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# A bound for the Euclidean distance between restricted and unrestricted estimators of parametric functions in the general linear model

Pawel R. Pordzik

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Abstract Let  $\hat{\kappa}$  and  $\hat{\kappa}_r$  denote the best linear unbiased estimators of a given vector of parametric functions  $\kappa = K\beta$  in the general linear models  $\mathcal{M} = \{y, X\beta, \sigma^2 V\}$ and  $\mathcal{M}_r = \{y, X\beta \mid R\beta = r, \sigma^2 V\}$ , respectively. A bound for the Euclidean distance between  $\hat{\kappa}$  and  $\hat{\kappa}_r$  is expressed by the spectral distance between the dispersion matrices of the two estimators, and the difference between sums of squared errors evaluated in the model  $\mathcal{M}$  and sub-restricted model  $\mathcal{M}_r^*$  containing an essential part of the restrictions  $R\beta = r$  with respect to estimating  $\kappa$ .

# 1 Introduction and preliminaries

For a given matrix A, let  $A^-$ ,  $A^+$ , C(A), r(A) and  $||A||_s$  denote a g-inverse, the Moore-Penrose inverse, the column space, the rank and the spectral norm of A, respectively. Let  $A^{\perp}$  denote a matrix of the maximum rank such that  $A'A^{\perp} = 0$ . Moreover, let  $Q_A = I - P_A$ , where I stands for an identity matrix and  $P_A = AA^+$ . For a nonnegative definite (n.n.d.) matrix A, let  $A^{\frac{1}{2}}$  denote a matrix such that  $(A^{\frac{1}{2}})^2 = A$ . Finally, for a given vector a, let ||a|| denote the Euclidean norm of a.

Consider the general linear model

$$\mathcal{M} = \{ \mathbf{y}, \, \mathbf{X}\boldsymbol{\beta}, \, \sigma^2 \mathbf{V} \} \tag{1.1}$$

in which y is an  $n \times 1$  observable random vector with expectation  $E(y) = X\beta$  and dispersion matrix  $D(y) = \sigma^2 V$ , where the matrices X and V are known, both allowed

P. R. Pordzik (🖂)

Department of Mathematical and Statistical Methods, Poznan University of Life Sciences, 60-637 Poznań, Poland e-mail: pordzik@up.poznan.pl to be rank-deficient, while the vector  $\boldsymbol{\beta}$  and the positive scalar  $\sigma^2$  are unknown parameters. The model  $\mathcal{M}$  is assumed to be consistent, that is,  $\boldsymbol{y} \in \mathcal{C}(\boldsymbol{X} : \boldsymbol{V})$ . Let  $\boldsymbol{\kappa} = \boldsymbol{K}\boldsymbol{\beta}$  denote a given vector of parametric functions linearly estimable in the model  $\mathcal{M}$ , i.e.  $\mathcal{C}(\boldsymbol{K}') \subseteq \mathcal{C}(\boldsymbol{X}')$ , and let  $\hat{\boldsymbol{\kappa}}$  stand for its best linear unbiased estimator (*blue*) in  $\mathcal{M}$ .

Furthermore, consider the restricted linear model

$$\mathcal{M}_{r} = \{ \mathbf{y}, \, \mathbf{X}\boldsymbol{\beta} \,|\, \mathbf{R}\boldsymbol{\beta} = \mathbf{r}, \, \sigma^{2} \mathbf{V} \}$$
(1.2)

obtained by supplementing the model  $\mathcal{M}$  with linear constraints specified by an  $m \times p$  known matrix  $\mathbf{R}$  and an  $m \times 1$  known vector  $\mathbf{r}$  such that  $\mathbf{r} \in \mathcal{C}(\mathbf{R})$ . Let  $\hat{\mathbf{k}}_r$  denote the *blue* of  $\mathbf{\kappa}$  in the model  $\mathcal{M}_r$ .

The aim of this note is to constrain the Euclidean distance between the estimators  $\hat{\kappa}$  and  $\hat{\kappa}_r$  in terms that allow a clear statistical interpretation. The bound involves two factors; the first one is the spectral distance between the dispersion matrices of the two estimators (measuring the sub-optimality of the *lue*  $\hat{\kappa}$  in  $\mathcal{M}_r$ , or conversely, the gain in matrix risk of the biased estimator  $\hat{\kappa}_r$  in the model  $\mathcal{M}$ ). The second factor, depending on *y* through goodness of fit statistics, is the difference between sums of squared errors evaluated in the model  $\mathcal{M}$  and sub-restricted model {*y*,  $X\beta | AR\beta = Ar$ ,  $\sigma^2 V$ } with implied restrictions being an essential part of  $R\beta = r$  with respect to estimating  $\kappa$ ; cf. Baksalary and Pordzik (1992).

Considering a linear model with nuisance parameters, the bound established in Sect. 2 allows to assess how sensitive the estimation of the main parameters might be with respect to possible overparametrization of the inference base. In this context, some improvement of the result by Baksalary (1984, Theorem 2.4) is presented in Sect. 3. A bound for the Euclidean distance between competing estimators is a natural tool to explore geodetic data. For numerical examples, concerned with the precise levelling problem, see Mäkinen (2002) and Mäkinen (2000); see also Schaffrin and Grafarend (1986) for application to Global Positioning System (GPS) data.

## 2 Results

Referring to the corner-stones of the inverse-partitioned-matrix method for statistical inference in the general linear model  $\mathcal{M}$ , assume that  $G_1$ ,  $G_3$  and  $G_4$  are any matrices such that

$$X'G_1(X : V) = 0, \quad VG_1X = 0, \quad (V - VG_1V)Q_X = 0,$$
 (2.1)

$$XG_3(X : VQ_X) = (X : 0)$$
 (2.2)

and

$$XG_4X' = V - VQ_X(Q_XVQ_X)^-Q_XV, \qquad (2.3)$$

i.e., the partitioned matrix  $((G'_1 : G'_3)' : (G_3 : -G'_4)')$  is a g-inverse of the bordered matrix  $((V : X)' : (X' : \mathbf{0})')$ ; cf. Rao (1971, 1972). Then the *blue* of  $\kappa$ , its dispersion

matrix and the sum of squared errors in the model  $\mathcal{M}$  can be expressed as  $\hat{\kappa} = KG_3 y$ ,  $D(\hat{\kappa}) = \sigma^2 KG_4 K'$  and  $SSE = y'G_1 y$ .

Let  $R_1\beta = r_1$  be an estimable part of the restrictions  $R\beta = r$  in the model  $\mathcal{M}$ , that is,  $R_1$  is a matrix such that  $\mathcal{C}(R'_1) = \mathcal{C}(R') \cap \mathcal{C}(X')$  and  $r_1 = R_1R^-r$ . Note that  $R_1\beta = r_1$  can be written as the implied restrictions  $LR\beta = Lr$ , where  $L = I - R_0R_0^-$  with  $R_0 = R(I - X^-X)$ ; for the proof, observe that, by the equality  $\mathcal{C}(R') \cap \mathcal{C}(X') = \mathcal{C}(R'(RX'^{\perp})^{\perp})$ , we have  $R_1 = LR$ . Baksalary and Pordzik (1989, Theorem 1) represented the consistency condition, the *blue* of  $\kappa$  and the sum of squared errors for the restricted model  $\mathcal{M}_r$  in terms referring to the model  $\mathcal{M}$  and a subset of estimable restrictions  $R_1\beta = r_1$ . The results useful for our purposes are given in the following lemma.

**Lemma 1** The restricted model  $\mathcal{M}_r = \{\mathbf{y}, \mathbf{X}\boldsymbol{\beta} | \mathbf{R}\boldsymbol{\beta} = \mathbf{r}, \sigma^2 \mathbf{V}\}$  is consistent if and only if the model  $\mathcal{M}$  is consistent, i.e.  $\mathbf{y} \in C(\mathbf{X} : \mathbf{V})$ , and

$$\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1 \in \mathcal{C}(\boldsymbol{S}), \tag{2.4}$$

where  $S = R_1 G_4 R'_1$  while  $\hat{\varrho}_1 = R_1 G_3 y$  is the blue of  $\varrho_1 = R_1 \beta$  in  $\mathcal{M}$ . If the model  $\mathcal{M}_r$  is consistent, then the blue of  $\kappa$  and its dispersion matrix are

$$\widehat{\boldsymbol{\kappa}}_r = \widehat{\boldsymbol{\kappa}} - \boldsymbol{C}\boldsymbol{S}^-(\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1) \quad and \quad D(\widehat{\boldsymbol{\kappa}}_r) = D(\widehat{\boldsymbol{\kappa}}) - \sigma^2 \boldsymbol{C}\boldsymbol{S}^- \boldsymbol{C}', \quad (2.5)$$

where  $C = K G_4 R'_1$  and  $\hat{\kappa}$  is the blue of  $\kappa$  in the model  $\mathcal{M}$ . Moreover, if SSE<sub>r</sub> and SSE are the sums of squared errors in the models  $\mathcal{M}_r$  and  $\mathcal{M}$ , respectively, then

$$SSE_r = SSE + (\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1)' \boldsymbol{S}^- (\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1).$$
(2.6)

Recall that, as far as only the estimation of  $\kappa$  in the model  $\mathcal{M}_r$  is concerned, some further reduction of the initial constraints is possible. Namely,  $\mathbf{R}_1 \boldsymbol{\beta} = \mathbf{r}_1$  can be reduced to a subset of implied restrictions which states the so-called essential part of  $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$  with respect to estimating  $\kappa = \mathbf{K}\boldsymbol{\beta}$ . Concerning the problem of reducing the linear constraints in the restricted model  $\mathcal{M}_r$ , Baksalary and Pordzik (1992, Theorem 2) showed that the *blue* of  $\kappa$  in the sub-restricted model { $\mathbf{y}, \mathbf{X}\boldsymbol{\beta} | \mathbf{A}\mathbf{R}\boldsymbol{\beta} = \mathbf{A}\mathbf{r}, \sigma^2 \mathbf{V}$ } continues to be the *blue* of  $\kappa$  in  $\mathcal{M}_r$  if and only if  $\mathcal{C}(\mathbf{C}') \subseteq \mathcal{C}(\mathbf{S}\mathbf{B}')$ , where  $\mathbf{B}$  is a matrix such that  $\mathcal{C}(\mathbf{R}'_1\mathbf{B}') = \mathcal{C}(\mathbf{R}'A') \cap \mathcal{C}(\mathbf{X}')$ . Hence an essential part of  $\mathbf{R}\boldsymbol{\beta} = \mathbf{r}$ with respect to  $\kappa$ , understood as a minimal set of the implied restrictions  $\mathbf{A}\mathbf{R}\boldsymbol{\beta} = \mathbf{A}\mathbf{r}$ satisfying the condition above, can be written as  $\mathbf{B}\mathbf{R}_1\boldsymbol{\beta} = \mathbf{B}\mathbf{r}_1$ , where  $\mathbf{B}$  is such that

$$\mathcal{C}(\mathbf{C}') = \mathcal{C}(\mathbf{S}\mathbf{B}'); \tag{2.7}$$

note that  $CS^-$  may be chosen as a representation of B; cf. Baksalary and Pordzik (1992, Corollary 2). The notion of essential restrictions occurs to be crucial for improving standard lines of majorization of the Euclidean distance between  $\hat{\kappa}_r$  and  $\hat{\kappa}$ .

**Theorem 1** Let  $\hat{\kappa}$  and  $\hat{\kappa}_r$  be the best linear unbiased estimators of  $\kappa = K\beta$  in the models  $\mathcal{M} = \{y, X\beta, \sigma^2 V\}$  and  $\mathcal{M}_r = \{y, X\beta | R\beta = r, \sigma^2 V\}$ , respectively. Then

$$\|\widehat{\boldsymbol{\kappa}} - \widehat{\boldsymbol{\kappa}}_r\|^2 \le \lambda \ (SSE_r^* - SSE), \tag{2.8}$$

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where  $\lambda$  is the largest eigenvalue of the matrix  $\sigma^{-2}[D(\hat{\kappa}) - D(\hat{\kappa}_r)]$ , SSE and SSE<sup>\*</sup><sub>r</sub> are the sums of squared errors in the models  $\mathcal{M}$  and  $\mathcal{M}_r^* = \{y, X\beta \mid BR_1\beta = Br_1, \sigma^2 V\}$ , wherein  $BR_1\beta = Br_1$  is an essential part of the restrictions  $R\beta = r$  with respect to  $\kappa$ .

*Proof* From Lemma 1 it follows that  $\hat{\kappa} - \hat{\kappa}_r = CS^-(\hat{\varrho}_1 - r_1) \in C(CS^-C')$  and, consequently,

$$\|\widehat{\boldsymbol{\kappa}} - \widehat{\boldsymbol{\kappa}}_r\|^2 \leq \|(\boldsymbol{C}\boldsymbol{S}^{-}\boldsymbol{C}')^{\frac{1}{2}}\|_s^2 \|(\boldsymbol{C}\boldsymbol{S}^{-}\boldsymbol{C}')^{\frac{1}{2}}(\boldsymbol{C}\boldsymbol{S}^{-}\boldsymbol{C}')^{-}\boldsymbol{C}\boldsymbol{S}^{-}(\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1)\|^2.$$

By definition of the spectral norm and invariance of the expression with respect to the choice of a g-inverse of the matrix S, this inequality can be written in the form

$$\|\widehat{\boldsymbol{\kappa}}-\widehat{\boldsymbol{\kappa}}_r\|^2 \leq \lambda (\widehat{\boldsymbol{\varrho}}_1-\boldsymbol{r}_1)'(\boldsymbol{S}^+)^{\frac{1}{2}} \boldsymbol{P}_{(\boldsymbol{S}^+)^{\frac{1}{2}}C'}(\boldsymbol{S}^+)^{\frac{1}{2}} (\widehat{\boldsymbol{\varrho}}_1-\boldsymbol{r}_1),$$

where  $\lambda$  is the largest eigenvalue of the matrix  $CS^-C' = \sigma^{-2}[D(\hat{k}) - D(\hat{k}_r)]$ . Furthermore, adopting the result stated in (2.6) to the restricted model  $\mathcal{M}_r^*$  one obtains

$$SSE_r^* = SSE + (\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1)'\boldsymbol{B}'(\boldsymbol{B}S\boldsymbol{B}')^{-}\boldsymbol{B}(\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1), \qquad (2.9)$$

with **B** satisfying (2.7). By the assumption of consistency of the model  $\mathcal{M}_r$ , it holds  $SS^+(\hat{\varrho}_1 - r_1) = (\hat{\varrho}_1 - r_1)$  and hence

$$SSE_{r}^{*} - SSE = (\hat{\boldsymbol{\varrho}}_{1} - \boldsymbol{r}_{1})'(\boldsymbol{S}^{+})^{\frac{1}{2}}\boldsymbol{P}_{(\boldsymbol{S}^{+})^{\frac{1}{2}}SB'}(\boldsymbol{S}^{+})^{\frac{1}{2}}(\hat{\boldsymbol{\varrho}}_{1} - \boldsymbol{r}_{1}).$$

Making use of the equality  $P_{(S^+)^{\frac{1}{2}}C'} = P_{(S^+)^{\frac{1}{2}}SB'}$ , implied by (2.7), completes the proof.

By the proof above, it is clear that  $SSE_r^*$  is invariant with respect to the choice of an essential part of  $R\beta = r$  for estimating  $\kappa$ ; as mentioned earlier, these essential restrictions can be represented by  $CS^-(R_1\beta - r_1) = 0$ .

Two other remarks are to be noted:

(*i*) The bound in (2.8) is equal to zero if and only if the estimator  $\hat{\kappa}$  continues to be the *blue* of  $\kappa$  under the model  $\mathcal{M}_r$ .

For the proof, let  $S_r$  denote the set of vectors y for which the restricted model  $M_r$  is consistent, then we have

$$\{\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1 : \boldsymbol{y} \in \mathcal{S}_r\} = \mathcal{C}(\boldsymbol{S}); \tag{2.10}$$

cf. Baksalary and Pordzik (1992, p. 242). From this and (2.9) it follows that  $SSE_r^* = SSE$  if and only if BS = 0. By (2.7), the latter equality is equivalent to C = 0, which is necessary and sufficient condition under which  $\hat{\kappa}_r = \hat{\kappa}$  for every  $y \in S_r$ ; cf. Baksalary and Pordzik (1992, Corollary 1). The equivalence of  $\lambda = 0$  and C = 0 is obvious.

(*ii*) The sums of squared errors  $SSE_r^*$  and  $SSE_r$  coincide if and only if all estimable restrictions are essential with respect to estimating  $\kappa$  in the model  $\mathcal{M}_r$ .

To prove this statement, first observe that, by the equalities (2.6) and (2.9), it holds

$$SSE_r - SSE_r^* = (\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1)'(\boldsymbol{S}^- - \boldsymbol{B}'(\boldsymbol{B}\boldsymbol{S}\boldsymbol{B}')^-\boldsymbol{B})(\widehat{\boldsymbol{\varrho}}_1 - \boldsymbol{r}_1).$$
(2.11)

Further, by (2.10), it follows that for every  $y \in S_r$  there exists a vector  $\alpha$  such that  $\widehat{\varrho}_1 - r_1 = S\alpha$  and, consequently, the right-hand side of (2.11) can be written as  $\alpha' S^{\frac{1}{2}} \mathcal{Q}_{S^{\frac{1}{2}}B'} S^{\frac{1}{2}} \alpha$ . This shows that  $SSE_r \geq SSE_r^*$ , wherein equality holds if and only if r(S) = r(SB'). In view of (2.7), we conclude that  $SSE_r = SSE_r^*$  if and only if r(S) = r(C), which means that  $R_1\beta = r_1$  form an essential part of  $R\beta = r$  with respect to  $\kappa$ ; this is the case, for instance, when the interest lies in estimating the expectation of y, that is, K = X.

#### **3** Applications

Consider the linear model  $\mathcal{M} = \{y, W\gamma + Z\delta, \sigma^2 I\}$  in which expectation consists of two parts:  $W\gamma$  and  $Z\delta$ , involving main and nuisance parameters, respectively. Let  $\eta = W' Q_Z W\gamma$  represent linearly estimable functions of the main parameters  $\gamma$  in  $\mathcal{M}$ . Clearly, the corresponding inference base without nuisance parameters can be written as the restricted model  $\mathcal{M}_r = \{y, W\gamma + Z\delta \mid Z\delta = 0, \sigma^2 I\}$ . Let  $\hat{\eta}$  and  $\hat{\eta}_r$ denote the best linear unbiased estimators of  $\eta$  obtained in the models  $\mathcal{M}$  and  $\mathcal{M}_r$ , respectively. Discussing the consequences of the presence of concomitant variables on estimating  $\eta$ , Baksalary (1984, Theorem 2.4) established a bound for the Euclidean distance between the two estimators, namely

$$\| \widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}_r \|^2 \le \lambda \cdot SSE_r, \tag{3.1}$$

where  $\lambda$  is the largest eigenvalue of the matrix  $\sigma^{-2}[D(\hat{\eta}) - D(\hat{\eta}_r)]$  and  $SSE_r$  is the sum of squared errors in the model  $\mathcal{M}_r$ . Some improvement of this bound can be obtained by the approach presented in Sect. 2. For applying the result of Theorem 1, we put X = (W : Z),  $\beta = (\gamma' : \delta')'$ , R = (0 : Z) and  $K = W'Q_Z X$ . By the algebraic property  $\mathcal{C}(A) \cap \mathcal{C}(B) = \mathcal{C}(A(A'B^{\perp})^{\perp})$ , it holds

$$\mathcal{C}(W:Z)' \cap \mathcal{C}(\mathbf{0}:Z)' = \mathcal{C}(\mathbf{0}:Q_WZ)'$$

which allows us to admit  $\mathbf{R}_1 = \mathbf{Z}' \mathbf{Q}_W \mathbf{X}$ . For  $\mathbf{V} = \mathbf{I}$ , a possible choice of  $\mathbf{G}_1$ ,  $\mathbf{G}_3$ and  $\mathbf{G}_4$  satisfying (2.1) to (2.3) is, respectively,  $\mathbf{Q}_X$ ,  $(\mathbf{X}'\mathbf{X})^-\mathbf{X}'$  and  $(\mathbf{X}'\mathbf{X})^-$ ; cf. Rao (1972). By this and the equality  $\mathbf{P}_{(W:Z)} = \mathbf{P}_W + \mathbf{P}_{Q_WZ}$ , we get  $\mathbf{C} = W' \mathbf{Q}_Z \mathbf{Q}_W \mathbf{Z}$ ,  $\mathbf{S} = \mathbf{Z}' \mathbf{Q}_W \mathbf{Z}$  and  $\hat{\mathbf{q}}_1 = \mathbf{Z}' \mathbf{Q}_W \mathbf{y}$ . Thus, referring to (2.8) and (2.9) with  $\mathbf{B} = \mathbf{C}\mathbf{S}^-$ , we obtain

$$\| \widehat{\boldsymbol{\eta}} - \widehat{\boldsymbol{\eta}}_r \|^2 \le \lambda \; (SSE_r^* - SSE), \tag{3.2}$$

where  $SSE = y' Q_{(W;Z)} y$  and  $SSE_r^* = SSE + y' P_{Q_W P_Z W} y$ . By the remarks following Theorem 1 and the equalities  $r(S) = r(Q_W Z)$  as well as

$$r(\boldsymbol{C}) = r(\boldsymbol{X}'\boldsymbol{Q}_{Z}\boldsymbol{P}_{X}\boldsymbol{Q}_{W}\boldsymbol{Z}) = r(\boldsymbol{Q}_{Z}\boldsymbol{P}_{X}\boldsymbol{Q}_{W}\boldsymbol{Z}) = r(\boldsymbol{Q}_{Z}\boldsymbol{Q}_{W}\boldsymbol{Z}), \quad (3.3)$$

it is clear that  $SSE_r^* = SSE_r$  if and only if  $C(Q_W Z) \cap C(Z) = \{0\}$ . Furtheron, note that the bound stated in (3.2) is equal to zero if and only if C = 0, that is,  $C(Q_W Z) \subseteq C(Z)$ ; the latter relation is the orthogonality condition for the two-way classification of data embraced by the model  $\mathcal{M}$ . Finally, observe that even if one gives up referring to an essential part of the restrictions  $Z\delta = 0$  with respect to  $\eta$ , it is still possible to improve (3.1) by (3.2) with  $SSE_r^*$  being replaced by  $SSE_r$ .

#### 4 Conclusions

Mäkinen (2002) obtained a bound for the Euclidean distance between the best linear unbiased estimator and any linear unbiased estimator in the general linear model. When applied to  $\hat{\kappa}$  and  $\hat{\kappa}_r$  under the model  $\mathcal{M}_r$ , this general approach leads to  $|| \hat{\kappa} - \hat{\kappa}_r ||^2 \leq \lambda \cdot SSE_r$ . The analogous bound was derived by Baksalary (1984) in the context of a standard linear model with nuisance parameters. Both results can be viewed as a tool for measuring the closeness of two competing estimators under a given inference base. Being oriented towards a comparison of two given estimators under the linear models  $\mathcal{M}$  and  $\mathcal{M}_r$ , our approach allows to improve the bound stated above. Making use of the relationships between statistical inference in both models, the new bound for the Euclidean norm of  $\hat{\kappa} - \hat{\kappa}_r$  is defined by two factors, based on the difference between the corresponding characteristics evaluated in  $\mathcal{M}$  and  $\mathcal{M}_r$ . Thus, the bound stated in Theorem 1 remains valid and allows a clear statistical interpretation independently of the fact whether  $\mathcal{M}$  or  $\mathcal{M}_r$  is assumed to be a true inference base.

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