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The Early Exercise Premium Representation for American Options on Multiply Assets

Tomasz Klimsiak · Andrzej Rozkosz

Abstract In the paper we consider the problem of valuation of American options written on dividend-paying assets whose price dynamics follow the classical multi-dimensional Black and Scholes model. We provide a general early exercise premium representation formula for options with payoff functions which are convex or satisfy mild regularity assumptions. Examples include index options, spread options, call on max options, put on min options, multiply strike options and power-product options. In the proof of the formula we exploit close connections between the optimal stopping problems associated with valuation of American options, obstacle problems and reflected backward stochastic differential equations.

Keywords American option · Multiply assets · Early exercise premium · Backward stochastic differential equation · Optimal stopping · Obstacle problem

Mathematics Subject Classification 91B28, 60H10, 65M06

1 Introduction

In the paper we study American options written on dividend-paying assets. We assume that the underlying assets dynamics follow the classical multidimensional Black and Scholes model. It is now well known that the arbitrage-free value of American options can be expressed in terms of the optimal stopping problem (Bensoussan [4], Karatzas

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[20]; see also [21] for nice exposition and additional references), in terms of variational inequalities (Jaillet et al. [19]) and in terms of solutions of reflected BSDEs (El Karoui and Quenez [14]). Although these approaches provide complete characterization of the option value (see Sect. 2 for a short review), the paper by Broadie and Detemple [7] shows that it is of interest to provide alternative representation, which expresses the value of an American option as the value of the corresponding European option plus the gain from early exercise. The main reason is that the representation, called the early exercise premium formula, gives useful information on the determinants of the option value. The formula was proved first by Kim [22] in the case of standard American put option on a single asset. Another important contributions in the case of single asset include Broadie and Detemple [6], El Karoui and Karatzas [13] and Jacka [18] (see also [10,21] and the references therein). The case of options on multiply dividend-paying assets is more difficult and has received rather little attention in the literature. In the important paper [7] and next in Detemple et al. [11] (see also [10]) the early exercise premium formula was established for concrete classes of options on multiply assets. Note that in the last paper call on min option, i.e. option with nonconvex payoff function is investigated. A subclass of call on min options consisting of capped options is studied in [6–8] (see also [10]).

In the present paper we provide a unified way of treating a wide variety of seemingly disparate examples. It allows us to prove a general exercise premium formula for options with convex payoff functions satisfying the polynomial growth condition or payoff function satisfying quite general condition considered in Laurence and Salsa [26]. Verifying the last condition requires knowledge of the payoff function and the structure of the exercise set. Therefore it is a complicated task in general. Fortunately, in most interesting cases one can easily check convexity of the payoff function or check some simpler condition implying the general condition from [26]. The class of options covered by our formula includes index options, spread options, call on max options, put on min options, multiply strike options, power-product options and others.

In the proof of the exercise premium formula we rely on some results on reflected BSDEs and their links with optimal stopping problems (see [14]) and with parabolic variational inequalities established in Bally et al. [2]. We also use classical results on regularity of the solution of the Cauchy problem for parabolic operator with constant coefficients, and in case of convex payoffs, some fine properties of convex functions. Perhaps it is worth mentioning that we do not use any regularity results on the free boundary problem for an American option. The basic idea of the proof comes from our earlier paper [25] devoted to standard American call and put options on single asset.

2 Preliminaries

We will assume that under the risk-neutral probability measure the underlying assets prices $X^{s,x,1}, \ldots, X^{s,x,n}$ evolve on the time interval $[s, T]$ according to stochastic differential equation of the form

$$X_t^{s,x,i} = x_i + \int_s^t (r - d_i) X_\theta^{s,x,i} \, d\theta + \sum_{j=1}^n \int_s^t \sigma_{ij} X_\theta^{s,x,i} \, dW_\theta^j, \quad t \in [s, T]. \quad (1)$$

Here $W = (W^1, \ldots, W^n)$ is a standard n-dimensional Wiener process, $r \geq 0$ is the rate of interest, $d_i \geq 0$ is the dividend rate of the asset i and $\sigma = \{\sigma_{ij}\}$ is the n-dimensional volatility matrix. We assume that $a = \sigma \cdot \sigma^*$ is positive definite. Since the distributions of the processes $X^{s,x,i}$ depend σ only through a, we may and will assume that σ is a symmetric square root of a. As for the payoff function ψ we will assume that it satisfies the assumptions:

> (A1) ψ is a nonnegative continuous function on $\mathbb{R}^n$ with polynomial growth,
> (A2) For every $t \in (0, T)$, ψ is a smooth function on $\{\psi = u\} \cap \bar{Q}_t$, i.e. there exists an open set $U \subset \mathbb{R}^n$ such that $\{u = \psi\} \cap \bar{Q}_t \subset [0, t] \times U$ and ψ is smooth on U (Here $Q_t \equiv [0, t) \times \mathbb{R}^n$, $\bar{Q}_t \equiv [0, t] \times \mathbb{R}^n$ and u is the value of an option with payoff ψ; see (5) and (9) below)

or

> (A3) ψ is a nonnegative convex function on $\mathbb{R}^n$ with polynomial growth.

Note that convex functions are locally Lipschitz, so assumption (A3) implies (A1). Assumption (A2) is considered in [26]. It is satisfied for instance if

> (A2′) The region where ψ is strictly positive is the union of several connected components in which ψ is smooth.

Following [26] let us also note that unlike (A2′) or (A3), condition (A2) cannot be verified by appealing to the structure of the payoff alone. Verifying (A2) requires additional knowledge of the structure of the exercise set $\{u = \psi\}$.

Let $\Omega = C([0, T]; \mathbb{R}^n)$ and let X be the canonical process on Ω. For $(s, x) \in Q_T$ let $P_{s,x}$ denote the law of the process $X^{s,x} = (X^{s,x,1}, \ldots, X^{s,x,n})$ defined by (1) and let $\{\mathcal{F}_t^s\}$ denote the completion of $\sigma(X_\theta; \theta \in [s, t])$ with respect to the family $\{P_{s,\mu}; \mu$ a finite measure on $\mathcal{B}(\mathbb{R}^n)\}$, where $P_{s,\mu}(\cdot) = \int_{\mathbb{R}^n} P_{s,x}(\cdot) \mu(dx)$. Then for each $s \in [0, T)$, $\mathbb{X} = (\Omega, (\mathcal{F}_t^s)_{t \in [s,T]}, X, P_{s,x})$ is a Markov process on $[0, T]$.

Let $I = \{0, 1\}^n$. For $\iota = (i_1, \ldots, i_n) \in I$ we set $D_\iota = \{x \in \mathbb{R}^n; (-1)^{i_k} x_k > 0, k = 1, \ldots, n\}$, $P = \bigcup_{\iota \in I} D_\iota$, $P_T = [0, T) \times P$. By Itô's formula,

$$X_t^{s,x,i} = x^i \exp\left((r - d_i - a_{ii})(t - s) + \sum_{j=1}^n \sigma_{ij}(W_t^j - W_s^j)\right), \quad t \in [s, T]. \quad (2)$$

Therefore if $s \in [0, T)$ and $x \in D_\iota$ for some $\iota \in I$ then $P_{s,x}(X_t \in D_\iota, t \geq s) = 1$. From this and the fact that a is positive definite it follows that if $x \in P_T$ then $\det \sigma(X_t) > 0$, $P_{s,x}$-a.s. for every $t \geq s$, where $\sigma(x) = \{\sigma_{ij}x_i\}_{i,j=1,\ldots,n}$. Moreover, $[s, T] \ni t \mapsto \sigma^{-1}(X_t)$ is a continuous process. Therefore, if $x \in P_T$ then by Lévy's theorem the process $B_{s,\cdot}$ defined as $B_{s,t} = \int_s^t \sigma^{-1}(X_\theta) \, dM_\theta$, where $M_t^i = X_t^i - X_0^i - \int_0^t (r - d_i) X_\theta^i \, d\theta$, $t \in [s, T]$, is under $P_{s,x}$ a standard n-dimensional $\{\mathcal{F}_t^s\}$-Wiener process on $[s, T]$ and

$$X_t^i - x^i = \int_s^t (r - d_i) X_\theta^i \, d\theta + \sum_{j=1}^n \int_s^t \sigma_{ij} X_\theta^i \, dB_{s,\theta}^j, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad (3)$$

i.e.

$$X_t^i = x^i \exp\left((r - d_i - a_{ii})(t - s) + \sum_{j=1}^n \sigma_{ij} B_{s,t}\right), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \quad (4)$$

The above forms of the assets price dynamics will be more convenient for us than (1) or (2). Note that from the definition of the process $B_{s,\cdot}$ and (4) it follows that $\sigma(X_\theta; \theta \in [s, t]) = \sigma(B_{s,\theta}; \theta \in [s, t])$ for $s \in [0, T)$, so for every $s \in [0, T)$ the filtration $\{\mathcal{F}_t^s\}$ is the completion of the Brownian filtration.

In Bensoussan [4] and Karatzas [20] (see also Sect. 2.5 in [21]) it is shown that under (A1) the arbitrage-free value V of an American option with payoff function ψ and expiration time T is given by the solution of the stopping problem

$$V(s, x) = \sup_{\tau \in \mathcal{T}_s} E_{s,x}\left(e^{-r(\tau - s)}\psi(X_\tau)\right), \quad (5)$$

where the supremum is taken over the set $\mathcal{T}_s$ of all $\{\mathcal{F}_t^s\}$-stopping times τ with values in $[s, T]$.

From the results proved in [12] it follows that under (A1) for every (s, x) there exists a unique solution $(Y^{s,x}, Z^{s,x}, K^{s,x})$, on the space $(\Omega, \mathcal{F}_T^s, P_{s,x})$, to the reflected BSDE with terminal condition $\psi(X_T)$, coefficient $f : \mathbb{R} \to \mathbb{R}$ defined as $f(y) = -ry$, $y \in \mathbb{R}$, and barrier $\psi(X)$ (RBSDE$_{s,x}(\psi, -ry, \psi)$ for short). This means that the processes $Y^{s,x}, Z^{s,x}, K^{s,x}$ are $\{\mathcal{F}_t^s\}$-progressively measurable, satisfy some integrability conditions and $P_{s,x}$-a.s.,

$$\begin{cases} Y_t^{s,x} = \psi(X_T) - \int_t^T rY_\theta^{s,x}\,d\theta + K_T^{s,x} - K_t^{s,x} - \int_t^T Z_\theta^{s,x}\,dB_{s,\theta}, & t \in [s, T], \\ Y_t^{s,x} \geq \psi(X_t), \quad t \in [s, T], \\ K^{s,x} \text{ is increasing, continuous, } K_s^{s,x} = 0, \ \int_s^T (Y_t^{s,x} - \psi(X_t))\,dK_t^{s,x} = 0. \end{cases} \quad (6)$$

In [12] it is also proved that for every $(s, x) \in Q_T$,

$$Y_t^{s,x} = u(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \quad (7)$$

where u is a viscosity solution to the obstacle problem

$$\begin{cases} \min(u(s, x) - \psi(x), -u_s - L_{BS}u(s, x) + ru(s, x)) = 0, & (s, x) \in Q_T, \\ u(T, x) = \psi(x), & x \in \mathbb{R}^n \end{cases} \quad (8)$$

with

$$L_{BS}u = \sum_{i=1}^n (r - d_i)x_i u_{x_i} + \frac{1}{2}\sum_{i,j=1}^n a_{ij}x_i x_j u_{x_i x_j}.$$

From [12,14] we know that V defined by (5) is equal to $Y_s^{s,x}$. Hence

$$V(s, x) = Y_s^{s,x} = u(s, x), \quad (s, x) \in [0, T] \times \mathbb{R}^n. \quad (9)$$

In the next section we analyze V via (9) but as a matter of fact instead of viscosity solutions of (8) we consider variational solutions which provide more information on the value function V.

3 Obstacle Problem for the Black and Scholes Equation

Assume that $\psi : \mathbb{R}^n \to \mathbb{R}_+$ is continuous and satisfies the polynomial growth condition. Let $L^2_\varrho = L^2(\mathbb{R}^n; \varrho^2 dx)$, $H^1_\varrho = \{u \in L^2_\varrho : \sum_{j=1}^n \sigma_{ij} x_i u_{x_j} \in L^2(\mathbb{R}^n; \varrho^2 dx), i = 1, \ldots, n\}$ and $W_\varrho = \{u \in L^2(0, T; H^1_\varrho) : u_t \in L^2(0, T; H^{-1}_\varrho)\}$, where u_t, u_{x_i} denote the partial derivatives in the distribution sense, $\varrho(x) = (1 + |x|^2)^{-\gamma}$ and $\gamma > 0$ is chosen so that $\int_{\mathbb{R}^n} \varrho^2(x) dx < \infty$ and $\int_{\mathbb{R}^n} \psi^2(x)\varrho^2(x) dx < \infty$. Following [2,25] we adopt the following definition.

Definition (a) A pair (u, μ) consisting of $u \in W_\varrho \cap C(\bar{Q}_T)$ and a Radon measure μ on Q_T is a variational solution to (8) if

$$u(T, \cdot) = \psi, \quad u \geq \psi, \quad \int_{Q_T} (u - \psi)\varrho^2 d\mu = 0$$

and the equation

$$u_t + L_{BS}u = ru - \mu$$

is satisfied in the strong sense, i.e. for every $\eta \in C_0^\infty(Q_T)$,

$$\langle u_t, \eta \rangle_{\varrho, T} + \langle L_{BS}u, \eta \rangle_{\varrho, T} = r\langle u, \eta \rangle_{2, \varrho, T} - \int_{Q_T} \eta \varrho^2 d\mu,$$

where

$$\langle L_{BS}u, \eta \rangle_{\varrho, T} = \sum_{i=1}^n \langle (r - d_i)x_i u_{x_i}, \eta \rangle_{2, \varrho, T} - \frac{1}{2} \sum_{i,j=1}^n a_{ij} \langle u_{x_i}, (x_i x_j \eta \varrho^2)_{x_j} \rangle_{2, T}.$$

Here $\langle \cdot, \cdot \rangle_{\varrho, T}$ stands for the duality pairing between $L^2(0, T; H^1_\varrho)$ and $L^2(0, T; H^{-1}_\varrho)$, $\langle \cdot, \cdot \rangle_{2, \varrho, T}$ is the usual scalar product in $L^2(0, T; L^2_\varrho)$ and $\langle \cdot, \cdot \rangle_{2, T} = \langle \cdot, \cdot \rangle_{2, \varrho, T}$ with $\varrho \equiv 1$.

(b) If μ in the above definition admits a density (with respect to the Lebesgue measure) of the form $\Phi_u(t, x) = \Phi(t, x, u(t, x))$ for some measurable $\Phi : \bar{Q}_T \times \mathbb{R} \to \mathbb{R}_+$, then we say that u is a variational solution to the semilinear problem

$$u_t + L_{BS}u = ru - \Phi_u, \quad u(T, \cdot) = \psi, \quad u \geq \psi. \tag{10}$$

In our main theorems below we show that if ψ satisfies (A1) and (A2) or (A3) then the measure μ is absolutely continuous with respect to the Lebesgue measure and its density has the form $\Phi_u(t, x) = \mathbf{1}_{\{u(t,x)=\psi(x)\}}\Psi^-(x)$, where $\Psi^- = \max\{-\Psi, 0\}$ and

Ψ is determined by ψ and the parameters r, d, a. In the next section we compute Ψ for some concrete options.

3.1 Payoffs Satisfying (A1), (A2)

Remark One can check that if u is a solution to (10) then v defined as

$$v(t, x) = u(T - t, (-1)^{i_1} e^{x_1}, \ldots, (-1)^{i_n} e^{x_n}) \equiv u(T - t, e^x)$$

for $t \in [0, T]$, $x = (x_1, \ldots, x_n) \in D_\iota$, $\iota \in I$ (D_ι is defined in Sect. 2) is a variational solution of the Cauchy problem

$$v_t - Lv = -rv + \bar{\Phi}, \quad v \geq \bar{\psi}, \quad v(0, \cdot) = \bar{\psi},$$

where

$$Lv = \sum_{i=1}^{n} \left(r - d_i - \frac{1}{2}\sigma_{ii}^2 \right) v_{x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} v_{x_i x_j}$$

and $\bar{\Phi}(t, x) = \Phi_u(T - t, e^x)$, $\bar{\psi}(t, x) = \psi(T - t, e^x)$. Furthermore, a simple calculation shows that if η is a smooth function on $\mathbb{R}^n$ with compact support and $U \subset \mathbb{R}^n$ is a bounded open set such that supp$[\eta] \subset U$ then $\tilde{v} = v\eta$ is a solution of the Cauchy-Dirichlet problem

$$\tilde{v}_t - \tilde{L}\tilde{v} = -r\tilde{v} + f, \quad \tilde{v}(0, \cdot) = \tilde{\psi}, \quad \tilde{v}_{|[0,T]\times\partial U} = 0,$$

where $\tilde{\psi} = \bar{\psi}\eta$, $\tilde{L}$ is some uniformly elliptic operator with smooth coefficients not depending on t and $f \in L^2(0, T; L^2(U))$. By classical regularity results (see, e.g., Theorem 5 in Sect. 7.1 in [15]), $\tilde{v} \in L^2(0, T; H^2(U)) \cap L^\infty(0, T; H_0^1(U))$ and $\tilde{v}_t \in L^2(0, T; L^2(U))$. From this and the construction of $\tilde{v}$ we infer that the regularity properties of $\tilde{v}$ are retained by u. It follows in particular that

$$u_t + L_{BS}u = ru - \Phi_u \quad \text{a.e. on } P_T. \tag{11}$$

Theorem 1 *Assume* (A1), (A2).

(i) *u defined by* (9) *is a variational solution of the semilinear Cauchy problem*

$$u_t + L_{BS}u = ru - \Phi_u^-, \quad u(T, \cdot) = \psi \tag{12}$$

 with

$$\Phi_u(t, x) = \mathbf{1}_{\{u(t,x)=\psi(x)\}}\Psi(x), \quad (t, x) \in Q_T,$$

where for $x \in \mathbb{R}^n$ *such that* $(t, x) \in \{u = \psi\}$,

$$\Psi(x) = -r\psi(x) + L_{BS}\psi(x).$$

(ii) *Set* $\sigma(x) = \{\sigma_{ij}x_i\}_{i,j=1,\dots,n}$ *and*

$$K_{s,t} = \int_s^t \Phi_u^-(\theta, X_\theta) \, d\theta, \quad t \in [s, T]. \tag{13}$$

Then for every $(s, x) \in P_T$ *the triple* $(u(\cdot, X), \sigma(X)u_x(\cdot, X), K_{s,\cdot})$ *is a unique solution of* $RBSDE_{s,x}(\psi, -ry, \psi)$.

Proof Fix $(s, x) \in P_T$. Let $(Y^{s,x}, Z^{s,x}, K^{s,x})$ be a solution of $RBSDE_{s,x}(\psi, -ry, \psi)$ and let u be a viscosity solution of (8). For $t_0 \in (s, T)$ let $U \subset \mathbb{R}^n$ be an open set of assumption (A2). Then there exists $\eta \in C^\infty(\mathbb{R}^n)$ such that $\eta \geq 0$, $\eta \equiv 1$ on $\{u = \psi\} \cap Q_{t_0}$ and $\eta \equiv 0$ on U^c (we make the convention that $\eta(t, x) = \eta(x)$). Of course $(Y^{s,x}, Z^{s,x}, K^{s,x})$ is a solution of $RBSDE_{s,x}(Y_{t_0}^{s,x}, -ry, \psi)$ on $[s, t_0]$. It is also a solution of $RBSDE_{s,x}(Y_{t_0}^{s,x}, -ry, \tilde{\psi})$ on $[s, t_0]$ with $\tilde{\psi}(x) = \eta(x)\psi(x)$, because $\tilde{\psi} \leq \psi$ and by (6) and (7),

$$\int_s^{t_0} (Y_t^{s,x} - \tilde{\psi}(X_t)) \, dK_t^{s,x} = \int_s^{t_0} (u(t, X_t) - \tilde{\psi}(X_t)) \mathbf{1}_{\{u(t,X_t)=\psi(X_t)\}} \, dK_t^{s,x}$$

$$= \int_s^{t_0} (u(t, X_t) - \psi(X_t)) \, dK_t^{s,x} = 0.$$

Since $\tilde{\psi}$ is smooth, applying Itô's formula yields

$$\tilde{\psi}(X_t) = \tilde{\psi}(X_s) + \sum_{i=1}^n \int_s^t \tilde{\psi}_{x_i}(X_\theta) \, dX_\theta^i + \frac{1}{2} \sum_{i,j=1}^n \int_s^t a_{ij} X_\theta^i X_\theta^j \tilde{\psi}_{x_i x_j}(X_\theta) \, d\theta.$$

From the above, (7) and [12, Remark 4.3] it follows that there exists a predictable process $\alpha^{s,x}$ such that $0 \leq \alpha^{s,x} \leq 1$ and

$$dK_t^{s,x} = \alpha_t^{s,x} \mathbf{1}_{\{u=\psi\}}(X_t) \left(-r\tilde{\psi}(X_t) + \sum_{i=1}^n (r - d_i) X_t^i \tilde{\psi}_{x_i}(X_t) \right.$$

$$\left. + \frac{1}{2} \sum_{i,j=1}^n a_{ij} X_t^i X_t^j \tilde{\psi}_{x_i x_j}(X_t) \right)^- dt$$

on $[s, t_0]$. Thus

$$dK_t^{s,x} = \alpha_t^{s,x} \mathbf{1}_{\{u(t,X_t)=\psi(X_t)\}} \Psi^-(X_t) \, dt \tag{14}$$

on $[s, t_0]$ for every $t_0 \in [s, T)$. Consequently, the above equation is satisfied on $[s, T]$. Since the coefficients of the stochastic differential equation (3) satisfy the assumptions

of the "equivalence of norm" result proved in [3] (see [3, Proposition 5.1]), it follows from [2, Theorem 3] that there exists a function α on Q_T such that $0 \leq \alpha \leq 1$ a.e. and for a.e. $(s, x) \in Q_T$,

$$\alpha_t^{s,x} = \alpha(t, X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \tag{15}$$

Moreover, $u \in C(\bar{Q}_T)$ by [12, Lemma 8.4] and from [2, Theorem 3] it follows that $u \in W_\varrho$ and u is a variational solution of the Cauchy problem

$$u_t + L_{BS}u = ru - \alpha \mathbf{1}_{\{u=\psi\}}\Psi^-, \quad u(T, \cdot) = \psi.$$

By the above and (11),

$$u_t + L_{BS}u = ru - \alpha \mathbf{1}_{\{u=\psi\}}\Psi^- \quad \text{a.e. on } Q_T,$$

so by Lemma A.4 in Chapter II in [23],

$$\psi_t + L_{BS}\psi = r\psi - \alpha\Psi^- \quad \text{a.e. on } \{u = \psi\}.$$

On the other hand, by the definition of Ψ,

$$\psi_t + L_{BS}\psi = L_{BS}\psi = r\psi + \Psi \quad \text{on } \{u = \psi\}.$$

Thus $\Psi = -\alpha\Psi^-$ a.e. on $\{u = \psi\}$, which implies that $\alpha\Psi = \Psi$ a.e. on $\{u = \psi\}$, and hence that

$$\mathbf{1}_{\{u=\psi\}}\alpha\Psi^- = \mathbf{1}_{\{u=\psi\}}\Psi^- \quad \text{a.e.} \tag{16}$$

Accordingly (12) is satisfied. From (2) it is clear that if $s \in [0, T)$ and $x \in D_\iota$ for some $\iota \in I$ then $P_{s,x}(X_t \in D_\iota, t \geq s) = 1$ and for every $t \in (s, T]$ the random variable X_t has strictly positive density on D_ι under $P_{s,x}$. From this and (16) it follows that

$$\mathbf{1}_{\{u=\psi\}}(t, X_t)\alpha(t, X_t)\Psi^-(X_t) = \mathbf{1}_{\{u=\psi\}}(t, X_t)\Psi^-(X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \tag{17}$$

for every $(s, x) \in P_T$. In [24] it is proved that the function $\mathbf{1}_{\{u=\psi\}}\alpha$ is a weak limit in $L^2(Q_T)$ of some sequence $\{\alpha_n\}$ of nonnegative functions bounded by 1 and such that $\alpha_n(t, X_t) \to \alpha_t^{s,x}$ weakly in $L^2([0, T] \times \Omega; dt \otimes P_{s,x})$ for every $(s, x) \in Q_T$. Therefore using once again the fact that for every $(s, x) \in P_T$ the process X has a strictly positive transition density under $P_{s,x}$ we conclude that (15) holds for every $(s, x) \in P_T$, which when combined with (17) implies (13). What is left is to prove that for every $(s, x) \in P_T$,

$$Z_t^{s,x} = \sigma(X_t)u_x(t, X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \tag{18}$$

From the results proved in [12, Sect. 6] it follows that for every $(s, x) \in Q_T$,

$$E_{s,x} \sup_{s \leq t \leq T} |Y_t^{s,x,n} - Y_t^{s,x}|^2 + E_{s,x} \int_s^T |Z_t^{s,x,n} - Z_t^{s,x}|^2 \, dt \to 0, \tag{19}$$

where $(Y^{s,x,n}, Z^{s,x,n})$ is a solution of the BSDE

$$Y_t^{s,x,n} = \psi(X_T) - \int_t^T rY_\theta^{s,x,n}\, d\theta$$
$$+ \int_t^T n\left(Y_\theta^{s,x,n} - \psi(X_\theta)\right)^-\, d\theta - \int_t^T Z_\theta^{s,x,n}\, dB_{s,\theta}. \qquad (20)$$

It is known (see [27]) that

$$Y_t^{s,x,n} = u_n(t, X_t), \quad t \in [s, T], \quad P_{s,x}\text{-a.s.}, \qquad (21)$$

where u_n is a viscosity solution of the Cauchy problem

$$(u_n)_t + L_{BS}u_n = -ru_n + n(u_n - \psi)^-, \quad u_n(T, \cdot) = \psi.$$

We know that $P_{s,x}(X_t \in D_t, t \geq s) = 1$ if $x \in D_t$. Moreover, by classical regularity results (see, e.g., [17, Theorem 1.5.9] and Remark preceding Theorem 1), $u_n \in C^{1,2}(P_T)$. Therefore applying Itô's formula shows that (20) holds true with $Z_\theta^{s,x,n}$ replaced by $\sigma(X_\theta)(u_n)_x(\theta, X_\theta)$. Since (20) has a unique solution (see [12, Corollary 3.7]), it follows that

$$Z_t^{s,x,n} = \sigma(X_t)(u_n)_x(t, X_t), \quad dt \otimes P_{s,x}\text{-a.s.} \qquad (22)$$

for every $(s, x) \in P_T$. By (19) and (21), $u_n \to u$ pointwise in Q_T. Moreover, from (21), (22) and standard estimates for solutions of BSDEs (see, e.g., [12, Sect. 6]) it follows that there is $C > 0$ such that for any $(s, x) \in P_T$,

$$E_{s,x} \sup_{s \leq t \leq T} |u_n(t, X_t)|^2 + E_{s,x} \int_s^T |\sigma(X_t)(u_n)_x(t, X_t)|^2\, dt \leq CE_{s,x}|\psi(X_T)|^2,$$
$$\qquad (23)$$

while from (19), (22) it follows that

$$E_{s,x} \sup_{s \leq t \leq T} \int_s^T |\sigma(X_t)((u_n)_x - (u_m)_x)(t, X_t)|^2\, dt \to 0 \qquad (24)$$

as $n, m \to \infty$. From (23) one can deduce that $u_n \in L^2(0, T; H_\varrho)$ and then, by using (24), that $u_n \to u$ in $L^2(0, T; H_\varrho)$ (see the arguments following (2.12) in the proof of [25, Theorem 2.3]). From the last convergence and (19), (22) it may be concluded that

$$E_{s,x} \int_s^T |\sigma(X_t)(u_n)_x(t, X_t) - Z_t^{s,x}|^2\, dt = 0$$

for $(s, x) \in P_T$, which implies (18).

3.2 Convex Payoffs

Assume that $\psi : \mathbb{R}^n \to \mathbb{R}$ is convex. Let m denote the Lebesgue measure on $\mathbb{R}^n$, $\nabla_i \psi$ denote the usual partial derivative with respect to x_i, $i = 1, \ldots, n$, and let E be set of all $x \in \mathbb{R}^n$ for which the gradient

$$\nabla \psi(x) = (\nabla_1 \psi(x), \ldots, \nabla_n \psi(x))$$

exists. Since ψ is locally Lipschitz function, $m(E^c) = 0$ and $\nabla \psi = (\psi_{x_1}, \ldots, \psi_{x_n})$ a.e. (recall that ψ_{x_i} stands for the partial derivative in the distribution sense). Moreover, for a.e. $x \in E$ there exists an n-dimensional symmetric matrix $\{H(x) = \{H_{ij}(x)\}$ such that

$$\lim_{E \ni y \to x} \frac{\nabla \psi(y) - \nabla \psi(x) - H(x)(y - x)}{|y - x|} = 0, \tag{25}$$

i.e. $H_{ij}(x)$ are defined as limits through the set where $\nabla_i \psi$ exists (see, e.g., [1, Sect. 7.9]). By Alexandrov's theorem (see, e.g., [1, Theorem 7.10]), if $x \in E$ is a point where (25) holds then ψ has second order differential at x and $H(x)$ is the hessian matrix of ψ at x, i.e. $H(x) = \{\nabla_{ij}^2 \psi(x)\}$.

The second order derivative of ψ in the distribution sense $D^2 \psi = \{\psi_{x_i x_j}\}$ is a matrix of real-valued Radon measures $\{\mu_{ij}\}$ on $\mathbb{R}^n$ such that $\mu_{ij} = \mu_{ji}$ and for each Borel set B, $\{\mu_{ij}(B)\}$ is a nonnegative definite matrix (see, e.g., [16, Sect. 6.3]). Let $\mu_{ij} = \mu_{ij}^a + \mu_{ij}^s$ be the Lebesgue decomposition of μ_{ij} into the absolutely continuous and singular parts with respect to m. By Theorem 1 in Sect. 6.4 in [16],

$$\mu_{ij}^a(dx) = \nabla_{ij}^2 \psi(x) \, dx. \tag{26}$$

For $R > 0$ set $D_R = P \cap \{x \in \mathbb{R}^n : |x| < R\}$ and $\tau_R = \inf\{t \geq s : X_t \notin D_R\}$. Let $\tilde{L}_{BS}$ denote the operator formally adjoint to L_{BS}. By [28, Theorem 4.2.5] for a sufficiently large $\alpha > 0$ there exist the Green's functions G_R^α, $\tilde{G}_R^\alpha$ for $\alpha - L_{BS}$ and $\alpha - \tilde{L}_{BS}$ on D_R. Let A be a continuous additive functional of $\mathbb{X}$ and let ν denote the Revuz measure of A (see, e.g., [29]). By the theorem proved in Sect. V.5 of [29], for every nonnegative $f \in C_0(\mathbb{R}^d)$,

$$E_{s,x} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t^\nu = \int_{\mathbb{R}^n} G_R^\alpha(x, y) f(y) \nu(dy).$$

Since $G_R^\alpha(x, y) = \tilde{G}_R^\alpha(y, x)$ by [28, Corollary 4.2.6], it follows that

$$E_{s,g \cdot m} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t^\nu = \int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) \nu(dy) \tag{27}$$

for any nonnegative $g \in C_0(D_R)$, where $E_{s,g \cdot m}$ denotes the expectation with respect to the measure $P_{s,g \cdot m}(\cdot) = \int P_{s,x}(\cdot) g(x) \, dx$ and

$$\tilde{G}_R^\alpha g(y) = \int_{G_R} \tilde{G}_R^\alpha(y, x) g(x) \, dx.$$

Note that if g is not identically equal to zero then $\tilde{G}^\alpha_R g$ is strictly positive (see [28, Theorem 4.2.5]).

Set

$$\mathcal{L}_{BS} = \sum_{i=1}^{n} (r - d_i) x_i \nabla_i + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j \nabla^2_{ij}.$$

Theorem 2 *Assume* (A3). *Then assertions* (i), (ii) *of Theorem 1 hold true with L_{BS} replaced by $\mathcal{L}_{BS}$.*

Proof We use the notation of Theorem 1. Fix $s \in [0, T)$. Since ψ is a continuous convex function, from Itô's formula proved in [5] it follows that there exists a continuous increasing process A such that for $x \in \mathbb{R}^n$,

$$\psi(X_t) = \psi(X_s) + A_t + \int_s^t \nabla\psi(X_\theta) \, dX_\theta, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.} \qquad (28)$$

From (28) it follows that A is a positive continuous additive functional (PCAF for short) of $\mathbb{X}$. Let ν denote the Revuz measure of A. We are going to show that $\mathbf{1}_P \cdot \nu = \mathbf{1}_P \cdot \mu$ where μ is the measure on $\mathbb{R}^n$ defined as

$$\mu(dx) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \, \mu_{ij}(dx).$$

To this end, let us set

$$\mu^\varepsilon_{ij} = \frac{\partial^2 \psi}{\partial x_i \partial x_j} * \rho_\varepsilon, \quad \mu^\varepsilon(dx) = \sum_{i,j=1}^{n} a_{ij} x_i x_j \, \mu^\varepsilon_{ij}(dx),$$

where $\{\rho_\varepsilon\}_{\varepsilon>0}$ is some family of mollifiers. Fix a nonnegative $g \in C_0(D_R)$ such that $g(x) > 0$ for some $x \in D_R$ and denote by A^ε the PCAF of $\mathbb{X}$ in Revuz correspondence with μ_ε. Then for a sufficiently large $\alpha > 0$,

$$E_{s,g\cdot m} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA^\varepsilon_t = \int_{\mathbb{R}^n} \tilde{G}^\alpha_R g(y) f(y) \, \mu^\varepsilon(dy) \qquad (29)$$

for all nonnegative $f \in C_0(\mathbb{R}^d)$. By [9, Theorem 2], $E_{s,x} \sup_{t\geq s} |A^\varepsilon_{t\wedge\tau_R} - A_{t\wedge\tau_R}| \to 0$ as $\varepsilon \downarrow 0$ for every $x \in \mathbb{R}^d$. Hence $\int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA^\varepsilon_t \to \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t$ weakly under $P_{s,x}$ for $x \in \mathbb{R}^d$. Since

$$A^\varepsilon_{t\wedge\tau_R} = \psi_\varepsilon(X_{t\wedge\tau_R}) - \psi_\varepsilon(X_s) - \int_s^{t\wedge\tau_R} \nabla\psi_\varepsilon(X_\theta) \, dX_\theta$$

and $\sup_{\varepsilon>0} \sup_{|x|\leq R} |\nabla\psi_\varepsilon(x)| \leq C(R) < \infty$ by Lemma in [9], it follows that for every compact subset $K \subset \mathbb{R}^n$, $\sup_{x\in K} \sup_{\varepsilon>0} E_{s,x} |A^\varepsilon_{t\wedge\tau_R}|^2 < \infty$. Therefore

$$E_{s,g\cdot m} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t^\varepsilon \to E_{s,g\cdot m} \int_s^{\tau_R} e^{-\alpha t} f(X_t) \, dA_t \qquad (30)$$

as $\varepsilon \downarrow 0$. On the other hand, since $\mu_{ij}^\varepsilon \to \mu_{ij}$ weakly* for $i, j = 1, \ldots, n$ and, by [28, Theorem 4.2.5], $f \tilde{G}_R^\alpha g \in C_0(G_R)$, we have

$$\sum_{i,j=1}^n \int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) a_{ij} y_i y_j \, \mu_{ij}^\varepsilon(dy) \to \sum_{i,j=1}^n \int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) \, a_{ij} y_i y_j \mu_{ij}(dy).$$

Combining this with (27), (29), (30) we see that for every $f \in C_0(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) \mu(dy) = \int_{\mathbb{R}^n} \tilde{G}_R^\alpha g(y) f(y) \, \nu(dy).$$

Since $\tilde{G}_R^\alpha g$ is strictly positive on D_R, we conclude from the above that $\mu = \nu$ on D_R for each $R > 0$. Consequently, $\mu = \nu$ on P. For $x \in P$, $P_{s,x}(X_t \in \mathbb{R}^n \setminus P) = 0$ for $t \geq s$. Hence

$$A_t^\nu = \int_s^t \mathbf{1}_P(X_s) \, dA_s^\nu = A_t^{\mathbf{1}_P \cdot \nu} = A_t^{\mathbf{1}_P \cdot \mu}, \quad t \geq s, \quad P_{s,x}\text{-a.s.} \qquad (31)$$

for $x \in P$. Let μ^a denote the absolutely continuous part in the Lebesgue decomposition of $\mathbf{1}_P \cdot \mu$. By (26), $\mu^a(dx) = \sum_{i,j=1}^n \mathbf{1}_P(x) a_{ij} x_i x_j \nabla_{ij}^2 \psi(x) \, dx$. Hence

$$A_t^{\mu^a} = \sum_{i,j=1}^n \int_s^t a_{ij} X_\theta^i X_\theta^j \nabla_{ij}^2 \psi(X_\theta) \, d\theta, \quad t \geq s, \quad P_{s,x}\text{-a.s.} \qquad (32)$$

for $x \in P$. From (28), (31), (32) and [12, Remark 4.3] it follows that

$$dK_t^{s,x} = \alpha_t^{s,x} \mathbf{1}_{\{u = \psi\}}(X_t) \left(-r\psi(X_t) + \sum_{i=1}^n (r - d_i) X_t^i \nabla_i \psi(X_t) \right.$$
$$\left. + \frac{1}{2} \sum_{i,j=1}^n a_{ij} X_t^i X_t^j \nabla_{ij}^2 \psi(X_t) \right)^- dt.$$

Let u be a viscosity solution of (8). From the above and the results proved in [2] (see the reasoning following (14)) we conclude that $u \in W_\varrho \cap C(\bar{Q}_T)$ and there is a function α on Q_T such that $0 \leq \alpha \leq 1$ a.e., (15) is satisfied and u is a variational solution of the Cauchy problem

$$u_t + L_{BS} u = ru - \alpha \mathbf{1}_{\{u=\psi\}} \Psi^-, \quad u(T, \cdot) = \psi \qquad (33)$$

with

$$\Psi = -r\psi + \mathcal{L}_{BS}\psi \quad \text{on } \{u = \psi\}. \qquad (34)$$

By Remark preceding Theorem 1, $u(t, \cdot) \in H^2_{loc}(\mathbb{R}^n)$. Therefore by Remark (ii) following Theorem 4 in Sect. 6.1 in [16] the distributional derivatives u_{x_i}, $u_{x_i x_j}$ are a.e. equal to the approximate derivatives $\nabla^{ap}_i u$, $(\nabla^{ap})^2_{ij} u$. Let $\mathcal{L}^{ap}_{BS}$ denote the operator defined as $\mathcal{L}_{BS}$ but with ∇_i, ∇_{ij} replaced by ∇^{ap}_i, $(\nabla^{ap})^2_{ij}$. Then u is a variational solution of (33) with L_{BS} replaced by $\mathcal{L}^{ap}_{BS}$ and (11) holds with L_{BS} replaced by $\mathcal{L}^{ap}_{BS}$. Hence

$$u_t + \mathcal{L}^{ap}_{BS} u = ru - \alpha \mathbf{1}_{\{u = \psi\}} \Psi^- \quad \text{a.e. on } Q_T.$$

On the other hand, since ψ is convex, $\psi \in BV_{loc}(\mathbb{R}^n)$ as a locally Lipschitz continuous function and, by Theorem 3 in Sect. 6.3 in [16], $\psi_{x_i} \in BV_{loc}(\mathbb{R}^n)$, $i = 1, \ldots, n$. Therefore ψ is twice approximately differentiable a.e. by Theorem 4 in Sect. 6.1 in [16]. It follows now from Theorem 3 in Sect. 6.1 in [16] that $\mathcal{L}^{ap} u = \mathcal{L}^{ap} \psi$ a.e. on $\{u = \psi\}$. Consequently,

$$\mathcal{L}^{ap}_{BS} \psi = r\psi - \alpha \Psi^- \quad \text{a.e. on } \{u = \psi\}. \tag{35}$$

Moreover, since ψ is convex, $\mathcal{L}_{BS} \psi = \mathcal{L}^{ap}_{BS} \psi$ a.e. on $\mathbb{R}^n$ by Remark (i) following Theorem 4 in Sect. 6.1 in [16]. Therefore combining (34) with (35) we see that $\Psi = -\alpha \Psi^-$ a.e. on $\{u = \psi\}$ from which as in the proof of Theorem 1 we get (17). To complete the proof it suffices now to repeat step by step the arguments following (17) in the proof of Theorem 1. $\qquad \square$

4 The Early Exercise Premium Representation

Let ξ denote the payoff process for an American option with payoff function ψ, i.e.

$$\xi_t = e^{-r(t-s)} \psi(X_t), \quad t \in [s, T],$$

and let η denote the Snell envelope for ξ, i.e. the smallest supermartingale which dominates ξ. It is known (see, e.g., Sect. 2.5 in [21]) that

$$\eta_t = e^{-r(t-s)} V(t, X_t), \quad t \in [s, T].$$

Assume (A1), (A2) or (A3). Applying Itô's formula and using Theorem 1 or 2 we get

$$\eta_t = e^{-r(t-s)} Y^{s,x}_t = e^{-r(T-s)} \psi(X_T) + \int_t^T e^{-r(\theta-s)} \Phi^-(X_\theta, Y^{s,x}_\theta) \, d\theta$$

$$- \int_t^T e^{-r(\theta-s)} Z^{s,x}_\theta \, dW_\theta, \quad t \in [s, T], \quad P_{s,x}\text{-a.s.},$$

which leads to the following corollary.

Corollary 3 *For every* $(s, x) \in Q_T$ *the Snell envelope admits the representation*

$$\eta_t = E_{s,x} \left(e^{-r(T-s)} \psi(X_T) + \int_t^T e^{-r(\theta-s)} \Phi^-(X_\theta, Y_\theta^{s,x}) \, d\theta \, | \mathcal{F}_t \right), \quad t \in [s, T].$$
(36)

Taking $t = s$ in (36) and using (7) we get the early exercise premium representation for the value function.

Corollary 4 *For every* $(s, x) \in Q_T$ *the value function* V *admits the representation*

$$V(s, x) = V^E(s, x) + E_{s,x} \int_s^T e^{-r(t-s)} \mathbf{1}_{\{V(t,X_t)=\psi(X_t)\}} \Psi^-(X_t) \, dt,$$

where

$$V^E(s, x) = E_{s,x} \left(e^{-r(T-s)} \psi(X_T) \right)$$

is the value of the European option with payoff function ψ *and expiration time* T.

In closing this section we show by examples that for many options Ψ^- can be explicitly computed. Using results of Sects. 4 and 5 in [30] one can check that the payoff functions ψ in examples 1–4 below satisfy (A3). It is also easy to see that the payoff function ψ in example 5 satisfies (A2$'$). Note that the payoff function in example 1 also satisfies (A2$'$) and, by [7,26], the payoff functions in examples 2–4 satisfy (A2). We would like to stress that the last assertion is by no means evident. On the other hand, the convexity of ψ in examples 2–4 is readily checked.

In all the examples we have computed the corresponding functions Ψ^- on the region $\{u = \psi\}$. When computing Ψ we keep in mind that $\{u = \psi\} \subset [0, T] \times \{\psi > 0\}$.

1. Index options and spread options

$$\psi(x) = \left(\sum_{i=1}^n w_i x_i - K \right)^+, \quad \Psi^-(x) = \left(\sum_{i=1}^n w_i d_i x_i - rK \right)^+ \quad \text{(call)}$$

$$\psi(x) = \left(K - \sum_{i=1}^n w_i x_i \right)^+, \quad \Psi^-(x) = \left(rK - \sum_{i=1}^n w_i d_i x_i \right)^+ \quad \text{(put)}$$

(Here $w_i \in \mathbb{R}$ for $i = 1, \ldots, n$).

2. Max options

$$\psi(x) = (\max\{x_1, \ldots, x_n\} - K)^+ \quad \text{(call on max)}$$

$$\Psi^-(x) = \left(\sum_{i=1}^n d_i \mathbf{1}_{B_i}(x) x_i - rK \right)^+,$$

where $B_i = \{x \in \mathbb{R}^n; \, x_i > x_j, \, j \neq i\}$.

3. Min options

$$\psi(x) = (K - \min\{x_1, \ldots, x_n\})^+ \quad \text{(put on min)}$$

$$\Psi^-(x) = \left(rK - \sum_{i=1}^{n} d_i \mathbf{1}_{C_i}(x)x_i\right)^+,$$

where $C_i = \{x \in \mathbb{R}^n; x_i < x_j, \ j \neq i\}$.

4. Multiple strike options

$$\psi(x) = (\max\{x_1 - K_1, \ldots, x_n - K_n\})^+,$$

$$\Psi^-(x) = \left(\sum_{i=1}^{n} \mathbf{1}_{B_i}(x - K)(d_i x_i - rK_i)\right)^+,$$

where $K = (K_1, \ldots, K_n)$.

5. Power-product options

$$\psi(x) = (|x_1 \cdot \ldots \cdot x_n|^\gamma - K)^+ \quad \text{for some } \gamma > 0.$$

If $x \in D_\iota$ with $\iota = (i_1, \ldots, i_n) \in \{0, 1\}^n$ then

$$\Psi^-(x) = \left(\left(r - \gamma \sum_{i=1}^{n}(r - d_i - a_{ii}) - \gamma^2 \sum_{i,j=1}^{n} a_{ij}\right) f(x) - rK\right)^+,$$

where $f(x) = ((-1)^{|\iota|} x_1 \cdot \ldots \cdot x_n)^\gamma$ and $|\iota| = i_1 + \cdots + i_n$.

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