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RESEARCH ARTICLE

Arens regularity for totally ordered semigroups

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Abstract

Let S be a semigroup. We shall consider the centres of the semigroup $(\beta S, \Box)$ and of the algebra $(M(\beta S), \Box)$, where $M(\beta S)$ is the bidual of the semigroup algebra $(\ell^1(S), \star)$, and whether the semigroup and the semigroup algebra are Arens regular, strongly Arens irregular, or neither. We shall also determine subsets of S^* and of $M(S^*)$ that are 'determining for the left topological centre' (DLTC sets) of βS and $M(\beta S)$. It is known that, when the semigroup S is cancellative, $\ell^1(S)$ is strongly Arens irregular and that there is a DLTC set consisting of two points of S^*. In contrast, there is little that has been published about the Arens regularity of $\ell^1(S)$ when S is not cancellative. Totally ordered, abelian semigroups, with the map $(s, t) \to s \wedge t$ as the semigroup operation, provide examples which show that several possibilities can occur. We shall determine the centres of βS and of $M(\beta S)$ for all such semigroups, and give several examples, showing that the minimum cardinality of DTC sets may be arbitrarily large, and, in particular, we shall give an example of a countable, totally ordered, abelian semigroup S with this operation for which there is no countable DTC set for βS or for $M(\beta S)$. There was no previously-known example of an abelian semigroup S for which βS or $M(\beta S)$ did not have a finite DTC set.

Keywords Stone–Čech compactifications of semigroups · Totally ordered sets as semigroups · Arens products on the second dual of a Banach algebra · Topological centres · DTC sets

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1 Introduction

In this section, we shall recall certain standard definitions and results.

Let S be a semigroup, so that S is a non-empty set together with an associative binary operation, $S \times S \to S$; the operation is usually denoted by juxtaposition. In this case, we shall write

$$L_s(t) = st\,, \quad R_s(t) = ts \quad (t \in S)$$

for the left and right translation operators on S for each $s \in S$. The *centre* of S is the sub-semigroup $\mathfrak{Z}(S)$, where

$$\mathfrak{Z}(S) = \{t \in S : st = ts \quad (s \in S)\}\,,$$

and S is *abelian* when $\mathfrak{Z}(S) = S$, so that $st = ts \ \ (s, t \in S)$. An element $s \in S$ is *idempotent* if $s^2 = s$, and S is an *idempotent semigroup* if each element is idempotent. A semigroup is *cancellative* if each L_s and R_s is injective, and *weakly cancellative* if the equations $xs = t$ and $sx = t$ have only finitely-many solutions for x for each $s, t \in S$.

Let $(S, \leq)$ be a non-empty, partially ordered set, and suppose that $s \wedge t = \min\{s, t\}$ exists for all $s, t \in S$. Then $(S, \wedge)$ is an abelian, idempotent semigroup, called a *semi-lattice*, and these are the particular semigroups that we shall consider here. Conversely, suppose that S is an abelian, idempotent semigroup. Take $s, t \in S$, and set $s \leq t$ if $st = s$. Then $(S, \leq)$ is a semilattice and $s \wedge t = st \ \ (s, t \in S)$ [8, Proposition 1.3.2]. Hence $(S, \wedge)$ is a semigroup that can be identified with S.

A semigroup S which is also a topological space is: a *left* (respectively, *right*) *topological semigroup* if L_s (respectively, R_s) is continuous for each $s \in S$. In the case where the semigroup S is also compact (and Hausdorff) as a topological space, we say that S is a *compact, left* or *right topological semigroup*, respectively.

The Stone–Čech compactification βS of a (discrete) set S is a compact set containing S as a dense subset, and is characterized by the property that every continuous mapping from S into a compact space K has a continuous extension from βS into K. We regard the semigroup S as a subset of βS.

Now suppose that S is a discrete semigroup. We start by defining two products $\Box$ and $\Diamond$ on βS such that $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are also semigroups. Of course, the semigroups $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are very well-known; they are the main topic of the monograph [7].

For each $s \in S$, the map $L_s : S \to \beta S$ has a continuous extension to a map $L_s : \beta S \to \beta S$; for each $v \in \beta S$, define $s \Box v = L_s(v)$. Next, each map

$$R_v : s \mapsto s \Box v\,, \quad S \to \beta S\,,$$

has a continuous extension to a map $R_v : \beta S \to \beta S$. For $u, v \in \beta S$, set

$$u \Box v = R_v(u)\,.$$

Then $\Box$ is a binary operation on βS, and the restriction of $\Box$ to $S \times S$ is the original semigroup product. Further, for each $v \in \beta S$, the map $R_v : \beta S \to \beta S$ is continuous, and, for each $s \in S$, the map $L_s : \beta S \to \beta S$ is continuous.

Similarly, we can define a binary operation $\Diamond$ on βS by exchanging L_s and R_s.

For $u, v \in \beta S$, we see that

$$u \Box v = \lim_\alpha \lim_\beta s_\alpha t_\beta \,, \quad u \Diamond v = \lim_\beta \lim_\alpha s_\alpha t_\beta \tag{1.1}$$

whenever (s_α) and (t_β) are nets in S with $\lim_\alpha s_\alpha = u$ and $\lim_\beta t_\beta = v$ in βS. The maps $\Box$ and $\Diamond$ agree with the maps defined in several different ways in [7].

In the case where S is an abelian semigroup, we have

$$u \Diamond v = v \Box u \quad (u, v \in \beta S)\,.$$

It is immediately checked that both $\Box$ and $\Diamond$ are associative operations on βS, and so we obtain the following fundamental result; see [7, §4.1] for more details.

Theorem 1.1 *Let S be a discrete semigroup. Then $(\beta S, \Box)$ and $(\beta S, \Diamond)$ are semigroups containing S as a sub-semigroup. Further:*

(i) *for each $v \in \beta S$, the map $R_v : u \mapsto u \Box v$ is continuous, and $(\beta S, \Box)$ is a compact, right topological semigroup;*

(ii) *for each $s \in S$, the map $L_s : v \mapsto s \Box v$ is continuous.* ◻

In fact, $(\beta S, \Box)$ is the largest compactification of S which is a compact, right topological semigroup, in the sense that any other such compactification is a continuous homomorphic image of $(\beta S, \Box)$ [7, Theorem 4.8]. We set $S^* = \beta S \setminus S$, the *growth* of S.

The following definitions are well-known; see [4, Definition 6.11], [5, Definition 6.1.1], and [10], for example.

Definition 1.2 Let S be a discrete semigroup. The *left* and *right topological centres* of βS are

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \{u \in \beta S : u \Box v = u \Diamond v \ \ (v \in \beta S)\}$$

and

$$\mathfrak{Z}_t^{(r)}(\beta S) = \{u \in \beta S : v \Box u = v \Diamond u \ \ (v \in \beta S)\}\,,$$

respectively. The semigroup S is *Arens regular* if

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = \beta S\,;$$

S is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(\beta S) = S$, *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(\beta S) = S$, and *strongly Arens irregular* if it is both left and right strongly Arens irregular. A non-empty subset V of βS is *determining for the left topological centre* (a DLTC set) for βS if $u \in \mathfrak{Z}_t^{(\ell)}(\beta S)$ whenever $u \in \beta S$ and $u \Box v = u \Diamond v \ \ (v \in V)$.

Our set $\mathfrak{Z}_t^{(\ell)}(\beta S)$ is equal to the *topological centre*,

$$\Lambda(\beta S) = \{u \in \beta S : L_u \text{ is continuous on } \beta S\}\,,$$

as defined in [7, Definition 2.4]. Thus S is Arens regular if and only if $\Lambda(\beta S) = \beta S$. In the case where S is an abelian semigroup, we have

$$\mathfrak{Z}_t^{(\ell)}(\beta S) = \mathfrak{Z}_t^{(r)}(\beta S) = \mathfrak{Z}(\beta S)\,,$$

the centre of both the semigroups $(S, \Box)$ and $(S, \Diamond)$; we refer to a 'DTC set' when the semigroup S is abelian.

In [4, Example 6.12], there is an example of a semigroup S that is right, but not left, strongly Arens irregular.

The following theorem is [4, Theorem 12.20] and extends [7, Theorem 6.54]; the result also follows from a short argument in [13, Theorem 2.2].

Theorem 1.3 *Let S be an infinite, weakly cancellative semigroup. Then S is strongly Arens irregular, and there is a two-point subset of S^* that is a* DLTC *set for βS.* □

In §2, we shall determine the centre of βS for certain totally ordered, abelian, idempotent semigroups S with respect to the operation $(s, t) \mapsto s \wedge t$ (such semigroups S are usually not weakly cancellative), and shall consider their DTC sets in §3. We shall give in Example 2.16 an example of an infinite semilattice (that is not weakly cancellative) that is Arens regular.

We set $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. The absolutely convex hull of a subset C of a linear space is denoted by aco C. This is the set of all elements of the form

$$\left\{\sum_{j=1}^{n} \alpha_j s_j : \alpha_1, \dots, \alpha_n \in \mathbb{C} \text{ with } \sum_{j=1}^{n} \left|\alpha_j\right| \le 1,\ s_1, \dots, s_n \in C,\ n \in \mathbb{N}\right\}.$$

The centre of an algebra A is denoted by $\mathfrak{Z}(A)$.

Let E be a Banach space with a subset F. Then the closed unit ball of E is $E_{[1]}$ and $F_{[1]} = E_{[1]} \cap F$. The dual and bidual spaces of E are denoted by E' and E'', respectively; we regard E as a subset of E'', so that $E_{[1]}$ is dense in $E''_{[1]}$ with respect to the weak-$*$ topology, $\sigma(E'', E')$.

There are two products, $\Box$ and $\Diamond$, on the Banach space A'', called the *first* and *second Arens products*, that extend the module actions on A''. Indeed, take $\lambda \in A'$ and $\mathrm{M} \in A''$, and define $\lambda \cdot \mathrm{M}$ and $\mathrm{M} \cdot \lambda$ in A' by

$$\langle a, \lambda \cdot \mathrm{M}\rangle = \langle \mathrm{M}, a \cdot \lambda\rangle\,, \quad \langle a, \mathrm{M} \cdot \lambda\rangle = \langle \mathrm{M}, \lambda \cdot a\rangle \quad (a \in A)\,,$$

and then, for $\mathrm{M}, \mathrm{N} \in A''$, define

$$\langle \mathrm{M} \Box \mathrm{N}, \lambda\rangle = \langle \mathrm{M}, \mathrm{N} \cdot \lambda\rangle, \quad \langle \mathrm{M} \Diamond \mathrm{N}, \lambda\rangle = \langle \mathrm{N}, \lambda \cdot \mathrm{M}\rangle \quad (\lambda \in A')\,.$$

The basic theorem of Arens (see [1, §2.6] and [5, §2.3]) is that $(A'', \Box)$ and $(A'', \Diamond)$ are Banach algebras and that the embedding of A in A'' is an isometric algebra monomorphism in both cases. A Banach algebra A is *Arens regular* if the two products $\Box$ and $\Diamond$ agree on A''; for a commutative Banach algebra A this holds if and only if $(A'', \Box)$ is commutative. We shall write just A'' for $(A'', \Box)$.

For $\mathrm{M}, \mathrm{N} \in A''$, we see that

$$\mathrm{M} \Box \mathrm{N} = \lim_{\alpha} \lim_{\beta} a_\alpha b_\beta\,, \quad \mathrm{M} \Diamond \mathrm{N} = \lim_{\beta} \lim_{\alpha} a_\alpha b_\beta \tag{1.2}$$

whenever (a_α) and (b_β) are nets in A with $\lim_\alpha a_\alpha = \mathrm{M}$ and $\lim_\beta b_\beta = \mathrm{N}$, where all limits are in the weak-$*$ topology, $\sigma(A'', A')$.

Theorem 1.4 *Let A be a Banach algebra. For $\mathrm{N} \in A''$, the map*

$$R_{\mathrm{N}} : \mathrm{M} \mapsto \mathrm{M} \Box \mathrm{N}\,, \quad A'' \to A''\,,$$

is weak-$*$ continuous; for $a \in A$, the map $L_a : \mathrm{M} \mapsto a \cdot \mathrm{M}$, $A'' \to A''$, is weak-$*$ continuous. □

The map $L_{\mathrm{N}} : \mathrm{M} \mapsto \mathrm{N} \Box \mathrm{M}$ is not necessarily weak-$*$ continuous on A''; this holds for each $\mathrm{N} \in A''$ if and only if A is Arens regular.

The following definitions of $\mathfrak{Z}_t^{(\ell)}(A'')$ and $\mathfrak{Z}_t^{(r)}(A'')$ were first given in [11]; the notation is from [4, Definition 2.24]. The definition of a DLTC is a small variant of the one in [4, Definition 12.3]. See also [5, §6.1].

Definition 1.5 Let A be a Banach algebra. Then the *left* and *right topological centres* of A'' are

$$\mathfrak{Z}_t^{(\ell)}(A'') = \left\{\mathrm{M} \in A'' : \mathrm{M} \Box \mathrm{N} = \mathrm{M} \Diamond \mathrm{N} \;\; (\mathrm{N} \in A'')\right\}\,,$$
$$\mathfrak{Z}_t^{(r)}(A'') = \left\{\mathrm{M} \in A'' : \mathrm{N} \Box \mathrm{M} = \mathrm{N} \Diamond \mathrm{M} \;\; (\mathrm{N} \in A'')\right\}\,,$$

respectively. The Banach algebra A is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(\ell)}(A'') = A$ and *right strongly Arens irregular* if $\mathfrak{Z}_t^{(r)}(A'') = A$; the algebra A is *strongly Arens irregular* if it is both left and right strongly Arens irregular. A subset V of A'' is *determining for the left topological centre* (a DLTC set) of A'' if $\mathrm{M} \in \mathfrak{Z}_t^{(\ell)}(A'')$ whenever $\mathrm{M} \in A''$ and $\mathrm{M} \Box \mathrm{N} = \mathrm{M} \Diamond \mathrm{N}$ $(\mathrm{N} \in V)$.

Note that $A \subset \mathfrak{Z}_t^{(\ell)}(A'') \cap \mathfrak{Z}_t^{(r)}(A'')$, and that the empty set is a DLTC set when A is Arens regular.

In the case where the Banach algebra A is commutative, A is Arens regular when $\mathfrak{Z}(A'') = A''$ and strongly Arens irregular when $\mathfrak{Z}(A'') = A$, and we refer to DTC sets.

Let S be a semigroup, and consider the Banach space $(\ell^1(S), \|\cdot\|_1)$. There is a continuous product $\star$, called *convolution*, on the space $\ell^1(S)$. Indeed, for $f, g \in \ell^1(S)$, define

$$(f \star g)(t) = \sum \{f(r)g(s) : r, s \in S,\ rs = t\} \quad (t \in S)\,. \tag{1.3}$$

(If there are no elements $r, s \in S$ with $rs = t$, then $(f \star g)(t) = 0$.) Then the structure $(\ell^1(S), \|\cdot\|_1, \star)$ is a Banach algebra, called the *semigroup algebra* on S. This algebra is commutative if and only if S is abelian.

Let S be a semigroup. A *semi-character* on S is a map $\theta : S \to \overline{\mathbb{D}}$ such that $\theta(st) = \theta(s)\theta(t) \quad (s, t \in S)$ and $\theta \neq 0$. The space of semi-characters on S is denoted by Φ_S; it is a locally compact space with respect to the topology of pointwise convergence.

Let S be an abelian semigroup. Then the algebra $\ell^1(S)$ is semisimple if and only if Φ_S separates the points of S, in the sense that, for each $s, t \in S$ with $s \neq t$, there exists $\theta \in \Phi_S$ with $\theta(s) \neq \theta(t)$ [6, Theorem 3.5], and this holds if and only if S is separating, in the sense that $s = t$ whenever $s, t \in S$ with $st = s^2 = t^2$ [6, Theorem 5.8]. Thus $\ell^1(S)$ is algebraically isomorphic to a Banach function algebra, $A(\Phi_S)$, on Φ_S in this case, and all characters on the Banach function algebra $A(\Phi_S)$ are given by evaluation at a point of Φ_S. Certainly an abelian semigroup that is either cancellative or an idempotent semigroup is separating. See [5] for an account of Banach function algebras, including this example.

The commutative Banach algebra of all continuous functions that vanish at infinity on a non-empty, locally compact space K is denoted by $C_0(K)$, taken with the uniform norm, $|\cdot|_K$. By the Riesz representation theorem, the dual space of the Banach space $(C_0(K), |\cdot|_K)$ is $(M(K), \|\cdot\|)$, the space of all complex-valued, regular Borel measures on K; the Borel sets of K are denoted by $\mathfrak{B}_K$. The space of positive measures in $M(K)$ is $M(K)^+$, the total variation of a measure μ is $|\mu|$, we have $\|\mu\| = |\mu|(K)$, and the support of $\mu \in M(K)$ is denoted by $\operatorname{supp} \mu$. We have $M(K) = M_d(K) \oplus M_c(K)$, where $M_d(K)$ and $M_c(K)$ are the discrete and continuous measures on K, respectively, and we identify $M_d(K)$ with $\ell^1(K)$.

Recall that the total variation of a measure μ is defined by

$$|\mu|(B) = \sup \sum_{j=1}^{n} \left|\mu(B_j)\right| \quad (B \in \mathfrak{B}_K),$$

where the supremum is taken over all finite partitions $\{B_1, \ldots, B_n\}$ of B for which $B_1, \ldots, B_n \in \mathfrak{B}_K$. Thus $\mu \mid B = 0$ if and only if $|\mu|(B) = 0$.

We write $M_{\mathbb{R}}(K)$ for the space of real-valued measures on K. Thus each $\mu \in M(K)$ can be written uniquely as

$$\mu = \mu_1 + \mathrm{i}\mu_2,$$

where $\mu_1, \mu_2 \in M_{\mathbb{R}}(K)$, and each $\mu \in M_{\mathbb{R}}(K)$ can be written as $\mu = \mu^+ - \mu^-$, where $\mu^+ = \mu \vee 0$, $\mu^- = (-\mu) \vee 0$, and

$$|\mu| = \mu \vee (-\mu) = \mu^+ + \mu^-.$$

Let S be a discrete semigroup. The dual space of the Banach space $\ell^1(S)$ is $\ell^\infty(S)$, identified with $C(\beta S)$, and so the bidual space of $\ell^1(S)$ is $M(\beta S)$, and we have

$$M(\beta S) = \ell^1(S) \oplus_1 M(S^*) \tag{1.4}$$

as a Banach space, where $\ell^1(S)$ is identified with the measures on S. Then $(M(\beta S), \Box)$ and $(M(\beta S), \Diamond)$ denote the space $M(\beta S)$ taken with the two products $\Box$ and $\Diamond$ that are defined by identifying $M(\beta S)$ with the bidual space $\ell^1(S)''$, so that $\ell^1(S)$ is a closed subalgebra of $(M(\beta S), \Box)$ and $(M(\beta S), \Diamond)$. (For full details of this identification, see [4, Chapter 7] and [5].)

In the case where S is an abelian semigroup, the Banach algebra $\ell^1(S)$ is Arens regular whenever $\mathfrak{Z}(M(\beta S)) = M(\beta S)$ and strongly Arens irregular whenever $\mathfrak{Z}(M(\beta S)) = \ell^1(S)$.

We regard S and βS as subsets of $\ell^1(S)$ and $M(\beta S)$ by identifying $p \in \beta S$ with the point mass $\delta_p \in M(\beta S)$. Let (s_α) be a net in βS that converges to $p \in \beta S$. Then δ_{s_α} converges to δ_p in the weak-$*$ topology of $M(\beta S)$, and so convergence is consistent with the previous definitions. We also see that the notations $\Box$ and $\Diamond$ for products on βS and $M(\beta S)$ are consistent. For example, $\delta_{p\Box q} = \delta_p \Box \delta_q \ \ (p, q \in \beta S)$.

Clearly

$$\mathfrak{Z}_t^{(\ell)}(M(\beta S)) \cap \beta S \subset \mathfrak{Z}_t^{(\ell)}(\beta S)\,;$$

we do not know a semigroup S for which the above inclusion is proper.

The *augmentation character* on $(M(\beta S), \Box)$ is the map

$$\varphi_0 : \mu \mapsto \langle 1_{\beta S},\, \mu \rangle = \mu(\beta S) = \int_{\beta S} \mathrm{d}\mu\,.$$

Clearly $\varphi_0(\mu \Box \nu) = \varphi_0(\mu)\varphi_0(\nu) \ \ (\mu, \nu \in M(\beta S))$, so φ_0 is indeed a character on $(M(\beta S), \Box)$, and φ_0 is weak-$*$ continuous.

The following theorem is [4, Theorem 12.15] and [5, Theorem 6.3.10].

Theorem 1.6 *Let S be an infinite, cancellative semigroup. Then the semigroup algebra $\ell^1(S)$ is strongly Arens irregular, and there exist a and b in S^* such that the two-point set $\{\delta_a, \delta_b\}$ is determining for the left topological centre of $M(\beta S)$.* □

Analogues of the above theorem for 'weighted semigroup algebras' are given in [3] and [12].

We do not know whether the fact that $\ell^1(S)$ is strongly Arens irregular implies that S is strongly Arens irregular, even for abelian semigroups.

We shall give in §3 examples of abelian, idempotent semigroups S such that the semigroup algebras $\ell^1(S)$ are strongly Arens irregular. In one example, there is no finite DTC set for S, but $M(\beta S)$ has a two-element DTC set. In another example, there is no countable DTC set for S or for $M(\beta S)$.

Let S be a semigroup, and consider the centre $\mathfrak{Z} = \mathfrak{Z}(M(\beta S), \Box)$. We write $\mathfrak{Z}_{\mathbb{R}}$ for $\mathfrak{Z}(M_{\mathbb{R}}(\beta S), \Box)$. The following result is immediate.

Proposition 1.7 *Let S be a semigroup. Suppose that $\mu \in M(\beta S)$ and that $\mu = \mu_1 + \mathrm{i}\mu_2$, where $\mu_1, \mu_2 \in M_{\mathbb{R}}(\beta S)$. Then $\mu \in \mathfrak{Z}$ if and only if $\mu_1, \mu_2 \in \mathfrak{Z}_{\mathbb{R}}$.* □

2 Totally ordered semigroups

In this section, we shall introduce a class of totally ordered, abelian, idempotent semigroups, and obtain some results about their Arens regularity and the Arens regularity of their semigroup algebras.

Throughout the section, T will denote an infinite, totally ordered space. For $s, t \in T$, we set

$$s \wedge t = \min\{s, t\}, \quad s \vee t = \max\{s, t\},$$

so that T is a lattice and a semigroup with respect to the operations

$$(s, t) \mapsto s \wedge t, \quad (s, t) \mapsto s \vee t, \quad T \times T \to T.$$

Thus T is an abelian, idempotent semigroup with respect to both these operations; we shall just consider the operation $\wedge$. We further suppose that T has a minimum element, called 0, and a maximum element, called ∞, and that T is complete, in the sense that every non-empty subset of T has a supremum and an infimum. We give T its interval topology, so that the closed intervals provide a subbase for the closed sets, and the intervals of the form (a, b), $[0, a)$, $(a, \infty]$, and $[0, \infty]$ are a subbase for the open sets of T, and we shall refer to them as open intervals. The space T is then a compact topological semigroup. Further, every increasing or decreasing net in T converges to its supremum or infimum, respectively.

Certain preliminaries about the semigroup S are contained in the paper of Ross [14].

We shall denote by S an arbitrary, infinite subset of T, so that S is a sub-semigroup of $(T, \wedge)$ that is also an abelian, idempotent semigroup. For subsets A and B of S, we write $A \leq B$ if $s \leq t \ \ (s \in A, \ t \in B)$. The set S with the discrete topology is denoted by S_d, and we set $X = \beta S_d$ and $X^* = X \setminus S$. The closures of a subset A of S in T and X are $\mathrm{cl}_T A$ and $\mathrm{cl}_X A$, respectively. The map

$$\pi : X \to T$$

denotes the continuous extension of the inclusion map of S into T, so that $\pi(X) = \mathrm{cl}_T S$. We shall write F_t for the fibre $\{x \in X : \pi(x) = t\}$ for $t \in \mathrm{cl}_T S$. We set $F_t^* = F_t \cap X^*$ throughout, so that

$$F_t^* = F_t \quad (t \in T \setminus S) \quad \text{and} \quad F_t^* = F_t \setminus \{t\} \quad (t \in S).$$

We recall the standard facts, that, for every subset A of S, the set $\mathrm{cl}_X A$ is clopen in X, and that, for every subsets A and B of S that are disjoint, the two sets $\mathrm{cl}_X A$ and $\mathrm{cl}_X B$ are disjoint in X.

We let E denote the set of accumulation points of S in T, so that $E \neq \emptyset$. Take $t \in T$. Then F_t^* is a closed, and hence compact, subspace of X, and clearly $F_t^* \neq \emptyset$ if and only if $t \in E$.

For $\mu \in M(X)$, we define $\mu_\pi \in M(T)$ by

$$\mu_\pi(B) = \mu(\pi^{-1}(B)) \quad (B \in \mathfrak{B}_T)\,.$$

For example: set $T = \mathbb{N} \cup \{\infty\}$ and $S = \mathbb{N}$, so that $E = \{\infty\}$ and $F_\infty = \mathbb{N}^*$; set $T = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ and $S = \mathbb{Q}$, so that $E = T$; set $T = [0, \omega_1]$, where ω_1 is the first uncountable ordinal, and $S = [0, \omega_1)$, so that E is the collection of limit ordinals in T. These examples are not weakly cancellative.

Consider the space $\{0, 1\}^\kappa$, where κ is a non-zero cardinal and a generic element in $\{0, 1\}^\kappa$ is $(u_\alpha : \alpha \leq \kappa)$. Then $\{0, 1\}^\kappa$ is a complete lattice with respect to the product

$$(u_\alpha) \wedge (v_\alpha) = (u_\alpha \wedge v_\alpha) \quad ((u_\alpha), (v_\alpha) \in \{0, 1\}^\kappa)$$

and a compact topological semigroup.

Suppose that S is a semi-lattice, and hence a separating semigroup, so that Φ_S separates the points of S and $(S, \wedge)$ is a partially ordered semigroup. Every semi-character in Φ_S maps S into $\{0, 1\}$, and so $(S, \wedge)$ can be embedded as a semigroup in $C := (\{0, 1\}^\kappa, \wedge)$, where κ denotes the cardinality of any subset Ψ of Φ_S that separates the points of S; we regard S as a sub-semigroup of $\{0, 1\}^\kappa$. Note that we can always choose Ψ to have the cardinality $|S|$ by using the fact that each $s \in S$ defines an element $\theta \in \Phi_S$ by setting $\theta(t) = 1$ if and only if $s \leq t$. In particular, when $S = \mathbb{Q}$, we can set $C = \{0, 1\}^{\aleph_0}$, the Cantor set, whereas $\left|\Phi_{\mathbb{Q}}\right| = \mathfrak{c}$.

Consider the case where $(S, \wedge)$ is totally ordered, and define T to be the closure of S in C, so that T is also a complete lattice that is compact in its interval topology. We observe that every $\theta \in \Psi$ can be extended to a semi-character defined on C by putting

$$\theta(x) = \sup\{\theta(s) : s \in S, s \leq x\}$$

for every $x \in C$, where $\sup(\emptyset)$ is regarded as the minimum element of C. We *claim* that the sub-semigroup $(T, \wedge)$ is also totally ordered. To see this, given $s \in S$, take T_s to be

$$T_s = \{x \in C : \theta(x) \leq \theta(s) \;\; (\theta \in \Psi)\} \cup \{x \in C : \theta(s) \leq \theta(x) \;\; (\theta \in \Psi)\}\,.$$

Then T_s is a closed subset of C that contains S, and so it contains T. The set T is contained in T_s for every $s \in S$. Now take $x \in T$. Then, for every $s \in S$, we have $\theta(x) \leq \theta(s) \;\; (\theta \in \Psi)$ or $\theta(s) \leq \theta(x) \;\; (\theta \in \Psi)$. Hence, for each $x, y \in T$, we have $x \leq y$ or $y \leq x$. This shows that $(T, \wedge)$ is also totally ordered, giving the claim.

Thus each totally ordered semigroup $(S, \wedge)$ can be embedded in a complete, compact, totally ordered topological semigroup $(T, \wedge)$, as described above; this fact is well-known.

Take $t \in E$. Throughout we shall write

$$A_t = S \cap [0, t) \quad \text{and} \quad B_t = S \cap (t, \infty]\,;$$

at least one of these sets is non-empty. It follows that

$$F_t^* \subset \mathrm{cl}_X A_t \cup \mathrm{cl}_X B_t\,, \tag{2.1}$$

and hence the sets $F_t^* \cap \mathrm{cl}_X A_t$ and $F_t^* \cap \mathrm{cl}_X B_t$ are disjoint, compact subspaces of F_t^* whose union is F_t^*.

Lemma 2.1 (i) *Take a subset A of S, and suppose that $\mu \in M(X)_{[1]}$ with* $\mathrm{supp}\,\mu \subset \mathrm{cl}_X A$. *Then the measure μ belongs to the weak-$*$ closure of* $\mathrm{aco}\{\delta_s : s \in A\}$.

(ii) *Suppose that A and B are subsets of S such that $A \le B$. Take $\mu, \nu \in M(X)$ with* $\mathrm{supp}\,\mu \subset \mathrm{cl}_X A$ *and* $\mathrm{supp}\,\nu \subset \mathrm{cl}_X B$. *Then*

$$\mu \,\Box\, \nu = \nu \,\Box\, \mu = \varphi_0(\nu)\mu\,.$$

(iii) *Take $p, q \in X$ with $\pi(p) < \pi(q)$. Then $p \,\Box\, q = q \,\Box\, p = p$.*

Proof (i) Certainly, by [2, Corollary 4.4.16], μ is in the weak-$*$ closure of the set

$$\mathrm{aco}\{\delta_u : u \in \mathrm{cl}_X A\}\,,$$

and each δ_u for $u \in \mathrm{cl}_X A$ is the weak-$*$ limit of a net in $\{\delta_s : s \in A\}$.

(ii) We may suppose that $\mu, \nu \in M(X)_{[1]}$. Take $\sigma \in \mathrm{aco}\{\delta_s : s \in A\}$ and $\tau \in \mathrm{aco}\{\delta_t : t \in B\}$. We have

$$\delta_s \star \delta_t = \delta_t \star \delta_s = \delta_s \quad (s \in A,\ t \in B)\,,$$

and so $\sigma \star \tau = \tau \star \sigma = \varphi_0(\tau)\sigma$. Using (i) and equation (1.2), we can take weak-$*$ limits to see that

$$\mu \,\Box\, \nu = \lim_{\sigma \to \mu} \lim_{\tau \to \nu} \sigma \star \tau = \lim_{\sigma \to \mu} \lim_{\tau \to \nu} \varphi_0(\tau)\sigma = \lim_{\sigma \to \mu} \varphi_0(\nu)\sigma = \varphi_0(\nu)\mu\,.$$

Similarly, $\nu \,\Box\, \mu = \varphi_0(\nu)\mu$, and so $\mu \,\Box\, \nu = \nu \,\Box\, \mu$.

(iii) We may suppose that there exists $t \in [\pi(p), \pi(q)]$ such that $p \in \mathrm{cl}_X A_t$ and $q \in \mathrm{cl}_X B_t$, and so this follows from clause (ii). ☐

Lemma 2.2 *Take $p \in F_t^*$, where $t \in E$. Then*

$$p \in \mathrm{cl}_X(S \cap ([0, t) \cup (t, \infty])) = \mathrm{cl}_X(S \setminus \{t\}).$$

Proof The element p is a point of accumulation of S in X, and so $\pi(p) = t$ is a point of accumulation of S in T, giving the result. ☐

Lemma 2.3 *Take $p, q \in X^*$, and $t \in E$.*

(i) *Suppose that $p, q \in F_t^* \cap \mathrm{cl}_X A_t$. Then $p \,\Box\, q = p$.*

(ii) *Suppose that $p, q \in F_t^* \cap \mathrm{cl}_X B_t$. Then $p \,\Box\, q = q$.*

(iii) *Suppose that $p \in F_t^* \cap \mathrm{cl}_X A_t$ and that $q \in F_t^* \cap \mathrm{cl}_X B_t$. Then $p \,\Box\, q = q \,\Box\, p = p$.*

Proof To prove (i), choose nets $(s_\alpha)_{\alpha\in I}$ and $(s_\beta)_{\beta\in J}$ in A_t which converge to p and q, respectively, in X. Now $\pi(s_\alpha) = s_\alpha \to \pi(p) = t$ and $\pi(s_\beta) = s_\beta \to \pi(q) = t$. So, for each $\alpha \in I$, the net $(s_\beta)_{\beta\in J, s_\beta > s_\alpha}$ converges to q in X. Since $p\Box q = \lim\limits_{s_\alpha\to p}\lim\limits_{s_\beta\to q} s_\alpha \wedge s_\beta$ and $s_\alpha \wedge s_\beta = s_\alpha$ if $s_\beta > s_\alpha$, it follows that $p\Box q = \lim\limits_{s_\alpha\to p} s_\alpha = p$.

The proof of (ii) is similar, and (iii) is a special case of Lemma 2.1(ii). □

We have seen that, for $p, q \in X$ with $p \neq q$, we have $p\,\Box\,q = q\,\Box\,p$ unless p and q both belong to sets of the form $F_t^* \cap \mathrm{cl}_X A_t$ or of the form $F_t^* \cap \mathrm{cl}_X B_t$ for some $t \in E$. These sets will play a crucial rôle in the description of $\mathfrak{Z}(X)$ and $\mathfrak{Z}(M(X))$.

Lemma 2.4 *Take $t \in E$. Then $\left|F_t^*\right| \geq 2^{\mathfrak{c}}$.*

Proof We may suppose that $t \in \mathrm{cl}_T A_t$.

We first note that, since T is totally ordered, there is an infinite limit ordinal τ and a strictly increasing net $(s_\alpha : \alpha < \tau)$ in $S \cap [0, t)$ that converges to t in T.

Let (N_k) be a family of pairwise-disjoint, infinite subsets of $\mathbb{N}$. For each $k \in \mathbb{N}$, take E_k to be the set of s_α such that $\alpha = \lambda + n$, where λ is 0 or a limit ordinal and $n \in N_k$. The sequence $\{E_k : k \in \mathbb{N}\}$ partitions the set $\{s_\alpha : \alpha < \tau\}$ into an infinite number of disjoint subnets. The sets $\mathrm{cl}_X E_k$ are pairwise-disjoint in X and $(\mathrm{cl}_X E_k) \cap F_t \neq \emptyset \ \ (k \in \mathbb{N})$. Thus F_t is infinite.

Since F_t^* is an infinite, compact subspace of X, we have $\left|F_t^*\right| \geq 2^{\mathfrak{c}}$ by [7, Theorem 3.59]. □

Theorem 2.5 *The semigroup $(S, \wedge)$ is strongly Arens irregular, and the semigroup algebra $(\ell^1(S), \star)$ is not Arens regular.*

Proof Consider a point $p \in X^*$, say $p \in F_t^*$, where $t \in E$. We may suppose that $p \in \mathrm{cl}_X A_t$, and so there exists $q \in F_t^* \cap \mathrm{cl}_X A_t$ with $q \neq p$ by Lemma 2.4. By Lemma 2.3(i), it follows that $p\,\Box\,q \neq q\,\Box\,p$, and so $p \notin \mathfrak{Z}(X)$. Hence $\mathfrak{Z}(X) = S$, showing that S is strongly Arens irregular.

Clearly $\delta_p \notin \mathfrak{Z}(M(X))$, and so $\ell^1(S)$ is not Arens regular. □

It follows that, in the special case that we are considering, we have $\mathfrak{Z}(M(X)) \cap X = \mathfrak{Z}(X)$.

We shall now consider when the semigroup algebra $(\ell^1(S), \star)$ is strongly Arens irregular.

Lemma 2.6 *Let $\mu \in M(X)$, and take $t \in E$. Suppose that*

$$\mu \mid (F_t^* \cap U) = 0\,,$$

where $U = \mathrm{cl}_X A_t$ or $U = \mathrm{cl}_X B_t$. Then $\mu\,\Box\,p = p\,\Box\,\mu \ \ (p \in F_t^* \cap U)$.

Proof We may suppose that $\mu \in M_{\mathbb{R}}(K)$ because $\mu = \nu_1 + \mathrm{i}\nu_2$, where $\nu_1, \nu_2 \in M_{\mathbb{R}}(K)$. We may also suppose that $U = \mathrm{cl}_X A_t$, and that there exists $p \in F_t^* \cap U$. Furthermore, we may suppose that $\mu(\{s\}) = 0$ for every $s \in S$ because we can replace μ by $\mu - \sum_{s\in S} \mu(\{s\})\delta_s$ as $\ell^1(S) \subseteq \mathfrak{Z}(M(X))$.

Now $\bigcap_{u\in A_t} \mathrm{cl}_X(S\cap[u,t])$ is contained in $\{t\}\cup(F_t^*\cap U)$ if $t\in S$, and is contained in $F_t^*\cap U$ if $t\in E\setminus S$. In either case, we have $|\mu|(\bigcap\{\mathrm{cl}_X(S\cap[u,t]) : u\in A_t\})=0$.

Choose $\varepsilon>0$. Since $|\mu|$ is regular, there exists $u\in A_t$ such that $|\mu|(\mathrm{cl}_X(S\cap [u,t]))<\varepsilon$. So $\|\mu \mid \mathrm{cl}(S\cap[u,t])\|<\varepsilon$. Let $\mu_1=\mu\mid \mathrm{cl}_X A_u$ and $\mu_2=\mu\mid\mathrm{cl}_X B_t$, so that $\mu_1\,\Box\, p=p\,\Box\,\mu_1$ and $\mu_2\,\Box\,p=p\,\Box\,\mu_2$ by Lemma 2.1(ii).

Now $\mu=\mu\mid\mathrm{cl}_X(S\cap[u,t])+\mu_1+\mu_2$ because

$$X=\mathrm{cl}_X(S\cap[u,t])+\mathrm{cl}_X A_u+\mathrm{cl}_X B_t$$

and these are disjoint subsets of X. It follows that

$$\|\mu\,\Box\,p-p\,\Box\,\mu\|<2\varepsilon\,.$$

This holds for each $\varepsilon>0$, and so $\mu\,\Box\,p=p\,\Box\,\mu$. □

Proposition 2.7 *Let $\mu\in M(X)_{[1]}$. Take $t\in E$, and set*

$$U=F_t^*\cap\mathrm{cl}_X A_t\quad\text{or}\quad U=F_t^*\cap\mathrm{cl}_X B_t\,.$$

Suppose that $p\in U$ and that $\mu\,\Box\,p=p\,\Box\,\mu$. Then $\mu\mid U=zp$ for some $z\in\overline{\mathbb{D}}$. Suppose also that $q\in U$ with $q\neq p$ and that $\mu\,\Box\,q=q\,\Box\,\mu$. Then $\mu\mid U=0$.

Proof We suppose that $U=F_t^*\cap\mathrm{cl}_X A_t$.

Set $\nu=\mu\mid U$. Since $(\mu-\nu)\mid U=0$, it follows from Lemma 2.6 that $(\mu-\nu)\,\Box\,p=p\,\Box\,(\mu-\nu)$. By hypothesis, $\mu\,\Box\,p=p\,\Box\,\mu$, and so $\nu\,\Box\,p=p\,\Box\,\nu$.

Take $s\in[0,t)$. By Lemma 2.1(i), the measure ν belongs to the weak-$*$ closure of $C_s:=\mathrm{aco}\{\delta_r : r\in S\cap(s,t)\}$. Each $\sigma\in C_s$ has the form $\sigma=\zeta_1\delta_{s_1}+\cdots+\zeta_m\delta_{s_m}$, where $m\in\mathbb{N}$, $\zeta_1,\dots,\zeta_m\in\mathbb{C}$ with $|\zeta_1|+\cdots+|\zeta_m|\le 1$, and $s_1,\dots,s_m\in(s,t)$. We have $s\star\sigma=\varphi_0(\sigma)s$, and so

$$p\,\Box\,\nu=\lim_{s\to p}\lim_{\sigma\to\nu} s\star\sigma=\lim_{s\to p}\varphi_0(\nu)s=zp\,,$$

where $z=\varphi_0(\nu)\in\overline{\mathbb{D}}$.

On the other hand, take $\sigma\in\mathrm{aco}\{\delta_r : r\in A_t\}$ of the above form. Then $\sigma\star s=\sigma$ for $s>s_i\ \ (i\in\mathbb{N}_m)$, and so

$$\nu\,\Box\,p=\lim_{\sigma\to\nu}\lim_{s\to p}\sigma\star s=\lim_{\sigma\to\nu}\sigma=\nu\,.$$

We conclude that $\nu=\nu\,\Box\,p=p\,\Box\,\nu=zp$, as required.

Now suppose also that $q\in U$ with $q\neq p$ and that $\mu\,\Box\,q=q\,\Box\,\mu$. Then there exist $z,w\in\overline{\mathbb{D}}$ such that $\nu=zp$ and $\nu=wq$. Since $q\neq p$, we have $zw=0$, and so $\nu=0$. □

Lemma 2.8 *Let $[a,b]$ be a closed interval in T. Suppose that $\mathcal{F}$ is a finite family of open intervals in T whose union contains $[a,b]$. Then there exist $n\in\mathbb{N}$ and $t_0,\dots,t_n\in T$ such that*

$$a=t_0<t_1<\cdots<t_{n-1}<t_n=b$$

and such that each interval $[t_i, t_{i+1}]$ *is contained in a member of* $\mathcal{F}$.

Proof This is an immediate induction on the cardinality of the family $\mathcal{F}$. $\square$

Proposition 2.9 *Let* $\mu \in M(X)$ *be such that* $\mu \mid F_t^* = 0 \ \ (t \in E)$. *Then* $\mu \in \mathfrak{Z}(M(X))$.

Proof Since $\ell^1(S) \subset \mathfrak{Z}(M(X))$, we may suppose that $\mu(\{s\}) = 0$ for every $s \in S$, because we can replace μ by $\mu - \sum_{s \in S} \mu(\{s\})\delta_s$. If $t \in E \setminus S$, then $\mu_\pi(\{t\}) = \mu(F_t^*) = 0$. It follows that $\mu_\pi(t) = 0$ for every $t \in T$.

Take $\nu \in M(X)_{[1]}$ and $\varepsilon > 0$.

Each $t \in T$ is contained in an open interval, say U_t, of T such that $|\mu|_\pi\,(U_t) < \varepsilon$, and, since T is compact, we can suppose that the union of finitely many sets of the form U_t is T. By Lemma 2.8, there exist $n \in \mathbb{N}$ and $t_0, \dots, t_n \in T$ with $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = \infty$ such that $|\mu|_\pi\,(I_i) < \varepsilon$ for $i = 0, 1, \dots, n-1$, where $I_i := [t_i, t_{i+1}]$.

Set $\mu_i = \mu \mid \pi^{-1}(I_i)$ and $\nu_i = \nu \mid \pi^{-1}(I_i)$ for $i = 0, 1, \dots, n-1$, so that $\mu = \sum_{i=0}^{n-1} \mu_i$ and $\nu = \sum_{i=0}^{n-1} \nu_i$. It follows from Lemma 2.1(ii) that $\mu_i \,\Box\, \nu_j = \nu_j \,\Box\, \mu_i$ whenever $i, j \in \{0, 1, \dots, n-1\}$ and $i \neq j$. Take $i \in \{0, 1, \dots, n-1\}$. Since $|\mu|_\pi\,(I_i) < \varepsilon$, we have $\|\mu_i \,\Box\, \nu_i\| < \varepsilon \,\|\nu_i\|$, and so

$$\left\| \sum_{i=0}^{n-1} \mu_i \,\Box\, \nu_i \right\| < \varepsilon \quad \text{and} \quad \left\| \sum_{i=0}^{n-1} \nu_i \,\Box\, \mu_i \right\| < \varepsilon\,.$$

Hence $\|\mu \,\Box\, \nu - \nu \,\Box\, \mu\| < 2\varepsilon$.

This holds for each $\varepsilon > 0$, and so $\mu \,\Box\, \nu = \nu \,\Box\, \mu$. We conclude that $\mu \in \mathfrak{Z}(M(X))$. $\square$

Proposition 2.10 *Let* $\mu \in M(X)$, *and suppose that there exists* $t \in E$ *such that* $\mu \mid F_t^* \neq 0$. *Then* $\mu \notin \mathfrak{Z}(M(X))$.

Proof We fix $t \in E$ as specified.

We again set $U = F_t^* \cap \mathrm{cl}_X A_t$, and we may suppose that $\mu \mid U \neq 0$, using equation (2.1). By Lemma 2.4, there are two distinct points, say p and q, in U. It follows from Proposition 2.7 that either $\mu \,\Box\, p \neq p \,\Box\, \mu$ or $\mu \,\Box\, q \neq q \,\Box\, \mu$, and hence $\mu \notin \mathfrak{Z}(M(X))$. $\square$

Theorem 2.11 *Let* $\mu \in M(X)$. *Then* $\mu \in \mathfrak{Z}(M(X))$ *if and only if* $\mu \mid F_t^* = 0 \ \ (t \in E)$.

Proof This now follows from Propositions 2.9 and 2.10. $\square$

Corollary 2.12 (i) *Let* $\mu \in M(X)$. *Then* $\mu \in \mathfrak{Z}(M(X))$ *if and only if* $|\mu| \in \mathfrak{Z}(M(X))$.

(ii) *Let* $\mu \in M_{\mathbb{R}}(X)$. *Then the following are equivalent:*

(a) $\mu \in \mathfrak{Z}(M(X))$;

(b) $|\mu| \in \mathfrak{Z}(M(X))$;

(c) $\mu^+, \mu^- \in \mathfrak{Z}(M(X))$.

Proof (i) This is immediate from Theorem 2.11.

(ii) Clearly (a) $\Leftrightarrow$ (b), and (c) $\Rightarrow$ (a).

Suppose that (a) and (b) hold. Then $\mu^+ = (\mu + |\mu|)/2 \in \mathfrak{Z}(M(X))$ and $\mu^- = (\mu - |\mu|)/2 \in \mathfrak{Z}(M(X))$, giving (c). □

Let E be a Banach space, and denote by $\mathcal{B}(E)$ the Banach algebra of bounded linear operators on E. An element $P \in \mathcal{B}(E)$ is a *projection* if $P^2 = P$. A closed linear subspace F of E is *complemented* if there is a closed linear subspace G of E such that $E = F \oplus G$. Suppose that P is a projection in $\mathcal{B}(E)$. Then the spaces $P(E)$ and $\ker P$ are complemented in E and $E = P(E) \oplus \ker P$; both are *1-complemented* if $\|P\| = 1$.

Let E be a (complex) Banach lattice, so that E is the complexification of the real Banach lattice $E_{\mathbb{R}}$. A Banach lattice is *Dedekind complete* if every increasing net in $E^+ = E_{\mathbb{R}}^+$ that is bounded above has a supremum.

Let K be a non-empty, compact space. Then $M(K)$ is a Dedekind complete Banach lattice. Let $B \in \mathfrak{B}_K$, and set

$$M_B = \{\mu \in M(K) : \mu \mid B = 0\}\,.$$

Since $|\mu \vee \nu| \leq |\mu| + |\nu| \quad (\mu, \nu \in M(K))$, it follows that M_B is a sublattice of $M(K)$. Also M_B is Dedekind complete because, for any increasing, bounded net (μ_α) in M_B^+, we have $\mu := \bigvee_\alpha \mu_\alpha$ exists in $M(K)^+$, and clearly $\mu \in M_B$. Thus $\{\mu \in M(K) : \mu \mid B = 0 \;\; (B \in \mathcal{F})\}$ is a Dedekind complete lattice for each family $\mathcal{F}$ in $\mathfrak{B}_K$.

(We remark that, for each compact space K, the Banach lattice $M(K)$ is an AL-space; that each AL-space is order-continuous; and that, in an order-continuous Banach lattice, a set is order-closed if and only if it is norm-closed. Thus every norm-closed sublattice of $M(K)$ is Dedekind complete. For definitions and proofs of these statements, see [17], for example.)

Corollary 2.13 *The space $\mathfrak{Z}(M(X))$ is 1-complemented in $M(X)$, and $\mathfrak{Z}(M(X))$ is a Banach sublattice of $M(X)$ that is Dedekind complete.*

Proof Take $\mu \in M(X)$. The set $E_0 := \{t \in E : |\mu|(F_t^*) > 0\}$ is countable because $\{F_t^* : t \in E\}$ is a pairwise-disjoint family of sets. Define $P : M(X) \to M(X)$ by

$$P\mu = \sum \{\mu \mid F_t^* : t \in E\} = \sum \{\mu \mid F_t^* : t \in E_0\}\,.$$

The sum converges in $M(X)$, and P is a projection in $\mathcal{B}(M(X))$ with $\|P\| = 1$. By Theorem 2.11, $\ker P = \mathfrak{Z}(M(X))$, and hence $\mathfrak{Z}(M(X))$ is 1-complemented in $M(X)$.

It follows from the above remarks that $\mathfrak{Z}(M(X))$ is a Banach sublattice of $M(K)$ that is Dedekind complete. □

A topological space Y is *scattered* if each non-empty subset of Y contains a point that is isolated in the subset. In the case where K is a compact space, the following conditions are equivalent:

(a) K is scattered;

(b) $f(K)$ is countable for each $f \in C(K)$;
(c) $\ell^1(K) = M(K)$.
For this, see [9, §5], where several other equivalent conditions are given. For example, $\mathbb{Z}^+ \cup \{\infty\}$ and $[0, \omega_1]$ are scattered, but the spaces $\mathbb{Q}$ and $\{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ are not scattered.

Theorem 2.14 *The semigroup algebra* $(\ell^1(S), \star)$ *is strongly Arens irregular if and only if* $\mathrm{cl}_T S$ *is scattered.*

Proof First, suppose that $\mathrm{cl}_T S$ is scattered. Take $\mu \in \mathfrak{Z}(M(X))$. Then $|\mu|_\pi \in \ell^1(\mathrm{cl}_T S)$. By Theorem 2.11, $\mu \mid F_t^* = 0 \ \ (t \in E)$, and so $|\mu|_\pi(\{t\}) = |\mu|(\{t\}) \ \ (t \in \mathrm{cl}_T S)$. Also, $|\mu|_\pi(\{t\}) = 0 \ \ (t \in \mathrm{cl}_T S \setminus S)$ because $\pi^{-1}(\{t\}) \subset X^*$. Thus $|\mu|_\pi \in \ell^1(S)$. Since $\left\| \, |\mu|_\pi \right\|_1 = \| \, |\mu| \, \|_1$, it follows that $|\mu| \in \ell^1(S)$, and so $\mu \in \ell^1(S)$. Hence $\mathfrak{Z}(M(X)) = \ell^1(S)$, showing that $\ell^1(S)$ is strongly Arens irregular.

Conversely, suppose that $\mathrm{cl}_T S$ is not scattered. Choose a continuous probability measure, say μ, on $\mathrm{cl}_T S$, and set

$$F = \{f \circ \pi : f \in C(\mathrm{cl}_T S)\}\,,$$

so that F is a closed linear subspace of $C(X)$. We define a continuous linear functional ν on F by setting

$$\nu(f \circ \pi) = \mu(f) \quad (f \in C(\mathrm{cl}_T S))\,,$$

so that $\|\nu\| = \nu(1_{\mathrm{cl}_T S}) = 1$. By the Hahn–Banach theorem, we can extend ν to be a continuous linear functional on $C(X)$, still with $\|\nu\| = \nu(1_{\mathrm{cl}_T S}) = 1$; we regard ν as a probability measure on X. Since $\nu(F_t^*) = 0 \ \ (t \in E)$, it follows from Proposition 2.9 that $\nu \in \mathfrak{Z}(M(X))$. Since $\nu \notin \ell^1(S)$, the Banach algebra $\ell^1(S)$ is not strongly Arens irregular. □

The following corollary includes the case where $S = (\mathbb{Q}, \wedge)$. Analogous results for the algebra $L^1([0, 1])$, where $[0, 1]$ is a compact, totally ordered semigroup with respect to the operation $\wedge$, are given by Saghafi in [15].

Corollary 2.15 *Suppose that* $\mathrm{cl}_T S$ *is not scattered. Then*

$$\ell^1(S) \subsetneq \mathfrak{Z}(M(X)) \subsetneq M(X)\,,$$

and so the algebra $(\ell^1(S), \star)$ *is neither Arens regular nor strongly Arens irregular.*□

Example 2.16 We conclude this section by noting that, in Theorem 2.5, we cannot take T to be just partially ordered. Indeed, set $T = \{0, 1\}^{\aleph_0}$, as above, so that T is a partially ordered set that is a lattice. Then take S to consist of the elements of T that have at most one non-zero component, so that $(S, \wedge)$ is an abelian, idempotent semigroup, and S is not weakly cancellative. The zero sequence is denoted by 0. It follows from equation (1.1) that $u \,\Box\, v = 0$ for $u, v \in \beta S$, save when $u = v \in S$. Thus S and $\ell^1(S)$ are both Arens regular. □

3 DTC sets

We recall from Theorems 1.3 and 1.6 that an infinite, weakly cancellative semigroup is strongly Arens irregular and has a two-point DLTC set, and an infinite, cancellative semigroup S is such that the semigroup algebra $\ell^1(S)$ is strongly Arens irregular and has a two-point DLTC set contained in S^*. We now consider some related results for the semigroups $(S, \wedge)$ considered above.

Let S, T, X, and E be as above. We can obtain a DTC set V for $M(X)$ that is contained in X^* as follows.

For each $t \in E$, we choose two distinct points in $F_t^* \cap \mathrm{cl}_X A_t$ and two distinct points in $F_t^* \cap \mathrm{cl}_X B_t$ whenever the respective sets are non-empty. The collection of these points is called V. Now take $\mu \in M(X)$, and suppose that $\mu \,\Box\, p = p \,\Box\, \mu$ for each $p \in V$. It follows from Proposition 2.7 that $\mu \mid F_t^* \cap \mathrm{cl}_X A_t = \mu \mid F_t^* \cap \mathrm{cl}_X B_t = 0$ for each $t \in E$. Thus $\mu \mid F_t^* = 0$. By Theorem 2.11, this implies that $\mu \in \mathfrak{Z}(M(X))$, and hence V is a DTC set for $M(X)$. It follows that, if E is infinite, $M(X)$ has a DTC set consisting of at most 2^κ points of X, where $\kappa = |S|$; this is a small subset of X because $|X| = 2^{2^\kappa}$.

Suppose that E is finite. Then the above DTC set V is also finite.

Suppose that E is infinite, so that $|V| = |E|$. Then there cannot be a finite DTC set for the semigroup S. For suppose that V is a finite subset in X, and choose $t \in E \setminus \pi(V)$, and then choose $p \in F_t^*$. We have $p \,\Box\, v = v \,\Box\, p \quad (v \in V)$ by Lemma 2.1(iii), but $p \notin S$. Since S is strongly Arens irregular by Theorem 2.5, this shows that V is not a DTC set for S. Similarly, there is no countable DTC set for S when E is uncountable.

Thus we obtain the following result.

Proposition 3.1 *Suppose that the set E is finite. Then there is a finite* DTC *set for the semigroup S.*

Suppose that the set E is infinite or uncountable. Then there is no finite or countable DTC *set for the semigroup S, respectively.* □

Theorem 3.2 *Suppose that the set E is countable. Then the semigroup algebra $\ell^1(S)$ is strongly Arens irregular and has a* DTC *set consisting of at most four measures in $M(X^*)^+$.*

Proof The set E is scattered because it is a countable, compact space, and so $\mathrm{cl}_T S$ is scattered. By Theorem 2.14, $\ell^1(S)$ is strongly Arens irregular.

Set

$$A = \{t \in E : F_t^* \cap \mathrm{cl}_X A_t \neq \emptyset\}$$

and

$$B = \{t \in E : F_t^* \cap \mathrm{cl}_X B_t \neq \emptyset\}\,.$$

(One of these two sets might be empty, but at least one is non-empty.) Take $\{s_n : n \in I\}$ and $\{t_n : n \in J\}$ to be enumerations of A and B, respectively, where I and J are subsets of $\mathbb{N}$. For each $n \in I$, choose two distinct points, say $p_{1,n}$ and $p_{2,n}$, in $F^*_{s_n} \cap \mathrm{cl}_X A_{s_n}$, and, for each $n \in J$, choose two distinct points, say $q_{1,n}$ and $q_{2,n}$, in $F^*_{t_n} \cap \mathrm{cl}_X B_{t_n}$.

Consider the two measures

$$\mu_j = \sum_{n \in I} \frac{1}{2^n} \delta_{p_{j,n}} \quad (j = 1, 2)\,,$$

defined when $A \neq \emptyset$, and similarly μ_3, μ_4, defined when $B \neq \emptyset$. We obtain at most four measures in $M(X^*)+$, forming a set V.

We *claim* that V is a DTC set for $M(X)$. Indeed, suppose that $\mu \in M(X)$ and that $\mu \,\Box\, \nu = \nu \,\Box\, \mu \;\; (\nu \in V)$.

Take $n \in I$, and set $\nu_n = \mu \mid (F^*_{s_n} \cap A_{s_n})$. It follows from Lemma 2.6 that $(\mu - \nu_n) \,\Box\, p_{j,n} = p_{j,n} \,\Box\, (\mu - \nu_n) \;\; (j = 1, 2)$.

Set $\nu = \sum_{n \in I} \nu_n$. Then, for each $n \in I$, we have

$$(\mu - \nu) \mid (F^*_{s_n} \cap A_{s_n}) = (\mu - \nu_n) \mid (F^*_{s_n} \cap A_{s_n})$$

because $\nu_m \mid (F^*_{s_n} \cap A_{s_n}) = 0 \;\; (m \in I \setminus \{n\})$. Now take $m, n \in I$ with $m \neq n$. Again by Lemma 2.6, we have

$$\nu_m \,\Box\, p_{j,n} = p_{j,n} \,\Box\, \nu_m \quad (j = 1, 2)\,. \tag{3.1}$$

It follows that $(\mu - \nu_m) \,\Box\, p_{j,n} = p_{j,n} \,\Box\, (\mu - \nu_m) \quad (j = 1, 2)$. This shows that

$$(\mu - \nu) \,\Box\, p_{j,n} = p_{j,n} \,\Box\, (\mu - \nu) \quad (n \in I,\; j = 1, 2)\,,$$

and hence

$$(\mu - \nu) \,\Box\, \mu_j = \mu_j \,\Box\, (\mu - \nu) \quad (j = 1, 2)\,.$$

By the hypothesis, we have

$$\nu \,\Box\, \mu_j = \mu_j \,\Box\, \nu \quad (j = 1, 2)\,.$$

It now follows from equation (3.1) that

$$\nu_n \,\Box\, p_{j,n} = p_{j,n} \,\Box\, \nu_n \quad (j = 1, 2)\,.$$

By Proposition 2.7, $\nu_n = 0 \;\; (n \in I)$, i.e., $\mu \mid (F^*_{s_n} \cap A_{s_n}) = 0 \;\; (n \in I)$.

Similarly $\mu \mid (F^*_{t_n} \cap \mathrm{cl}_X B_{t_n}) = 0 \;\; (n \in J)$.

It then follows that $\mu \mid F^*_t = 0 \;\; (t \in E)$, and so, by Theorem 2.11, $\mu \in \mathfrak{Z}(M(X)) = \ell^1(S)$, completing the proof. ☐

Corollary 3.3 *Suppose that the semigroup S is countable and the semigroup algebra $\ell^1(S)$ is strongly Arens irregular. Then $\ell^1(S)$ has a* DTC *set consisting of at most four measures in $M(X^*)^+$.*

Proof Consider the space $\mathrm{cl}_T S$. Since S is countable, this space is second countable because open intervals in $\mathrm{cl}_T S$ are unions of open intervals of the form $[0, s)$, (s, t), and $(s, \infty]$, where $s, t \in S$. Since $\ell^1(S)$ is strongly Arens irregular, the set $\mathrm{cl}_T S$ is scattered by Theorem 2.14. By [16, Proposition 8.5.5], $\mathrm{cl}_T S$ is countable, and so the set E is countable. Hence the result follows from the theorem. □

Proposition 3.4 *Suppose that* $|E| = \kappa$, *where* $\kappa \geq \aleph_1$. *Then there is no* DTC *set for* $M(X)$ *with cardinality less than* κ.

Proof Assume that V is a DTC set for $M(X)$ with $|V| < \kappa$. For each $\nu \in V$, the set $\{t \in E : \nu \mid F_t^* \neq 0\}$ is countable because $\{F_t^* : t \in E\}$ is a family of pairwise-disjoint, non-empty, compact sets. Hence there exists $t \in E$ such that $\nu \mid F_t^* = 0 \ \ (\nu \in V)$. Choose $p \in F_t^*$. It follows from Lemma 2.6 that $p \,\Box\, \nu = \nu \,\Box\, p \ \ (\nu \in V)$. By the assumption, $p \in \mathfrak{Z}(M(X))$, a contradiction of Proposition 2.10.

It follows that there is no DTC set for $M(X)$ with $|V| < \kappa$. □

Corollary 3.5 *Suppose that the set* E *is uncountable. Then there is no countable* DTC *set for* $M(X)$. □

Example 3.6 (i) Consider the semigroup $(\mathbb{N}, \wedge)$. Then we see that

$$\mu \,\Box\, \nu = \varphi_0(\nu)\, \mu \quad (\mu \in M(\beta\,\mathbb{N}),\ \nu \in M(\mathbb{N}^*))\,.$$

It follows that any two distinct points in $\mathbb{N}^*$ form a two-point DTC set for $(\mathbb{N}, \wedge)$. Further, $\ell^1(\mathbb{N}, \wedge)$ is strongly Arens irregular and any two distinct points in $\mathbb{N}^*$ form a DTC set for $M(\beta\,\mathbb{N})$.

(ii) Consider the semigroup $(\mathbb{Q}, \wedge)$. By Proposition 3.1, there is no countable DTC set for this semigroup. By Corollary 2.15, $\ell^1(\mathbb{Q}, \wedge)$ is neither Arens regular nor strongly Arens irregular, and, by Corollary 3.5, there is no countable DTC for $M(X)$.

(iii) Consider the subset S of $T := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ that consists of numbers of the form $n - x$, where $n \in \mathbb{Z}$ and $x \in \{1/2, 1/4, 1/8, \dots\}$. Then the corresponding set E is equal to $\{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$, a countable set, and so, by Theorem 3.2, the semigroup algebra $\ell^1(S)$ is strongly Arens irregular. By Proposition 3.1, there is no finite DTC set for the semigroup S, but, by the argument of Theorem 3.2, $M(X)$ has a two-element DTC set in $M(X^*)^+$.

(iv) Consider the semigroup $S = T = ([0, \kappa], \wedge)$, where κ is a cardinal with $\kappa \geq \aleph_1$, so that the corresponding set E has cardinality κ. Since T is scattered, the algebra $\ell^1(S)$ is strongly Arens irregular by Theorem 2.14. By Proposition 3.4, there is no DTC set for $M(X)$ with cardinality strictly less than κ. This shows that the cardinality of a DTC set can be arbitrarily large, even when $\ell^1(S)$ is strongly Arens irregular. □

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