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Mechanical Balance Laws for Boussinesq Models of Surface Water Waves

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Abstract Depth-integrated long-wave models, such as the shallow-water and Boussinesq equations, are standard fare in the study of small amplitude surface waves in shallow water. While the shallow-water theory features conservation of mass, momentum and energy for smooth solutions, mechanical balance equations are not widely used in Boussinesq scaling, and it appears that the expressions for many of these quantities are not known. This work presents a systematic derivation of mass, momentum and energy densities and fluxes associated with a general family of Boussinesq systems. The derivation is based on a reconstruction of the velocity field and the pressure in the fluid column below the free surface, and the derivation of differential balance equations which are of the same asymptotic validity as the evolution equations. It is shown that all these mechanical quantities can be expressed in terms of the principal dependent variables of the Boussinesq system: the surface excursion η and the horizontal velocity w at a given level in the fluid.

Keywords Water waves · Boussinesq systems · Conservation laws · Pressure

Mathematics Subject Classification (2000) 35Q35 · 35Q53 · 76B15 · 76M45

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1 Introduction

Long-wave models for weakly nonlinear surface water waves were first developed by Boussinesq (1872) to describe surface gravity waves of small amplitude and long wavelength, propagating in a horizontal channel of uniform depth. These models are derived under the assumption that there is an approximate balance between nonlinear steepening effects and dispersive spreading. In mathematical terms, this balance is expressed by introducing two small parameters α and β measuring the wave amplitude and the wavelength, respectively, by comparison with the undisturbed depth of the channel. If both α and β are small and of the same order of magnitude, then a general system of equations may be derived which models the evolution of surface waves which fall into this scaling regime. The situation was reviewed recently by Bona et al. (2002), and a general family of Boussinesq systems was put forward. The present article aims for a further development of physical properties of the Boussinesq systems derived in Bona et al. (2002). In particular, the focus will be on derivation of mass, momentum and energy densities, and the associated fluxes in terms of the dependent variables used in the model equations. Moreover, expressions for the pressure associated to the particular variables used in these systems will be derived. It is our hope that these quantities will prove useful in studies where Boussinesq models are used to understand the effect of surface waves on the fluid flow underneath the surface.

The surface water-wave problem is generally described by the Euler equations with no-flow conditions at the bottom, and kinematic and dynamic boundary conditions at the free surface. Assuming weak transverse effects, the unknowns are the surface elevation $\eta(x,t)$, the horizontal and vertical fluid velocities $u_1(x,z,t)$ and $u_2(x,z,t)$, respectively, and the pressure $P(x,z,t)$. If the assumption of irrotational flow is made, then a velocity potential $\phi(x,z,t)$ can be used. If one is aiming for disturbances which are localized near the observer, and a flat bottom is given, the problem may be posed on a domain $\{(x,z) \in \mathbb{R}^2 \mid -h_0 < z < \eta(x,t)\}$ which extends to infinity in the positive and negative x-direction, and where the parameter h_0 represents the undisturbed depth of the fluid. Due to the incompressibility of the fluid, the potential then satisfies Laplace's equation in this domain. The fact that the fluid cannot penetrate the bottom is expressed by a homogeneous Neumann boundary condition at the flat bottom. Thus we have

$$\begin{aligned} \triangle\phi &= 0 \quad \text{in } -h_0 < z < \eta(x,t), \\ \phi_z &= 0 \quad \text{on } z = -h_0. \end{aligned}$$

The pressure is eliminated with help of the Bernoulli equation, and the free-surface boundary conditions are formulated in terms of the potential and the surface excursion by

$$\left.\begin{aligned} &\eta_t + \phi_x \eta_x - \phi_z = 0, \\ &\phi_t + \tfrac{1}{2}(\phi_x^2 + \phi_z^2) + g\eta = 0, \end{aligned}\right\} \quad \text{on } z = \eta(x,t),$$

where g represents the gravitational acceleration. The geometric setup of the problem is illustrated in Fig. 1.

As is well known, the difficulty in this problem lies mainly in the fact that the fluid domain is not known a priori. The initial-value problem for this equation has

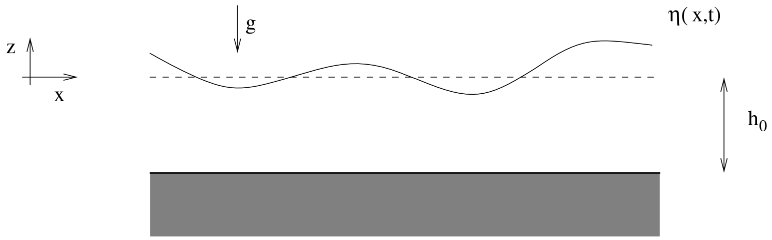

Fig. 1 The schematic elucidates the geometric setup of the problem. The free surface is described by a function $\eta(x,t)$. The undisturbed water depth is h_0, the gravitational acceleration is g, and the x-axis is aligned with the free surface at rest. The density of the fluid is ρ

been studied in several works, culminating in recent proofs of local well-posedness by Wu (1997) for the two-dimensional problem with infinite depth, and further improvements by Wu (1999) to treat the three-dimensional case. Lannes (2005) gave a simplified but more general proof in arbitrary dimensions which also covers the physically relevant case of finite depth and uneven bottom. If the main interest is in the evolution of the free surface, one may use a simplified model system which will be valid asymptotically for waves of small amplitude, for long wavelength, or both. Supposing that a is a representative amplitude, and ℓ represents a dominant wavelength, the Boussinesq scaling consists of assuming that $\alpha = a/h_0 \ll 1$, $\beta = h_0^2/\ell^2 \ll 1$, and $\alpha \sim \beta$. It was shown in Bona et al. (2002) that if the Boussinesq scaling is used then a general system of the form

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x + a w_{xxx} - b\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x + c\eta_{xxx} - d w_{xxt} &= 0, \end{aligned} \tag{1.1}$$

may be derived. Here η represents the excursion of the free surface as before, while w is a function of x and t only, which represents the horizontal velocity at a given height $h_0\theta$ in the fluid column. The parameters a, b, c and d are given by

$$a = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda h_0^3, \qquad b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda)h_0^2,$$

$$c = \frac{1}{2}\left(1-\theta^2\right)\mu g h_0^2, \qquad d = \frac{1}{2}\left(1-\theta^2\right)(1-\mu)h_0^2,$$

where $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$ are modeling parameters which can be chosen freely, and $0 \le \theta \le 1$. For a further discussion of the asymptotic validity of these model equations, the reader may consult (Bona et al. 2002) and the references contained therein. The work presented in Bona et al. (2002, 2004) was important in the sense that it gave a complete classification of systems of Boussinesq type, and provided a comprehensive study of existence and uniqueness of solutions for these systems. The validity of these and similar equations as asymptotic models for water waves has been under investigation in a number of mathematical works, starting with the work of Craig (1985), and in later work by Schneider and Wayne (2000), and Bona et al. (2005). The global existence of the water-wave problem in scaling regimes relevant for the system (1.1) has been proved by Alvarez-Samaniego and Lannes (2008).

The attention in the present paper is directed toward the physical quantities that characterize the water motion inside the channel, such as water pressure, and mass, momentum and energy densities and fluxes. It will turn out that all these quantities can be reconstructed from the two principal unknowns, η and w. This study was originally motivated by our desire to find accurate expressions for the energy associated with a surface wave field described by Boussinesq scaling, and to understand the energy conservation properties in undular bores propagating in narrow channels with even bottom. Regarding this problem, there has been an ongoing debate about the energy loss at the bore front which is predicted by the shallow-water theory. In Ali and Kalisch (2010), a preliminary version of an energy integral was used to show that energy losses exist only as a consequence of the approximate nature of the equations, and not because of some dissipation mechanism which is unaccounted for in Boussinesq scaling. This is only reasonable, of course, since the shallow-water system itself is purely hyperbolic, and an energy loss in this theory only occurs because of the loss of regularity due to hyperbolic wave breaking at the bore front.

To set the stage for the developments in the body of the paper, we consider the existing theory for the shallow-water approximation. The shallow-water approximation makes the assumption that there is no significant vertical acceleration of the fluid particles. Equivalently, it is assumed that the parameter β is small, so the waves to be described are long compared to the undisturbed depth h_0. In two dimensions, the shallow-water system is given by

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x &= 0, \\ w_t + g\eta_x + w w_x &= 0. \end{aligned} \tag{1.2}$$

Now this system is derived from physical principles of mass and momentum conservation in a control volume, such as shown in Fig. 2. In the shallow-water approximation these equations are vertically integrated, and are given by

$$\frac{\partial}{\partial t} M^0 + \frac{\partial}{\partial x} q_M^0 = 0, \tag{1.3}$$

$$\frac{\partial}{\partial t} I^0 + \frac{\partial}{\partial x} q_I^0 = 0, \tag{1.4}$$

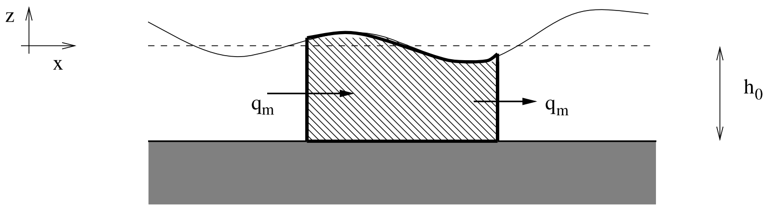

Fig. 2 A typical control volume used in the derivation of mass, momentum and energy balance laws. This control volume is a blend of a fixed and a material control volume. The bottom and lateral boundaries are held fixed while the upper boundary moves with the free surface. The mass flux is indicated through the lateral boundaries

where M^0 and I^0 are the mass and momentum density per unit span, respectively, while q_M^0 and q_I^0 are the corresponding fluxes. These quantities are given by

$$M^0 = \rho(h_0 + \eta),$$

$$q_M^0 = \rho(h_0 w + \eta w),$$

and

$$I^0 = \rho(h_0 + \eta)w, \tag{1.5}$$

$$q_I^0 = \rho(h_0 + \eta)w^2 + \frac{\rho}{2} g(h_0 + \eta)^2. \tag{1.6}$$

Note that the momentum flux contains a term which represents the pressure force. Equations (1.3) and (1.4) can then be written as

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x = 0, \\ \big[(h_0 + \eta)w\big]_t + \big[(h_0 + \eta)w^2\big]_x + g(h_0 + \eta)\eta_x = 0. \end{aligned} \tag{1.7}$$

For smooth solutions, the systems (1.7) and (1.2) are equivalent. The conservation of energy is not a separate principle in homogeneous fluids, but follows from the equations of motion (Kundu and Cohen 2008). The conservation of energy takes the form

$$\frac{\partial}{\partial t} E^0 + \frac{\partial}{\partial x} q_E^0 = 0, \tag{1.8}$$

where the energy density associated to the shallow-water approximation is given by

$$E^0 = \frac{\rho}{2} w^2 (h_0 + \eta) + \frac{\rho}{2} g\big(2h_0\eta + \eta^2\big). \tag{1.9}$$

Note that this quantity has been written in such a way that the potential energy is zero if there is no wave motion, i.e. if $\eta = 0$. The energy flux corrected for the work done by the pressure force is

$$q_E^0 = \frac{\rho}{2} w^3 (h_0 + \eta) + \rho g w (h_0 + \eta)^2. \tag{1.10}$$

The shallow-water system has additional conservation laws of the form (1.8), but these do not appear to describe any physical properties of the fluid (Whitham 1974). Finally, note that the conservation of angular momentum is encoded in the assumption of irrotational flow.

The main contribution of the present paper is the systematic derivation of expressions for the mass, momentum and energy densities and fluxes in the context of Boussinesq scaling. These expressions are similar to the ones recorded above, and in most cases correctly reduce to their shallow-water counterparts in the limit of very long waves, i.e. if the parameter β is taken to be zero. We first recall the derivation of the family (1.1) of systems in the next section. In Sect. 3, the pressure associated with a particular Boussinesq model is found. In Sects. 4, 5, and 6, the mass, momentum and energy balance equations are computed. In Sect. 7, these expressions are tabulated for some particular systems, and in Sect. 8, a numerical example is provided. Section 9 contains the results of a similar study pertaining to higher-order Boussinesq systems. Finally, a short conclusion is given in Sect. 10.

2 Derivation of the System

To set the stage for finding the mass, momentum and energy integrals, we recall the derivation of the general system (1.1). In order to identify the relevant terms in the equations, the variables are non-dimensionalized in the following way:

$$\tilde{x} = \frac{x}{\ell}, \qquad \tilde{z} = \frac{z+h_0}{h_0}, \qquad \tilde{\eta} = \frac{\eta}{a}, \qquad \tilde{t} = \frac{c_0 t}{\ell}, \qquad \tilde{\phi} = \frac{c_0}{ga\ell}\phi,$$

where $c_0 = \sqrt{gh_0}$. The free-surface boundary conditions then take the form

$$\left.\begin{aligned} &\tilde{\eta}_{\tilde{t}} + \alpha\tilde{\phi}_{\tilde{x}}\tilde{\eta}_{\tilde{x}} - \tfrac{1}{\beta}\tilde{\phi}_{\tilde{z}} = 0, \\ &\tilde{\eta} + \tilde{\phi}_{\tilde{t}} + \tfrac{1}{2}\big(\alpha\tilde{\phi}_{\tilde{x}}^2 + \tfrac{\alpha}{\beta}\tilde{\phi}_{\tilde{z}}^2\big) = 0, \end{aligned}\right\} \quad \text{on } \tilde{z} = 1 + \alpha\tilde{\eta}. \tag{2.1}$$

The standard approach consists of developing the potential ϕ in an asymptotic series, and using the Laplace equation and Neumann boundary condition at the bottom to write the non-dimensional velocity potential $\tilde{\phi}$ in the form

$$\tilde{\phi} = \sum_{m=0}^{\infty}(-1)^m \frac{\tilde{z}^{2m}}{(2m)!}\frac{\partial^{2m}\tilde{f}}{\partial\tilde{x}^{2m}}\beta^m = \tilde{f} - \frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}}\beta + \frac{\tilde{z}^4}{24}\tilde{f}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}\big(\beta^3\big).$$

Substituting this expression into the second boundary condition at the free surface yields the relation

$$\tilde{\eta} + \tilde{f}_{\tilde{t}} - \frac{\beta}{2}\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} + \frac{\alpha}{2}\tilde{f}_{\tilde{x}}^2 = \mathcal{O}\big(\alpha\beta, \beta^2\big), \tag{2.2}$$

which will be of use later in the derivation of the pressure. To find a closed system of two evolution equations, we insert the asymptotic expression for $\tilde{\phi}$ in the first equation in (2.1), and collect all terms of zeroth and first order in α and β. Then, we differentiate (2.2) and express the equations in terms of the non-dimensional horizontal velocity at the bottom $f_{\tilde{x}} = \tilde{v}$. This procedure yields the equations

$$\begin{aligned} \tilde{\eta}_{\tilde{t}} + \tilde{v}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{v})_{\tilde{x}} - \frac{1}{6}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}} &= \mathcal{O}\big(\alpha\beta, \beta^2\big), \\ \tilde{\eta}_{\tilde{x}} + \tilde{v}_{\tilde{t}} - \frac{1}{2}\beta\tilde{v}_{\tilde{x}\tilde{x}\tilde{t}} + \alpha\tilde{v}\tilde{v}_{\tilde{x}} &= \mathcal{O}\big(\alpha\beta, \beta^2\big). \end{aligned} \tag{2.3}$$

Now if we let $\tilde{w}$ be the non-dimensional velocity at a non-dimensional height $0 \le \theta \le 1$ in the fluid column, then Taylor's formula shows that

$$\tilde{\phi}_{\tilde{x}}|_{\tilde{z}=\theta} = \tilde{w} = \tilde{v} - \frac{\theta^2}{2}\tilde{v}_{\tilde{x}\tilde{x}}\beta + \frac{\theta^4}{24}\tilde{v}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}\big(\beta^3\big).$$

Then as shown in Bona et al. (2002), $\tilde{v}$ may be expressed in terms of $\tilde{w}$ by

$$\tilde{v} = \tilde{w} + \frac{1}{2}\beta\theta^2\tilde{w}_{\tilde{x}\tilde{x}} + \theta^4\frac{5}{24}\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}\beta^2 + \mathcal{O}(\beta^2). \tag{2.4}$$

Substituting this representation into the system (2.3) yields

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}(\alpha\beta, \beta^2),$$

$$\tilde{\eta}_{\tilde{x}} + \tilde{w}_{\tilde{t}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{1}{2}\beta(\theta^2 - 1)\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2).$$

Now for any real λ and μ, the previous system is a special case of the more general system

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda)\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{t}}$$
$$= \mathcal{O}(\alpha\beta, \beta^2),$$

$$\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} + \frac{1}{2}(1-\theta^2)\mu\beta\tilde{\eta}_{\tilde{x}\tilde{x}\tilde{x}} - \frac{1}{2}(1-\theta^2)(1-\mu)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}(\alpha\beta, \beta^2).$$

If terms of order $\mathcal{O}(\alpha\beta, \beta^2)$ are disregarded, the system takes the following form in dimensional variables:

$$\eta_t + h_0 w_x + (\eta w)_x + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\lambda h_0^3 w_{xxx} - \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)(1-\lambda)h_0^2\eta_{xxt}$$
$$= 0, \tag{2.5}$$

$$w_t + g\eta_x + ww_x + \frac{1}{2}(1-\theta^2)\mu g h_0^2\eta_{xxx} - \frac{1}{2}(1-\theta^2)(1-\mu)h_0^2 w_{xxt} = 0. \tag{2.6}$$

One important fact to notice about the systems (2.5) and (2.6) is that the higher-order terms do not contain summands of the order $\mathcal{O}(\alpha^2)$. This fact shows that any higher-order system derived along these lines will reduce to the shallow-water system (1.2) in the limit of very long waves, i.e. when $\beta \to 0$.

In the following sections, the Boussinesq system (2.5) and (2.6) is studied with respect to reconstruction of the pressure, and computation of mass, momentum and energy densities and fluxes from the dependent variables η and w. Considering for instance the total momentum contained in a control volume in the fluid reaching from the even bottom to the free surface, such as shown in Fig. 2, it is not immediately clear how to determine the correct order of approximation of such a quantity. However, after a moment of thought it appears that a natural way to proceed is to derive differential balance laws in analogy with (1.7) in the shallow-water theory. However, it will turn out that in Boussinesq scaling, these balance laws are valid only up to the same order as the evolution equations. Accordingly, to (2.5) and (2.6) derived above, which are valid to order $\mathcal{O}(\alpha\beta, \beta^2)$, there are associated non-dimensional densities $\tilde{X}$ and fluxes $\tilde{q}_X$ which satisfy the relation

$$\frac{\partial}{\partial \tilde{t}}\tilde{X} + \frac{\partial}{\partial \tilde{x}}\tilde{q}_X = \mathcal{O}\big(\alpha^2, \alpha\beta, \beta^2\big)$$

where X may represent the mass, momentum or energy density, and q_X the corresponding flux.

Various forms of energy integrals have been used in the literature. In particular, previous efforts of studying energy budgets in Boussinesq scaling include the incorporation of an energy equation into the evolution system. This approach was taken by Dutykh and Dias, and it yields a third evolution equation (Dutykh and Dias 2009) in addition to the two equations (2.5) and (2.6). In the work of Christov (2001), the focus was on deriving'energy-consistent' Boussinesq models, which were required to preserve a certain functional in time. In a long study, Keulegan and Patterson defined an energy integral associated to the single Boussinesq equation (Keulegan and Patterson 1940), which apparently represents the physical energy. Despite these efforts, a systematic study deriving these expressions for all Boussinesq systems of the type derived in Bona et al. (2002) seems to be unavailable at present.

3 Pressure

In this section the pressure associated with the general system (2.5) and (2.6) is found. While it is interesting for various applications to be able to reconstruct the pressure from the primary dependent variables of the equations of motion, an appropriate expression for the pressure is also essential for computation of the momentum and energy balances. Indeed the pressure force on a control volume and the work done by the pressure force appear prominently in these balance laws.

The starting point for obtaining the pressure is the Bernoulli equation,

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 = -\frac{P}{\rho} - gz + C. \tag{3.1}$$

In order to find the constant C, we evaluate this equation at the free surface. Assuming that the surface disturbance is sufficiently localized so that $\eta \to 0$ and $\phi \to$ const. as $x \to \infty$, the constant C is given by

$$C = \frac{P_{\text{atm}}}{\rho},$$

where P_{atm} denotes the atmospheric pressure. Therefore the pressure P in equation (3.1) can be obtained from

$$P - P_{\text{atm}} = -\rho g z - \rho\phi_t - \frac{\rho}{2}|\nabla\phi|^2.$$

As is customary, we introduce the dynamic pressure P' which measures the deviation from hydrostatic pressure. The dynamic pressure is defined by

$$P' = P - P_{\text{atm}} + \rho g z = -\rho\phi_t - \frac{\rho}{2}|\nabla\phi|^2.$$

The dynamic pressure is scaled by using a typical wave amplitude a. Accordingly, if we define $\rho a g \tilde{P}' = P'$, then $\tilde{P}'$ is given in terms of the velocity potential by

$$\begin{aligned}\tilde{P}' &= \frac{P - P_{\mathrm{atm}}}{ag\rho} + \frac{z}{a} \\ &= -\tilde{\phi}_{\tilde{t}} - \frac{1}{2}\alpha\tilde{\phi}_{\tilde{x}}^2 - \frac{1}{2}\frac{\alpha}{\beta}\tilde{\phi}_{\tilde{z}}^2.\end{aligned}$$

Substituting the expression for $\tilde{\phi}$ found in Sect. 2, we obtain

$$\tilde{P}' = -\tilde{f}_{\tilde{t}} + \beta\frac{\tilde{z}^2}{2}\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} - \frac{1}{2}\alpha\tilde{f}_{\tilde{x}}^2 + \mathcal{O}\big(\alpha\beta, \beta^2\big).$$

However, using the relation (2.2) in the previous section, we see that

$$\tilde{\eta} - \frac{1}{2}\beta\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} = -\tilde{f}_{\tilde{t}} - \frac{1}{2}\alpha\tilde{f}_{\tilde{x}}^2 + \mathcal{O}\big(\alpha\beta, \beta^2\big).$$

Recalling that $\tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} = \tilde{w}_{\tilde{x}\tilde{t}} + \mathcal{O}(\beta)$, the second-order dynamic pressure emerges in the form

$$\tilde{P}' = \tilde{\eta} + \frac{1}{2}\beta\big(\tilde{z}^2 - 1\big)\tilde{w}_{\tilde{x}\tilde{t}} + \mathcal{O}\big(\alpha\beta, \beta^2\big).$$

Using this expression, and converting to dimensional variables, the total pressure is given by

$$P = P_{\mathrm{atm}} + \rho g(\eta - z) + \frac{1}{2}\rho\big((z + h_0)^2 - h_0^2\big)w_{xt}.$$

Now if this expression is evaluated at the free surface it appears that the pressure is not equal to atmospheric pressure. Even though one may argue that the value of the pressure at the free surface is still correct to second order in α and β, this failure of the asymptotic method to represent the pressure at the free surface is certainly a drawback when the method is used in practice. On the other hand, the defect in the expression for the pressure may be alleviated by making the third-order correction

$$P_1 = P_{\mathrm{atm}} + \rho g(\eta - z) + \frac{1}{2}\rho\big((z + h_0)^2 - h_0^2\big)w_{xt} - \frac{1}{2}\rho\big(2h_0\eta + \eta^2\big)w_{xt}.$$

This approximation is correct to the same order as the previous expression, but it has the advantage that it takes on the appropriate value when evaluated at the free surface:

$$P_1|_{z=\eta} = P_{\mathrm{atm}}.$$

While it might be advantageous to use P_1 in practical situations, for the developments in the next few sections, the exact value of the pressure at the surface is immaterial. Moreover, as the atmospheric pressure is of a magnitude much smaller than the significant terms in the evolution equations, it will be assumed in the following that the atmospheric pressure is zero. Thus, we work with the expression

$$P = \rho g(\eta - z) + \frac{1}{2}\rho\big((z+h_0)^2 - h_0^2\big)w_{xt} \tag{3.2}$$

for the pressure. We note also that the definition of the pressure does not depend on the vertical level θh_0 where the horizontal velocity is modeled. Moreover, P does not depend on either of the modeling parameters λ and μ, and agrees with the expression for the pressure associated to the so-called 'classical' Boussinesq system (7.1), which was given in Peregrine (1972). The system (7.1) will be studied more closely in Sect. 7.

4 Mass Balance

In this section, mass conservation properties of the system (2.5) and (2.6) are explored. Consider the total mass of the fluid contained in a control volume of unit width, delimited by the interval $[x_1, x_2]$ on the lateral sides, and by the bottom and the free surface, such as indicated in Fig. 2. This mass is given by

$$\mathcal{M} = \int_{x_1}^{x_2}\int_{-h_0}^{\eta} \rho \,\mathrm{d}z\,\mathrm{d}x.$$

The physical principle of mass conservation implies that the change per unit time in the total mass in a control volume is equal to the net mass flux into the control volume. Since there is no mass flux through the bottom or through the free surface, mass conservation can be stated in terms of the flow variables as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2}\int_{-h_0}^{\eta} \rho \,\mathrm{d}z\,\mathrm{d}x = \left[\int_{-h_0}^{\eta} \rho\phi_x(x,z)\,\mathrm{d}z\right]_{x_2}^{x_1}.$$

In non-dimensional form this becomes

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\int_{0}^{1+\alpha\tilde{\eta}} \mathrm{d}\tilde{z}\,\mathrm{d}\tilde{x} = \alpha\left[\int_{0}^{1+\alpha\tilde{\eta}} \tilde{\phi}_{\tilde{x}}(\tilde{x},\tilde{z})\,\mathrm{d}\tilde{z}\right]_{x_2/\ell}^{x_1/\ell}.$$

Integrating with respect to $\tilde{z}$ and substituting the expression for $\tilde{\phi}_{\tilde{x}}$ in terms of $\tilde{v}$ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}(1+\alpha\tilde{\eta})\,\mathrm{d}\tilde{x} &= \alpha\left[\int_{0}^{1+\alpha\tilde{\eta}}\left\{\tilde{v} - \frac{\tilde{z}^2}{2}\beta\tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\beta^2\big)\right\}\mathrm{d}\tilde{z}\right]_{x_2/\ell}^{x_1/\ell} \\ &= \alpha\left[\tilde{v} + \alpha\tilde{v}\tilde{\eta} - \frac{\beta}{6}\tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha\beta,\beta^2\big)\right]_{x_2/\ell}^{x_1/\ell}.\end{aligned}$$

Using the approximation (2.4) leads to the first-order approximation

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}(1+\alpha\tilde{\eta})\,\mathrm{d}\tilde{x} = \alpha\left[\tilde{w} + \alpha\tilde{w}\tilde{\eta} + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha\beta,\beta^2\big)\right]_{x_2/\ell}^{x_1/\ell}. \tag{4.1}$$

We notice that the mass balance equation (4.1) contains terms of order $\mathcal{O}(\alpha^2\beta, \alpha\beta^2)$ on the right-hand side. In order to determine which terms should be kept to get an approximation of the same order as the evolution equations, it appears most natural to derive a differential form of the mass balance equation. Thus we divide by the length of the interval to obtain

$$\frac{1}{x_2/\ell - x_1/\ell}\int_{x_1/\ell}^{x_2/\ell} \tilde{\eta}_{\tilde{t}}\,d\tilde{x}$$
$$= \frac{1}{x_2/\ell - x_1/\ell}\left[\tilde{w} + \alpha\tilde{w}\tilde{\eta} + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha\beta, \beta^2\big)\right]_{x_2/\ell}^{x_1/\ell}.$$

Taking the limit as $x_2/\ell \to x_1/\ell$ yields the differential balance equation

$$\tilde{\eta}_{\tilde{t}} + \tilde{w}_x + \alpha(\tilde{w}\tilde{\eta})_{\tilde{x}} + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} = \mathcal{O}\big(\alpha\beta, \beta^2\big). \tag{4.2}$$

Denoting the non-dimensional mass density by

$$\tilde{M} = 1 + \alpha\tilde{\eta},$$

and the non-dimensional mass flux by

$$\tilde{q}_M = \alpha\tilde{w} + \alpha^2\tilde{\eta}\tilde{w} + \frac{1}{2}\alpha\beta\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}},$$

the mass balance is

$$\frac{\partial}{\partial\tilde{t}}\tilde{M} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_M = \mathcal{O}\big(\alpha\beta, \beta^2\big).$$

Using the scalings $M = \rho h_0\tilde{M}$ and $q_M = \rho h_0 c_0 \tilde{q}_M$, the dimensional forms of these quantities are

$$M = \rho(h_0 + \eta), \tag{4.3}$$

and

$$q_M = \rho h_0 w + \rho\eta w + \frac{\rho h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx}. \tag{4.4}$$

Incidentally, we see that dimensional form of (4.2) is the same as (2.5) with $\lambda = 1$. Thus the model (2.5) and (2.6) conserves mass exactly if the modeling parameter λ is chosen to be equal to one. For other choices of λ, mass is not exactly conserved, but only up to the same non-dimensional order as the equation is valid. A possible check on the viability of the expression for the mass is the behavior as the parameter β approaches zero. In this case, the waves are extremely long, and the shallow-water theory should be the result. Indeed, letting $\beta \to 0$ in the mass balance equation (4.2), and changing to dimensional variables, the equation reduces to the first equation in (1.7).

5 Momentum Balance

This section and the next are devoted to finding approximate expressions for momentum and energy contained in a control volume, and the respective fluxes. The total *horizontal* momentum of a fluid of constant density ρ, contained inside a control volume in a horizontal channel of unit width, reaching from the bottom to the free surface, and delimited by the interval $[x_1, x_2]$ in the x-direction is given by

$$\mathcal{I} = \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \phi_x \, \mathrm{d}z \, \mathrm{d}x.$$

Conservation of momentum is expressed by the requirement that the rate of change of $\mathcal{I}$ is equal to the net influx of momentum through the boundaries plus the net work done on the boundary of the control volume. This relation is expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho \phi_x \, \mathrm{d}z \, \mathrm{d}x = \left[\int_{-h_0}^{\eta} \rho \phi_x^2 \, \mathrm{d}z + \int_{-h_0}^{\eta} P \, \mathrm{d}z \right]_{x_2}^{x_1}.$$

The right-hand side of the last equality will be called the net momentum flux, corrected for pressure forces, or simply the momentum flux. Expressing the last relation in non-dimensional variables yields

$$\begin{aligned} &\alpha \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \tilde{\phi}_{\tilde{x}} \, \mathrm{d}\tilde{z} \, \mathrm{d}\tilde{x} \\ &\quad = \left[\alpha^2 \int_0^{1+\alpha\tilde{\eta}} \tilde{\phi}_{\tilde{x}}^2 \, \mathrm{d}\tilde{z} + \alpha \int_0^{1+\alpha\tilde{\eta}} \tilde{P}' \, \mathrm{d}\tilde{z} - \int_0^{1+\alpha\tilde{\eta}} (\tilde{z} - 1) \, \mathrm{d}\tilde{z} \right]_{x_2}^{x_1}. \end{aligned}$$

Substituting the values of $\tilde{\phi}_{\tilde{x}}$ and $\tilde{P}'$ found in Sects. 2 and 3 yields

$$\begin{aligned} &\alpha \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \left\{ \tilde{f}_{\tilde{x}} - \frac{\tilde{z}}{2} \beta \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \mathcal{O}\big(\beta^2\big) \right\} \mathrm{d}\tilde{z} \, \mathrm{d}\tilde{x} \\ &= \left[\alpha^2 \int_0^{1+\alpha\tilde{\eta}} \big\{ \tilde{f}_{\tilde{x}}^2 + \mathcal{O}(\beta) \big\} \, \mathrm{d}\tilde{z} \right. \\ &\quad \left. + \alpha \int_0^{1+\alpha\tilde{\eta}} \left\{ \tilde{\eta} + \frac{\beta}{2} \big(\tilde{z}^2 - 1\big) \tilde{f}_{\tilde{x}\tilde{x}\tilde{t}} + \mathcal{O}\big(\alpha\beta, \beta^2\big) \right\} \mathrm{d}\tilde{z} + \int_0^{1+\alpha\tilde{\eta}} (1 - \tilde{z}) \, \mathrm{d}\tilde{z} \right]_{x_2/\ell}^{x_1/\ell}. \end{aligned}$$

Using the non-dimensional horizontal velocity $\tilde{v} = \tilde{f}_{\tilde{x}}$ and integrating with respect to $\tilde{z}$ transforms this into

$$\begin{aligned} &\alpha \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \left\{ \tilde{v}(1 + \alpha\tilde{\eta}) - \frac{\beta}{6} \tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha\beta, \beta^2\big) \right\} \mathrm{d}\tilde{x} \\ &= \left[\alpha^2 \big\{ \tilde{v}^2 + \mathcal{O}(\alpha, \beta) \big\} + \left\{ \alpha\tilde{\eta} + \frac{\alpha^2}{2} \tilde{\eta}^2 - \frac{\alpha\beta}{3} \tilde{v}_{\tilde{x}\tilde{t}} + \frac{1}{2} + \mathcal{O}\big(\alpha^2\beta, \alpha\beta^2\big) \right\} \right]_{x_2/\ell}^{x_1/\ell}. \end{aligned} \tag{5.1}$$

Now expressing (5.1) in terms of $\tilde{w}$ using (2.4) yields

$$\alpha \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \left\{ \tilde{w}(1+\alpha\tilde{\eta}) + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}} \right\} \mathrm{d}\tilde{x}$$

$$= \left[\alpha^2 \tilde{w}^2 + \alpha\tilde{\eta} + \frac{\alpha^2}{2}\tilde{\eta}^2 - \frac{\alpha\beta}{3}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{1}{2} \right]_{x_2/\ell}^{x_1/\ell} + \mathcal{O}(\alpha^3, \alpha^2\beta, \alpha\beta^2). \quad (5.2)$$

Differentiating with respect to $\tilde{x}$ as in the previous section finally reveals the balance equation:

$$\tilde{w}_{\tilde{t}} + \alpha(\tilde{w}\tilde{\eta})_{\tilde{t}} + \frac{\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} + \tilde{\eta}_{\tilde{x}} + 2\alpha\tilde{w}\tilde{w}_{\tilde{x}} + \alpha\tilde{\eta}\tilde{\eta}_{\tilde{x}} - \frac{1}{3}\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}}$$

$$= \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

Again, this differential balance equation appears to be the most appropriate gauge of which terms are to be included in the expressions for momentum density and flux. Taking the appropriate terms in (5.2) which are of order zero or one in the differential momentum balance, we find the non-dimensional momentum density to be

$$\tilde{I} = \alpha\tilde{w} + \alpha^2\tilde{w}\tilde{\eta} + \frac{1}{2}\alpha\beta\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}},$$

and the non-dimensional momentum flux (corrected for the pressure force) as

$$\tilde{q}_I = \alpha\tilde{\eta} + \alpha^2\tilde{w}^2 + \frac{\alpha^2}{2}\tilde{\eta}^2 - \frac{\alpha\beta}{3}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{1}{2}.$$

Then the momentum balance is

$$\frac{\partial}{\partial\tilde{t}}\tilde{I} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_I = \mathcal{O}(\alpha^2, \alpha\beta, \beta^2).$$

Using the scaling $I = \rho c_0 h_0 \tilde{I}$ and $q_I = \rho c_0^2 h_0 \tilde{q}_I$, the dimensional forms of the momentum density and momentum flux per unit span are given by

$$I = \rho(h_0 + \eta)w + \frac{\rho}{2}\left(\theta^2 - \frac{1}{3}\right)h_0^3 w_{xx} \quad (5.3)$$

and

$$q_I = \rho h_0 w^2 + \frac{\rho g}{2}(h_0 + \eta)^2 - \frac{\rho h_0^3}{3} w_{xt}. \quad (5.4)$$

In order to compare these terms with the shallow-water theory we let $\beta \to 0$ in (5.2). Then the corresponding dimensional expressions for the momentum density and flux are

$$I = \rho(h_0 + \eta)w$$

and

$$q_I = \rho h_0 w^2 + \frac{\rho}{2} g (h_0 + \eta)^2.$$

It can be seen that the momentum density I reduces correctly to the shallow water form I^0. On the other hand, the flux does not reduce to the shallow-water momentum flux q_I^0. This is also borne out by the fact that (5.2) contains a correction term of order $\mathcal{O}(\alpha^3)$, namely the term $\alpha^3 \tilde{w}^2 \tilde{\eta}$. If agreement is desired here, one has to consider a higher-order theory. Such an approach will be presented in Sect. 9. Another way to obtain agreement is to use a higher-order correction for the momentum flux only by adding the term $\alpha^3 \tilde{w}^2 \tilde{\eta}$. This correction will keep the expression correct to the same order, but will facilitate comparison with the shallow-water theory.

6 Energy Balance

The total mechanical energy inside a control volume of the same type as used in the previous two sections is

$$\mathcal{E} = \frac{1}{2} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \rho |\nabla \phi|^2 \,\mathrm{d}z \,\mathrm{d}x + \int_{x_1}^{x_2} \int_0^{\eta} \rho g (z + h_0) \,\mathrm{d}z \,\mathrm{d}x,$$

where the first term represents the kinetic energy, and the second term is the potential energy. The potential energy has been defined in such a way that it is zero when no wave motion is present. Conservation of total mechanical energy is written as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} \int_{-h_0}^{\eta} \frac{\rho}{2} |\nabla \phi|^2 \,\mathrm{d}z \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t} \int_{x_1}^{x_2} \int_0^{\eta} \rho g (z + h_0) \,\mathrm{d}z \,\mathrm{d}x \\ &\quad = \left[\int_{-h_0}^{\eta} \left\{ \frac{\rho}{2} |\nabla \phi|^2 + \rho g (z + h_0) \right\} \phi_x \,\mathrm{d}z + \int_{-h_0}^{\eta} \phi_x P \,\mathrm{d}z \right]_{x_2}^{x_1}. \end{aligned}$$

This conservation equation follows from physical conservation of energy; a more detailed derivation may be found in Stoker (1957), Chap. 1. Converting to non-dimensional variables gives

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \left\{ \frac{\alpha^2}{2} \left(\tilde{\phi}_{\tilde{x}}^2 + \frac{1}{\beta} \tilde{\phi}_{\tilde{z}}^2 \right) \right\} \mathrm{d}\tilde{z} \,\mathrm{d}\tilde{x} + \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_1^{1+\alpha\tilde{\eta}} \tilde{z} \,\mathrm{d}\tilde{z} \,\mathrm{d}\tilde{x} \\ &= \alpha \left[\int_0^{1+\alpha\tilde{\eta}} \left\{ \frac{\alpha^2}{2} \left(\tilde{\phi}_{\tilde{x}}^3 + \frac{1}{\beta} \tilde{\phi}_{\tilde{z}}^2 \tilde{\phi}_{\tilde{x}} \right) + \tilde{z} \tilde{\phi}_{\tilde{x}} \right\} \mathrm{d}\tilde{z} \right. \\ &\qquad \left. + \alpha \int_0^{1+\alpha\tilde{\eta}} \tilde{P}' \tilde{\phi}_{\tilde{x}} \,\mathrm{d}\tilde{z} + \int_0^{1+\alpha\tilde{\eta}} (1 - \tilde{z}) \tilde{\phi}_{\tilde{x}} \,\mathrm{d}\tilde{z} \right]_{x_2/\ell}^{x_1/\ell}. \end{aligned} \tag{6.1}$$

We compute these integrals individually. Substituting the expressions for $\tilde{\phi}_{\tilde{x}}$ and $\tilde{\phi}_{\tilde{z}}$ yields

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \left\{ \frac{\alpha^2}{2} \left(\tilde{\phi}_{\tilde{x}}^2 + \frac{1}{\beta} \tilde{\phi}_{\tilde{z}}^2 \right) \right\} \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} + \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_1^{1+\alpha\tilde{\eta}} \tilde{z}\, \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \frac{\alpha^2}{2} \tilde{f}_{\tilde{x}}^2\, \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} + \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_1^{1+\alpha\tilde{\eta}} \tilde{z}\, \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} + \mathcal{O}\big(\alpha^2\beta\big) \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \left\{ \frac{\alpha^2}{2} \tilde{f}_{\tilde{x}}^2 + \frac{(1+\alpha\tilde{\eta})^2}{2} - \frac{1}{2} \right\} \mathrm{d}\tilde{x} + \mathcal{O}\big(\alpha^3, \alpha^2\beta\big).
\end{aligned}
$$

Recalling that $f_{\tilde{x}} = \tilde{v}$ is the velocity at the bottom, and using (2.4), there appears

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_0^{1+\alpha\tilde{\eta}} \left\{ \frac{\alpha^2}{2} \left(\tilde{\phi}_{\tilde{x}}^2 + \frac{1}{\beta} \tilde{\phi}_{\tilde{z}}^2 \right) \right\} \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} + \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \int_1^{1+\alpha\tilde{\eta}} \tilde{z}\, \mathrm{d}\tilde{z}\, \mathrm{d}\tilde{x} \\
&\quad = \frac{\mathrm{d}}{\mathrm{d}\tilde{t}} \int_{x_1/\ell}^{x_2/\ell} \left\{ \alpha\tilde{\eta} + \frac{\alpha^2}{2} \tilde{\eta}^2 + \frac{\alpha^2}{2} \tilde{w}^2 \right\} \mathrm{d}\tilde{x} + \mathcal{O}\big(\alpha^3, \alpha^2\beta\big).
\end{aligned}
$$

Treating the first integral on the right-hand side of (6.1) in a similar way results in the expression

$$
\begin{aligned}
&\alpha \int_0^{1+\alpha\tilde{\eta}} \left\{ \frac{\alpha^2}{2} \left(\tilde{\phi}_{\tilde{x}}^3 + \frac{1}{\beta} \tilde{\phi}_{\tilde{z}}^2 \tilde{\phi}_{\tilde{x}} \right) + \tilde{z}\tilde{\phi}_{\tilde{x}} \right\} \mathrm{d}\tilde{z} \\
&\quad = \alpha \int_0^{1+\alpha\tilde{\eta}} \left\{ \tilde{z} f_{\tilde{x}} - \frac{\tilde{z}^3}{2} \beta f_{\tilde{x}\tilde{x}\tilde{x}} \right\} \mathrm{d}\tilde{z} + \mathcal{O}\big(\alpha^3, \alpha\beta^2\big) \\
&\quad = \frac{\alpha}{2} \tilde{w} + \alpha^2 \tilde{\eta}\tilde{w} - \frac{\alpha\beta}{8} \tilde{w}_{\tilde{x}\tilde{x}} + \frac{\alpha\beta}{4} \theta^2 \tilde{w}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha^3, \alpha^2\beta, \alpha\beta^2\big)
\end{aligned}
$$

for the energy flux. The work done by the pressure force takes the following form:

$$
\begin{aligned}
&\alpha^2 \int_0^{1+\alpha\tilde{\eta}} \tilde{P}' \tilde{\phi}_{\tilde{x}}\, \mathrm{d}\tilde{z} + \alpha \int_0^{1+\alpha\tilde{\eta}} (1-\tilde{z}) \tilde{\phi}_{\tilde{x}}\, \mathrm{d}\tilde{z} \\
&\quad = \alpha^2 \int_0^{1+\alpha\tilde{\eta}} \big\{ \tilde{\eta} + \mathcal{O}(\beta) \big\} \big\{ \tilde{f}_{\tilde{x}} + \mathcal{O}(\beta) \big\}\, \mathrm{d}\tilde{z} \\
&\qquad + \alpha \int_0^{1+\alpha\tilde{\eta}} (1-\tilde{z}) \left\{ \tilde{f}_{\tilde{x}} - \frac{\beta}{2} \tilde{z}^2 \tilde{f}_{\tilde{x}\tilde{x}\tilde{x}} + \mathcal{O}\big(\beta^2\big) \right\} \mathrm{d}\tilde{z} \\
&\quad = \frac{\alpha}{2} \tilde{v} + \alpha^2 \tilde{v}\tilde{\eta} - \frac{\alpha\beta}{24} \tilde{v}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha^3, \alpha^2\beta, \alpha\beta^2\big) \\
&\quad = \frac{\alpha}{2} \tilde{w} + \alpha^2 \tilde{\eta}\tilde{w} + \frac{\alpha\beta}{4} \left(\theta^2 - \frac{1}{6} \right) \tilde{w}_{\tilde{x}\tilde{x}} + \mathcal{O}\big(\alpha^3, \alpha^2\beta, \alpha\beta^2\big).
\end{aligned}
$$

Collecting these terms, the energy balance equation transforms to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\int_{x_1/\ell}^{x_2/\ell}\left\{\alpha\tilde{\eta}+\frac{\alpha^2}{2}\tilde{\eta}^2+\frac{\alpha^2}{2}\tilde{w}^2\right\}\mathrm{d}\tilde{x}\\
&\quad=\left[\frac{\alpha}{2}\tilde{w}+\alpha^2\tilde{\eta}\tilde{w}+\frac{\alpha\beta}{4}\left(\theta^2-\frac{1}{2}\right)\tilde{w}_{\tilde{x}\tilde{x}}\right.\\
&\qquad\left.+\frac{\alpha}{2}\tilde{w}+\alpha^2\tilde{\eta}\tilde{w}+\frac{\alpha\beta}{4}\left(\theta^2-\frac{1}{6}\right)\tilde{w}_{\tilde{x}\tilde{x}}+\mathcal{O}\left(\alpha^3,\alpha^2\beta,\alpha\beta^2\right)\right]_{x_2}^{x_1}.
\end{aligned}\tag{6.2}$$

Thus the differential form of the energy balance equation is given by

$$\begin{aligned}
&\left(\tilde{\eta}+\frac{\alpha}{2}\tilde{w}^2+\frac{\alpha}{2}\tilde{\eta}^2\right)_{\tilde{t}}+\left(\tilde{w}+2\alpha(\tilde{w}\tilde{\eta})+\frac{\beta}{2}\left(\theta^2-\frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}}\right)_{\tilde{x}}\\
&\quad=\mathcal{O}\left(\alpha^2,\alpha\beta,\beta^2\right).
\end{aligned}\tag{6.3}$$

Taking the appropriate terms in the energy density and flux in (6.2) which are of order zero or one in the differential energy balance (6.3), we find the non-dimensional energy density to be

$$\tilde{E}=\alpha\tilde{\eta}+\frac{\alpha^2}{2}\tilde{\eta}^2+\frac{\alpha^2}{2}\tilde{w}^2,$$

and the non-dimensional energy flux (corrected for the work done by pressure force) as

$$\tilde{q}_E=\alpha\tilde{w}+2\alpha^2\tilde{w}\tilde{\eta}+\frac{\alpha\beta}{2}\left(\theta^2-\frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}}.$$

With these definitions, the energy balance is

$$\frac{\partial}{\partial\tilde{t}}\tilde{E}+\frac{\partial}{\partial\tilde{x}}\tilde{q}_E=\mathcal{O}\left(\alpha^2,\alpha\beta,\beta^2\right).$$

The dimensional forms of the energy density and energy flux per unit span are given by

$$E=\frac{\rho}{2}g\left(2h_0\eta+\eta^2\right)+\frac{\rho}{2}h_0w^2\tag{6.4}$$

and

$$q_E=\rho g\left(h_0^2+2h_0\eta\right)w+\frac{\rho}{2}\left(\theta^2-\frac{1}{3}\right)c_0^2h_0^3w_{xx}.$$

In order to compare these terms with the shallow-water theory we let $\beta\to 0$ in (6.2). Then the corresponding dimensional expression for the energy density is still (6.4). The corresponding dimensional expression for the energy flux is

$$q_E=\rho g\left(h_0^2+2h_0\eta\right)w.$$

It now appears that neither the energy density nor the energy flux agrees with the corresponding quantities in the shallow-water theory. Again, this can already be read off from the fact that the energy balance (6.3) contains correction terms of $\mathcal{O}(\alpha^2)$. It is also clear from noticing that the energy density in the shallow-water theory contains cubic terms, and the corresponding flux contains a quartic term. As was the case for the momentum balance, the energy density and flux can be corrected by higher-order terms without compromising the validity of the model to second order. Nevertheless, it might be more satisfactory to derive expressions corresponding to a higher-order Boussinesq system, as will be shown in Sect. 9.

7 Special Systems

In this section, a number of special cases of the general system (2.5) and (2.6) are examined, and the expressions for pressure and the mass, momentum, and energies densities and fluxes are tabulated. Some comments on boundary and initial conditions for physically relevant numerical studies are given.

Classical Boussinesq System This system emerges for $\theta^2 = \frac{1}{3}$, $\mu = 0$ and arbitrary λ in (2.5) and (2.6). In dimensional variables, the system has the form

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x &= 0, \\ w_t + g\eta_x + w w_x - \frac{h_0^2}{3} w_{xxt} &= 0. \end{aligned} \tag{7.1}$$

This system is not the one originally developed by Boussinesq (1872), but is still commonly known as the classical Boussinesq system. The original Boussinesq system featured the term of the form η_{xtt} in the second equation (Boussinesq 1872; Whitham 1974). In the case of (7.1), the mass density and mass flux per unit span are given by

$$M = \rho(h_0 + \eta), \tag{7.2}$$

and

$$q_M = \rho h_0 w + \rho \eta w,$$

respectively. The corresponding quantities for the momentum and energy balance laws are

$$I = \rho(h_0 + \eta) w$$

and

$$q_I = \rho h_0 w^2 + \frac{\rho}{2} g (h_0 + \eta)^2 - \frac{\rho}{3} h_0^3 w_{xt}, \tag{7.3}$$

as well as

$$E = \frac{\rho}{2} g \big(2 h_0 \eta + \eta^2\big) + \frac{\rho}{2} h_0 w^2 \tag{7.4}$$

and

$$q_E = \rho g\left(h_0^2 + 2h_0\eta\right)w.$$

The system (7.1) has been used in a number of numerical studies, such as Peregrine (1966). Well-posedness has been established for the Cauchy problem on the real line (Amick 1984; Schonbek 1981). The initial-boundary-value problem has been treated in Fokas and Pelloni (2005).

We note that the mass density M, the momentum flux q_I and the energy density E all have the same form for all choices of θ, μ and λ, so these terms are not tabulated in the following.

Coupled BBM System The coupled BBM system appears in (2.5) and (2.6) if $\lambda = 0$ and $\mu = 0$. This system takes the form

$$\begin{aligned} \eta_t + h_0 w_x + (\eta w)_x - \frac{h_0^2}{2}\left(\theta^2 - \frac{1}{3}\right)\eta_{xxt} &= 0, \\ w_t + g\eta_x + w w_x - \frac{h_0^2}{2}\left(1 - \theta^2\right)w_{xxt} &= 0. \end{aligned} \tag{7.5}$$

The mass flux associated to (7.5) is given by

$$q_M = \rho(h_0 + \eta)w + \frac{\rho h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx}. \tag{7.6}$$

The corresponding quantities for the momentum density and energy flux are

$$I = \rho(h_0 + \eta)w + \frac{\rho h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx}$$

and

$$q_E = \rho g\left(h_0^2 + 2h_0\eta\right)w + \frac{\rho c_0^2 h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx},$$

respectively. This system, which is similar to the BBM equation (Benjamin et al. 1972), was studied in Bona and Chen (1998), where the well-posedness of the boundary-value problem on a finite interval was established (see also Alazman et al. 2006). Based on previous work on the single BBM equation (Bona et al. 1981), Bona and Chen (1998) put forward a numerical method which is optimal in the sense that it has computational complexity of $\mathcal{O}(N)$ when N gridpoints are used. The system was also recently used in the study of wave breaking in undular bores (Bjørkavåg and Kalisch 2011). Because of its benign numerical behavior, this system appears particularly useful for computational studies of water waves. Moreover, for the particular value $\theta^2 = \frac{7}{9}$, the system features exact solitary-wave solutions (Chen 1998). These will be used later to provide a numerical example in Sect. 8.

Coupled KdV System This system appears if $\lambda = 1$ and $\mu = 1$. It has the form

$$\begin{aligned} &\eta_t + h_0 w_x + (\eta w)_x + \left(\theta^2 - \frac{1}{3}\right)\frac{h_0^3}{2} w_{xxx} = 0, \\ &w_t + g\eta_x + w w_x + \left(1 - \theta^2\right) g \frac{h_0^2}{2} \eta_{xxx} = 0. \end{aligned} \tag{7.7}$$

As long as the value of θ is the same as for the coupled BBM system, the associated quantities for the mass, momentum and energy densities and fluxes are also the same. The system has been shown to have global solutions in Bona et al. (2010), and a rigorous proof of the convergence of solutions of the water-wave problem to solutions of (7.7) in the limit as $\alpha \to 0$ and $\beta \to 0$ was given in Bona et al. (2005). A numerical study of this system was done by Bona et al. (2007).

Kaup System If $\theta^2 = 1$, $\lambda = 1$, and μ is arbitrary, the so-called Kaup system appears. It has the form

$$\begin{aligned} &\eta_t + h_0 w_x + (\eta w)_x + \frac{h_0^3}{3} w_{xxx} = 0, \\ &w_t + g\eta_x + w w_x = 0. \end{aligned} \tag{7.8}$$

Even though this system is not well-posed (Bona et al. 2004), it is important, because it has an integrable Hamiltonian structure (Kaup 1975). Moreover, (7.8) appears naturally when one bases the derivation of the long-wave system on approximating the Hamiltonian function (Craig and Groves 1994) of the full surface water-wave problem. A version of this system also appears in the context of interfacial waves if it is required that an approximate Hamiltonian function be conserved (Craig et al. 2005). In the current context, the mass flux is given by

$$q_M = \rho h_0 w + \rho \eta w + \frac{\rho h_0^3}{3} w_{xx},$$

and the momentum density is

$$I = \rho(h_0 + \eta) w + \frac{\rho}{3} h_0^3 w_{xx}.$$

The energy density is given by (6.4), and the energy flux is

$$q_E = \rho g\left(h_0^2 + 2h_0\eta\right) w + \frac{\rho}{3} c_0^2 h_0^3 w_{xx}.$$

8 A Numerical Example

In this section, the balance laws derived above are examined in a concrete situation. The coupled BBM system (7.5) features solitary-wave solutions in closed form if the value $\theta^2 = \frac{7}{9}$ is chosen (Chen 1998). Suppose the undisturbed depth h_0 and an amplitude η_0 are given. Then the solitary wave takes the form

$$\begin{aligned} \eta(x,t) &= \eta_0 \operatorname{sech}^2\big(\kappa(x - x_0 - C_s t)\big), \\ w(x,t) &= w_0 \operatorname{sech}^2\big(\kappa(x - x_0 - C_s t)\big), \end{aligned} \tag{8.1}$$

where

$$w_0 = \sqrt{\frac{3g}{\eta_0 + 3h_0}}\eta_0, \qquad C_s = \frac{3h_0 + 2h_0}{\sqrt{3h_0(\eta_0 + 3h_0)}}\sqrt{gh_0}, \qquad \kappa = \frac{3}{2h_0}\sqrt{\frac{\eta_0}{2\eta_0 + 3h_0}}.$$

The non-dimensional parameter α can now be defined in terms of the solitary-wave amplitude η_0 as $\alpha = \frac{\eta_0}{h_0}$. The parameter β may be defined by $\beta = \kappa^2 h_0^2$. There is some ambiguity in the definition of the parameter β in this example, but it is clear that $\beta < \alpha$, whatever definition is used. As discussed in Benjamin and Lighthill (1954), the solitary wave represents the limit case with respect to the ratio α/β for which the Boussinesq scaling is valid.

Consider a channel of depth $h_0 = 1$ m, and a control volume delimited by the interval $[-10, 10]$ in the x-axis. The mass per unit width contained in this control interval is defined in the Boussinesq scaling by $\int_{-10}^{10} M(x,t)\,\mathrm{d}x$, where M is given by (7.2). The mass flux through the boundaries of the control volume is defined by $q_M(-10,t)$ and $q_M(10,t)$, where q_M is given by (7.6) with $\theta^2 = \frac{7}{9}$. Thus to second order in α and β, the mass balance law takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{-10}^{10} M(x,t)\,\mathrm{d}x - q_M(-10,t) + q_M(10,t) = 0.$$

To test the conservation, we compute the quantities M and q_M during the passage of a solitary wave which is centered at $x_0 = -25$ m at time $t = 0$. The left panel of Fig. 3 shows the control volume at the instant $t = 8.5$ s. The right panel shows

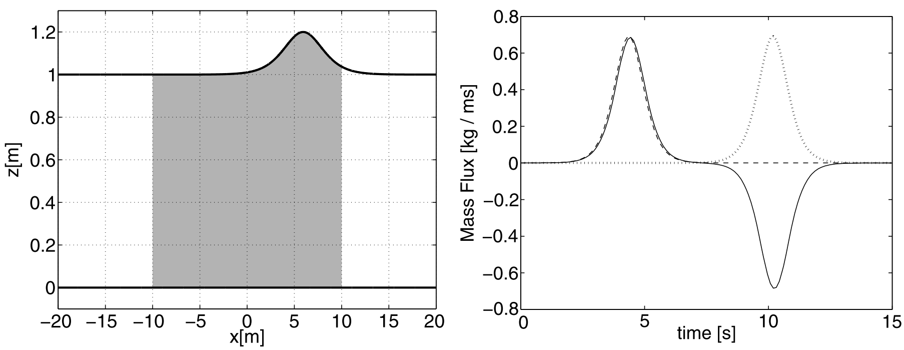

Fig. 3 *The left panel* shows the control volume above the interval $[-10, 10]$ at time $t = 8.5$ s. The top boundary is given by a solitary wave of amplitude 0.2 m whose crest was initially positioned at $x = -25$ m. *The right panel* shows plots of time series of the rate of change in the total mass (*solid curve*), the mass influx at $x = -10$ m (*dashed curve*) and the mass outflux at $x = 10$ m (*dotted curve*) per unit span

Table 1 This table shows the maximum error in conservation of the mechanical balance laws for mass, momentum, and energy for increasing solitary-wave amplitude. The second column represents the maximum error for the mass balance as defined in (8.2). The third and fourth columns are the maximum error for momentum and energy balances, respectively. The last column represents the model error which is $\mathcal{O}(\alpha^2)$

Amplitude	Err($\mathcal{M}$)	Err($\mathcal{I}$)	Err($\mathcal{E}$)	α^2
0.05	0.0001	0.0001	0.0001	0.0025
0.10	0.0002	0.0006	0.0003	0.0100
0.15	0.0007	0.0016	0.0009	0.0225
0.20	0.0013	0.0032	0.0020	0.0400
0.25	0.0023	0.0056	0.0035	0.0625
0.30	0.0035	0.0089	0.0056	0.0900
0.35	0.0050	0.0131	0.0084	0.1225

the quantities $\int_{-10}^{10} M(x,t)\,\mathrm{d}x$, $q_M(-10,t)$ and $q_M(10,t)$ as functions of time t. The figure suggests that mass conservation holds approximately. To further quantify the error in the conservation of mass, we define the non-dimensional error by

$$\mathrm{Err}(\mathcal{M}) = \frac{1}{c_0 h_0 \rho} \max_t \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_{-10}^{10} M\,\mathrm{d}x - q_M(-10,t) + q_M(10,t) \right|. \tag{8.2}$$

The results for various amplitudes of the solitary wave are displayed in Table 1. The first column shows the amplitude η_0 of the wave. The second column shows the error in mass conservation defined in (8.2). Using the expressions for I, q_I, E, and q_E defined above, corresponding expressions can be obtained for the conservation of the momentum and energy, and those are shown in columns 3 and 4 of Table 1.

The results in Table 1 confirm that in the case of the coupled BBM system, the maximum error in the conservation of mass, momentum, and energy is smaller than the error order $\mathcal{O}(\alpha^2, \alpha\beta, \beta^2)$ guaranteed by the analysis in the previous sections. Note that we have $\beta < \alpha$ for the solitary-wave solution, so that the model error is only determined by $\mathcal{O}(\alpha^2)$. In order to validate the more general case where the error is determined by both α and β, a further numerical study could be done where solutions of one of the systems (2.5) and (2.6) are computed by numerical discretization.

9 Higher-Order Models

In the derivation of the balance laws in the previous sections, it was noted that the momentum flux and the energy density and flux did not reduce to the corresponding shallow-water quantities (1.6), (1.9), and (1.10) in the limit $\beta \to 0$. However, in some cases, such as in the study of energy conservation in undular bores presented in Ali and Kalisch (2010), it is important that the integrals for the mass, momentum and energy fluxes through a control interval correctly reduce to the corresponding shallow-water expressions. One possible strategy is to include higher-order terms in the expressions for these quantities as this does not change the order of the approximation.

However, it might be preferable to have available an altogether higher-order theory in which both the evolution equations and the densities and fluxes are valid to higher order. Indeed, there is a vast literature on higher-order Boussinesq models, and other fully nonlinear equations, such as the Green–Naghdi equations (Green and Naghdi 1976) (also known as Su–Gardner equations, Su and Gardner 1969). For the sake of coherence, our focus will be on the higher-order models derived in Bona et al. (2002). These equations are of the form

$$\begin{aligned}
&\tilde{\eta}_{\tilde{t}} + \tilde{w}_{\tilde{x}} + \alpha(\tilde{\eta}\tilde{w})_{\tilde{x}} \\
&\quad + \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} + \frac{1}{2}(\theta^2 - 1)\alpha\beta(\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{x}})_{\tilde{x}} + \frac{5}{24}\left(\theta^2 - \frac{1}{5}\right)^2\beta^2\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{x}} \\
&\qquad = \mathcal{O}\big(\alpha^2\beta, \alpha\beta^2, \beta^3\big),
\end{aligned} \tag{9.1}$$

$$\begin{aligned}
&\tilde{w}_{\tilde{t}} + \tilde{\eta}_{\tilde{x}} + \alpha\tilde{w}\tilde{w}_{\tilde{x}} - \frac{1}{2}\big(1 - \theta^2\big)\beta\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} \\
&\quad - \alpha\beta\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{x}\tilde{t}} - \alpha\beta\tilde{\eta}_{\tilde{x}}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{1}{2}\big(\theta^2 + 1\big)\alpha\beta\tilde{w}_{\tilde{x}\tilde{x}}\tilde{w}_{\tilde{x}} + \frac{1}{2}\big(\theta^2 - 1\big)\alpha\beta\tilde{w}\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}} \\
&\quad + \frac{5}{24}\big(\theta^2 - 1\big)\left(\theta^2 - \frac{1}{5}\right)\beta^2\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}\tilde{t}} = \mathcal{O}\big(\alpha^2\beta, \alpha\beta^2, \beta^3\big).
\end{aligned} \tag{9.2}$$

More general equations can be derived by choosing model parameters such as λ and μ in Sect. 2. However, as became clear in the previous sections, these model parameters have no impact on the form of the physical quantities derived here, so we will take the system (9.1) and (9.2) as the basis for our investigations. Using similar ideas as before in Sect. 3, and a higher-order version of (2.2), the associated non-dimensional dynamic pressure is given by

$$\begin{aligned}
\tilde{P}' = {}& \tilde{\eta} + \frac{1}{2}\beta\big(\tilde{z}^2 - 1\big)\tilde{w}_{\tilde{x}\tilde{t}} - \alpha\beta\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{1}{4}\beta^2\big(\tilde{z}^2 - 1\big)\left(\theta^2 - \frac{(\tilde{z}^2 + 1)}{6}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{t}} \\
& + \frac{1}{2}\alpha\beta\big(\tilde{z}^2 - 1\big)\tilde{w}\tilde{w}_{\tilde{x}\tilde{x}} - \frac{1}{2}\alpha\beta\big(\tilde{z}^2 - 1\big)\tilde{w}_{\tilde{x}}^2 + \mathcal{O}\big(\alpha^2\beta, \alpha\beta^2, \beta^3\big).
\end{aligned}$$

Thus the dimensional pressure associated to (9.1) and (9.2) is

$$\begin{aligned}
P = {}& P_{\mathrm{atm}} + \rho g(\eta - z) + \frac{\rho}{2}\big(z^2 + 2h_0 z\big)w_{xt} - \rho\eta h_0 w_{xt} \\
& + \frac{\rho}{4}\big(z^2 + 2h_0 z\big)\left(h_0^2\theta^2 - \frac{(z^2 + 2h_0 z + 2h_0^2)}{6}\right)w_{xxxt} \\
& + \frac{\rho}{2}\big(z^2 + 2h_0 z\big)\big(w w_{xx} - w_x^2\big).
\end{aligned}$$

Using the same method as in Sect. 3, one may define the corrected pressure which is equal to atmospheric pressure when evaluated at the free surface. The dimensional form reads

$$\begin{aligned}P_1 = P_{\text{atm}} &+ \rho g(\eta - z) + \frac{\rho}{2}\big((z+h_0)^2 - (\eta+h_0)^2\big)w_{xt} \\ &+ \frac{\rho}{4}h_0^2\theta^2\big((z+h_0)^2 - (\eta+h_0)^2\big)w_{xxxt} \\ &- \frac{\rho}{24}\big((z+h_0)^4 - (\eta+h_0)^4\big)w_{xxxt} \\ &+ \frac{\rho}{2}\big((z+h_0)^2 - (\eta+h_0)^2\big)\big(ww_{xx} - w_x^2\big).\end{aligned}$$

Since the balance equations are derived to third order, this correction does not figure in the following analysis, and we continue to work with the expression $P - P_{\text{atm}}$. The mass density is given by the same formula as before, (4.3). The non-dimensional mass flux is

$$\begin{aligned}\tilde{q}_M = \alpha\tilde{w} + \alpha^2\tilde{\eta}\tilde{w} &+ \frac{\alpha\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}} + \frac{1}{2}\alpha^2\beta\big(\theta^2 - 1\big)\tilde{w}_{\tilde{x}\tilde{x}}\tilde{\eta} \\ &+ \frac{\alpha\beta^2}{12}\left(\frac{1}{10} - \theta^2 + \frac{5}{2}\theta^4\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}.\end{aligned}$$

Thus the mass balance equation is

$$\frac{\partial}{\partial\tilde{t}}\tilde{M} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_M = \mathcal{O}\big(\alpha\beta^2, \alpha^2\beta, \beta^3\big).$$

Using the scaling $\tilde{q}_M = \rho h_0 c_0 q_M$, the dimensional form is

$$\begin{aligned}q_M = \rho\bigg\{h_0 w + \eta w &+ \frac{h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx} \\ &+ \frac{h_0^2}{2}\left(\theta^2 - \frac{1}{3}\right)\eta w_{xx} + \frac{h_0^5}{12}\left(\frac{1}{10} - \theta^2 + \frac{5}{2}\theta^4\right)w_{xxxx}\bigg\},\end{aligned}$$

and it is also evident that the shallow-water mass flux q_M^0 is reached in the limit $\beta \to 0$. Moreover, note that (9.1) can be recognized as a mass conservation equation, so that mass conservation holds exactly also in the context of the higher-order system. The non-dimensional momentum density is

$$\begin{aligned}\tilde{I} = \alpha\tilde{w} + \alpha^2\tilde{\eta}\tilde{w} &+ \frac{\alpha\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}} + \frac{\alpha^2\beta}{2}\big(\theta^2 - 1\big)\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{x}} \\ &+ \frac{\alpha\beta^2}{12}\left(\frac{5\theta^4}{2} - \theta^2 + \frac{1}{10}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}},\end{aligned}$$

while the non-dimensional momentum flux is

$$\tilde{q}_I = \alpha^2\tilde{w}^2 + \alpha^3\tilde{\eta}\tilde{w}^2 + \frac{1}{2} + \alpha\tilde{\eta} + \frac{\alpha^2}{2}\tilde{\eta}^2 - \frac{\alpha\beta}{3}\tilde{w}_{\tilde{x}\tilde{t}}$$
$$+ \alpha^2\beta\left(\theta^2 - \frac{2}{3}\right)\tilde{w}\tilde{w}_{\tilde{x}\tilde{x}} - \alpha^2\beta\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{\alpha^2\beta}{3}\tilde{w}_{\tilde{x}}^2 - \alpha\frac{\beta^2}{6}\left(\theta^2 - \frac{1}{5}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{t}}.$$

Therefore the momentum balance is

$$\frac{\partial}{\partial\tilde{t}}\tilde{I} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_I = \mathcal{O}\left(\alpha\beta^2, \alpha^2\beta, \beta^3\right).$$

In particular, we see that there are no higher-order terms of the form $\mathcal{O}(\alpha^3)$. Therefore this approximation already entails the shallow-water approximation. Indeed, for very long waves (as $\beta \to 0$), this balance is reduced exactly to the conservation equation of momentum for the shallow-water system (1.4). Using the scaling $I = \rho h_0 c_0 \tilde{I}$, the dimensional momentum density is given by

$$I = \rho\left\{h_0 w + \eta w + \frac{h_0^3}{2}\left(\theta^2 - \frac{1}{3}\right)w_{xx} + \frac{h_0^2}{2}\left(\theta^2 - 1\right)\eta w_{xx}\right.$$
$$\left. + \frac{h_0^5}{12}\left(\frac{5\theta^4}{2} - \theta^2 + \frac{1}{10}\right)w_{xxxx}\right\}.$$

Using the scaling $q_I = \rho h_0 c_0^2 \tilde{q}_I$, the dimensional momentum flux appears as

$$q_I = \rho\left\{h_0 w^2 + \eta w^2 + \frac{g}{2}(h_0 + \eta)^2 - \frac{h_0^3}{3}w_{xt}\right.$$
$$\left. + h_0^3\left(\theta^2 - \frac{2}{3}\right)w w_{xx} - h_0^2\eta w_{xt} + \frac{h_0^3}{3}w_x^2 - \frac{h_0^5}{6}\left(\theta^2 - \frac{1}{5}\right)w_{xxxt}\right\}.$$

Finally, let us consider the energy balance. The non-dimensional energy density is

$$\tilde{E} = \frac{\alpha^2}{2}(1 + \alpha\tilde{\eta})\tilde{w}^2 + \frac{1}{2}\left(2\alpha\tilde{\eta} + \alpha^2\tilde{\eta}^2\right) + \frac{\alpha^2\beta}{6}\tilde{w}_{\tilde{x}}^2 + \frac{\alpha^2\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}\tilde{w}_{\tilde{x}\tilde{x}},$$

and the non-dimensional energy flux is

$$\tilde{q}_E = \frac{\alpha^3}{2}\tilde{w}^3 + \alpha\tilde{w}(1 + \alpha\tilde{\eta})^2 + \frac{\alpha\beta}{2}\left(\theta^2 - \frac{1}{3}\right)\tilde{w}_{\tilde{x}\tilde{x}}$$
$$+ \alpha^2\beta\left(\theta^2 - \frac{2}{3}\right)\tilde{\eta}\tilde{w}_{\tilde{x}\tilde{x}} - \frac{\alpha^2\beta}{3}\tilde{w}\tilde{w}_{\tilde{x}\tilde{t}} + \frac{\alpha\beta^2}{12}\left(\frac{5\theta^4}{2} - \theta^2 + \frac{1}{10}\right)\tilde{w}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}.$$

Therefore the energy balance is given by

$$\frac{\partial}{\partial\tilde{t}}\tilde{E} + \frac{\partial}{\partial\tilde{x}}\tilde{q}_E = \mathcal{O}\left(\alpha^3, \alpha\beta^2, \alpha^2\beta, \beta^3\right).$$

Using the scaling $E = \rho h_0 c_0^2 \tilde{E}$, the dimensional energy density is

$$E = \rho\left\{\frac{1}{2}(h_0+\eta)w^2 + \frac{g}{2}\big(2h_0\eta+\eta^2\big) + \frac{h_0^3}{6}w_x^2 + \frac{h_0^3}{2}\left(\theta^2-\frac{1}{3}\right)ww_{xx}\right\}.$$

The dimensional energy flux is given by

$$\begin{aligned} q_E = \rho\bigg\{&\frac{h_0}{2}w^3 + gw(h_0+\eta)^2 + \frac{gh_0^4}{2}\left(\theta^2-\frac{1}{3}\right)w_{xx} \\ &+ gh_0^3\left(\theta^2-\frac{2}{3}\right)\eta w_{xx} - \frac{h_0^3}{3}ww_{xt} + \frac{gh_0^6}{12}\left(\frac{5\theta^4}{2}-\theta^2+\frac{1}{10}\right)w_{xxxx}\bigg\}, \end{aligned}$$

where $q_E = \rho h_0 c_0^3 \tilde{q}_E$.

Note that as $\beta \to 0$, the energy balance does *not* reduce correctly to the shallow-water energy conservation. In order to achieve the correct limit, a further correction term of order $\mathcal{O}(\alpha^4)$ has to be included in the non-dimensional energy flux, namely the term $\frac{\alpha^4}{2}\tilde{\eta}\tilde{w}^3$. Alternatively, in order to obtain a model which correctly reduces to the shallow-water equations one would have to derive a seventh-order system of evolution equations. Such a derivation does not present a problem from a theoretical point of view, but appears to be irrelevant from a practical standpoint. Numerical integration of a system containing such high spatial derivatives would necessitate the imposition of at least 12 boundary conditions. Since these are not known in any practical situation, it appears a moot endeavor to pursue such a development.

10 Conclusion

Based on the derivation of a general class of Boussinesq models in Bona et al. (2002), a method of determining the corresponding mass, momentum and energy densities and fluxes has been developed. The pressure and all densities and fluxes are expressed in terms of the dependent variables η representing the surface elevation and w representing the horizontal fluid velocity at a given height $h_0\theta$ in the fluid column. The correct order of approximation of these quantities has been found by expressing the mass, momentum, and energy conservation as mechanical balance laws which are required to hold asymptotically to the same order as the evolution equations. As can be seen, the mass, momentum, and energy densities and fluxes depend only on the parameter θ, that is, on the depth at which the horizontal velocity is taken. No dependence has been found on the model parameters λ and μ which give the relation between terms of the type η_{xxt} and terms of type w_{xxx}. It has been found that some of the quantities found do not reduce to the corresponding shallow-water expressions in the limit of very long waves. As a remedy, higher-order expressions corresponding to fifth-order Boussinesq systems have also been derived.

A concrete example has been investigated which confirms the results of this paper in a special case of the coupled BBM equation. Nevertheless, it should be emphasized that the method employed here is strictly formal. If one wanted to attempt a mathematical proof that the balance laws derived above are valid to the same order as the equations, one might start with the Zakharov–Craig–Sulem formulation of the water-wave problem (Craig and Sulem 1993; Zakharov 1968) and develop the Dirichlet–Neumann operator with respect to the small parameters α and β. Indeed this technique has been used in Bona et al. (2005) to prove the convergence of solutions of the water-wave problem to solutions of (7.7), and it may also yield a rigorous proof of the validity of the mechanical balance laws.

Recently, two-dimensional Boussinesq models have also been derived and studied both numerically and with regard to well-posedness (Chen 2009; Dougalis et al. 2007, 2009), and we expect that our analysis can be extended to two-dimensional models. Of course, in this case questions about the geometry of the control volumes might become important.

There is an abundant literature on improvement of the Boussinesq equations as regards various aspects, such as the inclusion of bottom topography, three-dimensional channel geometry and underlying shear flow. A few examples of such work are contained in Kim et al. (2009), Kirby (1986), Lannes and Bonneton (2009), Madsen et al. (2006), Nachbin and Choi (2007), Teng and Wu (1997), Wahlen (2008). Major efforts have also been made toward extending the applicability of Boussinesq-type models to waves of larger amplitude, relaxing the shallow-water assumption, and improving dispersion properties (Agnon et al. 1999; Chazel et al. 2009; Green and Naghdi 1976; Kennedy et al. 2001; Madsen and Schäffer 1998; Lannes and Bonneton 2009; Shi et al. 2001; Su and Gardner 1969; Wei et al. 1995). In contrast to these studies, we have found expressions for mechanical quantities connected with the original Boussinesq scaling. While there is no improvement of the model in any of the directions named above, our results make it possible to use the Boussinesq equations in studies where the focus is on aspects of the fluid flow other than the shape of the free surface and the horizontal velocity.

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