Dynamics of an epidemic model with non-local infections for diseases with latency over a patchy environment

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Abstract   In this paper, with the assumptions that an infectious disease in a population has a fixed latent period and the latent individuals of the population may disperse, we formulate an SIR model with a simple demographic structure for the population living in an n-patch environment (cities, towns, or countries, etc.). The model is given by a system of delay differential equations with a fixed delay accounting for the latency and a non-local term caused by the mobility of the individuals during the latent period. Assuming irreducibility of the travel matrices of the infection related classes, an expression for the basic reproduction number $\mathcal{R}_0$ is derived, and it is shown that the disease free equilibrium is globally asymptotically stable if $\mathcal{R}_0 < 1$, and becomes unstable if $\mathcal{R}_0 > 1$. In the latter case, there is at least one endemic equilibrium and the disease will be uniformly persistent. When $n = 2$, two special cases allowing reducible travel matrices are considered to illustrate joint impact of the disease latency and population mobility on the disease dynamics. In addition to the existence of the disease free equilibrium and interior endemic equilibrium, the existence of a boundary equilibrium and its stability are discussed for these two special cases.

Keywords   Infectious disease · SIR model · Latent period · Patch · Non-local infection · Dispersal · Basic reproduction number

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1 Introduction

Globalization has made traveling more and more common. Classical deterministic epidemic models with the assumption that the space is homogeneous are simply not applicable any more. In order to understand how a disease invades a population, the population movements and the spatial structure of population habitats can not be neglected. For example, SARS was first reported in Guangdong Province of China in November 2002. This emerging disease spread very quickly, due to the travel of infected persons by airplanes, trains, and buses, to some other regions in the mainland of China, as well as to Hong Kong, Singapore, Vietnam, Canada and other places. By the end of June 2003, it had spread to 32 countries and regions, and caused about 800 deaths and more than 8,000 infections (see, Ruan et al. 2006; Wang and Ruan 2004). Thus, it is important to study how population movements, spatial structure and disease transmission interact with each other to determine the spread of diseases. One of the most interesting and important subjects in this field is to obtain a threshold condition that determines whether an infectious disease will persist or go to extinction. This threshold condition is usually in terms of the basic reproduction number, $\mathcal{R}_0$, which is the average number of secondary infections caused by a single infectious individual introduced into a totally susceptible population (Anderson and May 1991; Diekmann and Heesterbeek 2000). This parameter $\mathcal{R}_0$ is a key concept in the study of infectious diseases because it can provide guidance for the control of diseases.

We choose a spatially discrete environment consisting of n patches. Here, a patch may represent a city or a town, and population movements between patches may be justified by the migration and travel among patches. Along this line, the study of the effects of population dispersal on disease dynamics over a patchy environment has been extensive. Arino and van den Driessche (2003a,b, 2006) formulated epidemic models with populations traveling among cities in which the residences of individuals are maintained. Wang and Zhao (2004, 2005) considered epidemic models of multi-patches without any record of the residence of individuals. Wang and Zhao (2006) proposed an epidemic model with population dispersal and infection period. Salmani and van den Driessche (2006) discussed an SEIRS epidemic model for n patches to describe the dynamics of an infectious disease in a population in which individuals travel between patches. Hsieh et al. (2007), Brauer and van den Driessche (2001), Castillo-Chavez and Yakubu (2001), and Wang and Mulone (2003) are among other studies of epidemic models for meta-populations.

Based on the calculations of basic reproduction numbers, the aforementioned papers established the threshold dynamics for the diseases. The models in these papers captured the essence of SIR (susceptible → infectious → removed) or SIS (susceptible → infectious → susceptible) or SEIR (susceptible → exposed (latent) → infectious → removed) or SEIRS (susceptible → exposed (latent) → infectious → removed → susceptible). However, none of them considered both a fixed latent period and the mobility of the infected individuals during the fixed latent period together. As

we shall see later, the incorporation of both of these two factors will actually bring non-local infection terms into the model.

In our previous work (Li and Zou 2009), we generalized the classical Kermack–McKendrick SIR epidemic model to a 2-patch environment for a disease with latency. We obtained a system of delay differential equations with a fixed delay accounting for the latency and non-local terms caused by the mobility of the individuals during the latent period. We analytically showed that the model preserves some properties that the classic Kermack–McKendrick SIR epidemic model possesses: disease always dies out, leaving a certain portion of the susceptible population untouched (called final sizes). Although we could not determine the two final sizes, we were able to show that the ratio of the final sizes in the two patches is totally determined by the ratio of the dispersal rates of the susceptible individuals between the two patches. We also numerically explored the patterns by which the disease dies out, and found out that the new model may have very rich patterns for the disease to die out. In particular, it allows multiple outbreaks of the disease before if goes to extinction, strongly contrasting with the classic Kermack–McKendrick SIR epidemic model.

Kermack–McKendrick type epidemic models are only suitable for fast diseases, for which, the demographic structure can be ignored. But for a disease with long mean infection life time, we can not ignore the demographic structure. In order for a model to be more realistic for such disease with long infection time, in this project, we incorporate a demographic structure by adding recruitment (including births) and natural deaths. Unlike the previous work (Li and Zou 2009) in which the disease always dies out, the new model may allow existence of an endemic equilibrium. In such a situation, the disease free equilibrium may or may not be stable, and the basic reproduction number is sharply related to the stability of the disease free equilibrium as well as to the existence of an endemic equilibrium and the persistence of the disease. The purpose of the present paper is to incorporate a constant latent period for a general disease and a simple demographic structure into the population over n-patch environment and investigate the disease dynamics of the model in terms of the basic reproduction number.

We point out that many diseases have latency (see, e.g., Table 3.1 in Anderson and May 1991), and the length of the latent period differs from disease to disease; even for the same disease, it differs from individual to individual. Therefore, a more realistic way to incorporate latency into a model is by considering a general distribution function for the length of the latent period, as was done in van den Driessche et al. (2007). But this would increase difficulty for analyzing resulting model. Our choice of a fixed latent period can be considered as an approximation of the mean latent period, and this would be appropriate for those diseases whose latent periods vary only relatively slightly. For example, poliomyelitis has a latent period of 1–3 days (comparing to its much longer infectious period of 14–20 days), whooping cough has a latent period of 21–23 days (comparing to its infectious period of 7–10 days), and hepatitis B has a latent period of 13–17 days (comparing to its infectious period of 19–22 days) (see, e.g., Table 3.1 in Anderson and May 1991).

The rest of the paper is organized as follows. In the next section, we present the model formulation for an n-patch environment. Section 3 is devoted to proving the well-posedness of the model obtained in Sect. 2. Sections 4 and 5 deal with the situa-

tion when all dispersal rate matrices for the infected classes (latent and infectious) are irreducible, in which the stability analysis of the disease free equilibrium, the existence of endemic equilibrium and uniformly persistence of the disease are given. In Sect. 6, we are concerned with the situation when the irreducibility of the infection related dispersal matrices may not hold. We only consider two special cases for $n = 2$, and the lower dimension enable us to obtain more detailed results on the joint impact of the latency and the mobility of the infected individuals. Section 7 summarizes the main results of the paper, discusses the dependence of the basic reproduction number on some model parameters, and poses an open problem on the stability of the endemic equilibrium.

2 Model formulation

Consider a population that lives in n patches (e.g., cities). Let $S_i(t)$, $I_i(t)$, $R_i(t)$ be the sub-populations of the susceptible, infectious and removed classes on patch i, $i = 1, \ldots, n$ at time t, respectively. These patches are connected in the sense that individuals can move (migrate or travel) between these patches. Assume that an infectious disease with a fixed latent period denoted by τ, is brought into the population. Due to this latency and the mobility of the individuals during the latent period, the rate at which patch i gains new infectious individuals at time t depends on the new infections infected τ time units ago not only in patch i but also in other patches. To determine this dependence, we use the concept of the infection age, denoted by a. Let $l_i(t, a)$ be the density (with respect to the infection age a) of individuals at time t in patch i ($i = 1, \ldots, n$) with infection age a. Similar to the equation governing the growth of a population with natural age structure (see Metz and Diekmann 1986), the densities $l_i(t, a)$, $i = 1, \ldots, n$ are described by the following system of first-order partial differential equations:

$$\frac{\partial l_i(t, a)}{\partial t} + \frac{\partial l_i(t, a)}{\partial a} = -(d_i + \tilde{d}_i(a))l_i(t, a) + \sum_{j=1}^{n} D_{ij}(a)l_j(t, a) - \sum_{j=1}^{n} D_{ji}(a)l_i(t, a). \quad (2.1)$$

Here $D_{ij}(a)l_j(t, a)$ corresponds to the dispersal of the infected individuals at the infection age a from patch j to patch i; constant $d_i > 0$ denotes the natural death rate in patch i which is independent of the infection age and the disease status; and $\tilde{d}_i(a)$ represents the rate at which the infected individuals in patch i are removed (via deaths due to the disease and/or other possible means such as isolation or quarantine and recovery with permanent immunity by treatments). In addition, we have assumed that there is no delay in the dispersal between patches and there is no loss during migration from patch j to patch i, that is, all of those who leave patch j for patch i arrive at patch i safely.

By the meaning of $l_i(t, a)$, it is obvious that at a given time t, the total number of infectious individuals in patch i is given by

$$I_i(t) = \int_{\tau}^{\infty} l_i(t, a)da. \quad (2.2)$$

Obviously, $l_i(t, 0)$ corresponds to new infections in patch i which come from contacts between infectious and susceptible individuals. Mass action infection mechanism leads to

$$l_i(t, 0) = \lambda_i I_i(t) S_i(t), \tag{2.3}$$

where λ_i is the infection rate (effective contact rate or transmission rate) in patch i.

For convenience of showing the main idea to build up the new model, we further assume that

$$\tilde{d}_i(a) = \begin{cases} \tilde{d}_i^l(a) = \delta_i, & \text{for } 0 \leq a \leq \tau \quad \text{and } i = 1, \ldots, n, \\ \tilde{d}_i^I(a) = \gamma_i + \mu_i, & \text{for } \quad a > \tau \quad \text{and } i = 1, \ldots, n, \end{cases} \tag{2.4}$$

and

$$D_{ij}(a) = \begin{cases} D_{ij}^l(a) = D_{ij}^l, & \text{for } 0 \leq a \leq \tau \quad \text{and } i, j = 1, \ldots, n, \quad i \neq j, \\ D_{ij}^I(a) = D_{ij}^I, & \text{for } \quad a > \tau \quad \text{and } i, j = 1, \ldots, n, \quad i \neq j, \end{cases} \tag{2.5}$$

where $\tilde{d}_i^l(a)$ is assumed be a constant δ_i, and denotes the removal rate of latent individuals due to possible means such as quarantine in patch i, $i = 1, \ldots, n$, and $\tilde{d}_i^I(a)$ represents the removal rate of infectious individuals in patch i, $i = 1, \ldots, n$, and is assumed be the summation of γ_i accounting for the recovery with permanent immunity and possible isolation, and μ_i standing for the disease mortality rate of infectious individuals in patch i, $i = 1, \ldots, n$. For simplicity of notation, we let $\sigma_i = \gamma_i + \mu_i$, $i = 1, \ldots, n$, in the sequel. Meanwhile, $D_{ij}^l(a)$ and $D_{ij}^I(a)$ represent the dispersal of latent and infectious individuals respectively from patch j to patch i, and are assumed to be independent of the infection age.

The fact that the removal rates $(d_i + \tilde{d}_i(a))$, $i = 1, \ldots, n$, are bounded away from zero for all $a \geq 0$, ensures that

$$l_i(t, \infty) = 0. \tag{2.6}$$

This condition is biologically realistic and can be mathematically verified by solving (using the method of characteristics) (2.1) and (2.3) with any given biologically reasonable initial distribution (e.g., $l_i(0, a) \geq 0, i = 1, 2, \ldots, n$). Now, integrating (2.1) with respect to a from τ to ∞ and making use of (2.6) leads to

$$\begin{aligned} \frac{dI_i(t)}{dt} &= -\int_\tau^\infty \frac{\partial l_i(t, a)}{\partial a} da - \int_\tau^\infty (d_i + \tilde{d}_i^l(a)) l_i(t, a) da \\ &\quad + \int_\tau^\infty \sum_{j=1}^n D_{ij}(a) l_j(t, a) da - \int_\tau^\infty \sum_{j=1}^n D_{ji}(a) l_i(t, a) da \\ &= l_i(t, \tau) - (d_i + \sigma_i) I_i(t) + \sum_{j=1}^n D_{ij}^I I_j(t) - \sum_{j=1}^n D_{ji}^I I_i(t), \quad i, j = 1, \ldots, n. \end{aligned} \tag{2.7}$$

We adopt the simplest demographic structure of the population under consideration, in which we assume that there is a constant recruitment of susceptible individuals denoted by K_i in patch i, $i = 1, \ldots, n$, and constant natural death rate for each class denoted still by d_i and assume that the disease does not transmit vertically. With these assumptions, the disease dynamics can be described by the following equations:

$$\frac{dS_i(t)}{dt} = K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t) - \sum_{j=1}^{n} D_{ji}^S S_i(t) - \lambda_i I_i(t) S_i(t),$$

$$\frac{dI_i(t)}{dt} = -(d_i + \sigma_i) I_i(t) + \sum_{j=1}^{n} D_{ij}^I I_j(t) - \sum_{j=1}^{n} D_{ji}^I I_i(t) + l_i(t, \tau), \qquad (2.8)$$

$$\frac{dR_i(t)}{dt} = -d_i R_i(t) + \delta_i \int_0^\tau l_i(t, a) da + \gamma_i I_i(t) + \sum_{j=1}^{n} D_{ij}^R R_j(t) - \sum_{j=1}^{n} D_{ji}^R R_i(t),$$

where $D_{ij}^S \geq 0$ is the rate at which susceptible individuals migrate from patch j to patch i, $i \neq j$, and $D_{ij}^R \geq 0$ is the rate at which removed individuals migrate from patch j to patch i, $i \neq j$. With these assumptions, we know that $D_{ii}^S \equiv 0$, $D_{ii}^I \equiv 0$, $D_{ii}^I \equiv 0$ and $D_{ii}^R \equiv 0$, for $i = 1, \ldots, n$. Note that the equations for the removed class R_i, $i = 1, \ldots, n$ are decoupled from the equations for S_i and I_i, $i = 1, \ldots, n$. Thus we only need to consider the $2n$ equations for S_i and I_i, $i = 1, \ldots, n$ in (2.8).

Obviously, $l_i(t, \tau)$ is the rate at which patch i gains infectious individuals, which can be determined below in terms of $S_j(t)$ and $I_j(t)$ for all $j = 1, \ldots, n$.

For fixed $\xi \geq 0$, let

$$V_i^\xi(t) = l_i(t, t - \xi), \quad \text{for } \xi \leq t \leq \xi + \tau \quad \text{and} \quad i = 1, \ldots, n.$$

Then for $1 \leq i \neq j \leq n$,

$$\frac{d}{dt} V_i^\xi(t) = \frac{\partial}{\partial t} l_i(t, a)|_{a=t-\xi} + \frac{\partial}{\partial a} l_i(t, a)|_{a=t-\xi}$$

$$= -\left(d_i + \tilde{d}_i(t - \xi)\right) l_i(t, t - \xi) + \sum_{j=1}^{n} D_{ij}(t - \xi) l_j(t, t - \xi)$$

$$\quad - \sum_{j=1}^{n} D_{ji}(t - \xi) l_i(t, t - \xi)$$

$$= -\left(d_i + \tilde{d}_{l_i}(t - \xi)\right) V_i^\xi(t) + \sum_{j=1}^{n} D_{ij}^I(t - \xi) V_j^\xi(t) - \sum_{j=1}^{n} D_{ji}^I(t - \xi) V_i^\xi(t)$$

$$= -(d_i + \delta_i) V_i^\xi(t) + \sum_{j=1}^{n} D_{ij}^I V_j^\xi(t) - \sum_{j=1}^{n} D_{ji}^I V_i^\xi(t). \qquad (2.9)$$

Denote $\mathbf{V}^\xi(t) = \left(V_1^\xi(t), \cdots, V_n^\xi(t) \right)^T$, where T represents the transpose of a vector. Then $\mathbf{V}^\xi(t)$ satisfies

$$\frac{d}{dt}\mathbf{V}^\xi(t) = \mathbf{B}\mathbf{V}^\xi(t), \tag{2.10}$$

where

$$\mathbf{B} = \begin{bmatrix} -(d_1 + \delta_1) - \sum_{j=1}^n D_{j1}^l & D_{12}^l & \cdots & D_{1n}^l \\ D_{21}^l & -(d_2 + \delta_2) - \sum_{j=1}^n D_{j2}^l & \cdots & D_{2n}^l \\ \vdots & \vdots & \ddots & \vdots \\ D_{n1}^l & D_{n2}^l & \cdots & -(d_n + \delta_n) - \sum_{j=1}^n D_{jn}^l \end{bmatrix}.$$

Integrating (2.10) with respect to t from ξ to t, we have

$$\mathbf{V}^\xi(t) = \exp(\mathbf{B}(t-\xi)) \left(V_1^\xi(\xi), \ldots, V_n^\xi(\xi) \right)^T, \quad \xi \le t \le \xi + \tau. \tag{2.11}$$

By using the definition of $V_i^\xi(t)$ and (2.3),

$$\mathbf{V}^\xi(t) = \exp(\mathbf{B}(t-\xi)) \left(l_1(\xi, 0), \ldots, l_n(\xi, 0) \right)^T, \quad \xi \le t \le \xi + \tau,$$
$$= \exp(\mathbf{B}(t-\xi)) \left(\lambda_1 I_1(\xi) S_1(\xi), \ldots, \lambda_n I_n(\xi) S_n(\xi) \right)^T. \tag{2.12}$$

For $t \ge \tau$ (hence $t - \tau \ge 0$), letting $\mathbf{l}(t, \tau) = (l_1(t, \tau), \ldots, l_n(t, \tau))^T$, we obtain

$$\mathbf{l}(t, \tau) = \mathbf{V}^{t-\tau}(t)$$
$$= \exp(\mathbf{B}\tau) \left(\lambda_1 I_1(t - \tau) S_1(t - \tau), \ldots, \lambda_n I_n(t - \tau) S_n(t - \tau) \right)^T. \tag{2.13}$$

Denoting $[b_{ij}(\tau)]_{n \times n} := \exp(\mathbf{B}\tau)$, it follows that

$$l_i(t, \tau) = \sum_{j=1}^n b_{ij}(\tau) \lambda_j I_j(t - \tau) S_j(t - \tau). \tag{2.14}$$

Substituting $l_i(t, \tau)$ back into the I_i equations in (2.8) and taking out the first $2n$ equations for S_i, and I_i, $i = 1, \ldots, n$, results in the following new model:

$$\frac{d}{dt} S_i(t) = K_i - d_i S_i(t) + \sum_{j=1}^n D_{ij}^S S_j(t) - \sum_{j=1}^n D_{ji}^S S_i(t) - \lambda_i I_i(t) S_i(t),$$

$$\frac{d}{dt} I_i(t) = -(d_i + \sigma_i) I_i(t) + \sum_{j=1}^n D_{ij}^I I_j(t) - \sum_{j=1}^n D_{ji}^I I_i(t) \tag{2.15}$$

$$+ \sum_{j=1}^n b_{ij}(\tau) \lambda_j I_j(t - \tau) S_j(t - \tau), \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad t \ge \tau.$$
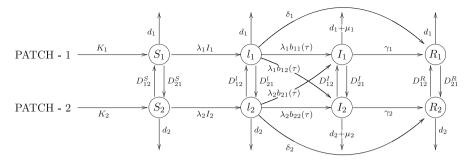
Fig. 1 Flow chart for 2-patch environment

For $t < \tau$, there is no infected individual entering infectious class, and the dynamics of S and I classes are governed by the following system of ordinary differential equations:

$$\frac{d}{dt} S_i(t) = K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^{S} S_j(t) - \sum_{j=1}^{n} D_{ji}^{S} S_i(t) - \lambda_i I_i(t) S_i(t),$$

$$\frac{d}{dt} I_i(t) = -(d_i + \sigma_i) I_i(t) + \sum_{j=1}^{n} D_{ij}^{I} I_j(t)$$

$$- \sum_{j=1}^{n} D_{ji}^{I} I_i(t) \quad \text{for} \quad i = 1, \dots, n, \ t \in [0, \tau]. \tag{2.16}$$

The last term on the right side of the I_i equation in (2.15) accounts for non-local infections, reflecting how the infections infected τ time units ago in all patches contribute to the growth of the infectious population in patch i. As is clearly from the structure of the matrix **B** and the expression (2.14), such an effect of non-local infections is caused by the mobility of the individuals in the latent period. To help understand this effect, let us consider a simpler case: $n = 2$. The two-patch version of the model is demonstrated by the flow chart in Fig. 1.

In this two-patch case, if we further assume $d_1 = d_2 = d$, and $\delta_1 = \delta_2 = \delta$, then

$$\mathbf{B} = \begin{bmatrix} -(d + \delta) - D_{21}^{I} & D_{12}^{I} \\ D_{21}^{I} & -(d + \delta) - D_{12}^{I} \end{bmatrix}$$

$$= \begin{bmatrix} -(d + \delta) & 0 \\ 0 & -(d + \delta) \end{bmatrix} + \begin{bmatrix} -D_{21}^{I} & D_{12}^{I} \\ D_{21}^{I} & -D_{12}^{I} \end{bmatrix},$$

and we can obtain $[b_{ij}(\tau)] = \exp(\mathbf{B}\tau)$ explicitly as

$$b_{11}(\tau) = e^{-(d+\delta)\tau} (1 - \alpha_1(\tau)), \quad b_{12}(\tau) = e^{-(d+\delta)\tau} \alpha_2(\tau),$$
$$b_{22}(\tau) = e^{-(d+\delta)\tau} (1 - \alpha_2(\tau)), \quad b_{21}(\tau) = e^{-(d+\delta)\tau} \alpha_1(\tau), \tag{2.17}$$

where

$$\alpha_i(\tau) = \frac{D_{ji}^l}{D_{ji}^l + D_{ij}^l}\left(1 - e^{-(D_{ji}^l + D_{ij}^l)\tau}\right), \quad \text{for} \quad 1 \le i \ne j \le 2. \qquad (2.18)$$

Hence the model becomes

$$\frac{dS_1(t)}{dt} = K_1 - d_1 S_1(t) - D_{21}^S S_1(t) + D_{12}^S S_2(t) - \lambda_1 I_1(t) S_1(t),$$

$$\frac{dS_2(t)}{dt} = K_2 - d_2 S_2(t) - D_{12}^S S_2(t) + D_{21}^S S_1(t) - \lambda_2 I_2(t) S_2(t),$$

$$\frac{dI_1(t)}{dt} = -(d + \sigma_1)I_1(t) - D_{21}^I I_1(t) + D_{12}^I I_2(t) \qquad (2.19)$$

$$+ e^{-(d+\delta)\tau}(1 - \alpha_1(\tau))\lambda_1 I_1(t - \tau) S_1(t - \tau)$$

$$+ e^{-(d+\delta)\tau}\alpha_2(\tau)\lambda_2 I_2(t - \tau) S_2(t - \tau),$$

$$\frac{dI_2(t)}{dt} = -(d + \sigma_2)I_2(t) - D_{12}^I I_2(t) + D_{21}^I I_1(t)$$

$$+ e^{-(d+\delta)\tau}\alpha_1(\tau)\lambda_1 I_1(t - \tau) S_1(t - \tau)$$

$$+ e^{-(d+\delta)\tau}(1 - \alpha_2(\tau))\lambda_2 I_2(t - \tau) S_2(t - \tau).$$

From this simpler version of the model, it is seen that the dispersion of the individuals in latent period plays a different role from that of the susceptible and infectious individuals. The explanation for those instantaneous terms in (2.19) are quite straightforward, and we now explain those delayed terms in the model. The probability that an individual infected in patch 1 can survive the latent period is $e^{-(d+\delta)\tau}$. Due to the mobility during the latent period between the two patches, τ time units later, a survived infected individual infected in patch 1, may be in patch 1 with probability $(1 - \alpha_1(\tau))$ or in patch 2 with probability $\alpha_1(\tau)$. This explains the term $e^{-(d+\delta)\tau}(1 - \alpha_1(\tau))\lambda_1 I_1(t - \tau) S_1(t - \tau)$ in the I_1 equation and the term $e^{-(d+\delta)(\tau)}\alpha_1(\tau)\lambda_1 I_1(t-\tau) S_1(t-\tau)$ in the I_2 equation. The terms $e^{-(d+\delta)\tau}(1-\alpha_2(\tau))$ $\lambda_2 I_2(t-\tau) S_2(t-\tau)$ in I_2 equation and the term $e^{-(d+\delta)\tau}\alpha_2(\tau)\lambda_2 I_2(t-\tau) S_2(t-\tau)$ in I_1 equation are explained similarly. Alternatively, we may explain these terms in light of fractions as below: among the individuals infected in the first patch τ time units ago, a fraction $e^{-(d+\delta)\tau}$ can survive the latent period, a fraction $(1 - \alpha_1(\tau))$ of which is now still in patch 1 while a fraction $\alpha_1(\tau)$ of which has now moved to patch 2.

For (2.15) with general n, the elements $b_{ij}(\tau)$ can be explained in a similar way (probabilities or fractions) and hence, one should expect the relation $0 \le b_{ij}(\tau) \le 1$, and this relation will be used later in proving the well-posedness of the new model. Although for the general case, we are unable to find the precise expressions for $b_{ij}(\tau)$, we are able to prove the above expectation by a comparison argument and properties of nonnegative matrices.

Lemma 2.1 *Let*

$$\underline{d} = \min_{1 \le i \le n} d_i, \quad \underline{\delta} = \min_{1 \le i \le n} \delta_i \quad and \quad \bar{d} = \max_{1 \le i \le n} d_i, \quad \bar{\delta} = \max_{1 \le i \le n} \delta_i.$$

Then,

$$e^{-(\bar{d}+\bar{\delta})\tau} \le \sum_{i=1}^{n} b_{ij}(\tau) \le e^{-(\underline{d}+\underline{\delta})\tau}, \quad for \ j = 1, \ldots, n. \tag{2.20}$$

Proof Choose a constant $K > 0$ sufficiently large such that

$$K > \max \left\{ (d_1 + \delta_1)\tau + \sum_{j=1}^{n} D^l_{j1}\tau, \ldots, (d_n + \delta_n)\tau + \sum_{j=1}^{n} D^l_{jn}\tau \right\}.$$

Write $\mathbf{B}\tau$ as $\mathbf{B}\tau = -K\mathbf{E} + K\mathbf{E} + \mathbf{D}_0 + \mathbf{D}_l$, where $\mathbf{E}$ is the $n \times n$ identity matrix and

$$\mathbf{D}_0 := \begin{bmatrix} -(d_1 + \delta_1)\tau & 0 & \cdots & 0 \\ 0 & -(d_2 + \delta_2)\tau & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -(d_n + \delta_n)\tau \end{bmatrix},$$

$$\mathbf{D}_l := \begin{bmatrix} -\sum_{j=1}^{n} D^l_{j1}\tau & D^l_{12}\tau & \cdots & D^l_{1n}\tau \\ D^l_{21}\tau & -\sum_{j=1}^{n} D^l_{j2}\tau & \cdots & D_{l_{2n}}\tau \\ \vdots & \vdots & \ddots & \vdots \\ D^l_{n1}\tau & D_{l_{n2}}\tau & \cdots & -\sum_{j=1}^{n} D^l_{jn}\tau \end{bmatrix}.$$

Note that a scalar matrix is communicative with any $n \times n$ matrix. Let $\underline{\mathbf{D}} = -(\underline{d}+\underline{\delta})\tau\mathbf{E}$. Then both $K\mathbf{E} + \mathbf{D}_0 + \mathbf{D}_l$ and $K\mathbf{E} + \underline{\mathbf{D}} + \mathbf{D}_l$ are nonnegative matrices and

$$K\mathbf{E} + \mathbf{D}_0 + \mathbf{D}_l \le K\mathbf{E} + \underline{\mathbf{D}} + \mathbf{D}_l.$$

Thus,

$$\begin{aligned} \exp(\mathbf{B}\tau) &= \exp(-K\mathbf{E} + K\mathbf{E} + \mathbf{D}_0 + \mathbf{D}_l) \\ &= \exp(-K\mathbf{E}) \exp(K\mathbf{E} + \mathbf{D}_0 + \mathbf{D}_l) \\ &\le \exp(-K\mathbf{E}) \exp(K\mathbf{E} + \underline{\mathbf{D}} + \mathbf{D}_l) \\ &= \exp(\mathbf{D}_l) \exp(\underline{\mathbf{D}}). \end{aligned} \tag{2.21}$$

Let $V = (1, \ldots, 1)$. It is easy to verify that $V\mathbf{D}_l = \mathbf{0}$, and hence $V \exp(\mathbf{D}_l) = V\mathbf{E}$. Therefore,

$$V \exp(\mathbf{B}\tau) \le V \exp(\mathbf{D}_l) \exp(\underline{\mathbf{D}}) = V \exp(\underline{\mathbf{D}}), \tag{2.22}$$

leading to the right side inequalities in (2.20). The left side inequalities in (2.20) can be similarly proved, and the proof of the lemma is completed. □

3 Well-posedness

Our new model consists of two parts: a system of ODEs (2.16) for $t \in [0, \tau]$ and a system of DDEs (2.15) for $t \geq \tau$. For biological reasons, the following non-negative initial value conditions should be posed for the model:

$$S_i(0) \geq 0, \quad \text{and} \quad I_i(0) \geq 0, \quad \text{for } i = 1, \ldots, n. \tag{3.1}$$

In order for the model to be biologically well-posed, we need to make sure that the model (2.16)–(2.15) with (3.1) has a unique solution which remains non-negative and bounded. The following theorem confirms this.

Theorem 3.1 *The initial value problem* (2.16)–(2.15)–(3.1) *has a unique solution which exists globally (i.e., for all $t \geq 0$), remains non-negative and is bounded.*

Proof The standard theory of ODEs ensures that the initial value problem (2.16)–(3.1) has a unique solution $(S_1^0(t), \ldots, S_n^0(t), I_1^0(t), \ldots, I_n^0(t))$ which exists globally, remains non-negative and is bounded. Consider the restriction of this solution on $[0, \tau]$ and denote its components by

$$\phi_i(\theta) = S_i^0(\theta), \quad \text{and} \quad \psi_i(\theta) = I_i^0(\theta), \quad \text{for } i = 1, \ldots, n, \quad \text{and} \quad \theta \in [0, \tau].$$

Then, $\phi_i(\theta)$ and $\psi_i(\theta)$ are continuous and non-negative functions on $[0, \tau]$. By the fundamental theory of delay differential equations (see, e.g., Hale and Verduyn Lunel (1993)), we know that the DDE system (2.15) with the initial conditions

$$S_i(\theta) = \phi_i(\theta) \quad \text{and} \quad I_i(\theta) = \psi_i(\theta), \quad \text{for } i = 1, \ldots, n, \tag{3.2}$$

has a unique solution $(S(t, \phi, \psi), I(t, \phi, \psi))$, which is well-defined on its maximal interval of existence $[\tau, t_{\max}(\phi, \psi))$, where

$$\begin{aligned}
(S(t, \phi, \psi), I(t, \phi, \psi)) &:= (S_1(t, \phi, \psi), \ldots, S_n(t, \phi, \psi), \\
&\qquad I_1(t, \phi, \psi), \ldots, I_n(t, \phi, \psi)), \\
(\phi, \psi) &:= (\phi_1(\theta), \ldots, \phi_n(\theta), \psi_1(\theta), \ldots, \psi_n(\theta)).
\end{aligned}$$

Firstly, we show the non-negativity of the solution for $t \in [\tau, t_{\max}(\phi, \psi))$. For this purpose, let us rewrite the system (2.15) as follows:

$$\frac{d}{dt}\mathbf{S}(t) = \mathbf{K} + \mathbf{D}(t)\mathbf{S}(t), \tag{3.3}$$

$$\frac{d}{dt}\mathbf{I}(t) = \mathbf{C}\mathbf{I}(t) + \mathbf{A}(t)\mathbf{I}(t - \tau), \quad t \geq \tau, \tag{3.4}$$

where $\mathbf{S}(t)=(S_1(t),\dots,S_n(t))^T, \mathbf{I}(t)=(I_1(t),\dots,I_n(t))^T$ and $\mathbf{K}\triangleq(K_1,\dots,K_n)^T$ and

$$
\mathbf{D}(t)=\begin{bmatrix}
-g_1(t) & D_{12}^S & \cdots & D_{1n}^S \\
D_{21}^S & -g_2(t) & \cdots & D_{2n}^S \\
\vdots & \vdots & \ddots & \vdots \\
D_{n1}^S & D_{n2}^S & \cdots & -g_n(t)
\end{bmatrix},\quad
\mathbf{C}=\begin{bmatrix}
-h_1 & D_{12}^I & \cdots & D_{1n}^I \\
D_{21}^I & -h_2 & \cdots & D_{2n}^I \\
\vdots & \vdots & \ddots & \vdots \\
D_{n1}^I & D_{n2}^I & \cdots & -h_n
\end{bmatrix},
$$

$$
\mathbf{A}(t)=\begin{bmatrix}
b_{11}(\tau)\lambda_1 S_1(t-\tau) & b_{12}(\tau)\lambda_2 S_2(t-\tau) & \cdots & b_{1n}(\tau)\lambda_n S_n(t-\tau) \\
b_{21}(\tau)\lambda_1 S_1(t-\tau) & b_{22}(\tau)\lambda_2 S_2(t-\tau) & \cdots & b_{2n}(\tau)\lambda_n S_n(t-\tau) \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}(\tau)\lambda_1 S_1(t-\tau) & b_{n2}(\tau)\lambda_2 S_2(t-\tau) & \cdots & b_{nn}(\tau)\lambda_n S_n(t-\tau)
\end{bmatrix},
$$

with $g_i(t)=d_i+\sum_{j=1}^n D_{ji}^S+\lambda_i I_i(t)$, and $h_i=d_i+\sigma_i+\sum_{j=1}^n D_{ji}^I$, for $i=1,\dots,n$.
Noting that the off-diagonal elements of matrix $\mathbf{D}(t)$ are non-negative, we conclude that the entries of the matrix $\exp(\int_\tau^t \mathbf{D}(\xi)d\xi)$ are all nonnegative. Indeed, let $G(t)=\max\{d_1+\sum_{j=1}^n D_{j1}^S+\lambda_1 I_1(t)+1,\dots,d_n+\sum_{j=1}^n D_{jn}^S+\lambda_n I_n(t)+1\}$ and rewrite $\mathbf{D}(t)$ as

$$
\mathbf{D}(t)=\begin{bmatrix}
-G(t) & 0 & \cdots & 0 \\
0 & -G(t) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & -G(t)
\end{bmatrix}
$$

$$
+\begin{bmatrix}
G(t)-g_1(t) & D_{12}^S & \cdots & D_{1n}^S \\
D_{21}^S & G(t)-g_2(t) & \cdots & D_{2n}^S \\
\vdots & \vdots & \ddots & \vdots \\
D_{n1}^S & D_{n2}^S & \cdots & G(t)-g_n(t)
\end{bmatrix}
$$

$$
\triangleq -G(t)\mathbf{E}+\bar{\mathbf{D}}(t).
$$

Then all entries of $\bar{\mathbf{D}}(t)$ are nonnegative, and hence, so are the entries of $e^{\int_\tau^t \bar{\mathbf{D}}(\xi)d\xi}$. It is obvious that

$$
e^{\int_\tau^t(-G(\xi)\mathbf{E})d\xi}=\begin{bmatrix}
e^{\int_\tau^t -G(\xi)d\xi} & 0 & \cdots & 0 \\
0 & e^{\int_\tau^t -G(\xi)d\xi} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & e^{\int_\tau^t -G(\xi)d\xi}
\end{bmatrix}.
$$

Noting that the scalar matrix $-G(t)\mathbf{E}$ commutes with any $n \times n$ matrix (hence with $\bar{\mathbf{D}}(t)$), we have

$$e^{\int_\tau^t \mathbf{D}(\xi)d\xi} = e^{\int_\tau^t -G(\xi)\mathbf{E}d\xi}e^{\int_\tau^t \bar{\mathbf{D}}(\xi)d\xi},$$

implying that all entries of $e^{\int_\tau^t \mathbf{D}(\xi)d\xi}$ are nonnegative. Now from (3.3), we have

$$\mathbf{S}(t) = e^{\int_\tau^t \mathbf{D}(\xi)d\xi}\mathbf{S}(\tau) + \int_\tau^t \mathbf{K}e^{\int_\tau^{t-s} \mathbf{D}(\xi)d\xi}ds \geq 0, \quad \text{for} \quad t \in [\tau, t_{\max}(\phi, \psi)). \quad (3.5)$$

Similarly, for any $t \geq \tau$, all entries of $\exp(\mathbf{C}t)$ are nonnegative. Moreover, by the non-negativity of $\mathbf{S}(t)$ established above, we know that all entries of $A(t)$ are all non-negative. Now, (3.4) leads to

$$\mathbf{I}(t) = e^{\mathbf{C}t}\mathbf{I}(\tau) + \int_\tau^t e^{\mathbf{C}(t-s)}\mathbf{A}(s)\mathbf{I}(s-\tau)ds, \quad \text{for } t \geq \tau, \quad (3.6)$$

implying $\mathbf{I}(t) \geq 0$ for $t \in [\tau, 2\tau]$ from the initial condition $I_i(\theta) \geq 0$ for $\theta \in [0, \tau]$ and $i = 1, \ldots, n$. This and (3.6) ensure $\mathbf{I}(t) \geq 0$ for $t \in [2\tau, 3\tau]$. By induction, we then conclude that $\mathbf{I}(t) \geq 0$ for $t \in [\tau, t_{\max}(\phi, \psi))$.

Now, we show that $S_i(t)$ and $I_i(t)$ are bounded for $t \in [\tau, t_{\max}(\phi, \psi))$ and $i = 1, \ldots, n$. Let $N(t) = S_1(t-\tau) + \cdots + S_n(t-\tau) + I_1(t) + \cdots + I_n(t)$. By Lemma 2.1 and non-negativity of $S_i(t)$ and $I_i(t)$, we have

$$\begin{aligned}
\frac{d}{dt}N(t) = {} & (K_1 + \cdots + K_n) - d_1 S_1(t-\tau) - \cdots - d_n S_n(t-\tau) \\
& -\lambda_1 I_1(t-\tau)S_1(t-\tau) - \cdots - \lambda_n I_n(t-\tau)S_n(t-\tau) \\
& -(d_1 + \sigma_1)I_1(t) - \cdots - (d_n + \sigma_n)I_n(t) \\
& + \sum_{i=1}^n \sum_{j=1}^n b_{ij}(\tau)\lambda_j I_j(t-\tau)S_j(t-\tau) \\
\leq {} & (K_1 + \cdots + K_n) - d_1 S_1(t-\tau) - \cdots - d_n S_n(t-\tau) \\
& -(d_1 + \sigma_1)I_1(t) - \cdots - (d_n + \sigma_n)I_n(t) \\
\leq {} & (K_1 + \cdots + K_n) - \min\{d_1, \ldots, d_n\}N(t). \quad (3.7)
\end{aligned}$$

This implies that $N(t)$ is bounded, and so are $S_i(t)$ and $I_i(t)$ for $i = 1, \ldots, n$ and $t \in [\tau, t_{\max}(\phi, \psi))$. By the theory of continuation of solutions (see, e.g., Hale and Verduyn Lunel 1993), we conclude that $t_{\max}(\phi, \psi) = \infty$, which means the solution $(S(t, \phi, \psi), I(t, \phi, \psi))$ exists globally. This together with the results on $S_i^0(t)$ and $I_i^0(t)$ on $t \in [0, \tau)$ implies that all of the above results actually hold for all $t \geq 0$. This completes the proof of Theorem 3.1. □

Remark 3.2 From the proof of this theorem, we see that the S-components of the solution to (2.15) with (3.1) actually remain positive. If we further assume $\psi_i(0) > 0$ for

$i = 1, \ldots, n$, then the I- components of the solution also remain positive. Moreover, from (3.6), one concludes that if either the matrix $[b_{ij}(\tau)]$ or the matrix $\exp(\mathbf{C})$ is irreducible, then $\psi(0) > 0$ (i.e., $\psi_i(0) > 0$ for at least one $i \in \{1, \ldots, n\}$) also guarantees that all I-components of the solution are positive. Note that the irreducibility of $[b_{ij}(\tau)]$ is implied by that of $[D_{ij}^I]$, and the irreducibility of $\exp(\mathbf{C})$ is implied by that of $[D_{ij}^I]$. The biological meaning of this is that once inoculated in one patch, the disease will spread to all other patches, due to the mobility of the infectious or latent individuals.

Remark 3.3 Although the new model consists of two systems, (2.16) only plays a role of generating the necessary initial functions on $[0, \tau]$ for (2.15). The long term behavior of the solution to (2.16)–(2.15)–(3.1) is indeed determined by (2.15). Therefore in the rest of this paper, we only consider (2.15) since we are only interested in the the long term disease dynamics.

4 Disease free equilibrium and its stability

In this section, we assume that the travel rate matrices $[D_{ij}^S]$, $[D_{ij}^I]$ and $[D_{ij}^I]$ are irreducible. As usual, we start by investigating disease free equilibrium. A disease free equilibrium (DFE) is a steady state solution of the system (2.15) with all infectious variables being zeros. A DFE for the model (2.15) is thus given by $\mathcal{E}^{(0)} = (S_1^{(0)}, \ldots, S_n^{(0)}, 0, \ldots, 0)$ with $\mathbf{S}^{(0)} = (S_1^{(0)}, \ldots, S_n^{(0)})^T$ satisfying the linear system $\mathbf{M}\mathbf{S}^{(0)} = \mathbf{K}$, where

$$
\mathbf{M} = \begin{bmatrix} d_1 + \sum_{j=1}^n D_{j1}^S & -D_{12}^S & \cdots & -D_{1n}^S \\ -D_{21}^S & d_2 + \sum_{j=1}^n D_{j2}^S & \cdots & -D_{2n}^S \\ \vdots & \vdots & \ddots & \vdots \\ -D_{n1}^S & -D_{n2}^S & \cdots & d_n + \sum_{j=1}^n D_{jn}^S \end{bmatrix}.
$$

Note that matrix $\mathbf{M}$ is irreducible, has positive column sums and negative off-diagonal entries. Thus $\mathbf{M}$ is a non-singular M-Matrix (Berman and Plemmons 1979, page 141) with $\mathbf{M}^{-1} > 0$, and therefore, the linear system has a unique solution given by $\mathbf{S}^{(0)} = \mathbf{M}^{-1}\mathbf{K} > 0$. This shows the existence of a unique disease free equilibrium $\mathcal{E}^{(0)}$.

In order to discuss the stability of $\mathcal{E}^{(0)}$, we introduce the following matrices:

$$
\mathcal{F} \triangleq \begin{bmatrix} b_{11}(\tau)\lambda_1 S_1^{(0)} & b_{12}(\tau)\lambda_2 S_2^{(0)} & \cdots & b_{1n}(\tau)\lambda_n S_n^{(0)} \\ b_{21}(\tau)\lambda_1 S_1^{(0)} & b_{22}(\tau)\lambda_2 S_2^{(0)} & \cdots & b_{2n}(\tau)\lambda_n S_n^{(0)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(\tau)\lambda_1 S_1^{(0)} & b_{n2}(\tau)\lambda_2 S_2^{(0)} & \cdots & b_{nn}(\tau)\lambda_n S_n^{(0)} \end{bmatrix},
$$

and

$$
\mathcal{V} \triangleq
\begin{bmatrix}
d_1 + \sigma_1 + \sum_{j=1}^n D_{j1}^I & -D_{12}^I & \cdots & -D_{1n}^I \\
-D_{21}^I & d_2 + \sigma_2 + \sum_{j=1}^n D_{j2}^I & \cdots & -D_{2n}^I \\
\vdots & \vdots & \ddots & \vdots \\
-D_{n1}^I & -D_{n2}^I & \cdots & d_n + \sigma_n + \sum_{j=1}^n D_{jn}^I
\end{bmatrix}.
$$

Note that $\mathcal{F}$ is a non-negative matrix, $\mathcal{V}$ is irreducible, and has positive column sums and non-positive off-diagonal entries. Thus $\mathcal{V}$ is a non-singular M-Matrix (Berman and Plemmons 1979, page 141) with $\mathcal{V}^{-1} > 0$. Hence, $\mathcal{F}\mathcal{V}^{-1}$ is a positive matrix, and its spectral radius $\rho(\mathcal{F}\mathcal{V}^{-1})$ is a simple eigenvalue of $\mathcal{F}\mathcal{V}^{-1}$, corresponding to which there is a positive eigenvector. The following theorem shows that the disease dies out if $\rho(\mathcal{F}\mathcal{V}^{-1}) < 1$.

Theorem 4.1 *If $\rho(\mathcal{F}\mathcal{V}^{-1}) < 1$, the disease free equilibrium $\mathcal{E}^{(0)} \triangleq (S_1^{(0)}, \ldots, S_n^{(0)}, 0, \ldots, 0)$ is globally asymptotically stable. If $\rho(\mathcal{F}\mathcal{V}^{-1}) > 1$, the disease free equilibrium $\mathcal{E}^{(0)}$ becomes unstable.*

Proof Firstly, we need to show that the disease free equilibrium $\mathcal{E}^{(0)} \triangleq (S_1^{(0)}, \ldots, S_n^{(0)}, 0, \ldots, 0)$ is locally asymptotically stable (unstable) under the condition $\rho(\mathcal{F}\mathcal{V}^{-1}) < 1$ (> 1). To this end, we consider the linearization of (2.15) at $\mathcal{E}^{(0)}$:

$$
\frac{dS_i(t)}{dt} = -d_i S_i(t) + \sum_{j=1}^n D_{ij}^S S_j(t) - \sum_{j=1}^n D_{ji}^S S_i(t) - \lambda_i S_i^{(0)} I_i(t),
$$

$$
\frac{dI_i(t)}{dt} = -(d_i + \sigma_i) I_i(t) + \sum_{j=1}^n D_{ij}^I I_j(t) - \sum_{j=1}^n D_{ji}^I I_i(t) \tag{4.1}
$$

$$
+ \sum_{j=1}^n b_{ij}(\tau) \lambda_j S_j^{(0)} I_j(t - \tau),
$$

which can be rewritten as the following matrix form,

$$
\frac{d}{dt}
\begin{bmatrix} \mathbf{S}(t) \\ \mathbf{I}(t) \end{bmatrix}
=
\begin{bmatrix} -\mathbf{M} & \mathbf{N} \\ \mathbf{0} & -\mathcal{V} \end{bmatrix}
\begin{bmatrix} \mathbf{S}(t) \\ \mathbf{I}(t) \end{bmatrix}
+
\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathcal{F} \end{bmatrix}
\begin{bmatrix} \mathbf{S}(t - \tau) \\ \mathbf{I}(t - \tau) \end{bmatrix}
\tag{4.2}
$$

where, $\mathbf{S}(t)$ and $\mathbf{I}(t)$ are defined in the proof of Theorem 3.1, matrices $\mathbf{M}$, $\mathcal{F}$ and $\mathcal{V}$ are given above, and

$$
\mathbf{N} =
\begin{bmatrix}
\lambda_1 S_1^{(0)} & 0 & \cdots & 0 \\
0 & \lambda_2 S_2^{(0)} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n S_n^{(0)}
\end{bmatrix}.
$$

The characteristic equation of (4.2) is given by

$$\det \begin{bmatrix} z\mathbf{E} + \mathbf{M} & -\mathbf{N} \\ \mathbf{0} & z\mathbf{E} + \mathcal{V} - \mathcal{F}e^{-z\tau}, \end{bmatrix} = 0 \qquad (4.3)$$

where $\mathbf{E}$ is $n \times n$ identity matrix. The equation (4.3) is equivalent to

$$|z\mathbf{E} + \mathbf{M}| \cdot |z\mathbf{E} + \mathcal{V} - \mathcal{F}e^{-z\tau}| = 0. \qquad (4.4)$$

Let

$$\Delta_1(z) = |z\mathbf{E} + \mathbf{M}|, \quad \Delta_2(z, \tau) = |z\mathbf{E} + \mathcal{V} - \mathcal{F}e^{-z\tau}|.$$

Note that for matrix $\mathbf{M} = [M_{ij}]_{n \times n}$,

$$\sum_{j \neq i} |-M_{ji}| = \sum_{j=1}^{n} D_{ji}^S < d_i + \sum_{j=1}^{n} D_{ji}^S = \left| -d_i - \sum_{j=1}^{n} D_{ji}^S \right| = |-M_{ii}|. \quad (4.5)$$

By Gershgorin circle theorem (see, e.g., Golub and Van Loan 1996) we know that each eigenvalue of $-\mathbf{M}$ lies in the union of circles

$$|z + M_{ii}| \leq \sum_{j \neq i} |-M_{ji}|, \quad i = 1, \dots, n. \qquad (4.6)$$

This together with (4.5) implies that the real part of each eigenvalue of $-\mathbf{M}$ is negative.

Next, we show that the real part of each root of $\Delta_2(z, \tau) = 0$ is negative. For this purpose, we notice that the I equations of the linearization (4.1) is decoupled from the S equations and $\Delta_2(z, \tau) = 0$ is the characteristic equation of the I equations. Write the I-equations as

$$\frac{d}{dt}\mathbf{I}(t) = -\mathcal{V}\mathbf{I}(t) + \mathcal{F}\mathbf{I}(t - \tau). \qquad (4.7)$$

Note that $\mathcal{F}$ and $-\mathcal{V}$ are quasi-positive and irreducible matrices. Thus, a cooperative and irreducible system of ordinary differential equations can be associated with the system (4.7) by simply setting $\tau = 0$ in (4.7). This leads to the system

$$\frac{d}{dt}\mathbf{I}(t) = (\mathcal{F} - \mathcal{V})\mathbf{I}(t). \qquad (4.8)$$

By using the stability criteria for the cooperative and irreducible systems (see Theorem 5.1 and Corollary 5.2 in Smith 1995), we know that the linear stability of the trivial equilibrium for system (4.7) is equivalent to that for system (4.8). Therefore,

$$\max\{\text{Re}(z) : \Delta_2(z, \tau) = 0\} < 0 \ (> 0) \quad \text{if and only if } s(\mathcal{F} - \mathcal{V}) < 0 \ (> 0).$$

Here, $s(\mathcal{F} - \mathcal{V})$ is the stability modulus of $\mathcal{F} - \mathcal{V}$ defined as the maximal real part of all eigenvalues of the matrix $\mathcal{F} - \mathcal{V}$. However, by using the proof of Theorem 2 in van den Driessche and Watmough (2002), we know that $s(\mathcal{F} - \mathcal{V}) < 0$ (> 0) if and only if $\rho(\mathcal{F}V^{-1}) < 1$ (> 1). Thus, we have shown that if $\rho(\mathcal{F}V^{-1}) < 1$, then the real parts of all the roots of $\Delta_2(z, \tau) = 0$ are negative; if $\rho(\mathcal{F}V^{-1}) > 1$, then there is a root of $\Delta_2(z, \tau) = 0$ that has positive real part, concluding the local stability or instability of $\mathcal{E}^{(0)}$ for (2.15).

To complete the proof of the theorem, it remains to prove the global attractivity of $\mathcal{E}^{(0)}$ for (2.15) under the condition $\rho(\mathcal{F}V^{-1}) < 1$. From the S-equations in system (2.15) and the non-negativity of the solutions to the system (2.15) with (3.1), we have

$$
\begin{aligned}
\frac{d}{dt} S_i(t) &= K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t) - \sum_{j=1}^{n} D_{ji}^S S_i(t) - \lambda_i I_i(t) S_i(t), \\
&\leq K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t) - \sum_{j=1}^{n} D_{ji}^S S_i(t).
\end{aligned}
\tag{4.9}
$$

This suggests the following comparison system for the S-equations of (2.15):

$$
\frac{du_i(t)}{dt} = K_i - d_i u_i(t) + \sum_{j=1}^{n} D_{ij}^S u_j(t) - \sum_{j=1}^{n} D_{ji}^S u_i(t).
\tag{4.10}
$$

We have seen from the above arguments in this session (results on matrix -$\mathbf{M}$) that System (4.10) admits a unique positive equilibrium $(S_1^{(0)}, \ldots, S_n^{(0)})$ which is globally asymptotically stable (in a linear system, local stability is equivalent to global stability). By the comparison theorem for cooperative systems (see, e.g., Smith 1995; Smith and Waltman 1995), it follows that

$$
S_i^{\infty} \triangleq \limsup_{t \to \infty} S_i(t) \leq \lim_{t \to \infty} u_i(t) = S_i^{(0)}, \quad \text{for } i = 1, \ldots, n.
\tag{4.11}
$$

Thus, for any constant $\epsilon > 0$, there is a large enough T such that $S_i(t) \leq S_i^{(0)} + \epsilon$, all $t \geq T$.

Set

$$
\mathcal{F}_\epsilon \triangleq
\begin{bmatrix}
b_{11}(\tau)\lambda_1(S_1^{(0)} + \epsilon) & b_{12}(\tau)\lambda_2(S_2^{(0)} + \epsilon) & \cdots & b_{1n}(\tau)\lambda_n(S_n^{(0)} + \epsilon) \\
b_{21}(\tau)\lambda_1(S_1^{(0)} + \epsilon) & b_{22}(\tau)\lambda_2(S_2^{(0)} + \epsilon) & \cdots & b_{2n}(\tau)\lambda_n(S_n^{(0)} + \epsilon) \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}(\tau)\lambda_1(S_1^{(0)} + \epsilon) & b_{n2}(\tau)\lambda_2(S_2^{(0)} + \epsilon) & \cdots & b_{nn}(\tau)\lambda_n(S_n^{(0)} + \epsilon)
\end{bmatrix}.
$$

Since the spectral radius of $\mathcal{F}_\epsilon V^{-1}$ is continuous in ϵ, we can restrict $\epsilon > 0$ small enough such that $\rho(\mathcal{F}_\epsilon V^{-1}) < 1$. For $t \geq T$, we construct the following comparison

linear system for the I equations in (2.15):

$$\frac{d}{dt}I_i(t) = -(d_i + \sigma_i)I_i(t) + \sum_{j=1}^{n} D_{ij}^I I_j(t) - \sum_{j=1}^{n} D_{ji}^I I_i(t)$$

$$+ \sum_{j=1}^{n} b_{ij}(\tau)\lambda_j I_j(t-\tau)(S_j^{(0)} + \epsilon), \quad \text{for } i = 1, \ldots, n, \qquad (4.12)$$

i.e.

$$\frac{d\mathbf{I}(t)}{dt} = -\mathcal{V}\mathbf{I}(t) + \mathcal{F}_\epsilon \mathbf{I}(t-\tau). \qquad (4.13)$$

By the same argument as that for the stability of (4.7), we know that the trivial solution this system is globally asymptotically stable, implying that all solutions of the linear system (4.12) tend to the trivial solution as $t \to \infty$. By the comparison theory for monotone dynamical systems of delayed type, we conclude that all I components of the solution to (2.15) with (3.1) also tend to zeros as $t \to \infty$. This in return implies that the S equation in (2.15) has (4.10) as its limiting system, which has the dynamics of global convergence to $(S_1^{(0)}, \ldots, S_n^{(0)})$. Finally by the theory of asymptotically autonomous systems (see, e.g., Castillo-Chaves and Thieme 1995; Mischaikow et al. 1995), we conclude that the S component of the solution to (2.15) with (3.1) also converges to $(S_1^{(0)}, \ldots, S_n^{(0)})$. This confirms the global attractivity of $\mathcal{E}^{(0)}$ for (2.15) under the condition $\rho(\mathcal{F}V^{-1}) < 1$, and hence completes the proof. $\qquad \square$

5 Disease persistence and endemic equilibrium

In Sect. 4, under the assumption that the travel rate matrices $[D_{ij}^S]$, $[D_{ij}^I]$ and $[D_{ij}^I]$ are irreducible, we have shown that $\rho(\mathcal{F}V^{-1}) = 1$ is the critical value that distinguishes the stability and instability of the DFE. One naturally wonders what happens if $\rho(\mathcal{F}V^{-1}) > 1$. In this section, we still assume the irreducibility of all travel rate matrices, and show that if $\rho(\mathcal{F}V^{-1}) > 1$, the disease will persist in all patches. This conclusion together with a well-known result for persistent systems actually implies the existence of an endemic equilibrium for the model (2.15).

For the convenience of stating and proving the main results, we first introduce some notations. Let $C := C([-\tau, 0], \mathbb{R}^n)$ denote the set of all continuous functions from $[-\tau, 0]$ to $\mathbb{R}^n$. As is customary, $C_+ := ([-\tau, 0], \mathbb{R}_+^n)$ denote the subset of C consisting of all non-negative functions. By Theorem 3.1 and Remark 3.2, for any $(\phi, \psi) \in C_+ \times C_+$ with $\psi(0) > 0$, there is a unique solution to (2.15), denoted by

$$(S(t, \phi, \psi), I(t, \phi, \psi)) = (S_1(t, \phi, \psi), \ldots, S_n(t, \phi, \psi),$$
$$I_1(t, \phi, \psi), \ldots, I_n(t, \phi, \psi)),$$

whose components are all positive and bounded for $t > 0$.

Theorem 5.1 *Assume that $\rho(\mathcal{F}\mathcal{V}^{(-1)}) > 1$. Then there is an $\bar{\epsilon} > 0$ such that for every $(\phi, \psi) \in C_+ \times C_+$ with $\psi(0) > 0$, the solution $(S(t), I(t)) = (S(t, \phi, \psi), I(t, \phi, \psi))$ of (2.15) satisfies*

$$\liminf_{t \to \infty} I_i(t, \phi, \psi) \geq \bar{\epsilon}, \quad i = 1, 2, \ldots, n.$$

Moreover, the model (2.15) *admits at least one* (*componentwise*) *positive equilibrium.*

Proof Define $X := \{(\phi, \psi) \in C_+ \times C_+\}$, $X_0 := \{(\phi, \psi) \in X : \psi_i(0) > 0, i = 1, 2, \ldots, n\}$ and $\partial X_0 = X \backslash X_0$. It then suffices to show that (2.15) is uniformly persistent with respect to $(X_0, \partial X_0)$.

Let $\Phi(t) : X \to X$ be the solution semiflow of (2.15), that is, $\Phi(t)(\phi, \psi) = (S_t(\phi, \psi), I_t(\phi, \psi))$. By Theorem 3.1 and Remark 3.2, X and X_0 are positively invariant for $\Phi(t)$. Clearly, $\partial X_0 = \{(\phi, \psi) \in X : \psi_i(0) = 0$ for at least one $i \in \{1, 2, \ldots, n\}\}$ and it is relatively closed in X. Furthermore, system (2.15) is point dissipative in $\mathbb{R}_+^n$ since nonnegative solutions of (2.15) are ultimately bounded (see Theorem 3.1).

Define $\Omega_\partial = \{(\phi, \psi) \in X : (S_t(\phi, \psi), I_t(\phi, \psi)) \in \partial X_0, \quad \forall t \geq 0\}$. We now show that

$$\Omega_\partial = \{(\phi, \psi) \in \partial X_0 : I(t, \phi, \psi) = 0, \quad \forall t \geq 0\}. \tag{5.1}$$

Assume $(\phi, \psi) \in \Omega_\partial$. It suffices to show that $I(t, \phi, \psi) = 0, \forall t \geq 0$. For the sake of contradiction, assume that there is an i_0, $1 \leq i_0 \leq n$, and a $t_0 \geq 0$ such that $I_{i_0}(t_0) > 0$. We partition $\{1, 2, \ldots, n\}$ into two sets Z_1 and Z_2 such that

$$I_i(t_0, \phi, \psi) = 0, \quad \forall i \in Z_1; \qquad I_i(t_0, \phi, \psi) > 0, \quad \forall i \in Z_2.$$

Obviously Z_1 is non-empty due to the definition of Ω_∂. Z_2 is also non-empty since $I_{i_0}(t_0, \phi, \psi) > 0$. For any $j \in Z_1$, by irreducibility of the matrix $[b_{ij}(\tau)]$, there is an $i_1 \in Z_2$ such that $b_{ji_1} > 0$. This leads to

$$\frac{d}{dt} I_j(t, \phi, \psi)|_{t=t_0} = -(d_j + \sigma_j) I_j(t_0, \phi, \psi) + \sum_{i=1}^{n} D_{ji}^I I_i(t_0, \phi, \psi)$$

$$- \sum_{i=1}^{n} D_{ij}^I I_j(t_0, \phi, \psi)$$

$$+ \sum_{i=1}^{n} b_{ji}(\tau) \lambda_i I_i(t_0 - \tau, \phi, \psi) S_i(t_0 - \tau, \phi, \psi)$$

$$\geq b_{ji_1}(\tau) I_{i_1}(t_0, \phi, \psi) > 0. \tag{5.2}$$

It follows that there is an $\epsilon_0 > 0$ such that $I_j(t, \phi, \psi) > 0$ for $j \in Z_1$ and $t_0 < t < t_0 + \epsilon_0$. Clearly, we can restrict $\epsilon_0 > 0$ small enough such that $I_i(t, \phi, \psi) > 0$ for $i \in Z_2$ for $t_0 < t < t_0 + \epsilon_0$. This means that $(S_t(\phi, \psi), I_t(\phi, \psi)) \notin \partial X_0$ for $t_0 < t < t_0 + \epsilon_0$, which contradicts the assumption that $(\phi, \psi) \in \Omega_\partial$. This proves (5.1).

Choose $\xi > 0$ small enough such that $\rho(\mathcal{F}_\xi \mathcal{V}^{-1}) > 1$ where

$$
\mathcal{F}_\xi \triangleq \begin{bmatrix}
b_{11}(\tau)\lambda_1 S_1^{(0)} - \xi\lambda_1 & b_{12}(\tau)\lambda_2 S_2^{(0)} & \cdots & b_{1n}(\tau)\lambda_n S_n^{(0)} \\
b_{21}(\tau)\lambda_1 S_1^{(0)} & b_{22}(\tau)\lambda_2 S_2^{(0)} - \xi\lambda_2 & \cdots & b_{2n}(\tau)\lambda_n S_n^{(0)} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n1}(\tau)\lambda_1 S_1^{(0)} & b_{n2}(\tau)\lambda_2 S_2^{(0)} & \cdots & b_{nn}(\tau)\lambda_n S_n^{(0)} - \xi\lambda_n
\end{bmatrix}.
$$

Let us consider the following linear system

$$
\begin{aligned}
\frac{d}{dt} S_i(t) &= K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t) - \sum_{j=1}^{n} D_{ji}^S S_i(t) - \lambda_i \bar\epsilon S_i(t) \\
&= K_i - \left(d_i + \sum_{j=1}^{n} D_{ji}^S + \lambda_i \bar\epsilon \right) S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t), \quad i = 1, \ldots, n. \quad (5.3)
\end{aligned}
$$

which is a perturbation of (4.10). Restrict $\bar\epsilon > 0$ small enough such that (5.3), just as (4.10), has a unique positive equilibrium $(S_1^{(0)}(\bar\epsilon), \ldots, S_n^{(0)}(\bar\epsilon))$ which is globally asymptotically stable. By the implicit function theorem, it follows that $(S_1^{(0)}(\bar\epsilon), \ldots, S_n^{(0)}(\bar\epsilon))$ is continuous in $\bar\epsilon$. Thus, we can further restrict $\bar\epsilon$ small enough such that $(S_1^{(0)}(\bar\epsilon), \ldots, S_n^{(0)}(\bar\epsilon)) > (S_1^{(0)} - \xi, \ldots, S_n^{(0)} - \xi)$.

Next for the solution $(S(t, \phi, \psi), I(t, \phi, \psi))$ of (2.15) through (ϕ, ψ), we claim that

$$
\limsup_{t \to \infty} \max_i \{I_i(t, \phi, \psi)\} > \bar\epsilon, \quad \text{for all} \quad (\phi, \psi) \in X_0. \quad (5.4)
$$

Otherwise, there is a $T_1 > 0$ such that $0 < I_i(t, \phi, \psi) \leq \bar\epsilon$, $i = 1, \ldots, n$, for all $t \geq T_1$. Then for $t \geq T_1$, we have

$$
\begin{aligned}
\frac{d}{dt} S_i(t) &\geq K_i - d_i S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t) - \sum_{j=1}^{n} D_{ji}^S S_i(t) - \lambda_i \bar\epsilon S_i(t) \\
&= K_i - \left(d_i + \sum_{j=1}^{n} D_{ji}^S + \lambda_i \bar\epsilon \right) S_i(t) + \sum_{j=1}^{n} D_{ij}^S S_j(t), \quad i = 1, \ldots, n.
\end{aligned}
$$
$$(5.5)$$

Since the equilibrium $(S_1^{(0)}(\bar\epsilon), \ldots, S_n^{(0)}(\bar\epsilon))$ of (5.3) is globally asymptotically stable and $S^{(0)}(\bar\epsilon) > S^{(0)} - \xi$, there is a T_2 such that $S(t) > S^{(0)} - \xi$ for $t \geq T_1 + T_2$.

Consequently, for $t \geq T_1 + T_2$,

$$\frac{d}{dt} I_i(t) \geq -(d_i + \sigma_i) I_i(t) + \sum_{j=1}^{n} D_{ij}^I I_j(t) - \sum_{j=1}^{n} D_{ji}^I I_i(t)$$

$$+ \sum_{j=1}^{n} b_{ij}(\tau) \lambda_j I_j(t - \tau)(S_j^{(0)} - \xi), \quad i = 1, \ldots, n. \quad (5.6)$$

By the same arguments as that for the stability and instability of the ODE (4.8) and the DDE (4.7) in Sect. 4, we know that the assumption $\rho(\mathcal{F}_\xi \mathcal{V}^{-1}) > 1$ implies that the trivial solution of the linear system

$$\frac{d}{dt} \mathbf{I}(t) = -\mathcal{V} \mathbf{I}(t) + \mathcal{F}_\xi \mathbf{I}(t - \tau) \quad (5.7)$$

is unstable. This together with (5.6) and the comparison theorem implies that there is at least one $i \in \{1, \ldots, n\}$ such that $I_i(t) \to \infty$ as $t \to \infty$, a contradiction to the boundedness of solutions. Therefore (5.4) holds.

Note that $(S_1^{(0)}, \ldots, S_n^{(0)})$ is globally asymptotically stable in $\mathbb{R}_+^n \backslash \{\mathbf{0}\}$ for system (4.10). By the afore-mentioned claim, it then follows that $\mathcal{E}^{(0)}$ is an isolated invariant set in X, and $W^s(\mathcal{E}^{(0)}) \cap X_0 = \emptyset$. Clearly, every orbit in Ω_∂ converges to $\mathcal{E}^{(0)}$, and $\mathcal{E}^{(0)}$ is the only invariant set in Ω_∂. By Theorem 4.6 in Thieme (1993) and Theorem 4.3 and Remark 4.3 in Hirsch et al. (2001) for a stronger repelling property of ∂X_0, we conclude that system (2.15) is indeed uniformly persistent with respect to $(X_0, \partial X_0)$. Moreover, by Theorem 2.4 in Zhao (1995), system (2.15) has an equilibrium $(S_1^*, \ldots, S_n^*, I_1^*, \ldots, I_n^*) \in X_0$, implying that $(S_1^*, \ldots, S_n^*) \in \mathbb{R}_+^n$ and $(I_1^*, \ldots, I_n^*) \in \text{int}(\mathbb{R}_+^n)$. We further claim that $(S_1^*, \ldots, S_n^*) \in \mathbb{R}_+^n \backslash \{\mathbf{0}\}$. Suppose that $(S_1^*, \ldots, S_n^*) = \mathbf{0}$. By the I-equations in (2.15), we then obtain

$$0 = -(d_i + \sigma_i) I_i^* + \sum_{j=1}^{n} D_{ij}^I I_j^* - \sum_{j=1}^{n} D_{ji}^I I_i^*, \quad \text{for} \quad i = 1, \ldots, n, \quad (5.8)$$

and hence

$$0 = -\sum_{i=1}^{n} (d_i + \sigma_i) I_i^*. \quad (5.9)$$

Therefore, $I_i^* = 0$, for $i = 1, \ldots, n$, a contradiction. By the S-equation in (2.15) and the irreducibility of the cooperative matrix $[D_{ij}^S]$, it follows that $S^* = S(t, S^*, I^*) \in \text{int}(\mathbb{R}_+^n)$ with $S^* := (S_1^*, \ldots, S_n^*)$ and $I^* := (I_1^*, \ldots, I_n^*)$, for $\forall t > 0$. Then (S^*, I^*) is a componentwise positive equilibrium of system (2.15). $\qquad \square$

Remark 5.2 From the main results in Sects. 4 and 5, we see that if the travel rate matrices for the three classes are irreducible, then $\rho(\mathcal{F} \mathcal{V}^{-1}) = 1$ plays a threshold role in

determining whether the disease will die out or persist. In the case of $\tau = 0$, by the next generation matrix argument (see, Diekmann et al. 1990; van den Driessche and Watmough 2002), one knows that $\rho(\mathcal{F}\mathcal{V}^{-1})$ gives the basic reproduction number for the reduced ODE model, accounting for "the average number of secondary infectious individuals produced by a single infectious individual introduced into a host population where everyone is susceptible" (see e.g., Anderson and May 1991; Diekmann et al. 1990; van den Driessche and Watmough 2002). Note that $\rho(\mathcal{F}\mathcal{V}^{-1})$ depends on τ continuously. By a continuity argument and the above established threshold property (Theorems 4.1 and 5.1), together with the biological meaning of the basic reproduction number, we claim that the quantity $\rho(\mathcal{F}\mathcal{V}^{-1})$ is the basic reproduction number for the new model (2.15), denoted by $\mathcal{R}_0$.

6 Allowing reducible travel rate matrices—two-patch case

The results in Sects. 4 and 5 are obtained under the assumption that the travel rate matrices are all irreducible. In reality, these assumptions may not be satisfied. For example, when an infectious disease is reported in one or more cities, the health authorities in some or all cities may implement a ban against travel by the infected individuals. Such a measure may make some travel rate matrices reducible. In this section, we deal with cases allowing reducible rate matrices. The high dimension of model (2.15) with general n increases the difficulty in obtaining more information on the dynamics of the model. In order to show the main idea to obtain the dynamics of the model, which might be obscured by the complicated computation for the higher dimensional case, we simply focus on the case $n = 2$. We shall see that this minimum dimension choice for patchy environment enables us to do some more detailed calculations and rigorous analysis, by which some new phenomena can be revealed.

When $n = 2$, the model (2.15) becomes

$$
\begin{aligned}
\frac{dS_1(t)}{dt} &= K_1 - d_1 S_1(t) - D_{21}^S S_1(t) + D_{12}^S S_2(t) - \lambda_1 I_1(t) S_1(t), \\
\frac{dS_2(t)}{dt} &= K_2 - d_2 S_2(t) - D_{12}^S S_2(t) + D_{21}^S S_1(t) - \lambda_2 I_2(t) S_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1) I_1(t) - D_{21}^I I_1(t) + D_{12}^I I_2(t) \\
&\quad + b_{11}(\tau) \lambda_1 I_1(t - \tau) S_1(t - \tau) + b_{12}(\tau) \lambda_2 I_2(t - \tau) S_2(t - \tau), \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2) I_2(t) - D_{12}^I I_2(t) + D_{21}^I I_1(t) \\
&\quad + b_{21}(\tau) \lambda_1 I_1(t - \tau) S_1(t - \tau) + b_{22}(\tau) \lambda_2 I_2(t - \tau) S_2(t - \tau).
\end{aligned}
\tag{6.1}
$$

The linear system $\mathbf{M}\mathbf{S}^{(0)} = \mathbf{K}$ for determining the disease free equilibrium is simplified to

$$
\begin{bmatrix} d_1 + D_{21}^S & -D_{12}^S \\ -D_{21}^S & d_2 + D_{12}^S \end{bmatrix} \begin{bmatrix} S_1^{(0)} \\ S_2^{(0)} \end{bmatrix} = \begin{bmatrix} K_1 \\ K_2 \end{bmatrix},
\tag{6.2}
$$

which yields the disease free equilibrium $\mathcal{E}^{(0)} = (S_1^{(0)}, S_2^{(0)}, 0, 0)$ with

$$S_1^{(0)} = \frac{D_{12}^S K_2 + D_{12}^S K_1 + d_2 K_1}{d_1 d_2 + d_1 D_{12}^S + d_2 D_{21}^S}, \quad \text{and} \quad S_2^{(0)} = \frac{D_{21}^S K_1 + D_{21}^S K_2 + d_1 K_2}{d_1 d_2 + d_1 D_{12}^S + d_2 D_{21}^S}, \quad (6.3)$$

regardless of the irreducibility of the matrix $[D_{ij}^S]$ (hence, of matrix **M**). Moreover, for this simple case, the corresponding matrices $\mathcal{F}$ and $\mathcal{V}$ reduce to

$$\mathcal{F} = \begin{bmatrix} b_{11}(\tau)\lambda_1 S_1^{(0)} & b_{12}(\tau)\lambda_2 S_2^{(0)} \\ b_{21}(\tau)\lambda_1 S_1^{(0)} & b_{22}(\tau)\lambda_2 S_2^{(0)} \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} d_1 + \sigma_1 + D_{21}^I & -D_{12}^I \\ -D_{21}^I & d_2 + \sigma_2 + D_{12}^I \end{bmatrix},$$

by which one can compute $\rho(\mathcal{F}\mathcal{V}^{-1})$ to obtain the following formula:

$$\rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\mathcal{P} + \sqrt{\mathcal{P}^2 - 4\mathcal{Q}}}{2\det(\mathcal{V})}, \quad (6.4)$$

where

$$\mathcal{P} = b_{11}(\tau)\lambda_1 S_1^{(0)}(d_2 + \sigma_2 + D_{12}^I) + b_{12}(\tau)\lambda_2 S_2^{(0)} D_{21}^I$$
$$+ b_{21}(\tau)\lambda_1 S_1^{(0)} D_{12}^I + b_{22}(\tau)\lambda_2 S_2^{(0)}(d_1 + \sigma_1 + D_{21}^I),$$
$$\mathcal{Q} = \det(\mathcal{V})\lambda_1\lambda_2 S_1^{(0)} S_2^{(0)}[b_{11}(\tau)b_{22}(\tau) - b_{12}(\tau)b_{21}(\tau)].$$

For the convenience of comparison later, we first consider the case when the two patches are fully disconnected by setting all dispersal rates to zero, implying that

$$b_{11}(\tau) = e^{-(d_1+\delta_1)\tau} =: \epsilon_1, \quad b_{12}(\tau) = 0, \quad b_{21}(\tau) = 0, \quad b_{22}(\tau) = e^{-(d_2+\delta_2)\tau} =: \epsilon_2. \quad (6.5)$$

Thus, (6.1) is decoupled to

$$\frac{dS_1(t)}{dt} = K_1 - d_1 S_1(t) - \lambda_1 I_1(t) S_1(t),$$
$$\frac{dI_1(t)}{dt} = -(d_1 + \sigma_1)I_1(t) + \epsilon_1\lambda_1 I_1(t-\tau)S_1(t-\tau), \quad (6.6)$$

for patch 1, and

$$\frac{dS_2(t)}{dt} = K_2 - d_2 S_2(t) - \lambda_2 I_2(t) S_2(t),$$
$$\frac{dI_2(t)}{dt} = -(d_2 + \sigma_2)I_2(t) + \epsilon_2\lambda_2 I_2(t-\tau)S_2(t-\tau). \quad (6.7)$$

for patch 2. By the results in the recent work of van den Driessche et al. (2007), the disease dynamics in each patch in such a disconnected case is described by the corresponding basic reproduction number

$$\mathcal{R}_{i0}^{(0)} \triangleq \frac{K_i}{d_i} \frac{\epsilon_i \lambda_i}{d_i + \sigma_i} < 1, \quad i = 1, 2,$$

as summarized below.

Theorem 6.1 *If $\mathcal{R}_{i0}^{(0)} < 1$, then the disease dies out in Patch $i(i = 1, 2)$ in the sense that the disease free equilibrium $(\frac{K_i}{d_i}, 0)$ is globally asymptotically stable; if $\mathcal{R}_{i0}^{(0)} > 1$, then the disease will persist in the population in the sense that the disease free equilibrium is unstable and there is a unique endemic equilibrium*

$$(S_i^*, I_i^*) = \left(\frac{d_i + \sigma_i}{\epsilon_i \lambda_i}, \frac{K_i \epsilon_i \lambda_i - d_i(d_i + \sigma_i)}{\lambda_i(d_i + \sigma_i)} \right),$$

which is asymptotically stable.

In the rest of this section, we explore the impact of dispersals between the two patches on the disease dynamics of (6.1) in cases allowing reducible travel rate matrices. For simplicity, we only consider two simpler scenarios that make the two patches connected: (i) Only susceptible individuals disperse; (ii) The dispersals of infected individuals are unidirectional.

6.1 Sub-case 1: Only susceptible individuals travel

In this subsection, we assume that only susceptible individuals in the two patches travel. Such an assumption may account for the situation when all exposed and infectious individuals are prohibited (e.g., by health authorities) from traveling. This implies that D_{12}^S and D_{21}^S are positive, but $D_{12}^I = D_{21}^I = D_{12}^I = D_{21}^I = 0$. Accordingly, one can compute to obtain the following:

$$\mathbf{B} = \begin{bmatrix} -(d_1 + \delta_1) & 0 \\ 0 & -(d_2 + \delta_2) \end{bmatrix}, \quad \text{and} \quad [b_{ij}(\tau)] = \exp(\mathbf{B}\tau) = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix},$$

where ϵ_i, $i = 1, 2$ are defined in (6.5). In such a case, the model (6.1) reduces to

$$\begin{aligned}
\frac{dS_1(t)}{dt} &= K_1 - d_1 S_1(t) - D_{21}^S S_1(t) + D_{12}^S S_2(t) - \lambda_1 I_1(t) S_1(t), \\
\frac{dS_2(t)}{dt} &= K_2 - d_2 S_2(t) - D_{12}^S S_2(t) + D_{21}^S S_1(t) - \lambda_2 I_2(t) S_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1) I_1(t) + \epsilon_1 \lambda_1 I_1(t - \tau) S_1(t - \tau), \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2) I_2(t) + \epsilon_2 \lambda_2 I_2(t - \tau) S_2(t - \tau).
\end{aligned} \tag{6.8}$$

We have seen that the DFE $\mathcal{E}^{(0)}$ still exists and is given by (6.3), but its stability/instability can not be concluded from Theorem 4.1 as the irreducibility of $[D_{ij}^I]$ and $[D_{ij}^I]$ does not hold. Linearizing (6.8) at $\mathcal{E}^{(0)}$ leads to

$$
\begin{aligned}
\frac{dS_1(t)}{dt} &= -(d_1 + D_{21}^S)S_1(t) + D_{12}^S S_2(t) - \lambda_1 S_1^{(0)} I_1(t), \\
\frac{dS_2(t)}{dt} &= -(d_2 + D_{12}^S)S_2(t) + D_{21}^S S_1(t) - \lambda_2 S_2^{(0)} I_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1)I_1(t) + \epsilon_1 \lambda_1 S_1^{(0)} I_1(t - \tau), \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2)I_2(t) + \epsilon_2 \lambda_2 S_2^{(0)} I_2(t - \tau).
\end{aligned}
\tag{6.9}
$$

The characteristic equation of (6.9) is given by

$$
Q_1(z)Q_2(z)Q_3(z) = 0,
\tag{6.10}
$$

where

$$
\begin{aligned}
Q_1(z) &= z + (d_1 + \sigma_1) - \epsilon_1 \lambda_1 S_1^{(0)} e^{-z\tau}, \\
Q_2(z) &= z + (d_2 + \sigma_2) - \epsilon_2 \lambda_2 S_2^{(0)} e^{-z\tau}, \\
Q_3(z) &= z^2 + (d_1 + d_2 + D_{12} + D_{21})z + (d_1 d_2 + d_1 D_{21}^S + d_2 D_{21}^S).
\end{aligned}
$$

It is obvious that all roots of $Q_3(z)$ have negative real parts. By the results on Hayes equation (see the Appendix in Hale and Verduyn Lunel 1993), one knows that for $j = 1, 2$, all roots of $Q_j(z) = 0$ have negative real parts if and only if

$$
\mathcal{R}_{j0} \triangleq \frac{\epsilon_j \lambda_j S_j^{(0)}}{d_j + \sigma_j} < 1.
\tag{6.11}
$$

Therefore, the DFE $\mathcal{E}^{(0)}$ is asymptotically stable if $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} < 1$ and it is unstable if $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} > 1$. In the latter case, we expect other equilibrium or equilibria, which will be explored below.

We start with looking for possible boundary equilibria, that is, equilibrium of the from $\mathcal{E}^{(1)} = (S_1^{(1)}, S_2^{(1)}, I_1^{(1)}, 0)$ or $\mathcal{E}^{(2)} = (S_1^{(2)}, S_1^{(2)}, 0, I_2^{(2)})$ with $I_1^{(1)} > 0$ for the former or $I_2^{(2)} > 0$ for the latter. For $\mathcal{E}^{(1)}$, we need to solve the algebraical equations

$$
\begin{aligned}
K_1 - d_1 S_1^{(1)} - D_{21}^S S_1^{(1)} + D_{12}^S S_2^{(1)} - \lambda_1 I_1^{(1)} S_1^{(1)} &= 0, \\
K_2 - d_2 S_2^{(1)} - D_{12}^S S_2^{(1)} + D_{21}^S S_1^{(1)} &= 0, \\
-(d_1 + \sigma_1)I_1^{(1)} + \epsilon_1 \lambda_1 I_1^{(1)} S_1^{(1)} &= 0,
\end{aligned}
\tag{6.12}
$$

for positive $S_1^{(1)}$, $S_2^{(1)}$ and $I_1^{(1)}$ which are determined by

$$S_1^{(1)} = \frac{d_1 + \sigma_1}{\lambda_1 \epsilon_1}, \quad S_2^{(1)} = \frac{1}{d_2 + D_{12}^S}\left(K_2 + D_{21}^S \frac{d_1 + \sigma_1}{\lambda_1 \epsilon_1}\right), \tag{6.13}$$

$$I_1^{(1)} = \frac{\epsilon_1}{d_1 + \sigma_1}\left[K_1 - \frac{(d_1 + D_{21}^S)(d_1 + \sigma_1)}{\lambda_1 \epsilon_1} + \frac{D_{12}^S}{d_2 + D_{12}^S}\left(K_2 + D_{21}^S \frac{d_1 + \sigma_1}{\lambda_1 \epsilon_1}\right)\right]$$

$$= \frac{d_1 d_2 + d_1 D_{21}^S + d_2 D_{21}^S}{\lambda_1 (d_2 + D_{12}^S)}(\mathcal{R}_{10} - 1). \tag{6.14}$$

Thus, $\mathcal{E}^{(1)}$ exists ($I_1^{(1)} > 0$) if and only if

$$\mathcal{R}_{10} = \frac{\epsilon_1 \lambda_1 S_1^{(0)}}{d_1 + \sigma_1} > 1. \tag{6.15}$$

Similarly, for $\mathcal{E}^{(2)}$ we have

$$S_1^{(2)} = \frac{1}{d_1 + D_{21}^S}\left(K_1 + D_{12}^S \frac{d_2 + \sigma_2}{\lambda_2 \epsilon_2}\right), \quad S_2^{(2)} = \frac{d_2 + \sigma_2}{\lambda_2 \epsilon_2}, \tag{6.16}$$

$$I_2^{(2)} = \frac{\epsilon_2}{d_2 + \sigma_2}\left[K_2 - \frac{(d_2 + D_{12}^S)(d_2 + \sigma_2)}{\lambda_2 \epsilon_2} + \frac{D_{21}^S}{d_1 + D_{21}^S}\left(K_1 + D_{12}^S \frac{d_2 + \sigma_2}{\lambda_2 \epsilon_2}\right)\right]$$

$$= \frac{d_1 d_2 + d_2 D_{12}^S + d_1 D_{12}^S}{\lambda_2 (d_1 + D_{21}^S)}(\mathcal{R}_{20} - 1). \tag{6.17}$$

Hence, $\mathcal{E}^{(2)}$ exists ($I_2^{(2)} > 0$) if and only if

$$\mathcal{R}_{20} = \frac{\epsilon_2 \lambda_2 S_2^{(0)}}{d_2 + \sigma_2} > 1. \tag{6.18}$$

Finally, an interior equilibrium is an equilibrium of the form $\mathcal{E}^{(*)} = (S_1^*, S_1^*, I_1^*, I_2^*)$ with all components positive, which can be determined from the following equations,

$$\begin{aligned}
K_1 - d_1 S_1^* - D_{21}^S S_1^* + D_{12}^S S_2^* - \lambda_1 I_1^* S_1^* &= 0, \\
K_2 - d_2 S_2^* - D_{12}^S S_2^* + D_{21}^S S_1^* - \lambda_2 I_2^* S_2^* &= 0, \\
-(d_1 + \sigma_1)I_1^* + \epsilon_1 \lambda_1 I_1^* S_1^* &= 0, \\
-(d_2 + \sigma_2)I_2^* + \epsilon_2 \lambda_2 I_2^* S_2^* &= 0.
\end{aligned} \tag{6.19}$$

Solving these equations for positive components leads to

$$S_1^* = \frac{d_1 + \sigma_1}{\epsilon_1 \lambda_1}, \quad S_2^* = \frac{d_2 + \sigma_2}{\epsilon_2 \lambda_2},$$

$$I_1^* = \frac{\epsilon_1}{d_1 + \sigma_1} \left[K_1 - \frac{(d_1 + D_{21}^S)(d_1 + \sigma_1)}{\epsilon_1 \lambda_1} + \frac{D_{12}^S(d_2 + \sigma_2)}{\epsilon_2 \lambda_2} \right], \quad (6.20)$$

$$I_2^* = \frac{\epsilon_2}{d_2 + \sigma_2} \left[K_2 - \frac{(d_2 + D_{12}^S)(d_2 + \sigma_2)}{\epsilon_2 \lambda_2} + \frac{D_{21}^S(d_1 + \sigma_1)}{\epsilon_1 \lambda_1} \right].$$

Define

$$\hat{\mathcal{R}}_{10} = \frac{\epsilon_1 \lambda_1 S_1^{(2)}}{d_1 + \sigma_1}, \quad \hat{\mathcal{R}}_{20} = \frac{\epsilon_2 \lambda_2 S_2^{(1)}}{d_2 + \sigma_2}. \quad (6.21)$$

By straightforward calculations we can further express I_1^* and I_2^* in terms of $\hat{\mathcal{R}}_{10}$ and $\hat{\mathcal{R}}_{20}$ as the following:

$$I_1^* = \frac{\epsilon_1 [K_1 \lambda_2 \epsilon_2 + D_{12}^S(d_2 + \sigma_2)]}{(d_1 + \sigma_1)\epsilon_2 \lambda_2} \left(1 - \frac{1}{\hat{\mathcal{R}}_{10}} \right) \quad (6.22)$$

$$I_2^* = \frac{\epsilon_2 [K_2 \lambda_1 \epsilon_1 + D_{21}^S(d_1 + \sigma_1)]}{(d_2 + \sigma_2)\epsilon_1 \lambda_1} \left(1 - \frac{1}{\hat{\mathcal{R}}_{20}} \right) \quad (6.23)$$

Thus, the interior equilibrium $\mathcal{E}^{(*)}$ exists if and only if

$$\hat{\mathcal{R}}_{10} > 1 \quad \text{and} \quad \hat{\mathcal{R}}_{20} > 1. \quad (6.24)$$

The following theorem summarizes the structure of equilibria and addresses stability of the DFE and the two boundary equilibria.

Theorem 6.2 *Consider the system* (6.8).

(i)   *If* $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} < 1$, *then the DFE* $\mathcal{E}^{(0)}$ *is locally asymptotically stable; if* $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} > 1$, *then the DFE* $\mathcal{E}^{(0)}$ *becomes unstable.*
(ii) *If* $\mathcal{R}_{10} > 1$, *then the boundary equilibrium* $\mathcal{E}^{(1)}$ *exists. Moreover,* $\mathcal{E}^{(1)}$ *is asymptotically stable if* $\hat{\mathcal{R}}_{20} < 1$, *and is unstable if* $\hat{\mathcal{R}}_{20} > 1$.
(iii) *If* $\mathcal{R}_{20} > 1$, *then the boundary equilibrium* $\mathcal{E}^{(2)}$ *exists. Moreover,* $\mathcal{E}^{(2)}$ *is asymptotically stable if* $\hat{\mathcal{R}}_{10} < 1$, *and is unstable if* $\hat{\mathcal{R}}_{10} > 1$.
(iv) *If* $\hat{\mathcal{R}}_{10} > 1$ *and* $\hat{\mathcal{R}}_{20} > 1$, *then there is the interior equilibrium* $\mathcal{E}^{(*)}$.

Remark 6.3 Direct computations show that

$$\hat{\mathcal{R}}_{10} < \mathcal{R}_{10} \Leftrightarrow 1 < \mathcal{R}_{20}, \quad (6.25)$$

$$\hat{\mathcal{R}}_{20} < \mathcal{R}_{20} \Leftrightarrow 1 < \mathcal{R}_{10}. \quad (6.26)$$

Moveover, $\hat{\mathcal{R}}_{10} < 1 < \mathcal{R}_{10}$ and $\hat{\mathcal{R}}_{20} < 1 < \mathcal{R}_{20}$ can not hold simultaneously, implying that the two boundary equilibria, if both exist, can not be simultaneously stable.

Proof Part (i) and Part (iv) and the existence of the boundary equilibrium in both Parts (ii) and (iii) have already been obtained above. Thus, we only need to prove the stability/instability of the boundary equilibria stated in Parts (ii) and (iii).

Assume that $\mathcal{R}_{10} > 1$. Linearizing (6.8) at the $\mathcal{E}^{(1)}$ leads to

$$
\begin{aligned}
\frac{dS_1(t)}{dt} &= -(d_1 + D_{21}^S + \lambda_1 I_1^{(1)})S_1(t) + D_{12}^S S_2(t) - \lambda_1 S_1^{(1)} I_1(t), \\
\frac{dS_2(t)}{dt} &= -(d_2 + D_{12}^S)S_2(t) + D_{21}^S S_1(t) - \lambda_2 S_2^{(1)} I_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1)I_1(t) + \epsilon_1\lambda_1 S_1^{(1)} I_1(t-\tau) + \epsilon_1\lambda_1 I_1^{(1)} S_1(t-\tau), \quad (6.27) \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2)I_2(t) + \epsilon_2\lambda_2 S_2^{(1)} I_2(t-\tau),
\end{aligned}
$$

from which, one can obtain the characteristic equation given by

$$
\begin{vmatrix}
z + d_1 + D_{21}^S + \lambda_1 I_1^{(1)} & -D_{12}^S & \lambda_1 S_1^{(1)} & 0 \\
-D_{21}^S & z + d_2 + D_{12}^S & 0 & \lambda_2 S_2^{(1)} \\
\epsilon_1\lambda_1 I_1^{(1)} e^{-z\tau} & 0 & z + d_1 + \sigma_1 - \epsilon_1\lambda_1 S_1^{(1)} e^{-z\tau} & 0 \\
0 & 0 & 0 & z + d_2 + \sigma_2 - \epsilon_2\lambda_2 S_2^{(1)} e^{-z\tau}
\end{vmatrix}
$$
$$
= 0.
$$

The above equation can be written as

$$
\Delta_3(z, \tau)\Delta_4(z, \tau) = 0, \quad (6.28)
$$

where

$$
\Delta_3(z, \tau) = \begin{vmatrix}
z + d_1 + D_{21}^S + \lambda_1 I_1^{(1)} & -D_{12}^S & \lambda_1 S_1^{(1)} \\
-D_{21}^S & z + d_2 + D_{12}^S & 0 \\
\epsilon_1\lambda_1 I_1^{(1)} e^{-z\tau} & 0 & z + d_1 + \sigma_1 - \epsilon_1\lambda_1 S_1^{(1)} e^{-z\tau}
\end{vmatrix},
$$

$$
\Delta_4(z, \tau) = z + d_2 + \sigma_2 - \epsilon_2\lambda_2 S_2^{(1)} e^{-z\tau}.
$$

By the well-known results on Hayes equation (see, e.g., the Appendix in Hale and Verduyn Lunel 1993), one knows that all roots of $\Delta_4(z, \tau) = 0$ have negative real parts if and only if $\hat{\mathcal{R}}_{20} < 1$.

Next, we re-write $\Delta_3(z, \tau) = 0$ as

$$
P(z, \tau) := P_3(z) + P_2(z)e^{-z\tau} = 0, \quad (6.29)
$$

where

$$P_3(z) := z^3 + a_1 z^2 + a_2 z + a_3, \quad \text{and} \quad P_2(z) := a_4 z^2 + a_5 z + a_6,$$

with

$$a_1 = 2d_1 + d_2 + \sigma_1 + D_{21}^S + D_{12}^S + \lambda_1 I_1^{(1)},$$

$$a_2 = (d_1 + \lambda_1 I_1^{(1)} + D_{21}^S)(d_2 + D_{12}^S) - D_{21}^S D_{12}^S$$
$$+ (d_1 + \sigma_1)(d_1 + D_{21}^S + \lambda_1 I_1^{(1)} + d_2 + D_{12}^S),$$

$$a_3 = (d_1 + \sigma_1)[(d_1 + \lambda_1 I_1^{(1)} + D_{21}^S)(d_2 + D_{12}^S) - D_{21}^S D_{12}^S],$$

and

$$a_4 = -\epsilon_1 \lambda_1 S_1^{(1)},$$

$$a_5 = -\epsilon_1 \lambda_1 S_1^{(1)}(d_1 + D_{21}^S + d_2 + D_{12}^S),$$

$$a_6 = -(d_1 d_2 + d_1 D_{12}^S + d_2 D_{21}^S)\epsilon_1 \lambda_1 S_1^{(1)}.$$

Note that $S_1^{(1)}$ and $I_1^{(1)}$ depend on τ, so do a_1, a_2, a_3, a_4, a_5 and a_6.

For $\tau = 0$, let $\tilde{a}_i = a_i(\tau)|_{\tau=0}$, $i = 1, 2, 3, 4, 5, 6$, and

$$\begin{aligned}
P(z, 0) &= z^3 + \tilde{a}_1 z^2 + \tilde{a}_2 z + \tilde{a}_3 + \tilde{a}_4 z^2 + \tilde{a}_5 z + \tilde{a}_6 \\
&= z^3 + (\tilde{a}_1 + \tilde{a}_4)z^2 + (\tilde{a}_2 + \tilde{a}_5)z + (\tilde{a}_3 + \tilde{a}_6) \\
&\triangleq z^3 + C_1 z^2 + C_2 z + C_3,
\end{aligned} \tag{6.30}$$

where $C_1 = \tilde{a}_1 + \tilde{a}_4$, $C_2 = \tilde{a}_2 + \tilde{a}_5$, $C_3 = \tilde{a}_3 + \tilde{a}_6$. The Routh–Hurwitz Theorem (Gantmacher 1959, p194) for the cubic polynomial $z^3 + C_1 z^2 + C_2 z + C_3$ is applicable. Let $\tilde{I}_1^{(1)} = I_1^{(1)}(\tau)|_{\tau=0}$, $\tilde{S}_1^{(1)} = S_1^{(1)}(\tau)|_{\tau=0}$, and note that $\tilde{I}_1^{(1)}, \tilde{S}_1^{(1)} > 0, \epsilon_1 = \epsilon_2 = 1$, and

$$\begin{aligned}
C_1 &= \tilde{a}_1 + \tilde{a}_4 \\
&= 2d_1 + d_2 + \sigma_1 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)} - \lambda_1 \tilde{S}_1^{(1)} \\
&= 2d_1 + d_2 + \sigma_1 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)} - \lambda_1 \frac{(d_1 + \sigma_1)}{\lambda_1} \\
&= d_1 + d_2 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)} > 0,
\end{aligned} \tag{6.31}$$

and

$$
\begin{aligned}
C_1 C_2 - C_3 &= (\tilde{a}_1 + \tilde{a}_4)(\tilde{a}_2 + \tilde{a}_5) - (\tilde{a}_3 + \tilde{a}_6) \\
&= [(2d_1 + d_2 + \sigma_1 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)}) - \epsilon_1 \lambda_1 \tilde{S}_1^{(1)}][(d_1 + \lambda_1 \tilde{I}_1^{(1)} \\
&\quad + D_{21}^S)(d_2 + D_{12}^S) - D_{21}^S D_{12}^S \\
&\quad + (d_1 + \sigma_1)(d_1 + D_{21}^S + \lambda_1 \tilde{I}_1^{(1)} + d_2 + D_{12}^S) \\
&\quad - \lambda_1 \tilde{S}_1^{(1)}(d_1 + D_{21}^S + d_2 + D_{12}^S)] - (d_1 + \sigma_1)(d_2 + D_{12}^S)\lambda_1 \tilde{I}_1^{(1)} \\
&= (d_1 + d_2 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)})[(d_1 + \lambda_1 \tilde{I}_1^{(1)} \\
&\quad + D_{21}^S)(d_2 + D_{12}^S) - D_{21}^S D_{12}^S \\
&\quad + (d_1 + \sigma_1)(d_1 + D_{21}^S + \lambda_1 \tilde{I}_1^{(1)} + d_2 + D_{12}^S) \\
&\quad - \lambda_1 \frac{d_1 + \sigma_1}{\lambda_1}(d_1 + D_{21}^S + d_2 + D_{12}^S)] - (d_1 + \sigma_1)(d_2 + D_{12}^S)\lambda_1 \tilde{I}_1^{(1)} \\
&= (d_1 + d_2 + D_{21}^S + D_{12}^S + \lambda_1 \tilde{I}_1^{(1)})[(d_1 + \lambda_1 \tilde{I}_1^{(1)} \\
&\quad + D_{21}^S)(d_2 + D_{12}^S) \\
&\quad - D_{21}^S D_{12}^S + (d_1 + \sigma_1)\lambda_1 \tilde{I}_1^{(1)}] - (d_1 + \sigma_1)(d_2 + D_{12}^S)\lambda_1 \tilde{I}_1^{(1)} \\
&= [(d_2 + D_{12}^S) + (d_1 + D_{21}^S + \lambda_1 \tilde{I}_1^{(1)})][(d_1 + \lambda_1 \tilde{I}_1^{(1)})(d_2 + D_{12}^S) \\
&\quad + D_{21}^S(d_2 + D_{12}^S) \\
&\quad - D_{21}^S D_{12}^S + (d_1 + \sigma_1)\lambda_1 \tilde{I}_1^{(1)}] - (d_1 + \sigma_1)(d_2 + D_{12}^S)\lambda_1 \tilde{I}_1^{(1)} \\
&= (d_2 + D_{12}^S)(d_1 + \lambda_1 \tilde{I}_1^{(1)})(d_2 + D_{12}^S) + (d_2 + D_{12}^S)D_{21}^S d_2 \\
&\quad + (d_2 + D_{12}^S)(d_1 + \sigma_1)\lambda_1 \tilde{I}_1^{(1)} \\
&\quad + (d_1 + D_{21}^S + \lambda_1 \tilde{I}_1^{(1)})[(d_1 + \lambda_1 \tilde{I}_1^{(1)})(d_2 + D_{12}^S) \\
&\quad + D_{21}^S d_2 + (d_1 + \sigma_1)\lambda_1 \tilde{I}_1^{(1)}] - (d_1 + \sigma_1)(d_2 + D_{12}^S)\lambda_1 \tilde{I}_1^{(1)} \\
&= (d_2 + D_{12}^S)(d_1 + \lambda_1 \tilde{I}_1^{(1)})(d_2 + D_{12}^S) + (d_2 + D_{12}^S)D_{21}^S d_2 \\
&\quad + (d_1 + D_{21}^S + \lambda_1 \tilde{I}_1^{(1)})[(d_1 + \lambda_1 \tilde{I}_1^{(1)})(d_2 + D_{12}^S) + D_{21}^S d_2 \\
&\quad + (d_1 + \sigma_1)\lambda_1 \tilde{I}_1^{(1)}] \\
&> 0.
\end{aligned}
\tag{6.32}
$$

Hence, all zeros of $P(z, 0)$ have negative real parts.

It is clear that all zeros of $P(z, \tau)$ depend on τ continuously (see Busenberg and Cooke 1993, p 163). Notice also that the assumption (ii) of Beretta and Kuang (2002) (see p 1146) holds, which ensures that $\text{Re}(z) < +\infty$ for any zero of $P(z, \tau)$. Therefore, as τ increases, a zero of $P(z, \tau)$ can enter the right half of the complex plane

only by crossing the imaginary axis. Assume that $z = i\omega$ with $\omega \geq 0$ is a zero of $P(z, \tau)$. Then

$$-i\omega^3 - a_1\omega^2 + ia_2\omega + a_3 + (-a_4\omega^2 + ia_5\omega + a_6)e^{-i\omega\tau} = 0.$$

Taking the moduli to the above equation yields

$$|-i\omega^3 - a_1\omega^2 + ia_2\omega + a_3| = |a_4\omega^2 - ia_5\omega - a_6|,$$

which leads to

$$\omega^6 + (a_1^2 - 2a_2 - a_4^2)\omega^4 + (a_2^2 - 2a_1a_3 + 2a_4a_6 - a_5^2)\omega^2 + (a_3^2 - a_6^2) = 0.$$

Letting $y = \omega^2$, the above equation becomes

$$p(y) := y^3 + q_1 y^2 + q_2 y + q_3 = 0,$$

where $q_1 = a_1^2 - 2a_2 - a_4^2$, $q_2 = a_2^2 - 2a_1a_3 + 2a_4a_6 - a_5^2$, $q_3 = a_3^2 - a_6^2$. Detailed calculations show that

$$\begin{aligned}
q_1 &= (a_1 - a_4)(a_1 + a_4) - 2a_2 \\
&= (d_1 + D_{21}^S + \lambda_1 I_1^{(1)})^2 + (d_2 + D_{12}^S)^2 + 2D_{21}^S D_{12}^S > 0, \\
q_2 &= (a_2 + a_5)(a_2 - a_5) - 2a_1a_3 + a_4a_6 \\
&= (d_1d_2 + d_1 D_{12}^S + d_2 D_{21}^S)^2 + (d_1d_2 + d_1 D_{12}^S + d_2 D_{21}^S)\lambda_1 I_1^{(1)}(d_2 + D_{12}^S) \\
&\quad + \lambda_1 I_1^{(1)}(d_2 + D_{12}^S)(d_1d_2 + d_1 D_{12}^S + d_2 D_{21}^S) + (\lambda_1 I_1^{(1)})^2(d_2 + D_{12}^S)^2 \\
&\quad + (d_1 + \sigma_1)^2(\lambda_1 I_1^{(1)})^2 + 2(d_1 + \sigma_1)^2\lambda_1 I_1^{(1)}(d_1 + D_{21}^S) > 0,
\end{aligned}$$

and

$$\begin{aligned}
q_3 &= (a_3 + a_6)(a_3 - a_6) \\
&= (d_1 + \sigma_1)^2(d_2 + D_{12}^S)\lambda_1 I_1^{(1)}\left[2(d_1d_2 + d_1 D_{12}^S + d_2 D_{21}^S) + \lambda_1 I_1^{(1)}(d_2 + D_{12}^S)\right] \\
&> 0.
\end{aligned}$$

This implies that $p(y) = 0$ has no nonnegative real root. Therefore, for any $\tau > 0$, there is no root of the form $z = i\omega$ with $\omega \geq 0$ for $P(z, \tau) = 0$, and hence all roots of $P(z, \tau) = 0$ have nonnegative real parts for all $\tau \geq 0$.

Summarizing the above, we have proved that under $\mathcal{R}_{10} > 1$, the boundary equilibrium $\mathcal{E}^{(1)}$ is asymptotically stable if $\hat{\mathcal{R}}_{20} < 1$, and is unstable if $\hat{\mathcal{R}}_{20} > 1$, completing the proof of Part (ii).

By a similar and symmetric argument, we can prove Part (iii). The proof of the theorem is completed. $\qquad\square$

In the above theorem, we have only proved the local asymptotical stability of the DFE $\mathcal{E}^{(0)}$ when $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} < 1$. By using the fluctuation lemma (see, e.g., Hirsch et al. 1985) and a comparison argument, we actually can prove that $\mathcal{E}^{(0)}$ is indeed globally asymptotically stable for this case, as demonstrated below.

Theorem 6.4 *If* $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} < 1$, *then the disease free equilibrium* $\mathcal{E}^{(0)}$ *is globally asymptotically stable for* (6.8).

Proof We only need to show that every nonnegative solution of (6.8) converges to $\mathcal{E}^{(0)}$. Following the convention, we use the following notations: for a continuous and bounded function $f(t)$ defined on $[0, \infty)$,

$$f^\infty \triangleq \limsup_{t \to \infty} f(t), \quad \text{and} \quad f_\infty \triangleq \liminf_{t \to \infty} f(t).$$

Now, let $(S_1(t), S_2(t), I_1(t), I_2(t))$ be any non-negative solution of (6.8). Comparison theorem leads to (see (4.11) in Sect. 4)

$$\begin{aligned} 0 \le S_{1\infty} \le S_1^\infty \le S_1^{(0)}, \\ 0 \le S_{2\infty} \le S_2^\infty \le S_2^{(0)}. \end{aligned} \quad (6.33)$$

Also, by Theorem 3.1, we know that

$$\begin{aligned} 0 \le I_{1\infty} \le I_1^\infty < \infty, \\ 0 \le I_{2\infty} \le I_2^\infty < \infty. \end{aligned} \quad (6.34)$$

On the other hand, by the fluctuation lemma (see, e.g., Hirsch et al. 1985), there is a sequence t_n with $t_n \to \infty$ as $n \to \infty$ such that

$$I_1(t_n) \to I_1^\infty \quad \text{and} \quad I_1'(t_n) \to 0, \quad \text{as } n \to \infty.$$

Substituting the sequence t_n into the third equation of (6.8), letting $n \to \infty$ and making use of (6.33), we obtain

$$(d_1 + \sigma_1) I_1^\infty \le \epsilon_1 \lambda_1 I_1^\infty S_1^\infty \le \epsilon_1 \lambda_1 I_1^\infty S_1^{(0)}. \quad (6.35)$$

In a similar way, we can establish

$$(d_2 + \sigma_2) I_2^\infty \le \epsilon_2 \lambda_2 I_2^\infty S_2^\infty \le \epsilon_2 \lambda_2 I_2^\infty S_2^{(0)}. \quad (6.36)$$

Under $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} < 1$, (6.35)–(6.36) leads to $I_i^\infty = 0$, $i = 1, 2$. This together with (6.34) implies $\lim_{t \to \infty} I_i(t) = I_{i\infty} = I_i^\infty = 0$ for $i = 1, 2$. Finally, applying the theory of asymptotically autonomous systems (see, e.g., Castillo-Chaves and Thieme (1995)) to the first and second equations of (6.8), we conclude that $\lim_{t \to \infty} S_i(t) = S_i^{(0)}$, $i = 1, 2$. This completes the proof. □

Table 1 Stability of equilibria for (6.8). Globally asymptotically stable, locally asymptotically stable and unstable are denoted by G.A.S., L.A.S., U.S., respectively

Patch 2	Patch 1		
	$\mathcal{R}_{10} < 1$	$\hat{\mathcal{R}}_{10} < 1 < \mathcal{R}_{10}$ $[\hat{\mathcal{R}}_{20} < \mathcal{R}_{20} \Leftrightarrow R_{10} > 1]$	$1 < \hat{\mathcal{R}}_{10} < \mathcal{R}_{10}$ $[\hat{\mathcal{R}}_{20} < \mathcal{R}_{20} \Leftrightarrow R_{10} > 1]$
$\mathcal{R}_{20} < 1$	$\mathcal{E}^{(0)}$ ∃ and G.A.S.	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∃ and L.A.S. $\mathcal{E}^{(2)}$ ∄	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∃ and L.A.S. $\mathcal{E}^{(2)}$ ∄
$\hat{\mathcal{R}}_{20} < 1 < \mathcal{R}_{20}$ $[\hat{\mathcal{R}}_{10} < \mathcal{R}_{10} \Leftrightarrow 1 < R_{20}]$	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∄ $\mathcal{E}^{(2)}$ ∃ and L.A.S.	Impossible	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∃ and L.A.S. $\mathcal{E}^{(2)}$ ∃ and U.S.
$1 < \hat{\mathcal{R}}_{20} < \mathcal{R}_{20}$ $[\hat{\mathcal{R}}_{10} < \mathcal{R}_{10} \Leftrightarrow 1 < \mathcal{R}_{20}]$	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∄ $\mathcal{E}^{(2)}$ ∃ and L.A.S.	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∃ and U.S. $\mathcal{E}^{(2)}$ ∃ and L.A.S.	$\mathcal{E}^{(0)}$ ∃ and U.S. $\mathcal{E}^{(1)}$ ∃ and U.S. $\mathcal{E}^{(2)}$ ∃ and U.S. $\mathcal{E}^{(*)}$ ∃

Based on Theorems 6.2 and 6.4 and Remark 6.3, we can summarize the stability of equilibria in Table 1.

From Table 1, we see that $\mathcal{R}_{i0} = 1$ is the threshold value for the disease to persist in Patch-i. It is interesting to compare these two values ($\mathcal{R}_{10}$ and $\mathcal{R}_{20}$) with $\mathcal{R}_{10}^{(0)}$ and $\mathcal{R}_{20}^{(0)}$, the basic reproduction numbers for patch 1 and patch 2 respectively when the two patches are disconnected. Indeed, it is easily seen that

$$
\begin{aligned}
\mathcal{R}_{10} &= \frac{\lambda_1 \epsilon_1}{d_1 + \sigma_1} \cdot \frac{K_1}{d_1} \cdot \frac{d_2 + D_{12}^S + \frac{K_2}{K_1} D_{12}^S}{d_2 + D_{12}^S + \frac{d_2}{d_1} D_{21}^S} \\
&= \mathcal{R}_{10}^{(0)} \cdot \frac{d_2 + D_{12}^S + \frac{K_2}{K_1} D_{12}^S}{d_2 + D_{12}^S + \frac{d_2}{d_1} D_{21}^S},
\end{aligned}
\tag{6.37}
$$

and

$$
\begin{aligned}
\mathcal{R}_{20} &= \frac{\lambda_2 \epsilon_2}{d_2 + \sigma_2} \cdot \frac{K_2}{d_2} \cdot \frac{d_1 + D_{21}^S + \frac{K_1}{K_2} D_{21}^S}{d_1 + D_{21}^S + \frac{d_1}{d_2} D_{12}^S} \\
&= \mathcal{R}_{20}^{(0)} \cdot \frac{d_1 + D_{21}^S + \frac{K_1}{K_2} D_{21}^S}{d_1 + D_{21}^S + \frac{d_1}{d_2} D_{12}^S}.
\end{aligned}
\tag{6.38}
$$

It is obvious from the above formulas that $\mathcal{R}_{10}$ and $\mathcal{R}_{20}$ reflect the influence of travel of susceptible individuals between the two patches, and hence may be called the *travel*

modified basic reproduction numbers for patch 1 and patch 2 respectively. The following observations are direct consequences of (6.37)–(6.38) and their verifications are straightforward and thus, are omitted.

(O1) Assume $\mathcal{R}_{10}^{(0)} < 1$ and $\mathcal{R}_{20}^{(0)} < 1$. If $D_{12}^S > 0$ and $D_{21}^S > 0$ satisfy either

$$D_{12}^S > \frac{d_2(1 - \mathcal{R}_{10}^{(0)}) + \frac{d_2}{d_1} \cdot D_{21}^S}{\mathcal{R}_{10}^{(0)} \cdot (1 + \frac{K_2}{K_1}) - 1}$$

$$\text{with} \quad 1 > \mathcal{R}_{10}^{(0)} > \frac{K_1}{K_1 + K_2}; \tag{6.39}$$

or

$$D_{21}^S < \frac{d_1}{d_2} \left[\left(\mathcal{R}_{10}^{(0)} - 1 \right) (d_2 + D_{12}^S) + \mathcal{R}_{10}^{(0)} D_{12}^S \frac{K_2}{K_1} \right]$$

$$\text{with} \quad 1 > \mathcal{R}_{10}^{(0)} > \frac{d_2 + D_{12}^S}{d_2 + D_{12}^S + \frac{K_2}{K_1} D_{12}^S}, \tag{6.40}$$

then $\mathcal{R}_{10} > 1$ and $\mathcal{R}_{20} < 1$. By symmetry, the conditions parallel to the above can lead to $\mathcal{R}_{10} < 1$ and $\mathcal{R}_{20} > 1$. Here and in the sequel, we omit such parallel conditions.

(O2) Assume $\mathcal{R}_{10}^{(0)} > 1$ and $\mathcal{R}_{20}^{(0)} > 1$. If $D_{12}^S > 0$ and $D_{21}^S > 0$ satisfy either

$$D_{12}^S < \frac{d_2(1 - \mathcal{R}_{10}^{(0)}) + \frac{d_2}{d_1} \cdot D_{21}^S}{\mathcal{R}_{10}^{(0)} \cdot (1 + \frac{K_2}{K_1}) - 1},$$

$$\text{with} \quad 1 < \mathcal{R}_{10}^{(0)} < 1 + \frac{D_{21}^S}{d_1}; \tag{6.41}$$

or

$$D_{21}^S > \frac{d_1}{d_2} \left[\left(\mathcal{R}_{10}^{(0)} - 1 \right) (d_2 + D_{12}^S) + \mathcal{R}_{10}^{(0)} D_{12}^S \frac{K_2}{K_1} \right], \tag{6.42}$$

then $\mathcal{R}_{10} < 1$ but $\mathcal{R}_{20} > 1$.

(O3) Assume $\mathcal{R}_{10}^{(0)} < 1$ and $\mathcal{R}_{20}^{(0)} > 1$. If $D_{12}^S > 0$ and $D_{21}^S > 0$ satisfy either

$$\frac{d_2(1 - \mathcal{R}_{10}^{(0)}) + \frac{d_2}{d_1} \cdot D_{21}^S}{\mathcal{R}_{10}^{(0)} \cdot (1 + \frac{K_2}{K_1}) - 1} < D_{12}^S < \frac{d_2}{d_1} \left[\left(\mathcal{R}_{20}^{(0)} - 1 \right) (d_1 + D_{21}^S) + \mathcal{R}_{20}^{(0)} D_{21}^S \frac{K_2}{K_1} \right],$$

$$\text{with} \quad 1 > \mathcal{R}_{10}^{(0)} > \frac{K_1}{K_1 + K_2}; \tag{6.43}$$

or

$$\frac{d_1(1 - \mathcal{R}_{20}^{(0)}) + \frac{d_1}{d_2} \cdot D_{12}^S}{\mathcal{R}_{20}^{(0)} \cdot (1 + \frac{K_1}{K_2}) - 1} < D_{21}^S < \frac{d_1}{d_2} \left[(\mathcal{R}_{10}^{(0)} - 1)(d_2 + D_{12}^S) + \mathcal{R}_{10}^{(0)} D_{12}^S \frac{K_2}{K_1} \right],$$

with $1 > \mathcal{R}_{10}^{(0)} > \dfrac{d_2 + D_{12}^S}{d_2 + D_{12}^S + \frac{K_2}{K_1} D_{12}^S}$ and $1 < \mathcal{R}_{20}^{(0)} < 1 + \dfrac{D_{12}^S}{d_2}$, (6.44)

then $\mathcal{R}_{10} > 1$ but $\mathcal{R}_{20} > 1$.

(O4) Assume $\mathcal{R}_{10}^{(0)} < 1$ and $\mathcal{R}_{20}^{(0)} > 1$. If $D_{12}^S > 0$ and $D_{21}^S > 0$ satisfy either

$$\frac{d_2}{d_1} \left[(\mathcal{R}_{20}^{(0)} - 1)(d_1 + D_{21}^S) + \mathcal{R}_{20}^{(0)} D_{21}^S \frac{K_2}{K_1} \right] < D_{12}^S < \frac{d_2(1 - \mathcal{R}_{10}^{(0)}) + \frac{d_2}{d_1} \cdot D_{21}^S}{\mathcal{R}_{10}^{(0)} \cdot \left(1 + \frac{K_2}{K_1}\right) - 1},$$

with $1 > \mathcal{R}_{10}^{(0)} > \dfrac{K_1}{K_1 + K_2}$; (6.45)

or

$$\frac{d_1}{d_2} \left[(\mathcal{R}_{10}^{(0)} - 1)(d_2 + D_{12}^S) + \mathcal{R}_{10}^{(0)} D_{12}^S \frac{K_2}{K_1} \right] < D_{21}^S < \frac{d_1(1 - \mathcal{R}_{20}^{(0)}) + \frac{d_1}{d_2} \cdot D_{12}^S}{\mathcal{R}_{20}^{(0)} \cdot \left(1 + \frac{K_1}{K_2}\right) - 1},$$

with $1 > \mathcal{R}_{10}^{(0)} > \dfrac{d_2 + D_{12}^S}{d_2 + D_{12}^S + \frac{K_2}{K_1} D_{12}^S}$ and $1 < \mathcal{R}_{20}^{(0)} < 1 + \dfrac{D_{12}^S}{d_2}$, (6.46)

then $\mathcal{R}_{10} < 1$ but $\mathcal{R}_{20} < 1$.

The biological meanings of **(O1)–(O4)** can be obtained from Theorem 6.2 or Table 1 in terms of $\mathcal{R}_{10}$ and $\mathcal{R}_{20}$. For example, **(O1)** implies that travel of the susceptible individuals can help an otherwise dying out disease persist locally. Roughly speaking, larger inflow of susceptible individuals is in favor of the persistence of the disease in the patch. **(O2)** implies that travel of the susceptible individuals can also help drive an otherwise globally persistent disease out of one patch. **(O3)** and **(O4)** shows that appropriate travel rates may either cause an otherwise partially persistent disease to go to extinction, or help it persist globally in both patches.

We note that if $\mathcal{R}_{10}^{(0)} < 1$ and $\mathcal{R}_{20}^{(0)} < 1$ hold, then it is impossible to have $\mathcal{R}_{10} > 1$ and $\mathcal{R}_{20} > 1$ simultaneously; if $\mathcal{R}_{10}^{(0)} > 1$ and $\mathcal{R}_{20}^{(0)} > 1$ hold, then it is also impossible to have both $\mathcal{R}_{10} < 1$ and $\mathcal{R}_{20} < 1$.

Remark 6.5 From Table 1, we see that $\max\{\mathcal{R}_{10}, \mathcal{R}_{20}\} = 1$ is the critical value that distinguishes persistence (could be local or global) from extinction for the disease. This shows that the basic reproduction number for the model (6.8) is

$$\mathcal{R}_0 = \max\{\mathcal{R}_{10}, \mathcal{R}_{20}\}. \tag{6.47}$$

6.2 Sub-case 2: Travel of infected individuals is unidirectional

In this subsection, we still assume positive D_{12}^S and D_{21}^S. We consider a scenario that travel of the infected individuals is unidirectional. Without loss of generality, we assume that infected individuals can travel from Patch 2 to Patch 1, but can not travel from Patch 1 to Patch 2. That is, we assume that $D_{21}^I = D_{21}^I = 0$, but $D_{12}^I > 0$ and $D_{12}^I > 0$. If the two patches are two cities, such a situation may occur when the two cities have different public health systems, or the health officials in the two cities disagree on the severity of an infectious disease, resulting in one city implementing a ban against arrival of the infected individuals from the other city but not vice-versa.

In this case, the matrix $\mathbf{B}$ is upper triangular, and so is $[b_{ij}(\tau)] = \exp(\mathbf{B}\tau)$, given by

$$b_{11}(\tau) = e^{-(d_1+\delta_1)\tau} = \epsilon_1, \quad b_{22}(\tau) = e^{-(d_2+\delta_2+D_{12}^I)\tau},$$

$$b_{12}(\tau) = \frac{D_{12}^I \left[e^{-(d_2+\delta_2+D_{12}^I)\tau} - e^{-(d_1+\delta_1)\tau} \right]}{(d_1+\delta_1) - (d_2+\delta_2+D_{12}^I)}, \quad b_{21}(\tau) = 0. \qquad (6.48)$$

Thus, the model reduces to

$$\frac{dS_1(t)}{dt} = K_1 - d_1 S_1(t) - D_{21}^S S_1(t) + D_{12}^S S_2(t) - \lambda_1 I_1(t) S_1(t),$$

$$\frac{dS_2(t)}{dt} = K_2 - d_2 S_2(t) - D_{12}^S S_2(t) + D_{21}^S S_1(t) - \lambda_2 I_2(t) S_2(t),$$

$$\frac{dI_1(t)}{dt} = -(d_1+\sigma_1)I_1(t) + D_{12}^I I_2(t) + b_{11}(\tau)\lambda_1 I_1(t-\tau)S_1(t-\tau) \qquad (6.49)$$

$$\qquad + b_{12}(\tau)\lambda_2 I_2(t-\tau)S_2(t-\tau),$$

$$\frac{dI_2(t)}{dt} = -(d_2+\sigma_2)I_2(t) - D_{12}^I I_2(t) + b_{22}(\tau)\lambda_2 I_2(t-\tau)S_2(t-\tau).$$

The DFE $\mathcal{E}^{(0)}$ is still given by (6.3). A possible boundary equilibrium of the form $\mathcal{E}^{(1)} = (S_1^{(1)}, S_2^{(1)}, I_1^{(1)}, 0)$ is still given by (6.13) and (6.14). Hence, as is seen in Subsection 6.1, $\mathcal{E}^{(1)}$ exists if and only if $\mathcal{R}_{10} > 1$ where $\mathcal{R}_{10}$ is defined in Subsection 6.1. However, since $b_{12}(\tau) > 0$, a boundary equilibrium of the form $\mathcal{E}^{(2)} = (S_1^{(2)}, S_2^{(2)}, 0, I_2^{(2)})$ becomes impossible.

For the convenience of discussing stability of the equilibria, we define

$$\mathcal{R}_{20}' = \frac{b_{22}(\tau)\lambda_2 S_2^{(0)}}{d_2+\sigma_2+D_{12}^I}, \quad \text{and} \quad \hat{\mathcal{R}}_{20}' = \frac{b_{22}(\tau)\lambda_2 S_2^{(1)}}{d_2+\sigma_2+D_{12}^I}. \qquad (6.50)$$

Linearizing (6.49) at $\mathcal{E}^{(0)} = (S_1^{(0)}, S_2^{(0)}, 0, 0)$ leads to

$$
\begin{aligned}
\frac{dS_1(t)}{dt} &= -(d_1 + D_{21}^S)S_1(t) + D_{12}^S S_2(t) - \lambda_1 S_1^{(0)} I_1(t), \\
\frac{dS_2(t)}{dt} &= -(d_2 + D_{12}^S)S_2(t) + D_{21}^S S_1(t) - \lambda_2 S_2^{(0)} I_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1)I_1(t) + D_{12}^I I_2 + b_{11}(\tau)\lambda_1 S_1^{(0)} I_1(t - \tau) \\
&\quad + b_{12}(\tau)\lambda_2 S_2^{(0)} I_2(t - \tau), \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2 + D_{12}^I)I_2(t) + b_{22}(\tau)\lambda_2 S_2^{(0)} I_2(t - \tau),
\end{aligned}
\tag{6.51}
$$

the characteristic equation of which is

$$
Q_1(z)\hat{Q}_2(z)Q_3(z) = 0, \tag{6.52}
$$

where $Q_1(z)$ and $Q_3(z)$ are as in Sect. 6.1, but $\hat{Q}_2(z)$ is a modification of $Q_2(z)$ by the following formula:

$$
\hat{Q}_2(z) = z + (d_2 + \sigma_2 + D_{12}) - b_{22}(\tau)\lambda_2 S_2^{(0)} e^{-z\tau},
$$

which is a result of replacing ϵ_2 and $d_2 + \sigma_2$ in $Q_2(z)$ in Sect. 6.1 by $b_{22}(\tau)$ and $d_2 + \sigma_2 + D_{12}$, respectively. Thus, by a similar argument to that for the stability/instability of $\mathcal{E}^{(0)}$ in Sect. 6.1, we conclude that $\mathcal{E}^{(0)}$ is locally asymptotically stable if $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} < 1$, and it becomes unstable if $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} > 1$. Actually, we can also further prove that $\mathcal{E}^{(0)}$ is globally asymptotically stable if $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} < 1$, again by using the fluctuation lemma. In fact, for any non-negative solution $(S_1(t), S_2(t), I_1(t), I_2(t))$ of (6.49), by argument similar to that in proof of Theorem 6.4 and from the fourth equation of (6.49), we have

$$
(d_2 + \sigma_2 + D_{12}^I)I_2^\infty \leq b_{22}(\tau)\lambda_2 I_2^\infty S_2^\infty \leq b_{22}(\tau)\lambda_2 I_2^\infty S_2^{(0)},
$$

which results in $I_2^\infty = 0$ by noting that $\mathcal{R}'_{20} < 1$. Thus, $\lim_{t\to\infty} I_2(t) = I_{2\infty} = I_2^\infty = 0$. Then applying this to the third equation of (6.49), we get

$$
(d_1 + \sigma_1)I_1^\infty \leq b_{11}(\tau)\lambda_1 I_1^\infty S_1^\infty + b_{22}(\tau)\lambda_2 I_2^\infty S_2^\infty \leq b_{11}(\tau)\lambda_1 I_1^\infty S_1^{(0)},
$$

which implies that $I_1^\infty = 0$ by using $\mathcal{R}_{10} < 1$. Therefore, $\lim_{t\to\infty} I_1(t) = 0$. Finally, applying the theory of asymptotically autonomous systems (see, e.g., Castillo-Chaves and Thieme (1995)) to the first and second equations of (6.49), we conclude that $\lim_{t\to\infty} S_i(t) = S_i^{(0)}, i = 1, 2$. This gives the globally asymptotically stability of $\mathcal{E}^{(0)}$ for (6.49). Thus we have proved the following

Theorem 6.6 *The disease free equilibrium $\mathcal{E}^{(0)}$ of (6.49) is globally asymptotically stable if $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} < 1$; it is unstable if $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} > 1$.*

Next, we investigate what happens when $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} > 1$.

Case 1 $\mathcal{R}_{10} > 1$. We have seen above that in this case there is the boundary equilibrium $\mathcal{E}^{(1)}$. The stability of $\mathcal{E}^{(1)}$ is described in the following theorem.

Theorem 6.7 *Assume that* $\mathcal{R}_{10} > 1$. *Then* $\mathcal{E}^{(1)}$ *is locally asymptotically stable if* $\hat{\mathcal{R}}'_{20} < 1$; *it becomes unstable if* $\hat{\mathcal{R}}'_{20} > 1$. *In the latter case, there is an interior equilibrium* $\mathcal{E}^+ = (S_1^+, S_2^+, I_1^+, I_2^+)$ *(i.e., with* $S_i^+ > 0$, $I_i^+ > 0$, $i = 1, 2$).

Proof The linearization of (6.49) at $\mathcal{E}^{(1)}$ is

$$
\begin{aligned}
\frac{dS_1(t)}{dt} &= -(d_1 + D_{21}^S + \lambda_1 I_1^{(1)})S_1(t) + D_{12}^S S_2(t) - \lambda_1 S_1^{(1)} I_1(t), \\
\frac{dS_2(t)}{dt} &= -(d_2 + D_{12}^S)S_2(t) + D_{21}^S S_1(t) - \lambda_2 S_2^{(1)} I_2(t), \\
\frac{dI_1(t)}{dt} &= -(d_1 + \sigma_1)I_1(t) + D_{12}^I I_2 + b_{11}(\tau)\lambda_1 S_1^{(1)} I_1(t - \tau) \qquad (6.53) \\
&\quad + b_{11}(\tau)\lambda_1 I_1^{(1)} S_1(t - \tau) + b_{12}(\tau)\lambda_2 S_2^{(1)} I_2(t - \tau), \\
\frac{dI_2(t)}{dt} &= -(d_2 + \sigma_2 + D_{12}^I)I_2(t) + b_{22}(\tau)\lambda_2 S_2^{(1)} I_2(t - \tau),
\end{aligned}
$$

whose characteristic equation is given by,

$$
\Delta_3(z, \tau)\tilde{\Delta}_4(z, \tau) = 0, \qquad (6.54)
$$

where $\Delta_3(z, \tau)$ is the same as that in the Eq. (6.28), but $\tilde{\Delta}_4(z, \tau)$ is a modification of $\Delta_4(z, \tau)$ in (6.28) by the following expression,

$$
\tilde{\Delta}_4(z, \tau) = z + (d_2 + \sigma_2 + D_{12}^I) - b_{22}(\tau)\lambda_2 S_2^{(1)} e^{-z\tau}. \qquad (6.55)
$$

Since the roots of $\Delta_3(z, \tau) = 0$ always have negative real parts, the stability of $\mathcal{E}^{(1)}$ is determined by the root of $\tilde{\Delta}_4(z, \tau) = 0$, for which $\hat{\mathcal{R}}'_{20} = 1$ is the critical value: when $\hat{\mathcal{R}}'_{20} < 1$ all roots have negative real parts; when $\hat{\mathcal{R}}'_{20} = 1$, $z = 0$ is a root; when $\hat{\mathcal{R}}'_{20} > 1$ there is a positive real root. This confirms the statements for stability and instability of $\mathcal{E}^{(1)}$. The above also implies that $\hat{\mathcal{R}}'_{20} = 1$ is critical value for equilibrium bifurcation, meaning that there will be another non-negative equilibrium when $\hat{\mathcal{R}}'_{20} > 1$. Since there is no other boundary equilibrium, this newly bifurcated equilibrium must be an interior one, completing the proof of the theorem.   □

Case 2 $\mathcal{R}_{10} < 1$ but $\mathcal{R}'_{20} > 1$. Going back to (6.52), we know that in this case, all roots of $Q_1(z) = 0$ and $Q_3(z) = 0$ have negative real parts. Thus, the stability of $\mathcal{E}^{(0)}$ is totally determined by $\hat{Q}_2(z) = 0$. Note that $\mathcal{R}'_{20} = 1$ is a critical value for $\hat{Q}_2(z) = 0$: when $\mathcal{R}'_{20} < 1$, all roots of $\hat{Q}_2(z) = 0$ have negative real parts; at $\mathcal{R}'_{20} = 1$, $z = 0$ is a root of $\hat{Q}_2(z) = 0$ and all other roots have negative real parts; when $\mathcal{R}'_{20} > 1$, $\hat{Q}_2(z) = 0$ has positive real root. Thus, when $\mathcal{R}'_{20}$ increases to pass the

critical value 1, the DFE $\mathcal{E}^{(0)}$ looses its stability to another non-negative equilibrium. Since there is no boundary equilibrium, this newly bifurcated equilibrium must be an interior one. This analysis leads to the following

Theorem 6.8 *Assume $\mathcal{R}_{10} < 1$ and $\mathcal{R}'_{20} > 1$. Then there is an interior equilibrium for* (6.49).

Remark 6.9 From the above theorems, we know that $\max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} = 1$ is the threshold value that determines whether or not the disease can persist (either locally or globally), and hence,

$$\mathcal{R}_0 = \max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\} \qquad (6.56)$$

defines the basic reproduction number for the model (6.49).

7 Conclusion and discussion

We have derived a new epidemic model to describe the dynamics of disease with a fixed latency in a n-patch environment. Starting from the classical SIR model with demographical structure, making use of a system of first-order linear partial differential equations for the evolution of disease with infection age and time, and tracking the dispersal of latent individuals, we have obtained a new model in the form of a system of delay differential equations which, in addition to the linear dispersion terms, contains *non-local infection terms*. The patches can be communities, cities, regions and even countries; and the population dispersal among patches can be interpreted as the movements by which people travel or migrate between patches.

For this new model (2.15), we have justified the well-posedness by proving the positivity and boundedness of solutions. There are two infection related travel rate matrices for the model, one for travel of latent individuals and the other for the travel of infectious individuals, When both of the travel rate matrices are assumed to be irreducible, we have identified the basic reproduction number $\mathcal{R}_0$ for the model with general n, which are given by the spectral radius of a matrix. We have shown that the model demonstrates a threshold behavior in the sense that when $\mathcal{R}_0 < 1$, the disease dies out and when $\mathcal{R}_0 > 1$, the disease persists *globally* (i.e., in all patches).

When allowing infection related travel rate matrices to be reducible, the above results remain unproved for general n. However, we have considered two special cases in Sect. 6 under $n = 2$. One important difference is that without irreducibility of the travel rate matrices, the model may allow boundary equilibrium. For either of these two cases, we have also identified the basic reproduction number $\mathcal{R}_0$, and obtained similar threshold behavior. But now the persistence of the disease in these two cases when $\mathcal{R}_0 > 1$ can be either in the *local sense* or in the *global sense*, in contrast to the case when the travel rate matrices for the disease related individuals are irreducible.

Note that the formula for the basic reproduction $\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1})$ is identified for (2.15) with *irreducible* travel rate matrices for the infected individuals. However, it is interesting to notice that even in the special cases in Sect. 6.1 and 6.2 where the travel

rate matrices for latent and infectious individuals are indeed *reducible*, formal calculations by this formula also give the respective basic reproduction numbers. Indeed, for (6.8), simple calculation of (6.4) shows

$$
\begin{aligned}
\rho(\mathcal{F}\mathcal{V}^{-1}) &= \frac{\mathcal{P} + \sqrt{\mathcal{P}^2 - 4\mathcal{Q}}}{2\det(\mathcal{V})} \\
&= \frac{1}{2}(\mathcal{R}_{10} + \mathcal{R}_{20}) + \frac{1}{2}|\mathcal{R}_{10} - \mathcal{R}_{20}| \\
&= \max\{\mathcal{R}_{10}, \mathcal{R}_{20}\},
\end{aligned}
\tag{7.1}
$$

coinciding with (6.47). For (6.49), one can also compute (6.4) to obtain

$$
\begin{aligned}
\rho(\mathcal{F}\mathcal{V}^{-1}) &= \frac{\mathcal{P} + \sqrt{\mathcal{P}^2 - 4\mathcal{Q}}}{2\det(\mathcal{V})} = \frac{1}{2}(\mathcal{R}_{10} + \mathcal{R}'_{20}) + \frac{1}{2}\sqrt{(\mathcal{R}_{10} - \mathcal{R}'_{20})^2} \\
&= \frac{1}{2}(\mathcal{R}_{10} + \mathcal{R}'_{20}) + \frac{1}{2}|\mathcal{R}_{10} - \mathcal{R}_{20}| \\
&= \max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\},
\end{aligned}
\tag{7.2}
$$

agreeing with (6.56). These facts suggest that, regardless of irreducibility or reducibility of the travel rate matrices of the infection related individuals, the formula $\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1})$ is universal. The difference caused by irreducibility or reducibility seems to lie in that, when $\mathcal{R}_0 > 1$, the persistence for the former must be in the global sense, while the persistence for the latter can be local or global. With the above, we may explore a bit more about the impact of the travel of the latent individuals. From (6.4), if the infectious individuals are prevented from migrating/traveling between both patches, but the susceptible and latent individuals are allowed, that is, $D_{21}^S, D_{12}^S, D_{21}^I, D_{12}^I > 0$, but $D_{21}^I = D_{12}^I = 0$, then

$$
\begin{aligned}
\rho(\mathcal{F}\mathcal{V}^{-1}) &= \frac{\mathcal{P} + \sqrt{\mathcal{P}^2 - 4\mathcal{Q}}}{2\det(\mathcal{V})} \\
&= \frac{1}{2}(\mathcal{R}_{10} + \mathcal{R}_{20}) + \frac{1}{2}\sqrt{(\mathcal{R}_{10} - \mathcal{R}_{20})^2 + 4\mathcal{R}_{10}\mathcal{R}_{20}\frac{b_{12}(\tau)b_{21}(\tau)}{b_{11}(\tau)b_{22}(\tau)}} \\
&> \frac{1}{2}(\mathcal{R}_{10} + \mathcal{R}_{20}) + \frac{1}{2}|\mathcal{R}_{10} - \mathcal{R}_{20}| \\
&= \max\{\mathcal{R}_{10}, \mathcal{R}_{20}\},
\end{aligned}
\tag{7.3}
$$

that is,

$$
\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) > \max\{\mathcal{R}_{10}, \mathcal{R}_{20}\}.
\tag{7.4}
$$

This shows that the mobility of the latent individuals (described by $b_{ij}(\tau)$, $i = 1, 2$, causing non-local infections) increases the basic reproduction number, and hence, may cause an otherwise dying-out disease to persist. However, for (6.49), we know that $\mathcal{R}_0 = \max\{\mathcal{R}_{10}, \mathcal{R}'_{20}\}$ with $\mathcal{R}'_{20} = \mathcal{R}'_{20}(D_{12}^I)$. We notice that both $\mathcal{R}_0$ and $\mathcal{R}'_{20}$ are

decreasing with respect to D_{12}^l. In this case, $D_{12}^l > 0$ (the travel rate from Patch 2 to Patch 1 for the latent individuals) will decrease the basic reproduction number, which is in contrast to the roles of D_{12}^I, $D_{21}^I > 0$ on increasing the basic reproduction number in the above case in which only the infectious individuals are forbidden to travel.

The impact of travel rates of the susceptible individuals has been discussed in much details for (6.8) in Sect. 6.1, as is summarized in the observations (O1)–(O4). We now take a look at the impact of the travel of infectious individuals in the model (6.49). Notice that $\mathcal{R}_{20}' = \mathcal{R}_{20}'(D_{12}^I)$ is a decreasing function of D_{12}^I, so is $\mathcal{R}_0 = \max\{\mathcal{R}_{10}, \mathcal{R}_{20}'\}$. For example, when we have $\mathcal{R}_{10} < 1$ and $\mathcal{R}_{20}' > 1$ which gives $\mathcal{R}_0 > 1$, the increase of D_{12}^I (the unbalanced travel rate from Patch 2 to Patch 1 for the infectious class) will decrease $\mathcal{R}_{20}'$ to a value less than 1, which results in $\mathcal{R}_0 < 1$. Therefore, D_{12}^I indeed plays a role of decreasing the basic reproduction number, which is similar to the role of travel rate of the latent individuals in this case.

Finally, we point out that in both irreducible (general n) and reducible cases ($n = 2$), we are unable to prove the stability of the endemic equilibrium when it exists. This seems to be a very difficult mathematical problem due to the presence of the latent delay and the non-local infection terms in the model. We avoid to address it in this already lengthy paper, and have to leave it as a future project (a very interesting yet challenging one).

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