

Comment on the paper by A. H. Joarder

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The recent paper by Joarder (2007) studied the moments of the product and ratio of two correlated chi-square variables. The main results of the paper are presented in Theorems 2.1, 3.1 and 4.1. Theorem 2.1 derives the exact probability density function (pdf) of the two correlated chi-square variables in the form:

$$f(u_1, u_2) = A(u_1, u_2) \sum_{k=0}^{\infty} \left\{ 1 + (-1)^k \right\} \left(\frac{\rho \sqrt{u_1 u_2}}{1 - \rho^2} \right)^k \frac{\Gamma((k+1)/2)}{k! \Gamma((k+m)/2)}, \quad (1)$$

where

$$A(u_1, u_2) = \frac{2^{-m-1} (u_1 u_2)^{m/2-1} \exp\{- (u_1 + u_2) / (2(1 - \rho^2))\}}{\sqrt{\pi} \Gamma(m/2) (1 - \rho^2)^{m/2}}.$$

Theorem 3.1 derives the product moment of the two correlated chi-square variables in the form:

$$\begin{aligned} E(U_1^a U_2^b) &= B \sum_{k=0}^{\infty} \left\{ 1 + (-1)^k \right\} \frac{(2\rho)^k}{k! \Gamma((k+m)/2)} \Gamma\left(\frac{k+m}{2} + a\right) \\ &\quad \times \Gamma\left(\frac{k+m}{2} + b\right) \Gamma\left(\frac{k+1}{2}\right), \end{aligned} \quad (2)$$

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where

$$B = \frac{2^{a+b-1} (1 - \rho^2)^{a+b+m/2}}{\sqrt{\pi} \Gamma(m/2)}.$$

Theorem 4.1 derives the pdf of the correlation coefficient for a bivariate t population in the form:

$$h(r) = C(r) \sum_{k=0}^{\infty} \frac{(2\rho r)^k}{k!} \Gamma^2\left(\frac{k+m}{2}\right), \quad (3)$$

where

$$C(r) = \frac{2^{m-2} (1 - \rho^2)^{m/2} (1 - r^2)^{(m-3)/2}}{\pi \Gamma(m-1)}.$$

Each of (1), (2) and (3) involves an infinite sum. It appears that [Joarder \(2007\)](#) made no attempts to simplify the infinite sums. Here, I would like to point out that each of (1), (2) and (3) can be reduced to known hypergeometric functions. This can be handy since in-built routines for the hypergeometric functions are widely available.

First consider (1). Note that

$$\begin{aligned} f(u_1, u_2) &= 2A(u_1, u_2) \sum_{k=0}^{\infty} \left(\frac{\rho \sqrt{u_1 u_2}}{1 - \rho^2} \right)^{2k} \frac{\Gamma(k+1/2)}{(2k)! \Gamma(k+m/2)} \\ &= 2\sqrt{\pi} A(u_1, u_2) \sum_{k=0}^{\infty} \left(\frac{\rho^2 u_1 u_2}{4(1-\rho^2)^2} \right)^k \frac{1}{k! \Gamma(k+m/2)} \\ &= \frac{2\sqrt{\pi} A(u_1, u_2)}{\Gamma(m/2)} {}_0F_1\left(; \frac{m}{2}; \frac{\rho^2 u_1 u_2}{4(1-\rho^2)^2}\right), \end{aligned} \quad (4)$$

where ${}_0F_1(; a; x)$ is the hypergeometric function defined by

$${}_0F_1(; a; x) = \sum_{n=0}^{\infty} \frac{1}{(a)_n} \frac{x^n}{n!},$$

where $(f)_n = \Gamma(f+n)/\Gamma(f)$ denotes the ascending factorial. Now consider (2). Note that

$$\begin{aligned} E(U_1^a U_2^b) &= 2B \sum_{k=0}^{\infty} \frac{(2\rho)^{2k}}{(2k)! \Gamma(k+m/2)} \Gamma\left(k + \frac{m}{2} + a\right) \Gamma\left(k + \frac{m}{2} + b\right) \Gamma\left(k + \frac{1}{2}\right) \\ &= 2\sqrt{\pi} B \sum_{k=0}^{\infty} \frac{\rho^{2k}}{k! \Gamma(k+m/2)} \Gamma\left(k + \frac{m}{2} + a\right) \Gamma\left(k + \frac{m}{2} + b\right) \\ &= \frac{2\sqrt{\pi} B \Gamma(m/2+a) \Gamma(m/2+b)}{\Gamma(m/2)} {}_2F_1\left(\frac{m}{2} + a, \frac{m}{2} + b; \frac{m}{2}; \rho^2\right), \end{aligned} \quad (5)$$

where ${}_2F_1(a, b; c; x)$ is the Gauss hypergeometric function defined by

$${}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}.$$

Finally, consider (3). Note that

$$\begin{aligned} h(r) &= C(r) \left[\sum_{k=0}^{\infty} \frac{(2\rho r)^{2k}}{(2k)!} \Gamma^2\left(k + \frac{m}{2}\right) + \sum_{k=0}^{\infty} \frac{(2\rho r)^{2k+1}}{(2k+1)!} \Gamma^2\left(k + \frac{m+1}{2}\right) \right] \\ &= \sqrt{\pi} C(r) \left[\sum_{k=0}^{\infty} \frac{(\rho^2 r^2)^k}{k!} \frac{\Gamma^2(k+m/2)}{\Gamma(k+1/2)} + \rho r \sum_{k=0}^{\infty} \frac{(\rho^2 r^2)^k}{k!} \frac{\Gamma^2(k+(m+1)/2)}{\Gamma(k+3/2)} \right] \\ &= C(r) \left[\Gamma^2\left(\frac{m}{2}\right) {}_2F_1\left(\frac{m}{2}, \frac{m}{2}; \frac{1}{2}; \rho^2 r^2\right) \right. \\ &\quad \left. + 2\rho r \Gamma^2\left(\frac{m+1}{2}\right) {}_2F_1\left(\frac{m+1}{2}, \frac{m+1}{2}; \frac{3}{2}; \rho^2 r^2\right) \right]. \end{aligned} \quad (6)$$

The hypergeometric functions in (4), (5) and (6) are well known and well established, see Prudnikov et al. (1986) for detailed properties. The nine corollaries—Corollaries 3.1 to 3.9—of Joarder (2007) follow from (5) by using known properties of the Gauss hypergeometric function, see Sect. 7.3 of Prudnikov et al. (1986, vol 3).

References

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