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CONSTRUCTIVE APPROXIMATION

Representation of Polynomials by Linear Combinations of Radial Basis Functions

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Abstract Let $\mathcal{P}_n^d$ denote the space of polynomials on $\mathbb{R}^d$ of total degree n. In this work, we introduce the space of polynomials $\mathcal{Q}_{2n}^d$ such that $\mathcal{P}_n^d \subset \mathcal{Q}_{2n}^d \subset \mathcal{P}_{2n}^d$ and which satisfy the following statement: Let h be any fixed univariate even polynomial of degree n and $\mathcal{A}$ be a finite set in $\mathbb{R}^d$. Then every polynomial P from the space $\mathcal{Q}_{2n}^d$ may be represented by a linear combination of radial basis functions of the form $h(\|x+a\|)$, $a \in \mathcal{A}$, if and only if the set $\mathcal{A}$ is a uniqueness set for the space $\mathcal{Q}_{2n}^d$.

Keywords Radial basis functions · Polynomials · Linear representation

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1 Introduction

Let $\mathbb{R}^d$ be the real d-dimensional Euclidean space with norm $\|x\| = (\sum_{i=1}^d x_i^2)^{1/2}$. Denote by $\mathbb{B}^d(a,r) = \{x : \|x-a\| \le r\}$ the Euclidean ball with center a and radius r. Let $\mathbb{B}^d = \mathbb{B}^d(0,1)$ and $\mathbb{S}^{d-1} = \{\|x\| = 1\}$ be the unit ball and sphere in the space $\mathbb{R}^d$, respectively.

Let $h(t)$ be a function defined on $\mathbb{R}_+$. Consider the radial function $h(\|x\|)$ on $\mathbb{R}^d$. Given a point a in $\mathbb{R}^d$, we introduce the shifted radial function $h_a = h(\|x+a\|)$ with center $-a$. Let $\mathcal{A}$ be a subset in $\mathbb{R}^d$. Denote by $R(h,\mathcal{A})$ the class of functions of the form $h(\|x+a\|)$, where a runs over the set $\mathcal{A}$. Consider the class of functions

$$\mathcal{R}(h,\mathcal{A}) = \operatorname{span} R(h,\mathcal{A})$$

formed by all possible finite linear combinations of functions from the set $R(h,\mathcal{A})$.

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The problem of the representation of functions by linear combination of shifts of radial basis functions is of current interest in different applications of nonlinear approximation, including approximation by wave functions, learning theory, and tomography. Results about density of the spaces formed by linear combinations of shifts of fixed functions were obtained by Wiener (see Edwards [4]), Pinkus [11, 12], Schwartz [15], Agranovsky and Quinto [1], and many others. A series of important results on approximation by radial basis functions are obtained in [2, 8, 9, 13, 14].

Since a large class of functions may be approximated by polynomials, then the problem of the representation of polynomials by shifts of radial basis polynomials is closely connected with the above problems.

Let $s=(s_1,\dots,s_d)\in\mathbb{Z}^d$ be any vector with nonnegative integer coordinates. We set $x^s=x_1^{s_1}\cdots x_d^{s_d}$ and $|s|=s_1+\cdots+s_d$. Consider the space $\mathcal{P}^d=\operatorname{span}\{x^s: s\in\mathbb{Z}_+^d\}$ of all polynomials on $\mathbb{R}^d$, i.e., linear combinations of a finite number of monomials x^s. If $p(x)=\sum_s c_s x^s$ is a polynomial from $\mathcal{P}^d$, then the total degree of p is defined by $\deg p=\max\{|s|: c_s\neq 0\}$. Denote by $\mathcal{P}_n^d$ the space of all polynomials from $\mathcal{P}^d$ of degree at most n. The total dimension of $\mathcal{P}_n^d$ equals $D_n=\binom{n+d}{d}$.

2 Orthogonal System P of Polynomials on the Ball

In this section, we introduce the polynomial orthogonal basis $\mathbf{P}=\{P_I\}$ on the ball B^d which we will use in proofs of the main results (for a more detailed exposition of this basis, see [7]). For the construction of the basis $\mathbf{P}$, we use work by [3, 5, 6, 10]. A series of applications of the basis $\mathbf{P}$ for approximation by ridge and radial functions is contained in [6, 7].

First, we discuss some well-known results connected with orthogonal polynomials, which we use in this present work.

1. The Gegenbauer polynomials: The Gegenbauer polynomials (see [16–18, 20]) are usually defined via the generating function

$$\left(1-2tz+z^2\right)^{-\lambda}=\sum_{k=0}^{\infty}C_k^\lambda(t)z^k,$$

where $|z|<1$, $|t|<1$, and $\lambda>0$. The coefficients $C_k^\lambda(t)$ are algebraic polynomials of degree k and are termed the Gegenbauer polynomials associated with λ. The Gegenbauer polynomials possess the following properties:

(a) The family of polynomials $\{C_n^\lambda\}_0^\infty$ is a complete orthogonal system for the weighted space $L_2(\mathbb{Q},\sigma)$, $\mathbb{Q}=[-1,1]$, $\sigma_\lambda(t):=(1-t^2)^{\lambda-1/2}$, and

$$\int_{\mathbb{Q}}C_m^\lambda(t)C_n^\lambda(t)\sigma_\lambda(t)\,dt=\begin{cases}0, & m\neq n,\\ v_{n,\lambda}, & m=n,\end{cases}\tag{1}$$

where we use the usual notation $v_{n,\lambda}:=\frac{\pi^{1/2}(2\lambda)_n\Gamma(\lambda+1/2)}{(n+\lambda)n!\Gamma(\lambda)}$ and $(a)_0:=1$, $(a)_n:=a(a+1)\cdots(a+n-1)$.

Let d be a natural number. We set $U_n(t) = v_{n,d/2}^{-1/2} C_n^{d/2}(t)$. Then the family of polynomials $\{U_n\}_0^\infty$ is the complete orthonormal system in the weighted space $L_2(\mathbb{Q}, \sigma)$, where $\sigma(t) = \sigma_{d/2}(t) = (1 - t^2)^{(d-1)/2}$.

(b) Let $\mathcal{P}_n^d$ be the set of all algebraic polynomials of total degree n in d real variables. Let ξ be any point on the sphere $\mathbb{S}^{d-1}$. Then the polynomial $U_n(\xi \cdot x)$ is in $\mathcal{P}_n^d$ and is orthogonal to $\mathcal{P}_{n-1}^d$ in $L_2(B^d)$ (see [3, 10]), i.e.,

$$\int_{B^d} U_n(\xi \cdot x) p(x)\, dx = 0, \quad \forall p \in \mathcal{P}_{n-1}^d. \tag{2}$$

(c) For each $\xi, \eta \in S^{d-1}$ and $n \in \mathbb{Z}_+$, we have (see [10])

$$\int_{B^d} U_n(\xi \cdot x) U_n(\eta \cdot x)\, dx = \frac{U_n(\xi \cdot \eta)}{U_n(1)}. \tag{3}$$

(d) For each polynomial $p(x) \in \mathcal{P}_n$ such that $p(x) = (-1)^n p(-x)$ for all $x \in \mathbf{R}^d$, we have ([10, 16])

$$\int_{S^{d-1}} p(\xi) U_n(\xi \cdot \eta)\, d\xi = c_n p(\eta), \quad \text{where } c_n = \frac{2(2\pi)^{d-1} U_n(1)}{(n+1)_{d-1}}. \tag{4}$$

2. An Orthogonal System of Polynomials on the Sphere: Let j be a nonnegative integer number. Let $\mathcal{H}_j^{\text{hom}}$ be the subspace in $\mathcal{P}_j^d$ formed by all harmonic homogeneous polynomials (i.e., the spherical harmonics) of degree j. We know [16, 20] that the dimension l_j of $\mathcal{H}_j^{\text{hom}}$ equals $l_j = \binom{d+j-1}{j} - \binom{d+j-3}{j-2}$, $j \ge 2$, and $l_0 = 1$, $\dim l_1 = d$. It is easy to verify that the dimension $l_j = (1 + \frac{2}{(d-2)!} + c(s,d)) s(s+1) \cdots (s+d-3)$, where $0 \le c(s,d) \le 1$ is some constant depending only on s and d. Let $\boldsymbol{\Pi}_j = \{Y_{j,1}, \ldots, Y_{j,l_j}\}$ be an orthonormal basis in the space $\mathcal{H}_j^{\text{hom}}(\mathbb{S}^{d-1})$ of functions formed by restrictions of functions from $\mathcal{H}_j^{\text{hom}}$ to the sphere $\mathbb{S}^{d-1}$. Then the set $\bigcup_{j=0}^\infty \boldsymbol{\Pi}_j$ is an orthonormal basis in the space $L_2(S^{d-1})$, i.e., for any functions $Y_{j,k}$ and $Y_{j',k'}$ from $\bigcup_{j=0}^\infty \boldsymbol{\Pi}_j$, the following holds:

$$(Y_{j,k}, Y_{j',k'}) = \int_{S^{d-1}} Y_{j,k}(\xi) \overline{Y_{j',k'}(\xi)}\, d\xi = \delta_{j,j'} \delta_{k,k'},$$

where by $d\xi$ we denote the normalized Lebesgue measure on $\mathbb{S}^{d-1}$.

3. An Orthogonal System of Polynomials on the Ball: Consider the Hilbert space $L_2(\mathbb{S}^{d-1})$ of complex-valued square-integrable functions h on the sphere $\mathbb{S}^{d-1}$ with the inner product

$$(s_1, s_2) = \int_{\mathbb{S}^{d-1}} s_1(\xi) \overline{s_2(\xi)}\, d\xi, \quad s_1, s_2 \in L_2\big(\mathbb{S}^{d-1}\big).$$

Also consider the Hilbert space $L_2(\mathbb{Q}, \sigma)$ of real functions on the segment $\mathbb{Q} = [-1, 1]$ with the norm

$$\|g\|_{L_2(\mathbb{Q},\sigma)} = \left(\int_{\mathbb{Q}} \big|g(t)\big|^2 \sigma(t)\, dt \right)^{1/2}, \quad \text{where } \sigma(t) = \big(1 - t^2\big)^{(d-1)/2}.$$

Consider the system of normed Gegenbauer polynomials $\{U_i(t)\}$, $i = 0, 1, \ldots,$ on the segment $\mathbb{Q}$, forming a complete orthonormal system in the space $L_2(\mathbb{Q}, \sigma)$.

Introduce the set of triple indices

$$\mathbf{I} = \big\{ I := (i, j, k) : \ i \in \mathbb{Z}_+, \ j \in \{0, \ldots, i\}, \ j = i \,(\mathrm{mod}\, 2), \ k = 1, \ldots, l_j \big\}, \tag{5}$$

where l_j is the dimension of space $\mathcal{H}_j^{\mathrm{hom}}$ of harmonic homogeneous polynomials of degree j. For every index $I = (i, j, k)$ from $\mathbf{I}$, we construct the function on $\mathbb{R}^d$,

$$P_I(x) := P_{i,j,k}(x) := \nu_{ij} \int_{\mathbb{S}^{d-1}} U_i(x \cdot \xi)\, Y_{j,k}(\xi)\, d\xi, \tag{6}$$

where the coefficient ν_{ij} is the normalizing factor such that $\|P_I\|_{L_2} = 1$ and $x \cdot \xi = x_1\xi_1 + \cdots + x_d\xi_d$ is the inner product of the vectors x and ξ. From (6), we see that the function P_I is the polynomial on $\mathbb{R}^d$ of total degree i. Consider the system of polynomials

$$\mathbf{P} = \{P_I\}_{I \in \mathbf{I}}. \tag{7}$$

The system $\mathbf{P}$ is the complete orthonormal system of polynomials in the space $L_2(\mathbb{B}^d)$ (see [7]). In particular, for any natural n, the finite subsystem of polynomials

$$\mathbf{P}_n = \big\{ P_I : \ I = (i, j, k) \in \mathbf{I}, \ i \le n \big\} \tag{8}$$

in $\mathbf{P}$ forms ([7]) an orthogonal basis in the space $\mathcal{P}_n^d$ of all polynomials on $\mathbb{R}^d$ of total degree n.

3 Main Results

In this paper, we introduce another subspace of polynomials, which is defined by the following. Consider the subset of indices $\mathbf{I}_n = \{I = (i, j, k) \in \mathbf{I} : \ i + j \le 2n\}$ in the set $\mathbf{I}$ and the corresponding collection of polynomials from the system $\mathbf{P}$,

$$\mathbf{Q}_{2n} = \{P_I : \ I \in \mathbf{I}_n\}.$$

Definition Let $\mathcal{Q}_{2n}^d = \operatorname{span} \mathbf{Q}_{2n}$ be the linear space of polynomials formed by the linear span of polynomials from $\mathbf{Q}_{2n}$.

It is obvious that for every natural n, we have

$$\mathcal{P}_n^d \subset \mathcal{Q}_{2n}^d \subset \mathcal{P}_{2n}^d. \tag{9}$$

Denote by S_n the dimension of the space $\mathcal{Q}_{2n}^d$. From (9), we have $D_n \le S_n \le D_{2n}$. Since the collection of polynomials $\{P_I\}_{I \in \mathbf{I}_n}$ is the complete orthonormal system in the space $\mathcal{Q}_{2n}^d$, then S_n coincides with the cardinality of the set $\mathbf{I}_n$.

Let h be any fixed univariate even polynomial of degree $2n$. We show that every polynomial p_n from the space $\mathcal{Q}_{2n}^d$ may be represented by the linear combination of some S_n shifts $h(\|x + a_i\|)$, $i = 1, \ldots, S_n$, of the radial polynomial $h(\|x\|)$. Moreover, we show that the representation is valid for any collection of points $a_1, \dots, a_{S_n}$ from $\mathbb{R}^d$ forming a uniqueness set for the space $\mathcal{Q}_{2n}^d$.

Let n be any natural number. Let $h(t) = b_{2n}t^{2n} + \dots + b_0$, $b_{2n} \neq 0$, be any univariate even polynomial of degree $2n$; that is, all coefficients b_m with m odd are equal to zero.

A set of points $\mathcal{A}_n = \{a_1, \dots, a_{S_n}\}$ is said to be a *h-basis set* for the space $\mathcal{Q}_{2n}^d$ if $\mathcal{R}(h, \mathcal{A}_n) = \mathcal{Q}_{2n}^d$; that is, every polynomial p from $\mathcal{Q}_{2n}^d$ can be represented as

$$p(x) = c_1 h\big(\|x + a_1\|\big) + \dots + c_{S_n} h\big(\|x + a_{S_n}\|\big), \quad c_i \in \mathbb{R}.$$

A set of points $\mathcal{A}_n$ is called *a uniqueness set* for the space $\mathcal{Q}_{2n}^d$ if, for any two polynomials p_1 and p_2 from $\mathcal{Q}_{2n}^d$, the relations $p_1(a_i) = p_2(a_i)$, $i = 1, \dots, S_n$, implies the equality $p_1(a) = p_2(a)$ for all $a \in \mathbb{R}^d$.

Theorem 3.1 *Let n be any natural number and h be any univariate even polynomial of degree $2n$. Then set $\mathcal{A}_n$ is an h-basis set for the space $\mathcal{Q}_{2n}^d$ if and only if $\mathcal{A}_n$ is the uniqueness set for $\mathcal{Q}_{2n}^d$.*

Consider the Hilbert space $L_2 = L_2(\mathbb{B}^d)$ of complex functions on the ball B^d with inner product

$$\langle f_1, f_2 \rangle = \int_{B^d} f(x)\overline{g(x)}\, dx, \quad f_1, f_2 \in L_2.$$

Let f and G be a function and a subspace in L_2, respectively. Denote by $E(f, G) = \inf_{g \in G} \|f - g\|_{L_2}$ the distance of the function f from G. Let F be a function class in L_2. By $E(F, G) = \sup_{f \in F} \inf_{g \in G} \|f - g\|_{L_2}$, we denote the deviation of the class F from G. The following follows directly from Theorem 3.1 and the embedding (9).

Corollary 3.2 *Let n be any even number, $\mathcal{A}_n$ be a uniqueness set for $\mathcal{Q}_{2n}^d$, and h be any univariate even polynomial of degree n. Then for any function f from the space L_2, the following inequality holds*:

$$E\big(f, \mathcal{R}(h, \mathcal{A}_n)\big) = E\big(f, \mathcal{Q}_{2n}^d\big).$$

Let $s = (s_1, \dots, s_d) \in \mathbb{Z}_+^d$ and $|s| = s_1 + \dots + s_d$. Introduce the differential operator $D^s = \partial^{|s|}/\partial^{s_1} x_1 \cdots \partial^{s_d} x_d$. Let r be any natural number. In the space L_2, we consider the Sobolev class of functions

$$W_2^r\big(\mathbb{B}^d\big) := \Big\{ f : \|f\|_{W_2^r} = \|f\|_{L_2} + \max_{|s|=r} \big\|D^s f\big\|_{L_2} \le 1 \Big\}.$$

Corollary 3.3 *Let n be any even number, $\mathcal{A}_n$ be an uniqueness set for $\mathcal{Q}_{2n}^d$, and h be any univariate even polynomial of degree n. Then the inequalities*

$$c_1 n^{-r} \le E\big(W_2^r\big(\mathbb{B}^d\big), \mathcal{R}(h, \mathcal{A}_n)\big) \le c_2 n^{-r} \tag{10}$$

hold, where c_1 and c_2 depend only on r and d.

The upper bound for $E(W_2^r(\mathbb{B}^d), \mathcal{R}(h, \mathcal{A}_n))$ directly follows from Corollary 3.2 and Jackson's theorem. The lower bound follows from the estimate of the Kolmogorov n-widths of the class $W_2^r(\mathbb{B}^d)$ (see [19]).

4 Moments of Radial Functions

In this section, we establish some auxiliary results which we will use in the proof of the main theorem. Let f be a function from the space $L_2(\mathbb{B}^d)$. Denote by $M_I(f) = \langle f, P_I \rangle$ the I-moment of the function f. Let a be any point in $\mathbb{R}^d$ and s be a natural number. Introduce the functions $r(x) = \|x\|$ and $r_a(x) = \|x+a\|$. In the following lemma, we calculate the moments $M_I(r_a^{2s})$ of the shifted radial function $r_a^{2s} = \|x+a\|^{2s}$.

Given triple index $I = (i, j, k)$ from $\mathbf{I}$, we consider the normed Gegenbauer polynomial $U_i(t)$ on the segment $\mathbb{Q} = [-1, 1]$ of degree i and the spherical harmonic $Y_{j,k}(\theta)$ on the sphere $\mathbb{S}^{d-1}$ of degree j.

Lemma 4.1 *Let $I = (i, j, k)$ be any triple index from $\mathbf{I}$ and a be any point from $\mathbb{R}^d$. We write*

$$u_{\beta,i} := \left(t^\beta, U_i\right)_{\mathbb{Q}} := \int_{\mathbb{Q}} t^\beta U_i(t)\sigma(t)\,dt \tag{11}$$

and consider the function on $\mathbb{R}^d$

$$V_{i,j,k}(a) := \left((a\cdot\theta)^i, Y_{j,k}\right) := \int_{\mathbb{S}^{d-1}} (a\cdot\theta)^i\,\overline{Y_{j,k}}(\theta)\,d\theta. \tag{12}$$

Then, for every natural s, the I-moment of the function r_a^{2s} equals

$$M_I\left(r_a^{2s}\right) = \nu \sum_{\alpha=0}^{2s} \binom{2s}{\alpha} u_{2s-\alpha,i} V_{\alpha,j,k}(a),$$

where ν depends only on I, d, and s.

Proof Given s, we will use the equality

$$\|x\|^{2s} = c_s \int_{\mathbb{S}^{d-1}} (x\cdot\theta)^{2s}\,d\theta, \quad \text{where } c_s = \left(\int_{\mathbb{S}^{d-1}} (e\cdot\theta)^{2s}\,d\theta\right)^{-1}, \tag{13}$$

and where by e we denote the point $e = (0, \dots, 0, 1)$. Equality (13) directly follows from the homogeneity of the polynomials $\|x\|^{2s}$, $(x\cdot\theta)^{2s}$, and the invariance of the measure $d\theta$ with respect to the rotation operator in $\mathbb{R}^d$.

For a fixed point θ on $\mathbb{S}^{d-1}$, we consider the univariate polynomial $w_\theta(t) = (t + a\cdot\theta)^{2s}$ of degree $2s$. From (13), we have

$$\|x+a\|^{2s} = c_s \int_{\mathbb{S}^{d-1}} (x\cdot\theta + a\cdot\theta)^{2s}\,d\theta = c_s \int_{\mathbb{S}^{d-1}} w_\theta(x\cdot\theta)\,d\theta. \tag{14}$$

Using (14) and (6), we can represent the I-moment of the function r_a^{2s} by the following:

$$\begin{aligned}\langle r_a^{2s}, P_I\rangle &= \int_{\mathbb{B}^d} \|x+a\|^{2s} P_I(x)\,dx \\ &= c_{sij}\int_{\mathbb{B}^d} P_I(x)\,dx \int_{\mathbb{S}^{d-1}} w_\theta(x\cdot\theta)\,d\theta \\ &= c_{sij}\int_{\mathbb{B}^d} dx \int_{\mathbb{S}^{d-1}} U_i(x\cdot\xi)Y_{jk}(\xi)\,d\xi \int_{\mathbb{S}^{d-1}} w_\theta(x\cdot\theta)\,d\theta,\end{aligned}$$

where $c_{sij}=c_s\nu_{ij}$. Therefore, we have

$$\langle r_a^{2s}, P_I\rangle = c_{sij}\int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} Y_{jk}(\xi)\,d\xi\,d\theta \int_{\mathbb{B}^d} w_\theta(x\cdot\theta)\,U_i(x\cdot\xi)\,dx. \tag{15}$$

Given θ, we decompose the function w_θ by the orthogonal system of Gegenbauer polynomials $\{U_\alpha\}$:

$$w_\theta(t) = \sum_{\alpha=0}^{2s}(w_\theta, U_\alpha)_{\mathbb{Q}}\,U_\alpha(t).$$

Then the last integral in (15) equals

$$\int_{\mathbb{B}^d} w_\theta(x\cdot\theta)\,U_i(x\cdot\xi)\,dx = \sum_{\alpha=0}^{2s}(w_\theta, U_\alpha)_{\mathbb{Q}}\int_{\mathbb{B}^d} U_\alpha(x\cdot\theta)\,U_i(x\cdot\xi)\,dx. \tag{16}$$

The following identity holds (see (3)):

$$\int_{\mathbb{B}^d} U_\alpha(x\cdot\theta)\,U_i(x\cdot\xi)\,dx = \frac{U_i(\theta\cdot\xi)}{U_i(1)}\delta_{\alpha,i}.$$

Therefore, from (16), we obtain

$$\int_{\mathbb{B}^d} w_\theta(x\cdot\theta)\,U_i(x\cdot\xi)\,dx = (w_\theta, U_i)_{\mathbb{Q}}\frac{U_i(\theta\cdot\xi)}{U_i(1)}.$$

Substitute this expression into (15):

$$\begin{aligned}\langle r_a^{2s}, P_I\rangle &= \frac{c_{sij}}{U_i(1)}\int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} Y_{j,k}(\theta)(w_\theta, U_i)_{\mathbb{Q}}\,U_i(\theta\cdot\xi)\,d\xi\,d\theta \\ &= \frac{c_{sij}}{U_i(1)}\int_{\mathbb{S}^{d-1}}(w_\theta, U_i)_{\mathbb{Q}}\,d\theta\int_{\mathbb{S}^{d-1}} Y_{j,k}(\xi)U_i(\theta\cdot\xi)\,d\xi.\end{aligned}$$

By (4), we have

$$\int_{\mathbb{S}^{d-1}} Y_{j,k}(\xi)U_i(\theta\cdot\xi)\,d\xi = c_i' Y_{j,k}(\theta).$$

Hence, we obtain

$$\langle r_a^{2s}, P_I \rangle = \nu \int_{\mathbb{S}^{d-1}} (w_\theta, U_i)_{\mathbb{Q}} Y_{jk}(\theta)\, d\theta, \tag{17}$$

where $\nu = c_{sij} c_i' / U_i(1)$. The equality

$$w_\theta(t) = (t + a \cdot \theta)^{2s} = \sum_{\alpha=0}^{2s} \binom{2s}{\alpha} t^{2s-\alpha} (a \cdot \theta)^\alpha$$

implies

$$\begin{aligned} \int_{\mathbb{S}^{d-1}} (w_\theta, U_i)_{\mathbb{Q}} Y_{j,k}(\theta)\, d\theta &= \sum_{\alpha=0}^{2s} \binom{2s}{\alpha} \int_{\mathbb{S}^{d-1}} \big((\bullet)^{2s-\alpha} (a \cdot \theta)^\alpha, U_i(\bullet)\big)_{\mathbb{Q}} Y_{j,k}(\theta)\, d\theta \\ &= \sum_{\alpha=0}^{2s} \binom{2s}{\alpha} \big((\bullet)^{2s-\alpha}, U_i(\bullet)\big)_{\mathbb{Q}} \big((a \cdot \theta)^\alpha, Y_{j,k}\big). \end{aligned}$$

Thus, we obtain from (17),

$$\langle r_a^{2s}, P_I \rangle = \nu \sum_{\alpha=0}^{2s} \binom{2s}{\alpha} \big((\bullet)^{2s-\alpha}, U_i\big)_{\mathbb{Q}} \big((a \cdot \theta)^\alpha, Y_{j,k}\big).$$

Lemma 4.1 is complete. □

5 The Proof of Theorem 3.1

Let n be any natural number and $h(t) = \sum_{s=0}^{n} c_s t^{2s}$, $t \in \mathbb{R}$, be an univariate even polynomial of degree $2n$. Given a point a from $\mathbb{R}^d$, consider the polynomials $g(x) = h(\|x\|)$ and $g_a(x) = g(x + a)$. Let I be any index from the set $\mathbf{I}_n$. Calculate the moments $M_I(g_a)$ of the function g_a. Lemma 4.1 implies

$$M_I(g_a) = \sum_{s=0}^{n} c_s M_I\big(r_a^{2s}\big) = \nu \sum_{s=0}^{n} \sum_{\alpha=0}^{2s} c_s \binom{2s}{\alpha} u_{2s-\alpha,i} V_{\alpha,j,k}(a).$$

Set $c_{s,\alpha} = \nu c_s \binom{2s}{\alpha}$. By properties of orthogonality of the polynomials U_i, we have from (11),

$$u_{2s-\alpha,i} = \int_{\mathbb{Q}} t^{2s-\alpha} U_i(t) \sigma(t)\, dt = 0, \quad 2s - \alpha < i.$$

Analogously, by properties of orthogonality of spherical harmonics Y_{jk}, we have from (12),

$$V_{\alpha,j,k}(a) = \int_{\mathbb{S}^{d-1}} (a \cdot \theta)^\alpha \overline{Y_{j,k}}(\theta)\, d\theta = 0, \quad \alpha < j.$$

Also, we see that $V_{\alpha,j,k}(a)=0$ if $\alpha\neq j \pmod 2$. Thus,

$$M_I(g_a)=\nu\sum_{s=\frac{i+j}{2}}^{n}\sum_{\alpha=j}^{2s-i}c_{s,\alpha}u_{2s-\alpha,i}V_{\alpha,j,k}(a).$$

Note (see (5)) that $i=j \pmod 2$. Therefore, $i+j$ is always an even number. Interchanging the order summation, we obtain

$$M_I(g_a)=\sum_{\alpha=j}^{2n-i}\Big(\sum_s c_{s,\alpha}u_{2s-\alpha,i}\Big)V_{\alpha,j,k}(a), \tag{18}$$

where the index s runs over the set $\{\frac{\alpha+i}{2},\frac{\alpha+i}{2}+1,\dots,n\}$. Since $V_{\alpha,j,k}(a)=0$ if $\alpha\neq j \pmod 2$, then factually, α runs only over the set $\{j,\dots,2n-i\}$. Therefore, the equality $\alpha=j=i \pmod 2$ is always true; that is, $\alpha+i$ is always an even number.

Consider two collections of polynomials of the variable a:

$$\mathbf{M}_n=\big\{M_{i,j,k}(g_a):\ (i,j,k)\in\mathbf{I}_n\big\}\quad\text{and}\quad\mathbf{V}_n=\big\{V_{i,j,k}(a):\ (i,j,k)\in\mathbf{I}_n\big\}.$$

Lemma 5.1 *The linear spans*

$$\widehat{\mathbf{M}}_n=\operatorname{span}\mathbf{M}_n\quad\text{and}\quad\widehat{\mathbf{V}}_n=\operatorname{span}\mathbf{V}_n$$

of sets $\mathbf{M}_n$ and $\mathbf{V}_n$ coincide.

Proof Let $I=(i,j,k)\in\mathbf{I}_n$ be any index. From (18), we see that

$$M_I(g_a)\in\operatorname{span}\big\{V_{\alpha,j,k}(a):\ \alpha\in\{j,\dots,2n-i\}\big\}\in\operatorname{span}\big\{V_{\alpha,j,k}(a):\ (\alpha,j,k)\in\mathbf{I}_n\big\};$$

that is, $\widehat{\mathbf{M}}_n\subseteq\widehat{\mathbf{V}}_n$. We show that $\widehat{\mathbf{V}}_n\subseteq\widehat{\mathbf{M}}_n$. We fix j and k. The equality (18) we rewrite as

$$M_{i,j,k}(g_a)=\sum_{\alpha=j}^{2n-i}C_i^\alpha V_{\alpha,j,k}(a),\quad\text{where } C_i^\alpha=\sum_s c_{s,\alpha}u_{2s-\alpha,i}. \tag{19}$$

We show that the functions $V_{\alpha,j,k}$ for every $\alpha=j,\dots,2n$ belong to $\widehat{\mathbf{M}}_n$. We use induction on α. Let $\alpha=j$. We have from (19) with $i=2n-j$,

$$M_{n-j,j,k}(g_a)=C_{2n-j}^{j}V_{j,j,k}(a). \tag{20}$$

Hence, $V_{j,j,k}\in\widehat{\mathbf{M}}_n$. Now assume $V_{j,j,k},\dots,V_{j+\beta,j,k}\in\widehat{\mathbf{M}}_n$ with $\beta\geq 0$. Show that $V_{j+\beta+1,j,k}$ also belongs to $\widehat{\mathbf{M}}_n$. We have from (19) with $i=2n-j-\beta-1$,

$$\begin{aligned}M_{2n-j-\beta-1,j,k}(g_a)&=\sum_{\alpha=j}^{j+\beta+1}C_{2n-j-\beta-1}^{\alpha}V_{\alpha,j,k}(a)\\&=\sum_{\alpha=j}^{j+\beta}C_{2n-j-\beta-1}^{\alpha}V_{\alpha,j,k}(a)+C_{2n-j-\beta-1}^{j+\beta+1}V_{j+\beta+1,j,k}(a).\end{aligned}$$

Then, taking into consideration that the functions $V_{j,j,k},\dots,V_{j+\beta,j,k}$ and $M_{2n-j-\beta-1,j,k}(g_a)$ belong to $\widehat{\mathbf{M}}_n$, we obtain that $V_{j+\beta+1,j,k}$ also belongs to $\widehat{\mathbf{M}}_n$. Thus, the functions $V_{i,j,k}(a)$, for all $i\ge j$, belong to $\widehat{\mathbf{M}}_n$; that is, $\widehat{\mathbf{V}}_n\subseteq\widehat{\mathbf{M}}_n$. Lemma 5.1 is proved. □

Lemma 5.2 *The spaces $\widehat{\mathbf{V}}_n$ and $\mathcal{Q}^d_{2n}$ coincide.*

Proof The space $\widehat{\mathbf{V}}_n$ is the linear span of polynomials

$$V_{i,j,k}(a)=\bigl((a\cdot\theta)^i,Y_{jk}\bigr),\quad (i,j,k)\in\mathbf{I}_n. \tag{21}$$

Every polynomial $1,t,\dots,t^n$ is some linear combination of Gegenbauer polynomials $\{U_i(t)\}_{i=0}^n$. Therefore, $\widehat{\mathbf{V}}_n\subseteq\mathcal{Q}^d_{2n}$. We now prove that $\mathcal{Q}^d_{2n}\subseteq\widehat{\mathbf{V}}_n$. We show that the set $\mathbf{V}_n$ forms a linear basis in the space $\mathcal{Q}^d_{2n}$. Indeed, consider the system of functions $\mathbf{Q}_{2n}$ consisting of polynomials (6)

$$P_{i,j,k}(a)=\bigl(U_i(a\cdot\theta),Y_{jk}\bigr)=\sum_{\alpha=o}^{i}b_{i,\alpha}\bigl((a\cdot\theta)^\alpha,Y_{jk}\bigr),\quad (i,j,k)\in\mathbf{I}_n, \tag{22}$$

where $b_{i,\alpha}$ are the coefficients of the Gegenbauer polynomial U_i. The system $\mathbf{Q}_{2n}$ is a linear basis in the space $\mathcal{Q}^d_{2n}$. On the other hand, from (21) and (22), we see that $P_{i,j,k}\in\widehat{\mathbf{V}}_n$ for every index $(i,j,k)\in\mathbf{I}_n$. Thus, $\mathbf{V}_n$ is a linear basis in $\mathcal{Q}^d_{2n}$. □

Proof of Theorem 3.1 By Lemmas 5.1 and 5.2, the collection of functions of a-variable $\mathbf{M}_n=\{M_I(g_a)\}$, $I\in\mathbf{I}_n$, forms a basis in the space $\mathcal{Q}^d_{2n}$. Therefore, the following obvious statement holds. □

Proposition 5.3 *A set $\mathcal{A}_n=\{a_1,\dots,a_{S_n}\}$, where $S_n=\dim\mathcal{Q}^d_{2n}$, is a uniqueness set for the space of polynomials $\mathcal{Q}^d_{2n}$ if and only if the square matrix*

$$\bigl(M_I(g_{a_j})\bigr),\quad I\in\mathbf{I}_n,\ j=1,\dots,S_n,$$

is nondegenerate.

Proof Assume that a set $\mathcal{A}_n$ is a uniqueness set for the space $\mathcal{Q}^d_{2n}$. We show that any polynomial p from $\mathcal{Q}^d_{2n}$ may be represented by

$$p(x)=c_1h\bigl(\|x+a_1\|\bigr)+\cdots+c_{S_n}h\bigl(\|x+a_{S_n}\|\bigr). \tag{23}$$

Using moments $M_I(p)$ and $M_I(g_a)$ of functions p and $g_a=h(r_a)$, we can construct the system of linear equations

$$M_I(p)=c_1M_I(g_{a_1})+\cdots+c_{S_n}M_I(g_{a_{S_n}}),\quad I\in\mathbf{I}_n, \tag{24}$$

with respect to the unknowns coefficients $c_1,\dots,c_{S_n}$. By Proposition 5.3, the system (24) has a unique solution; that is, the polynomial p may be represented by (23).

Conversely, assume that every polynomial p from $\mathcal{Q}_{2n}^d$ may be represented by (23). Then the matrix $(M_I(g_{a_j}))$, $I \in \mathbf{I}_n$, $j = 1, \dots, S_n$, is nondegenerate. Hence, by Proposition 5.3, the set $\mathcal{A}_n$ is a uniqueness set for the space $\mathcal{Q}_{2n}^d$. □

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References

1. Agranovsky, M.L., Quinto, E.T.: Injectivity sets for the Radon transform over circles and complete systems of radial functions. J. Funct. Anal. **139**, 383–414 (1996)
2. Buhmann, M.D.: Approximation and interpolation with radial functions. In: Dyn, N., Leviatan, D., Levin, D., Pinkus, A. (eds.) Multivariate Approximation and Application, pp. 25–41. Cambridge Univ. Press, Cambridge (2001)
3. DeVore, R.A., Oskolkov, K., Petrushev, P.: Approximation by feed-forward neural networks. Ann. Numer. Math. **4**, 261–287 (1997)
4. Edwards, R.E.: Fourier Series. A Modern Introduction, vol. 1. Springer, Berlin (1979)
5. Logan, B., Shepp, L.: Optimal reconstruction of functions from its projections. Duke Math. J. **42**, 645–659 (1975)
6. Maiorov, V.: On best approximation by ridge functions. J. Approx. Theory **99**, 68–94 (1999)
7. Maiorov, V.: On best approximation of classes by radial functions. J. Approx. Theory **120**, 36–70 (2003)
8. Maiorov, V.: Almost optimal estimates for best approximation by translates on a torus. Constr. Approx. **21**, 337–349 (2005)
9. Maiorov, V.: Moments of shifted radial functions with respect to orthogonal system of polynomials on ball (submitted)
10. Petrushev, P.P.: Approximation by ridge functions and neural networks. SIAM J. Math. Anal. **30**, 155–189 (1999)
11. Pinkus, A.: Some density problems in multivariate approximation. In: Müller, M.W., Felten, M., Mache, D.H. (eds.) Approximation Theory: Proceedings of the International Dortmund Meeting IDOMAT 95, pp. 277–284. Akademie Verlag, Berlin (1995)
12. Pinkus, A.: TDI-subspaces of $C(\mathbb{R}^d)$ and some density problems from neural networks. J. Approx. Theory **85**, 269–287 (1996)
13. Powell, M.J.D.: The theory of radial basis function approximation in 1990. In: Light, W. (ed.) Advances in Numerical Analysis. Wavelets, Subdivisions, Algorithms, and Radial Basis Functions, vol. 2, pp. 105–210 (1992)
14. Schaback, R.: Approximation by radial basis functions with finitely many centers. Constr. Approx. **12**, 331–340 (1996)
15. Schwartz, L.: Sur certaines familles non fondamentales de fonctions continues. Bull. Soc. Math. Fr. **72**, 141–145 (1944)
16. Stein, E.M., Weiss, G.: Introduction to Fourier Analysis on Euclidean Spaces. Princeton Univ. Press, Princeton (1971)
17. Suetin, P.K.: Classical Orthogonal Polynomials. Nauka, Moscow (1979)
18. Szegö, G.: Orthogonal Polynomials. AMS, Providence (1939)
19. Temlyakov, V.N.: Approximation of Periodic Functions. Nova Science Publishers Inc., New York (1993)
20. Vilenkin, N.Ya.: Special Functions and a Theory of Representation of Groups. Fizmatgiz, Moscow (1965)