REGULAR ARTICLE

On nonparametric regression for bivariate circular long-memory time series

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Abstract

We consider nonparametric regression for bivariate circular time series with long-range dependence. Asymptotic results for circular Nadaraya–Watson estimators are derived. Due to long-range dependence, a range of asymptotically optimal bandwidths can be found where the asymptotic rate of convergence does not depend on the bandwidth. The result can be used for obtaining simple confidence bands for the regression function. The method is illustrated by an application to wind direction data.

Keywords Circular time series · Circular circular kernel regression · Long-range dependence · Gaussian subordination · Confidence interval

1 Introduction

Directional or circular time series arise in many scientific fields such as meteorology, oceanography, biology, neuroscience, bioinformatics, geoscience and cosmology. An overview of directional data analysis can be found for instance in books by Mardia (1972), Fisher (1993), Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001) and Ley and Verdebout (2017). Most literature deals with iid observations. More generally, for cases where more than one circular variable is observed, parametric and nonparametric circular-circular regression with iid residuals is discussed for instance in Gould (1969), Fisher and Lee (1992), Johnson and Wehrly (1978), Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001), Kato et al. (2008), Kim and SenGupta (2016) and Polsen and Taylor (2015). Nonparametric density estimation on the circle is discussed for instance by Watson (1983), Hall et al. (1987), Bai et al. (1988), Fisher (1989, 1993) and Taylor (2008). Also see Tsurata and Sagae (2017) for properties of higher order circular kernels. In practice, one often observes circular time series that

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exhibit serial dependence. Circular processes with weak serial dependence are considered for instance in Wehrly and Johnson (1980), Breckling (1989), Kato (2010), Modlin et al. (2012) and Wang and Gelfand (2014). Circular time series with long-range dependence are considered in Di Marzio et al. (2012) and Beran and Ghosh (2019). Di Marzio et al. (2012) discuss nonparametric trend estimation, Beran and Ghosh (2019) define models with long-range dependence using Gaussian subordination, and derive asymptotic results for parametric estimators where the mean direction depends on deterministic explanatory variables. In this paper, we extend the results in Beran and Ghosh (2019) to a nonparametric circular-circular regression model of the form

$$\vartheta_t = [\mu(\psi_t) + Z_t] \bmod 2\pi \tag{1}$$

where ϑ_t, $\psi_t \in [-\pi, \pi)$ ($t = 1, 2, \ldots$), ψ_t and Z_t are stationary long-memory processes defined by Gaussian subordination, and Z_t is such that

$$E(\sin Z_t) = 0, \ E(\cos Z_t) = R > 0 \tag{2}$$

(see e.g. Gould 1969). Equations (1) and (2) mean that, given $\psi_t = w^0$, the mean direction of ϑ_t is $\mu(w^0)$, i.e.

$$E\left[\exp(i\vartheta_t) \mid \psi_t = w^0\right] = R \exp\left(i\mu\left(w^0\right)\right) \ (w^0 \in [-\pi, \pi)). \tag{3}$$

We consider asymptotic properties of circular kernel estimators of $\mu(w^0)$. Limit theorems are derived using in particular general results in Mielniczuk and Wu (2004). Due to long-range dependence, a range of asymptotically optimal bandwidths can be found where the asymptotic rate of convergence does not depend on the bandwidth. The results can be used for obtaining simple simultaneous confidence bands for $\mu(w_1^0), \ldots, \mu(w_m^0)$ ($w_1^0, \ldots, w_m^0 \in [-\pi, \pi)$, $m \in \mathbb{N}$).

The paper is organized as follows. A general introduction and definitions are given in Sect. 2. Limit theorems are discussed in Sect. 3. In Sect. 4, asymptotic results from Sect. 3 are used to obtain confidence bands for $\mu(\psi)$. An application to wind direction data is discussed in Sect. 5. Proofs are given in the Appendix.

2 Definitions

2.1 Definition of the model

We extend the model introduced in Beran and Ghosh (2019) to bivariate circular time series with long-range dependence. First we recall the definition of long memory in the sense of second order dependence. A real-valued second order stationary time series with autocovariance function $\gamma(k)$ is said to exhibit long-range (or strong) dependence, if $\sum \gamma(k) = \infty$. Often it is assumed that

$$\gamma(k) \sim c_\gamma k^{2H-2} \ (k \to \infty), \tag{4}$$

or that the spectral density has a pole at the origin characterized by

$$f(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma(k) e^{-ik\lambda} \underset{\lambda \to 0}{\sim} c_f |\lambda|^{1-2H} \tag{5}$$

with $H \in (\frac{1}{2}, 1)$ and $0 < c_\gamma < \infty$. Here, "$\sim$" means that the ratio of the left and right hand side converges to 1. References to the extended literature on long-memory processes can be found for instance in Beran (1994), Giraitis et al. (2012) and Beran et al. (2013). For generating schemes for long memory processes also see e.g. Davidson and Sibbertsen (2005).

To obtain bivariate circular time series $(\psi_t, \vartheta_t) \in [0, 2\pi)^2$ $(t \in \mathbb{Z})$ with long-range dependence, the following assumptions will be used:

- (A1) Let Z_t, X_t $(t \in \mathbb{Z})$ be stationary Gaussian processes with $E(Z_t) = E(X_t) = 0$, $0 < \sigma_Z^2 = var(Z_t) < \infty$, $\sigma_X^2 = var(X_t) = 1$, and autocovariance functions γ_Z, γ_X and spectral densities f_Z, f_X such that

$$\gamma_Z(k) \underset{k \to \infty}{\sim} c_{\gamma,Z} k^{2H_Z - 2}, \tag{6}$$

$$f_Z(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_Z(k) e^{-ik\lambda} \underset{\lambda \to 0}{\sim} c_{f,Z} |\lambda|^{1-2H_Z}, \tag{7}$$

$$\gamma_X(k) \underset{k \to \infty}{\sim} c_{\gamma,X} k^{2H_X - 2}, \tag{8}$$

$$f_X(\lambda) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \gamma_X(k) e^{-ik\lambda} \underset{\lambda \to 0}{\sim} c_{f,X} |\lambda|^{1-2H_X} \tag{9}$$

  for some constants $0 < c_{\gamma,Z}, c_{f,Z}, c_{\gamma,X}, c_{f,X} < \infty$ and $\frac{1}{2} < H_Z, H_X < 1$. Moreover, assume that the two processes are independent from each other.
- (A2) Let $G_\psi(u) = \int_{-\pi}^{u} g_\psi(v) dv$ $(u \in [-\pi, \pi))$ be an absolutely continuous circular cumulative distribution function with density $g_\psi = G'_\psi$, and denote by Φ the standard normal distribution function. Define the circular time series ψ_t $(t \in \mathbb{Z})$ by

$$\psi_t = G_\psi^{-1}(\Phi(X_t)). \tag{10}$$

- (A3) Let $\mu : [-\pi, \pi) \to \mathbb{R}$ be a twice continuously differentiable function. The circular time series ϑ_t $(t \in \mathbb{Z})$ is defined by

$$\vartheta_t = [\mu(\psi_t) + Z_t] \bmod 2\pi. \tag{11}$$

Remark 1 Assumption (A1) implies

$$E(\sin Z_t) = 0, \quad E(\cos Z_t) = R_Z > 0, \tag{12}$$

$$E(\sin X_t) = 0, \quad E(\cos X_t) = R_X > 0. \tag{13}$$

Note also that Z_t and X_t can be written in the Wold representation with iid innovations $\varepsilon_t \sim N(0, \sigma_\varepsilon^2)$ and $\eta_t \sim N(0, \sigma_\eta^2)$ respectively, as

$$Z_t = \sum_{j=0}^{\infty} a_j \varepsilon_{t-j}, \ X_t = \sum_{j=0}^{\infty} c_j \eta_{t-j} \ (t \in \mathbb{Z}) \tag{14}$$

where $a_0 = c_0 = 1$ and

$$a_j \sim C_a j^{H_Z - 3/2}, c_j \sim C_c j^{H_X - 3/2} \ (j \to \infty) \tag{15}$$

for some suitable constants $0 < C_a, C_c < \infty$.

Remark 2 The circular time series $\psi_t = G_\psi^{-1}(\Phi(X_t))$ is said to be subordinated to the Gaussian process X_t (see e.g. Rosenblatt 1961, 1979; Taqqu 1975, 1979; Dobrushin and Major 1979; Dobrushin 1980). By definition, the marginal distribution of the series is the circular distribution G_ψ. Note that the circular density function $g_\psi = G'_\psi$ is arbitrary, i.e. using (10) any circular density function can be obtained as a marginal density via Gaussian subordination.

Assumption (8) means that X_t exhibits long-range dependence. For the circular time series ψ_t, a different definition of autocovariance and autocorrelation function has to be used. Let ν_ψ denote the mean direction of ψ_t, i.e. $E[\exp(i\psi_t)] = R \exp(i\nu_\psi)$. Jammalamadaka and Sarma (1988) proposed the circular autocorrelation function

$$\rho_{\text{circular}}(k) = \frac{E\left[\sin\left(\psi_t - \nu_\psi\right) \sin\left(\psi_{t+k} - \nu_\psi\right)\right]}{E\left[\sin^2\left(\psi_1 - \nu_\psi\right)\right]}. \tag{16}$$

To see whether long memory of X_t is inherited by ψ_t one needs to introduce the notion of Hermite rank. Denote by $\varphi(x) = (2\pi)^{-\frac{1}{2}} \exp(-\frac{1}{2}x^2)$ the standard normal density function, and by $L^2(\mathbb{R}; \varphi)$ the space of real valued functions h with $||h||^2 = \int |h(z)|^2 \varphi(z) dz < \infty$. Equipped with the scalar product $< h, \tilde{h} >= \int h(x) \tilde{h}(x) \varphi(x) dx$, $L^2(\mathbb{R}; \varphi)$ is a Hilbert space. Hermite polynomials

$$\mathcal{H}_q(z) = (-1)^q \exp\left(\frac{1}{2}z^2\right) \frac{d^q}{dz^q} \exp\left(-\frac{1}{2}z^2\right) \ (q = 0, 1, \dots)$$

build an orthogonal basis in $L^2(\mathbb{R}; \varphi)$. Thus, every function $h \in L^2(\mathbb{R}; \varphi)$ has an orthogonal L^2-representation

$$h(z) = \sum_{q=0}^{\infty} \frac{a_q}{q!} \mathcal{H}_q(z),$$

where the qth Hermite coefficient a_q is given by

$$a_q = \langle h, \mathcal{H}_q \rangle = \int_{-\infty}^{\infty} h(z) \mathcal{H}_q(z) \varphi(z) dz.$$

A function h with $a_0 = 0$ is said to have Hermite rank $m \geq 1$, if $a_m \neq 0$ and $a_j = 0$ ($j < m$) (Taqqu 1975). The following Lemma is derived in Beran and Ghosh (2019):

Lemma 1 *Let ψ_t ($t \in \mathbb{Z}$) be defined in* (10) *with γ_X characterized by* (8). *Denote by ν_ψ the mean direction of ψ_t, and define*

$$h(x) = \sin\left\{ G_\psi^{-1}(\Phi(x)) - \nu_\psi \right\}.$$

Suppose that the Hermite rank of h is m. Furthermore assume that

$$H_X > 1 - \frac{1}{2m}. \tag{17}$$

Then, setting

$$H_m := 1 + (H_X - 1)\, m, \tag{18}$$

we have

$$\rho_{circular}(k) \underset{k \to \infty}{\sim} c_m k^{2H_m - 2} \tag{19}$$

where

$$c_m = \frac{a_m^2 c_{\gamma, Z}^m}{m!\, E\left[\sin^2(\vartheta_1 - \mu)\right]}.$$

Remark 3 Note that, by definition of the mean direction ν_ψ,

$$E\left[h(X_t)\right] = E\left[\sin\left\{ G_\psi^{-1}(\Phi(X_t)) - \nu_\psi \right\}\right] = 0.$$

Lemma 1 means that, under assumption (17), the directional series ψ_t has long-range dependence in the sense that its circular autocorrelation $\rho_{\text{circular}}(k)$ has a hyperbolic decay and is not summable.

As for the second circular series ϑ_t, (11) implies

$$\vartheta_t = \left[\mu\left(G_\psi^{-1}(\Phi(X_t)) \right) + Z_t \right] \bmod 2\pi \tag{20}$$

$$= \left[\tilde{\mu}(X_t) + Z_t \right] \bmod 2\pi \tag{21}$$

where

$$\tilde{\mu}(x) = \mu\left(G_\psi^{-1}(\Phi(x)) \right).$$

Since the processes X_t and Z_t are stationary, ϑ_t is stationary with a constant mean direction ν_ϑ. By definition, the conditional mean direction of ϑ_t given $\psi_t = w^0$ is equal to $\mu(w^0)$ (see (3)). Moreover, (20) implies that long-range dependence in Z_t leads to long-range dependence in ϑ_t.

2.2 Kernel estimation of the conditional mean direction

Let $w^0 \in [0, 2\pi)$. The conditional mean direction $\mu(w^0)$ of ϑ_t can be estimated by a circular kernel estimator. In the context of iid observations, circular kernel estimators have been discussed for instance by Hall et al. (1987), Fisher (1989), Jammalamadaka and SenGupta (2001), Taylor (2008), Tsurata and Sagae (2017); Tsuruta and Sagae (2020), Bedouhene and Zougab (2020). Here, we will consider Nadaraya–Watson estimators of the form

$$\hat{\mu}_n(w^0) = \frac{\sum_{s=1}^{n} K_b \left(\psi_s - w^0 \right) \vartheta_s}{\sum_{s=1}^{n} K_b \left(\psi_s - w^0 \right)}. \tag{22}$$

For the kernel function K_b we adopt the definition introduced in Hall et al. (1987) and Tsurata and Sagae (2017) among others.

Definition 1 Let $L_\kappa(u) = L(\kappa(1 - \cos u))$ and $C_\kappa(L) = \int_{-\pi}^{\pi} L_\kappa(u) du$. Then, setting $\kappa = b^{-2}$ $(b > 0)$, we define circular kernels

$$K_b(u) = C_\kappa^{-1}(L) L_\kappa(u) \ (u \in [-\pi, \pi)). \tag{23}$$

Moreover, the lth moment of L is defined by

$$m_l(L) = \int_0^\infty L(r) r^{(l-1)/2} dr \ (l = 2k, k \in \{0, 1, 2, \ldots\}). \tag{24}$$

Remark 4 Note that by definition $\int_{-\pi}^{\pi} K_b(u) du = 1$. Thus, if $L \geq 0$, then K_b is a circular density function.

The following assumptions on L are used in Tsurata and Sagae (2017):

- (L1) $L'(r) = dL/dr$ is continuous.
- (L2)

$$\int_0^\infty L^2 \left(\frac{x^2}{2} \right) dx < \infty, \ \int_0^\infty L^2 \left(\frac{x^2}{2} \right) x^2 dx < \infty.$$

- (L3) For $j_0 = 2k_0$, we have $m_l(L) < \infty$ $(0 \leq l \leq j_0)$ and

$$m_l(L) = \int_0^y L(r) r^{(l-1)/2} dr + O \left(y^{-(j_0+1)/2} \right) \ (0 \leq l \leq j_0).$$

- (L4)

$$\lim_{r \to \infty} L(r) r^{(j_0+1)/2} = 0.$$

Note that, under assumptions (L1) to (L4), K_b has a Fourier series representation

$$K_b(u) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \alpha_s(b) \exp(-isu) \ (u \in [-\pi, \pi)) \tag{25}$$

with

$$\alpha_{-s}(b) = \alpha_s(b) = \int_{-\pi}^{\pi} K_b(u) \exp(isu) \, du.$$

The order of a circular kernel is defined as follows (Tsurata and Sagae 2017):

Definition 2 Let $p = 2k$ for some $k \in \{1, 2, \ldots\}$. Then $K_b(u) = C_\kappa^{-1}(L)L_\kappa(u)$ with $\kappa = b^{-2}$ is called a circular kernel of order p, if (L1) to (L4) hold with $j_0 \geq p + 2$, and

$$m_0(L) \neq 0, m_l(L) = 0 \ (l = 2, 4, \ldots, p - 2), m_p(L) \neq 0.$$

Remark 5 Tsurata and Sagae (2017) discuss methods for constructing higher order circular kernels from lower order kernels. For instance, let

$$L_2(r) = \exp(-r), L_2^{(1)}(r) = \frac{d}{dr} L_2(r),$$
$$r_\kappa(u) = \kappa(1 - \cos u),$$
$$L_{\kappa,2}(u) = L_2(r_\kappa(u)) = \exp(-\kappa(1 - \cos u)),$$

and denote by $C_k(L_{k,2})$ the corresponding normalizing constant. Then

$$m_l(L) = \int_0^{\infty} L_2(r) r^{(l-1)/2} dr = \Gamma\left(l + \frac{1}{2}\right) \ (l = 2k, k \in \{0, 1, 2, \ldots\})$$

(Tsurata and Sagae 2017). In particular, $m_0(L_2) = \Gamma(\frac{1}{2}) = \sqrt{\pi}$, and

$$m_2(L_2) = \Gamma\left(2 + \frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$$

Thus, $K_{b,2}(u) = C_\kappa^{-1}(L_{\kappa,2})L_{\kappa,2}(u)$ with $\kappa = b^{-2}$ is a circular kernel of order 2. Now, let

$$L_4(r) = \frac{3}{2} L_2(r) + r L_2^{(1)}(r),$$

set $L_{\kappa,4}(u) = L_4(r_\kappa(u))$ and denote by $C_k(L_{k,4})$ the corresponding normalizing constant. Then

$$K_{b,4}(u) = C_\kappa^{-1}\left(L_{\kappa,4}\right) L_{\kappa,4}(u)$$

is a kernel of order 4. Similarly, kernels $K_{b,p+2}$ of order $p + 2$ can be obtained recursively by setting

$$L_{p+2}(r) = \frac{p+1}{p} L_p(r) + \frac{2}{p} r L_p^{(1)}(r) \ (p = 2, 4, 6, \dots)$$

$L_{\kappa,p+2}(u) = L_{p+2}(r_k(u))$ and

$$K_{b,p+2}(u) = C_\kappa^{-1}\left(L_{\kappa,p+2}\right) L_{\kappa,p+2}(u).$$

3 Asymptotic results

The asymptotic rate of convergence of $\hat{\mu}_n(w^0) - \mu(w^0)$ is characterized by the following decomposition:

Theorem 1 *Let $w^0 \in [-\pi, \pi)$, suppose that (A1) to (A3) and (L1) to (L4) hold and K_b is a kernel of order $p = 2k$. Furthermore assume that $\tilde{\mu}(x) = \mu\left(G_\psi^{-1}(\Phi(x))\right)$ is $p+2$ times continuously differentiable, the derivatives $\tilde{\mu}^{(s)}$ and $g_\psi^{(s)}$ ($s = 1, 2, \dots, p$) are square integrable, $g_\psi(w) > 0$ ($w \in [-\pi, \pi)$) and*

$$n \to \infty, \ b \to 0, \ nb \to \infty.$$

Then,

$$\hat{\mu}_n\left(w^0\right) - \mu\left(w^0\right) = O\left(b^{2k}\right) + O_p\left(\frac{1}{\sqrt{nb}}\right) + O_p\left(n^{H_Z-1}\right) + O_p\left(b^2 n^{H_X-1}\right). \tag{26}$$

Remark 6 The first term in (26) is deterministic. It is due to the difference $\mu_n(w^0) - \mu(w^0)$ where $\mu_n(w^0) = E[K_b(\psi - w^0)\vartheta]/E[K_b(\psi - w^0)]$. The order of this error is smaller the higher the order of the kernel K_b is.

Equation (26) can be used to address the question of optimal bandwidth choice. Note first that the third term in (26) is the only one that does not depend on the bandwidth b. This means that an error of order $O_p(n^{H_Z-1})$ is always there, independently of the choice of b. A sequence of bandwidths b_n may therefore be called optimal, if

$$\max\left\{b_n^{2k}, \frac{1}{\sqrt{nb_n}}, b_n^2 n^{H_X-1}\right\} = o\left(n^{H_Z-1}\right). \tag{27}$$

The first question is whether (27) can be achieved. The answer is given by the following Lemma:

Lemma 2 *Under the assumptions of Theorem 1 the following statements are equivalent:*

(i) *The set of bandwidth sequences such that $b_n > 0$, $b_n \to 0$, $nb_n \to \infty$ and (27) holds is not empty.*

(ii) *The set of bandwidth sequences such that $b_n > 0$, $b_n \to 0$, $nb_n \to \infty$ and*

$$\lim_{n \to \infty} n^{(1-H_Z)/2k} b_n = 0, \ \lim_{n \to \infty} n^{(H_X - H_Z)/2} b_n = 0, \ \lim_{n \to \infty} n^{2H_Z - 1} b_n = \infty \quad (28)$$

is not empty.

(iii)

$$H_Z > \max \left\{ \frac{H_X + 2}{5}, \frac{2k + 1}{4k + 1} \right\}. \tag{29}$$

As a corollary we obtain:

Corollary 1 *Suppose that the assumptions of Theorem 1, and (29) hold. Then the set of optimal sequences is not empty. Moreover, for any such sequence b_n we have*

$$\hat{\mu}_n \left(w^0 \right) - \mu \left(w^0 \right) = O_p \left(n^{H_Z - 1} \right) \ (w^0 \in [0, 2\pi)). \tag{30}$$

A closer analysis of (30) leads to convergence in distribution:

Theorem 2 *Suppose that the assumptions of Theorem 1, and (29) hold. Then*

$$n^{1-H_Z} \left[\hat{\mu}_n \left(w^0 \right) - \mu \left(w^0 \right) \right] \underset{d}{\to} c_\mu \zeta \tag{31}$$

where

$$c_\mu = \sqrt{\frac{\sin \left(\pi H_Z - \frac{\pi}{2} \right) \Gamma \left(2 - 2H_Z \right) c_{f,Z}}{H_Z \left(H_Z - \frac{1}{2} \right)}},$$

$c_{f,Z}$ is defined in (7) and ζ is a standard normal random variable. More generally, for $w_1^0 < w_2^0 < \cdots < w_m^0$ ($w_j^0 \in [0, 2\pi)$),

$$n^{1-H_Z} \left[\hat{\mu}_n \left(w_1^0 \right) - \mu \left(w_1^0 \right), \ldots, \hat{\mu}_n \left(w_m^0 \right) - \mu \left(w_m^0 \right) \right] \underset{d}{\to} \left[c_\mu, \ldots, c_\mu \right] \cdot \zeta. \tag{32}$$

Remark 7 Equation (32) means in particular that the standardized random deviations of estimates at different values w_j^0, w_l^0 are asymptotically perfectly correlated. This is very much in contrast to usual behaviour in nonparametric regression. The degenerate limit theorem simplifies the construction of simultaneous confidence intervals for $\mu(w_1^0), \ldots, \mu(w_m^0)$.

Remark 8 It can also be shown that, if $H_X \leq \frac{1}{2}$, then (32) holds for any $H_Z > (2k + 1)/(4k+1)$. Note that in this case we also have $\max\{(H_X + 2)/5, (2k+1)/(4k+1)\} = (2k+1)/(4k+1)$.

Remark 9 For $k = 1$, we have $(2k + 1)/(4k + 1) = 3/5$. Now $H_X < 1$ implies $(H_X + 2)/5 < 3/5$ so that condition (29) reduces to

$$H_Z > \frac{3}{5}. \tag{33}$$

On the other hand, for $k \to \infty$, the ratio $(2k + 1)/(4k + 1)$ decreases monotonically to $1/2$. Therefore, in the limit (as $k \to \infty$), under the assumption that $H_X > 1/2$ condition (29) reduces to $H_Z > (H_X + 2)/5$.

Under simple additional conditions, uniform convergence of $\hat{\mu}_n(w^0)$ can be obtained:

Theorem 3 *Suppose that the assumptions of Theorem 1, and (29) hold. Moreover, assume*

$$\lim_{n \to \infty} \left(n^{H_Z + H_X - 2} \sum_{s=-\infty}^{\infty} |\alpha_s(b_n)| \right) = 0, \tag{34}$$

where $\alpha_s(b)$ are the coefficients in the Fourier series representation (25). Then

$$\sup_{w^0 \in [0, 2\pi)} \left| \hat{\mu}_n\left(w^0\right) - \mu\left(w^0\right) \right| \underset{p}{\to} 0.$$

Remark 10 The conditions in Theorem 3 are related to Lemma 3.2 and Corollary 3.2 in Ghosh (2014).

Typically, $\sum_{s=-\infty}^{\infty} |\alpha_s(b_n)|$ is asymptotically proportional to $b_n^{-\beta}$ for some $\beta > 0$. Theorem 3 then simplifies as follows.

Corollary 2 *Suppose that the assumptions of Theorem 1, and (29) hold. Moreover, assume that*

$$\sum_{s=-\infty}^{\infty} |\alpha_s(b)| = O\left(b^{-\beta}\right) (b \to 0) \tag{35}$$

for some $\beta > 0$, and b_n is a sequence of bandwidths such that $b_n > 0$, $b_n \to 0$, $nb_n \to \infty$, (27) holds and

$$\lim_{n \to \infty} n^{2 - H_Z - H_X} b_n^{\beta} = \infty. \tag{36}$$

Then

$$\sup_{w^0 \in [0, 2\pi)} \left| \hat{\mu}_n\left(w^0\right) - \mu\left(w^0\right) \right| \underset{p}{\to} 0.$$

Example 1 Denote by $I_s(\kappa)$ $(s = 0, 1, 2, \ldots)$ modified Bessel functions of order s. Consider a von Mises kernel with $\kappa = b^{-2}$,

$$K_b(u) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa \cos u).$$

The Fourier series representation of K_b is given by

$$K_b(u) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \alpha_s(\kappa) \exp(-isu)$$

where

$$\alpha_s(\kappa) = \frac{I_s(\kappa)}{I_0(\kappa)}.$$

Since $I_s(\kappa) > 0$, we have

$$\sum_{s=-\infty}^{\infty} |\alpha_s(\kappa)| = 2\pi K_b(0) = \frac{\exp(\kappa)}{I_0(\kappa)}.$$

Then, $I_0(\kappa) \sim \exp(\kappa)/\sqrt{2\pi\kappa}$ $(\kappa \to \infty)$ (see e.g. Robert 1990) implies

$$\sum_{s=-\infty}^{\infty} |\alpha_s(\kappa)| \underset{\kappa\to\infty}{\sim} \sqrt{2\pi\kappa} = \sqrt{2\pi}b^{-1}.$$

Thus, condition (36) with $\beta = 1$ holds. Similar results can be obtained for the generalized von Mises distribution introduced by Kim et al. (2013).

Example 2 For a wrapped normal kernel with $\sigma = b$,

$$K_b(u) = \frac{1}{\sqrt{2\pi}b} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{(u+2\pi j)^2}{2b^2}\right),$$

we have the Fourier series representation

$$K_b(u) = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \exp\left(-\frac{b^2}{2}s^2\right) \exp(-isu).$$

Thus,

$$\sum_{s=-\infty}^{\infty} |\alpha_s(b)| = 2\pi K_b(0)$$

$$= \frac{\sqrt{2\pi}}{b} \sum_{j=-\infty}^{\infty} \exp\left(-\frac{2\pi^2}{b^2}j^2\right),$$

and, for $b \leq 1$,

$$
\sum_{s=-\infty}^{\infty} |\alpha_s(b)| \leq b^{-1}\sqrt{2\pi} \sum_{j=-\infty}^{\infty} \exp\left(-2\pi^2 j^2\right)
$$
$$
= Cb^{-1}.
$$

Therefore, condition (36) with $\beta = 1$ holds.

4 Asymptotic confidence intervals

Theorem 2 can be used to obtain asymptotic confidence sets for $\mu(w_1^0),\ldots,\mu(w_m^0)$. Since condition (29) has to be checked first, a two-step procedure has to be applied. Suppose for instance that we use a kernel of order 2. Then we have to test the null hypothesis $H_0 : H_Z \leq 3/5$ against the alternative $H_1 : H_Z > 3/5$. If H_0 is rejected, then (32) implies that confidence intervals with asymptotic coverage probability $1-\alpha$ can be defined by

$$
\left[\hat{\mu}_n\left(w_1^0\right),\ldots,\hat{\mu}_n\left(w_m^0\right)\right] \pm n^{H_Z-1} q_{1-\alpha/2}\left[c_\mu\left(w_1^0\right),\ldots,c_\mu\left(w_m^0\right)\right] \qquad (37)
$$

where $q_{1-\alpha/2}$ is the $(1-\alpha/2)$-quantile of the standard normal distribution (see (32)).

Testing $H_Z \leq 3/5$ versus $H_Z > 3/5$ can be done as follows. Under the given conditions, $S_{Z,t} = \sin Z_t \bmod 2\pi$ has the same long-memory parameter H_Z as Z_t. Therefore a consistent estimator of H_Z can be based on the periodogram of the series $S_{\hat{Z},t} = \sin \hat{Z}_t$ $(t = 1, \ldots, n)$ where $\hat{Z}_t = [\vartheta_t - \hat{\mu}_n(\psi_t)] \bmod 2\pi$. There is a vast range of consistent methods for estimating the long-memory parameter (see e.g. Beran et al. 2013, chapter 5, and references therein). For instance, let $\hat{H}_Z$ be a local Whittle estimator based on the periodogram of $S_{\hat{Z},t}$ at the m lowest frequencies. Then, under mild regularity conditions (see Robinson 1995), as $m \rightarrow \infty$, $m/n \rightarrow 0$,

$$
\sqrt{m}\left(\hat{H}_Z - H_Z\right) \underset{d}{\rightarrow} \frac{1}{2}\zeta_1
$$

where ζ_1 is a standard normal random variable. Thus, given a level of significance $\alpha \in (0, 1)$, we reject $H_0 : H_Z \leq 3/5$, if

$$
\hat{H}_Z > \frac{3}{5} + \frac{1}{2\sqrt{m}} q_{1-\alpha}
$$

where $q_{1-\alpha}$ is the $(1-\alpha)$-quantile of the standard normal distribution.

Remark 11 Confidence intervals (37) are conditional, since they are computed under the condition that $H_0 : H_Z \leq 3/5$ is rejected. In principle this would call for an adjustment of quantiles, for instance using a Bonferroni correction. However, note that (32) and (37) are applicable only, if $H_Z > 3/5$. If this is the case (i.e. $H_Z > 3/5$),

then then H_0 is rejected with asymptotic probability one, provided that we use a consistent estimator $\hat{H}_Z$ of H_Z. Therefore, no correction for multiple testing needs to be applied asymptotically.

Remark 12 In practice, the constants $c_\mu(w_j^0)$ $(j = 1, \ldots, m)$ in (37) have to be estimated. This means that H_Z and $c_{f,z}$ have to be replaced by consistent estimates.

Remark 13 If one uses a kernel of order $p = 2k$ with $k \geq 2$ (instead of $k = 1$), a test of

$$H_0 : H_Z \leq \max\left\{ \frac{H_X + 2}{5}, \frac{2k + 1}{4k + 1} \right\}$$

against

$$H_1 : H_Z > \max\left\{ \frac{H_X + 2}{5}, \frac{2k + 1}{4k + 1} \right\}$$

has to be carried out. This is slightly more complicated, because the right hand side of the inequality involves the unknown parameter H_X. Since $\sin \psi_t$ has the same long-memory parameter H_X as ψ_t, H_X can be estimated based on the periodogram of the series $S_{\psi,t} = \sin \psi_t$ $(t = 1, \ldots, n)$. Suppose for instance that $\hat{H}_Z$ and $\hat{H}_X$ are local Whittle estimators based on $S_{\hat{Z},t} = \sin \hat{Z}_t$ (see above) and $S_{\psi,t}$ respectively. Then

$$\sqrt{m} \left(\hat{H}_Z - H_Z, \hat{H}_X - H_X \right) \xrightarrow{d} \frac{1}{2} (\zeta_1, \zeta_2)$$

where ζ_1, ζ_2 are independent standard normal random variables and m is the number of lowest frequencies used for estimation. Let

$$T = \hat{H}_Z - \max\left\{ \frac{\hat{H}_X + 2}{5}, \frac{2k + 1}{4k + 1} \right\}.$$

A rejection region at an asymptotic level of significance α is then defined by the condition $T > q_{1-\alpha}$ where $q_{1-\alpha}$ is the $(1 - \alpha)$-quantile of the random variable

$$W = W(H_Z, H_X) = H_Z + \frac{1}{2\sqrt{m}} \zeta_1 - \max\left\{ \frac{1}{5} H_X + \frac{2}{5} + \frac{1}{10\sqrt{m}} \zeta_2, \frac{2k + 1}{4k + 1} \right\}. \quad (38)$$

The distribution of W depends on H_Z and H_X. For each pair $(H_Z, H_X) \in \left(\frac{1}{2}, 1\right)^2$, $(1 - \alpha)$-quantiles $q_{1-\alpha} = q_{1-\alpha}(H_Z, H_X)$ of $W(H_Z, H_X)$ can be obtained by numeric integration or by simulations. If H_Z and H_X are replaced by their consistent estimates, approximate $(1 - \alpha)$-quantiles are given by $q_{1-\alpha}(\hat{H}_Z, \hat{H}_X)$.

(a) January to March

(b) April to June

(c) July to September
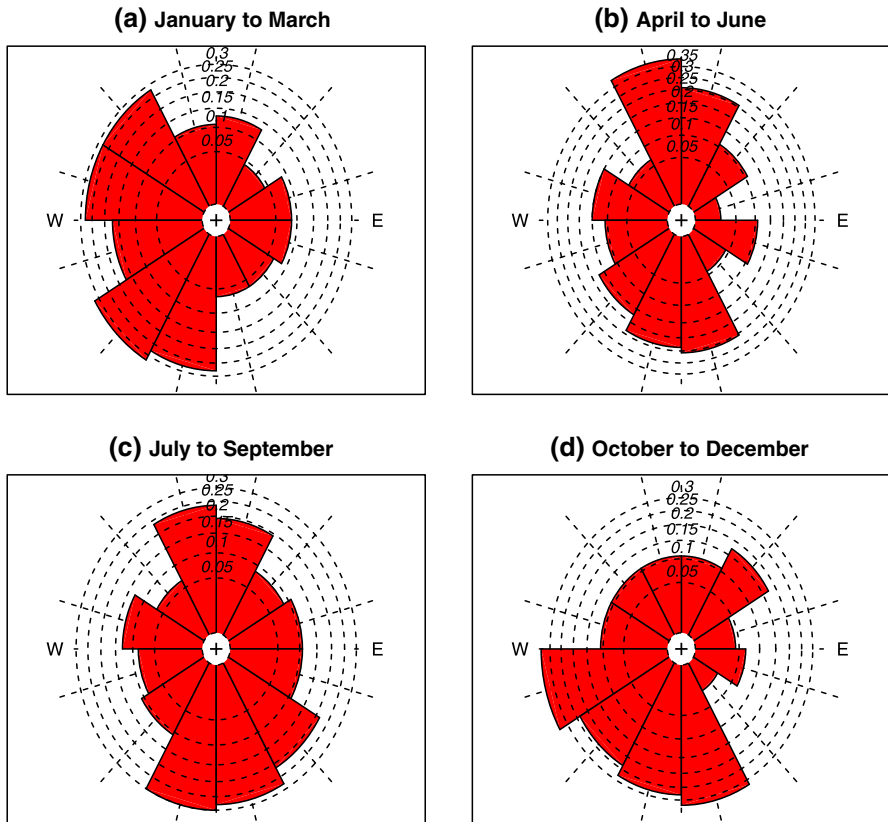

(d) October to December

Fig. 1 Windrose plots of daily wind directions in Chicago in 2015, split into four three-months periods: **a** January to March, **b** April to June, **c** July to September and **d** October to December

5 Data example

We consider daily average wind directions recorded in Chicago and Milwaukee between 1 January and 31 December, 2015 (data source https://www.glerl.noaa.gov/). Figure 1a–d show windrose plots of wind directions in Chicago (here given in degrees between 0° to $360°$), split into four three-months periods, i.e. (a) January to March, (b) April to June, (c) July to September and (d) October to December. The same plots are displayed for Milwaukee in Fig. 2a–d. These figures indicate typical seasonal patterns. In a first step we therefore remove a deterministic seasonal trend from both series. This is done using the deseasonalization method for circular data described in Beran and Ghosh (2019). Figure 3a shows the two series together with the fitted seasonal trend. The residual series are displayed in Fig. 3b. Note that for better visibility, the Milwaukee series was moved down vertically by $360°$.

Define now ϑ_t to be the residual series for Milwaukee (lower series in Fig. 3b), and ψ_t the residual series for Chicago. The plot of ϑ_t vs. ψ_t in Fig. 4 indicates a possible relationship between the two wind directions, though the circular nature of
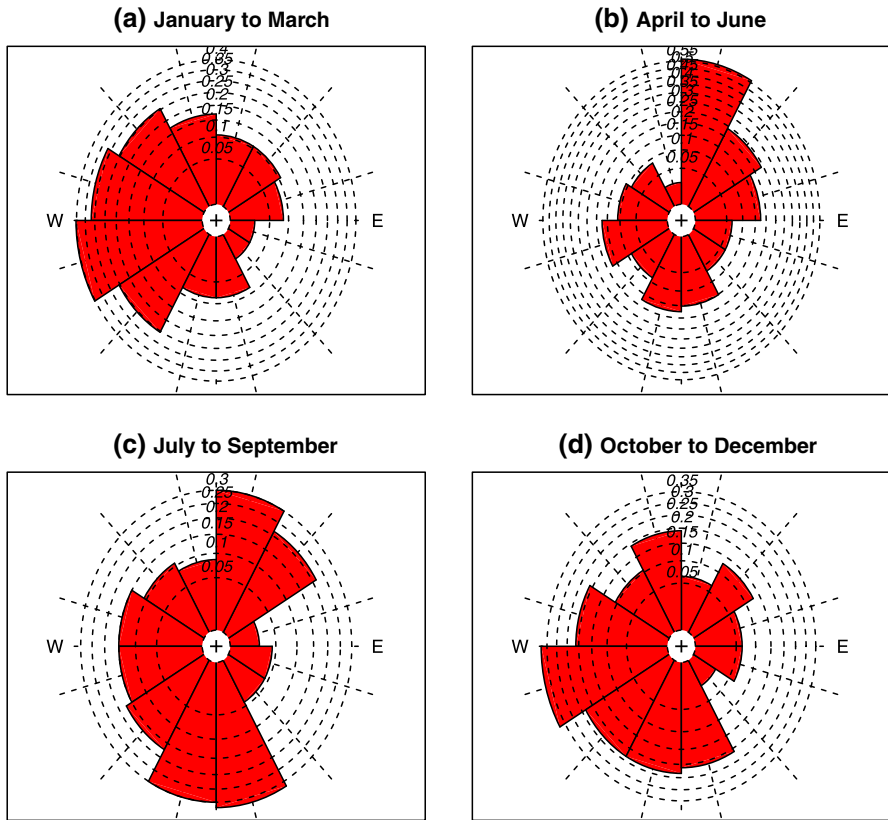
Fig. 2 Windrose plots of daily wind directions in Milwaukee in 2015, split into four three-months periods: **a** January to March, **b** April to June, **c** July to September and **d** October to December
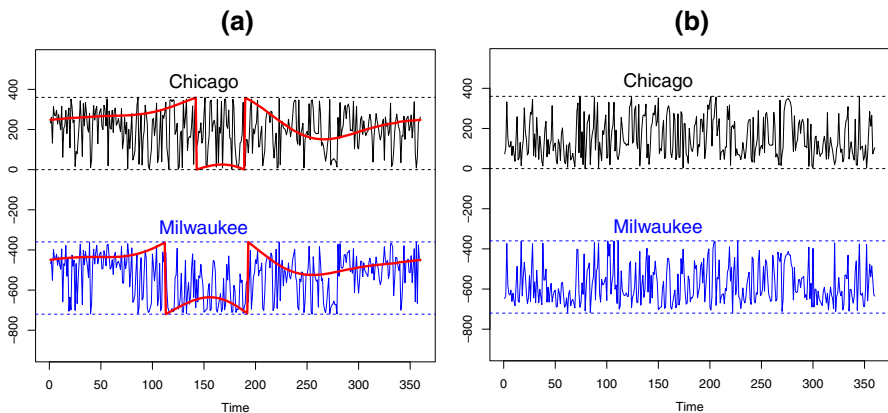

Fig. 3 **a** Daily wind directions in Chicago and Milwaukee, in 2015. Also plotted are fitted seasonal trends. For better visibility, the Milwaukee series is shifted vertically by $-360°$. **b** Residual series obtained after removing the seasonal trend
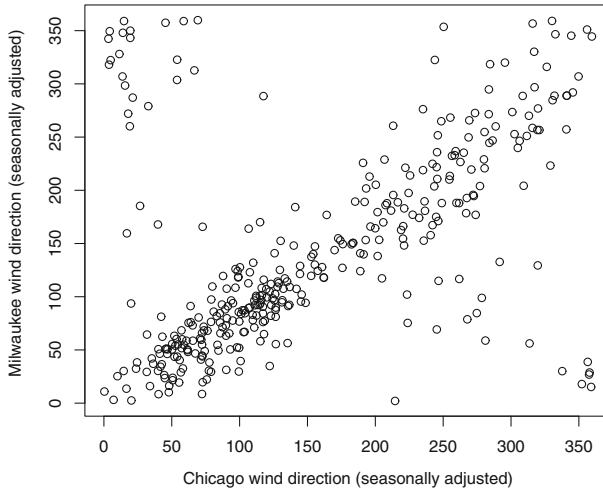
Fig. 4 Scatterplot of seasonally adjusted wind directions (Milwaukee vs. Chicago)
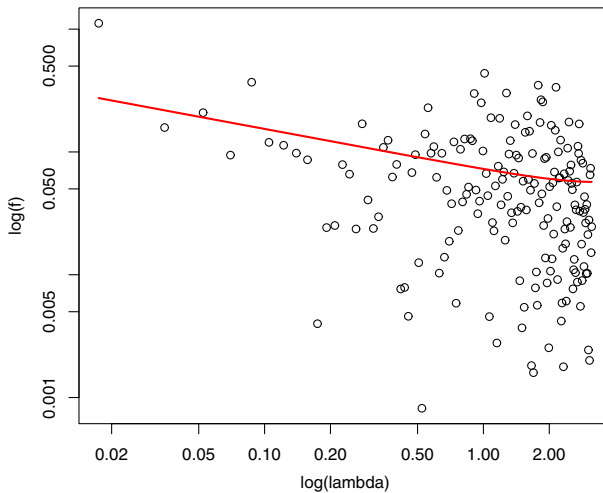

Fig. 5 Log-log-periodogram of $\sin \hat{Z}_t$

the variables makes a visual inspection more difficult. The Nadaraya–Watson estimate of $\mu(\psi)$, based on a von Mises kernel, is shown in Fig. 6. Since we are using a kernel of order 2, the value of H_X does not play any role (see Remark 9). With respect to estimation of H_Z, we consider a local Whittle estimator with $m = [n^{0.6}]$ frequencies. For $S_{\hat{Z},t} = \sin \hat{Z}_t$ where $\hat{Z}_t = \vartheta_t - \hat{\mu}_n(\psi_t) \mod 2\pi$, we obtain $\hat{H}_Z = 0.75$ with a 95%-confidence interval of $[0.58, 0.92]$. Thus, there is significant evidence for long memory in Z_t. The estimate does not change much, when other estimation methods are used. For instance, fitting fractional autoregressive models of orders $p = 0, 1, \ldots, 3$, and selecting the best model using the BIC (see e.g. Beran et al. 1998), leads to
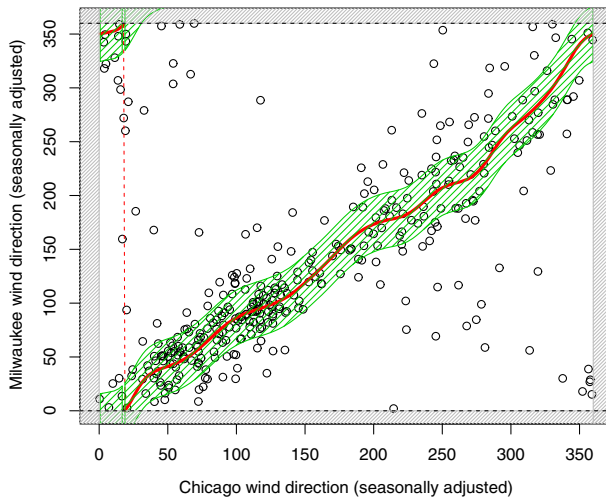
Fig. 6 Scatterplot of seasonally adjusted wind directions (Milwaukee vs. Chicago) together with $\hat{\mu}(\psi)$ and a $95\% -$ confidence band

$\hat{H}_Z = 0.67$ ([0.59, 0.75]). Figure 5 shows the log-log-periodogram $\sin \hat{Z}_t$, together with a fitted spectral density.

The next step is to test $H_Z \le 3/5$ against $H_Z > 3/5$. For the local Whittle estimator, the critical limit for the 5% level of significance is $3/5 + 1.645 \cdot \frac{1}{2} m^{-1/2} = 0.74$. Since $\hat{H}_Z = 0.75$ is above this threshold, there is evidence at the 5%-level of significance for condition (29). The same conclusion is reached when using estimates based on the BIC method. Confidence bands for $\mu(\psi)$ based on Theorem 2 and (37) are given in Fig. 6. As a cautionary remark one should bear in mind that, strictly speaking, the band is simultaneous for a finite - though possibly large - number of ψ-values only. Note also that due to the circular nature of the data, there is an apparent jump around $\psi = 17.6^o$. This is not a jump in the function μ, but rather due to plotting modulo $360°$. The same comment applies to the confidence band.

7 Appendix: proofs, figures

7.1 Proofs

Proof (of Theorem 1) The method of the proof of Theorem 1 follows similar lines as Proposition 1 and Section 3.2 in Mielniczuk and Wu (2004). For $\psi^0 \in [0, 2\pi)$, we have

$$\hat{\mu}_n \left(\psi^0 \right) = \frac{n^{-1} \sum\limits_{s=1}^{n} K_b \left(\psi_s - \psi^0 \right) \vartheta_s}{\hat{f}_n \left(\psi^0 \right)}$$

where

$$\hat{f}_n \left(\psi^0 \right) = n^{-1} \sum_{s=1}^{n} K_b \left(\psi_s - \psi^0 \right)$$

is a kernel density estimator of the density f_ψ at ψ^0. Using the notation

$$\mu_n \left(\psi^0 \right) = \frac{E \left[K_b \left(\psi_t - \psi^0 \right) \vartheta \right]}{E \left[K_b \left(\psi_t - \psi^0 \right) \right]}, \ D_n \left(\psi^0 \right) = \hat{\mu}_n \left(\psi^0 \right) - \mu_n \left(\psi^0 \right)$$

we have

$$\hat{\mu}_n \left(\psi^0 \right) - \mu \left(\psi^0 \right) = \mu_n \left(\psi^0 \right) - \mu \left(\psi^0 \right) + D_n \left(\psi^0 \right).$$

The first term is deterministic and does not depend on the dependence structure. Following arguments in Tsurata and Sagae (2017) (for the special case of $k = 1$ also see Taylor 2008), leads to

$$\mu_n \left(\psi^0 \right) - \mu \left(\psi^0 \right) = O \left(b^{2k} \right).$$

To study asymptotic properties of D_n, the following notation is useful:

$$\xi_t = \xi_t \left(\psi^0 \right) = K_b \left(\psi^0 - \psi_t \right) - E \left[K_b \left(\psi^0 - \psi \left(X_t \right) \right) \mid \eta_s, s \leq t - 1 \right],$$

$$A_n \left(\psi^0 \right) = \frac{1}{n} \sum_{t=1}^{n} \varepsilon_t K_b \left(\psi^0 - \psi \left(X_t \right) \right) + \frac{1}{n} \sum_{t=1}^{n} \left(\sum_{j=1}^{\infty} a_j \varepsilon_{t-j} \right) \xi_t,$$

and

$$J_t \left(\psi^0 \right) = Z_t K_b \left(\psi^0 - \psi_t \right).$$

Consider the decomposition

$$D_n \left(\psi^0 \right) \hat{f}_n \left(\psi^0 \right) = n^{-1} \sum_{s=1}^{n} K_b \left(\psi - \psi^0 \right) \left[\vartheta_s - \mu_n \left(\psi^0 \right) \right]$$

$$= A_n \left(\psi^0 \right) + B_n \left(\psi^0 \right) + C_n \left(\psi^0 \right) + r_n$$

with

$$A_n \left(\psi^0 \right) = \frac{1}{n} \sum_{t=1}^{n} \left\{ J_t \left(\psi^0 \right) - E \left[Z_t \mid \varepsilon_s, s \leq t - 1 \right] \right.$$

$$\times \left. E \left[K \left(\psi^0 - \psi_t \right) \mid \eta_s, s \leq t - 1 \right] \right\},$$

$$B_n \left(\psi^0 \right) = \frac{1}{n} \sum_{t=1}^{n} E \left[Z_t \mid \varepsilon_s, s \leq t - 1 \right]$$

$$\times E \left[K_b \left(\psi^0 - \psi_t \right) \mid \eta_s, s \leq t - 1 \right]$$

and

$$C_n \left(\psi^0 \right) = n^{-1} \sum_{t=1}^{n} K_b \left(\psi_t - \psi^0 \right) \left[\mu \left(\psi^0 \right) - \mu_n \left(\psi^0 \right) \right].$$

The asymptotic results in Mielniczuk and Wu (2004) are obtained for kernel regression with non-circular kernels. Taking into account the definition and properties of circular kernels, the arguments in Mielniczuk and Wu (2004) can be reformulated accordingly, to yield

$$A_n = O_p \left(\frac{1}{\sqrt{nb}} \right), \ B_n = O_p \left(n^{H_Z - 1} \right), \ C_n \left(\psi^0 \right) = O_p \left(b^2 n^{H_X - 1} \right)$$

and $r_n = o_p (\max\{A_n, B_n, C_n\})$. $\qquad \square$

Proof (of Lemma 2) From Theorem 1 and (26) we have

$$\hat{\mu}_n \left(\psi^0 \right) - \mu \left(\psi^0 \right) = O \left(b^{2k} \right) + O_p \left(\frac{1}{\sqrt{nb}} \right) + O_p \left(n^{H_Z - 1} \right) + O_p \left(b^2 n^{H_X - 1} \right).$$

In order that the term $O_p(n^{H_Z-1})$ dominates we need

$$\max \left\{ b^{2k}, \ (nb)^{-1/2}, \ b^2 n^{H_X - 1} \right\} = o \left(n^{H_Z - 1} \right).$$

This leads to

$$b = o \left(\min \left\{ n^{(H_Z - 1)/2k}, \ n^{(H_Z - H_X)/2} \right\} \right) \tag{39}$$

and

$$n^{1-2H_Z} = o\,(b)\,. \tag{40}$$

In order that (39) and (40) hold simultaneously, we need

$$n^{1-2H_Z} = o\left(\min\left\{n^{(H_Z-1)/2k},\, n^{(H_Z-H_X)/2}\right\}\right),$$

which leads to

$$H_Z > \max\left\{\frac{H_X+2}{5},\, \frac{2k+1}{4k+1}\right\}.$$

$\square$

Proof (of Corollary 1) The result follows directly from Lemma 2 and Theorem 1. $\square$

Proof (of Theorem 2) Writing

$$v_t = E\left[K_b\left(\psi^0 - \psi_t\right) \mid \eta_s,\, s \le t-1\right]$$

we have

$$
\begin{aligned}
B_n\left(\psi^0\right) &= \frac{1}{n}\sum_{t=1}^{n} E\left[Z_t \mid \varepsilon_s,\, s \le t-1\right] E\left[K_b\left(\psi^0 - \psi_t\right) \mid \eta_s,\, s \le t-1\right] \\
&= \frac{1}{n}\sum_{t=1}^{n}\left(\sum_{j=1}^{\infty} a_j \varepsilon_{t-j}\right) v_t \\
&= \frac{1}{n}\sum_{t=1}^{n} Z_t v_t + o_p\left(B_n\right).
\end{aligned}
$$

Now, define

$$\tilde{B}_n\left(\psi^0\right) = \left(K_b * f_\psi\left(\psi^0\right)\right)\frac{1}{n}\sum_{t=1}^{n} Z_t$$

where

$$K_b * f_\psi\left(\psi^0\right) = \int_{-\pi}^{\pi} K_b\left(\psi^0 - y\right) f_\psi\,(y)\,dy.$$

Since

$$E\,(v_t) = K_b * f_\psi\left(\psi^0\right),$$

we have

$$\frac{1}{n} \sum_{t=1}^{n} Z_t v_t - \tilde{B}_n \left(\psi^0 \right) = \frac{1}{n} \sum_{t=1}^{n} Z_t \left[v_t - E \left(v_t \right) \right]$$

and hence also

$$E \left[\frac{1}{n} \sum_{t=1}^{n} Z_t v_t - \tilde{B}_n \left(\psi^0 \right) \right] = 0.$$

Denote by $\gamma_v(k)$ the autocovariance function of v_t. Then

$$var \left(\frac{1}{n} \sum_{t=1}^{n} Z_t v_t - \tilde{B}_n \left(\psi^0 \right) \right) = n^{-2} \sum_{s,t=1}^{n} \gamma_Z \left(t - s \right) \gamma_v \left(t - s \right)$$

$$= n^{-1} \sum_{|k|<n} \left(1 - \frac{|k|}{n} \right) \gamma_Z \left(k \right) \gamma_v \left(k \right).$$

Now, as $k \to \infty$,

$$|\gamma_v \left(k \right)| \le c_{\gamma,v} \cdot k^{2H_X-2}, \gamma_Z \left(k \right) \sim c_{\gamma,Z} k^{2H_Z-2}$$

where $c_{\gamma,v}, c_{\gamma,Z}$ are suitable constants. Therefore,

$$var \left(\frac{1}{n} \sum_{t=1}^{n} Z_t v_t - \tilde{B}_n \left(\psi^0 \right) \right) \le const \cdot n^{2H_X+2H_Z-4} \sum_{|k|<n} \left(1 - \frac{|k|}{n} \right)$$

$$\times \left(\frac{|k|}{n} \right)^{2H_X+2H_Z-4} \frac{1}{n}$$

$$\sim C \cdot n^{2H_X+2H_Z-4}.$$

Now $H_X < 1$ implies $n^{2H_X+2H_Z-4} = o(n^{2H_Z-2})$. Thus, B_n may be replaced asymptotically by $\tilde{B}_n$. For $\tilde{B}_n$, the asymptotic distribution follows from $K_b * f_\psi (\psi^0) \to f_\psi (\psi^0)$ and well known limit theorems for sample means under long-range dependence (see e.g. Beran et al. 2013; Chapter 4.2 and Corollary 1.2). More specifically,

$$n^{H_Z-1} \left[\hat{\mu}_n \left(\psi^0 \right) - \mu \left(\psi^0 \right) \right] = n^{H_Z-1} \frac{\tilde{B}_n \left(\psi^0 \right)}{f_\psi \left(\psi^0 \right)} + o_p \left(n^{H_Z-1} \right)$$

$$= n^{H_Z-1} \frac{1}{n} \sum_{t=1}^{n} Z_t + o_p \left(n^{H_Z-1} \right)$$

$$= c_\mu W + o_p \left(n^{H_Z-1} \right).$$

$\square$

Proof (of Theorem 3) The line of proof is similar to Ghosh (2014). Under the given conditions, it is sufficient to prove that

$$S_n\left(\psi^0\right) = \frac{1}{n}\sum_{t=1}^{n} E\left[K_b\left(\psi^0 - \psi_t\right) \mid \eta_s, s \leq t-1\right] Z_t$$

converges uniformly to zero in probability. It is then enough to show

$$\lim_{n\to\infty} E\left[\sup_{\psi^0}\left|S_n\left(\psi^0\right)\right|\right] = 0.$$

By assumption,

$$K_b\left(w\right) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \alpha_s\left(b\right)\exp\left(-isw\right)$$

and Z_t is independent of η_s ($s \in \mathbb{Z}$). Thus,

$$S_n\left(\psi^0\right) = \frac{1}{n}\sum_{t=1}^{n} E\left[K_b\left(\psi_t - \psi^0\right) \mid \eta_j, j \leq t-1\right] Z_t$$

$$= \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \alpha_s\left(b\right)\left\{\frac{1}{n}\sum_{t=1}^{n} Z_t E\left[\exp\left(-is\psi_t\right) \mid \eta_j, j \leq t-1\right]\right\}\exp\left(is\psi^0\right),$$

and

$$\left|S_n\left(\psi^0\right)\right| \leq \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\left|\alpha_s\left(b\right)\right|\left|\frac{1}{n}\sum_{t=1}^{n} Z_t E\left[\exp\left(-is\psi_t\right) \mid \eta_j, j \leq t-1\right]\right|. \quad (41)$$

Now the right-hand side of (41) does not depend on ψ^0. Hence,

$$\sup_{\psi^0}\left|S_n\left(\psi^0\right)\right| \leq \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\left|\alpha_s\left(b\right)\right|\left|\frac{1}{n}\sum_{t=1}^{n} Z_t E\left[\exp\left(-is\psi_t\right) \mid \eta_j, j \leq t-1\right]\right|.$$

Using the notation

$$v_{t,s,\cos} = E\left[\cos\left(s\psi_t\right) \mid \eta_j, j \leq t-1\right], \ v_{t,s,\sin} = E\left[\sin\left(s\psi_t\right) \mid \eta_j, j \leq t-1\right],$$

we then have, for a suitable constant $0 < C < \infty$,

$$E\left[\sup_{\psi^0}\left|S_n\left(\psi^0\right)\right|\right] \leq \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\left\{\left|\alpha_s\left(b\right)\right|\right.$$

$$\times \sqrt{var\left(\frac{1}{n}\sum_{t=1}^{n}Z_t v_{t,s,\cos}\right) + var\left(\frac{1}{n}\sum_{t=1}^{n}Z_t v_{t,s,\sin}\right)} \Bigg\}$$

$$\leq Cn^{H_Z + H_X - 2}\sum_{s=-\infty}^{\infty}|\alpha_s(b)|.$$

Thus, assumption (34) implies

$$\lim_{n\to\infty} E\left[\sup_{\psi^0}\left|S_n\left(\psi^0\right)\right|\right] = 0.$$

$\square$

Proof (of Corollary 2) The result follows directly from Theorem 3, (35) and (36). $\square$

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