.14) does not hold for the function $f(\cos\theta, \sin\theta) = 1 + \varepsilon \sin\theta$ for small enough ε when $d = 2$. The mistake in the proof appeared on the line 6 of page 166, which states that $\|\tau(f)\| \leq (2|m_f| + 1)\|g\|_2^2$, but it should have been $\|\tau(f)\| \leq \|g\|_2^2 + 2|m_f|\|g\|_2$. As a consequence, the right-hand side of (4.14) has to be replaced by $c_d\|\tau(f)\|^2$. Since $\|\tau(f)\| \leq \|f\|_2^2$, the resulted inequality is then equivalent to

$$(1 - \|\tau(f)\|^2)\|\nabla_0 f\|_2^2 \geq c_d\|\tau(f)\|^2, \tag{1}$$

which was already known in the literature; see the discussion in [1] and references therein.

Since (4.14) no longer holds, an immediate question is whether the uncertainty principle in (4.11) and that in (1) are equivalent, assuming $\int_{\mathbb{S}^{d-1}} f(y)\,d\sigma(y) = 0$. That they are not equivalent is shown in the following proposition.

Proposition 1 *For $n \geq 3$, let $Y \in \mathcal{H}_n^d$, a spherical harmonic of degree n on $\mathbb{S}^{d-1}$, and let Q be a polynomial of degree at most $n-2$ such that $\int_{\mathbb{S}^{d-1}} Q(x)d\sigma = 0$. Assume that both $[Y(x)]^2$ and $[Q(x)]^2$ are even in every coordinate. Let*

$$f = b(Y + Q), \quad \text{where} \quad b^{-1} := \|Y + Q\|_2 > 0.$$

Then $\tau(f) = 0$. In particular, (1) becomes the trivial inequality $\|\nabla_0 f\|_2^2 \geq 0$, whereas (4.11) shows that $\|\nabla_0 f\|_2^2 \geq c > 0$.

Proof Since the degree of Q is at most $n-2$, it follows from the orthogonality of Y and the even parity of Y^2 and Q^2 that

$$\int_{\mathbb{S}^{d-1}} x_i|f(x)|^2 d\sigma = \int_{\mathbb{S}^{d-1}} x_i \left(Y(x)^2 + 2Y(x)Q(x) + Q(x)^2\right) d\sigma(x) = 0$$

for $1 \leq k \leq d$. Hence, $\tau(f) = 0$. By its definition, $\|f\|_2 = 1$, and, by the orthogonality of Y and the zero mean of Q, we see that $\int_{\mathbb{S}^{d-1}} f(x)d\sigma = 0$ so that (4.11) is applicable on f. □

As a simple example of the function f, we can choose $Q(x) = x_1^k$ and $Y(x) = C_n^\lambda(x_1)$ for $x = (x_1, \ldots, x_d) \in \mathbb{S}^{d-1}$, where $\lambda = (d-2)/2$ and $1 \leq k \leq n-2$.

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Reference

1. Dai, F., Xu, Y.: The Hardy–Rellich inequality and uncertainty principle on the sphere. Constr. Approx. **40**, 141–171 (2014)