Semigroup Forum (2022) 104:415–447
https://doi.org/10.1007/s00233-021-10247-1

RESEARCH ARTICLE

Density of type-dependent sets in Krull monoids with analytic structure

Maciej Radziejewski[1]

Received: 6 October 2021 / Accepted: 1 December 2021 / Published online: 30 December 2021

Abstract
We describe structural and quantitative properties of type-dependent sets in monoids with suitable analytic structure, including simple analytic monoids, introduced by Kaczorowski (Semigroup Forum 94:532–555, 2017. https://doi.org/10.1007/s00233-016-9778-9), and formations, as defined by Geroldinger and Halter-Koch (Non-unique factorizations, Chapman and Hall, Boca Raton, 2006. https://doi.org/10.1201/9781420003208). We propose the notions of rank and degree to measure the size of a type-dependent set in structural terms. We also consider various notions of regularity of type-dependent sets, related to the analytic properties of their zeta functions, and obtain results on the counting functions of these sets.

Keywords Semigroups with divisor theory · Analytic monoids · Formations · Type-dependent sets · Subsets defined by factorization properties

1 Introduction

The goal of quantitative factorization theory is to describe, as precisely as possible, the distribution of elements of a monoid subset A defined by factorization-related conditions. The monoid must be equipped with a norm function $\|\cdot\|$ making it possible to define the counting function

$$A(x) = |\{a \in X : \|a\| \le x\}| . \tag{1}$$

In the present paper we study the growth of $A(x)$ for a class of reduced monoids with divisor theory $\mathcal{S} \subseteq \mathcal{F}(\mathcal{P})$, called shifted formations, defined in Sect. 2. The inclusion

Communicated by László Márki.

✉ Maciej Radziejewski
maciejr@amu.edu.pl

[1] Faculty of Mathematics and Computer Science, Adam Mickiewicz University, Poznań, ul. Uniwersytetu Poznańskiego 4, 61-614 Poznan, Poland

of an element a in $A \subseteq \mathcal{F}(\mathcal{P})$ depends on the type of factorization of a in $\mathcal{F}(\mathcal{P})$. The kind of result that we expect is that $A(x)$ follows an asymptotic law of the type

$$A(x) \sim Cx^b (\log x)^c (\log\log x)^d, \qquad x \to \infty, \tag{2}$$

where the numbers b, c and d depend on the structural properties of A. The goal of this paper is to introduce appropriate functions describing the structure of A, to determine their properties, and to show that versions of (2) hold for certain classes of subsets of the divisor monoid, that we call *regular*, *almost-regular*, *semi-regular*, etc., cf. Theorem 35. We attempt to provide an easily extendible general framework to show regularity for a variety of sets.

As examples of applications we show: a technical, but still flexible result (Proposition 37) about the regularity of sets of a specific form, and the asymptotics for the counting functions of four specific sets, given below. The result on F_k is classical, although it was never considered in this particular setting.

Theorem 1 *Let $\mathcal{S}$ be a shifted formation with a principal shift $\lambda \geq 0$, S the set generated by absolutely irreducible elements of $\mathcal{S}$ (i.e. irreducibles which are powers of prime divisors* [3]*), S_1 the set of products of distinct absolutely irreducible elements of $\mathcal{S}$, and E_k the set of k-powerful divisors, where $k \in \mathbf{N}$. We have*

$$S(x) \sim Cx^{(1+\lambda)}(\log x)^{-1+1/h}, \qquad x \to \infty,$$
$$S_1(x) \sim C'x^{(1+\lambda)}(\log x)^{-1+1/h}, \qquad x \to \infty,$$

and

$$E_k(x) \sim C''x^{(1+\lambda)/k}, \qquad x \to \infty$$

for some $C, C', C'' > 0$.

Theorem 2 (cf. Narkiewicz [6]) *Let $\mathcal{S}$ be a shifted formation with a principal shift $\lambda \geq 0$ and F_k the set of elements of $\mathcal{S}$ with at most k distinct factorizations to irreducibles in $\mathcal{S}$, where $k \in \mathbf{N}$. We have*

$$F_k(x) \sim Cx^{(1+\lambda)}(\log x)^{-1+1/h}(\log\log x)^{N_k}, \qquad x \to \infty,$$

for some $C > 0$, and a non-negative integer N_k.

The setting of shifted formations, arguably somewhat artificial, was chosen, because the proofs work essentially without change for simple analytic monoids, as defined by Kaczorowski [5], and for formations, considered, among others, by Geroldinger and Halter-Koch [2, Section 8.3]. Shifted formations include both, allowing us to avoid repetition. Of course, more precise information on the counting functions (such as bounds for and oscillations of the error term) would require stronger assumptions, similar to those in the definition of an analytic monoid, or an L-semigroup [8]. On the other hand, weaker assumptions, such as to include non-simple analytic monoids, would make the quantitative theory more complex. The author hopes to address this problem in a future paper.

Quantitative factorization theory was initiated by Fogels [1] and further developed by Narkiewicz and other authors, cf. Geroldinger and Halter-Koch [2, Chapter 9] and the references cited there. Narkiewicz [6] was the first to treat type-dependent sets, including F_k, that do not necessarily belong to the smaller class of Ω-sets (see the end of Sect. 3 for a definition of Ω-sets). He introduced the notion of depth and proposed an inductive reasoning to find the asymptotics of the counting function of a type-dependent set of height 1 whose elements have bounded depth. In the language of the present paper the sets treated in [6], subsets of rings of integers in algebraic number fields, are algebraic products of: the set of elements with all prime divisors in the principal class, and an arbitrary set of height 1 and rank 0 whose elements have no principal divisors. The idea to use induction over depth is also used in the present paper (in Lemma 22). Unfortunately, Lemma 2 in [6] only applies to a set of elements of a single type. In the proof of Corollary 4 in [6], where sets with an infinite number of types may arise, the author only mentions that they can be dealt with in the same way. Kaczorowski [4, Theorem 3] obtained the asymptotics for the counting function of S (another "properly" type-dependent set), in the context of algebraic number fields, with C explicitly determined, and an upper bound for the error term.

Geroldinger and Halter-Koch [2, Theorem 9.1.2] gave a more general result on type-dependent sets. They introduced the height, i.e. the most important of the metrics of type-dependent sets, although not explicitly as set metrics, cf. Definition 9.1.1 and the formula for e_0 in Theorem 9.1.2. They developed a number of ideas from [6], adapting them to the setting of (quasi-)formations and sets of height greater than 1. Sets treated in [2, Theorem 9.1.2] are algebraic products of three type-dependent components (with elements in distinct components relatively prime) that we describe as follows: a set C of height e_0, a set B_1 of all elements with all prime divisors in a subset $U_1 \subseteq \mathrm{Cl}(\mathcal{S})$ and exponents divisible by e_1, and a set B_2 of all elements with all prime divisors in a subset $U_2 \subseteq \mathrm{Cl}(\mathcal{S})$ and exponents greater or equal to e_2. Moreover $e_2 > \min(e_0, e_1)$ and $\mathrm{rk}(C) = 0$ unless $e_0 > e_1$. Bearing in mind the different setting, Theorem 9.1.2 can be applied directly to the set F_k of Theorem 2, but not to S, S_1 or E_k of Theorem 1. Asymptotic lower and upper bounds for $S(x)$ of the correct order are given instead, cf. [2, p. 633]. The set E_k is of the same general shape as required in Theorem 9.1.2, with $B_2 = E_k$ and $C = B_1 = \{1\}$, however, it does not meet the technical assumption $e_2 > \min(e_0, e_1)$. In their proof of [2, Theorem 9.1.2] the authors attempted to close the gap left in [6]. Unfortunately, their argument also contains gaps, namely, on page 619, line 3, they do not take into account the repetitions that may occur when multiplying two Dirichlet series, and a similar problem occurs in the fourth display from bottom on the same page. For comparison, our Proposition 37 implies Theorems 1 and 2. It does not quite re-prove [2, Theorem 9.1.2], because we do not determine the constant factors in the asymptotic relationship $\sim$.

The paper is organized as follows. Section 2 contains known definitions and notation. Section 3 provides the language to describe the structural "size" of a type-dependent set. Notions of *rank* and *degree*, previously defined for Ω-sets, are extended to type-dependent sets. We show the basic properties of rank, degree and height, and how they behave under set operations like disjoint union and algebraic product. In

Sect. 4 we define regular type-dependent sets, whose zeta functions have appropriate analytic properties, agreeing with the set's height, rank and degree. We also consider other, weaker and stronger regularity properties, and show how they behave under basic set operations. In Sect. 5 we show that type-dependent sets of rank 0 are regular (Theorem 23). In Sect. 6 we construct a large family of regular sets (Theorem 30).

In Sect. 7 we apply the results of previous sections to obtain asymptotics for various counting functions. Theorem 35 gives asymptotics for the counting functions of almost regular sets and bounds for semi-regular sets. In the same theorem we also give upper and lower bounds for $A(x)$ for general type-dependent sets, using Theorems 30 and 23 respectively. It follows from Theorem 36 that these bounds cannot be improved in general, at least not using the language of the present paper. We also prove Proposition 37 and Theorems 1 and 2.

2 Preliminaries

We denote by $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{R}$ and $\mathbf{C}$ respectively the sets of positive integers, non-negative integers, integers, real numbers and complex numbers, and by $s = \sigma + it$ a complex variable with real part σ and imaginary part t. We make use of Landau's O and o, and Vinogradov's $\ll$ notation. We write $f \asymp g$ for $f \ll g$ and $g \ll f$. We write $f \sim g$ for $f(x) = g(x)(1 + o(1))$, $x \to +\infty$. Function support is denoted as Supp, symmetric difference of sets as $\triangle$, the cardinality of A as $|A|$ or $\#A$. The infimum of the empty set is assumed to equal $+\infty$. When G is a finite abelian group we let E denote the neutral element, $\mathrm{ord}(X)$ the order of $X \in G$, and $\widehat{G}$ the group of characters of G, with χ_0 for the trivial character. For $U \subseteq G$ the subgroup generated by U is denoted as $\langle U\rangle$, while $\langle \chi \mid U\rangle = |G|^{-1} \sum_{X \in U} \chi(X)$ for $\chi \in \widehat{G}$. We compare functions $\alpha : G \to \mathbf{N}_0$ using the product order, so $\beta \le \alpha$ means $\beta : G \to \mathbf{N}_0$ and $\beta(X) \le \alpha(X)$ for all $X \in G$. For comparing pairs and triples we use lexicographic order with the first term always being the most significant.

Suppose $\lambda \ge 0$ and $\mathcal{S}$ is a Krull monoid contained in a free semigroup $\mathcal{F}(\mathcal{P})$ generated by a set of primes $\mathcal{P}$ such that:

(i) for every $a, b \in \mathcal{S}$ divisibility $a \mid b$ in $\mathcal{S}$ is equivalent to $a \mid b$ in $\mathcal{F}(\mathcal{P})$,
(ii) every $p \in \mathcal{P}$ $\gcd(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in \mathcal{S}$.

Hence the inclusion map $\mathcal{S} \subseteq \mathcal{F}(\mathcal{P})$ is a divisor theory for $\mathcal{S}$. Let G be the quotient of the groups generated by $\mathcal{F}(\mathcal{P})$ and $\mathcal{S}$. The intersection of an element of G with $\mathcal{F}(\mathcal{P})$ is called a divisor class. We let $\mathrm{Cl}(\mathcal{S})$ denote the divisor class group, i.e. the set of divisor classes with multiplication induced by that in G. We let h denote the number of divisor classes and E the principal class. For $\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}$ and $a \in \mathcal{F}(\mathcal{P})$ we write $\chi(a)$ instead of $\chi([a])$. We assume:

(iii) the order h of the class group $\mathrm{Cl}(\mathcal{S})$ is finite.

Moreover, there is a norm function $\|\cdot\| : \mathcal{F}(\mathcal{P}) \to \mathbf{N}$ such that:

(iv) $\|\cdot\|$ is a multiplicative homomorphism, i.e. $\|1\| = 1$ and $\|ab\| = \|a\| \, \|b\|$ for all $a, b \in \mathcal{F}(\mathcal{P})$,
(v) $\|a\| > 1$ for all $a \in \mathcal{F}(\mathcal{P}) \setminus \{1\}$,

(vi) for every $x > 0$ and every $\varepsilon > 0$ the set $\{a \in \mathcal{F}(\mathcal{P}) : \|a\| \le x\}$ has $\ll_\varepsilon x^{1+\lambda+\varepsilon}$ elements.

In particular, for every $\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}$ the Dirichlet series

$$L(s, \mathcal{S}, \chi) = \sum_{a \in \mathcal{F}(\mathcal{P})} \frac{\chi(a)}{\|a\|^s}$$

is absolutely convergent for $\sigma > 1 + \lambda$. Moreover, we assume that

(vii) the functions $L(s, \mathcal{S}, \chi)$, for $\chi \ne \chi_0$, and the function $(s - 1 - \lambda)L(s, \mathcal{S}, \chi_0)$, have holomorphic, non-vanishing extensions to $\sigma \ge 1 + \lambda$.

If conditions $(i)-(vii)$ are satisfied, we call $\mathcal{S}$ a *shifted formation* with a principal shift λ. It follows from the properties of simple analytic monoids [5] that a simple analytic monoid with a principal shift λ is also a shifted formation with a principal shift λ. Every formation [2] is a shifted formation with a principal shift 0. We remark that condition (vii) may be relaxed, to include quasi-formations. In that case one needs to assume the same functions mentioned in (vii) to be holomorphic and non-vanishing in $\{1 + \lambda\} \cup \{s : \sigma > 1 + \lambda\}$ only, replace $\sim$ with $\asymp$ in Theorems 1, 2 and 35, moreover, in the proof of Theorem 35, one needs to claim that $\widetilde{H}_i(s)$ in (34) are holomorphic in $\{\sigma_0\} \cup \{s : \sigma > \sigma_0\}$ only.

When $A \subseteq \mathcal{F}(\mathcal{P})$ we consider the counting function (1) and the zeta function

$$Z_A(s) = \sum_{a \in A} \|a\|^{-s}, \qquad \sigma > 1 + \lambda.$$

For $f : A \to \mathbf{C}$ such that for every $\varepsilon > 0$ we have $f(a) \ll \|a\|^\varepsilon$ on A, we also put

$$Z_A(s, f) = \sum_{a \in A} f(a)\, \|a\|^{-s}, \qquad \sigma > 1 + \lambda.$$

We make use of auxiliary functions

$$P_X(s) = \sum_{p \in X \cap \mathcal{P}} \|p\|^{-s}, \qquad \sigma > 1 + \lambda,\ X \in \mathrm{Cl}(\mathcal{S}).$$

For $e \in \mathbf{N}$ we let $\mathcal{A}_e$ denote the set of Dirichlet series absolutely convergent for $\sigma > (1 + \lambda)/e$. We also use the classical function $\omega(a) = \sum_{\substack{p \in \mathcal{P} \\ p \mid a}} 1$. The following lemma shows, in particular, that $P_X \notin \mathcal{A}_2$, so the set $\mathcal{P} \cap X$ is infinite for every $X \in \mathrm{Cl}(\mathcal{S})$. Its countability follows from condition (vi).

Lemma 3 *Let* $X \in \mathrm{Cl}(\mathcal{S})$. *We have*

$$P_X(s) = \frac{1}{h} \sum_{\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\chi(X)} \log L(s, \mathcal{S}, \chi) + R_X(s), \qquad \sigma > 1 + \lambda,$$

for some $R_X \in \mathcal{A}_2$.

Proof We have

$$P_X(s) = \frac{1}{h} \sum_{\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\chi(X)} \sum_{p \in \mathcal{P}} \chi(p) \, \|p\|^{-s}, \qquad \sigma > 1 + \lambda$$

and

$$\log L(s, \mathcal{S}, \chi) = \sum \log \left(\left(1 - \chi(p) \, \|p\|^{-s}\right)^{-1} \right)_{p \in \mathcal{P}} \qquad \sigma > 1 + \lambda.$$

The assertion follows from $\log \left(\left(1 - \chi(p) \, \|p\|^{-s}\right)^{-1} \right) - \|p\|^{-s} \chi(p) \ll \|p\|^{-2\sigma}$, $\sigma > (1 + \lambda)/2$. □

Factorization-related properties of an element $a \in \mathcal{S}$, such as uniqueness of factorization to irreducibles, factorization lengths, etc., depend, in general, on its factorization in $\mathcal{F}(\mathcal{P})$. Each element $a \in \mathcal{F}(\mathcal{P})$ has a unique factorization

$$a = \prod_{p \in \mathcal{P}} p^{v_p(a)},$$

where almost all the $v_p(a)$ vanish. We say that elements $a, b \in \mathcal{F}(\mathcal{P})$ have the same factorization type, and write $a \approx b$, if for each $X \in \mathrm{Cl}(\mathcal{S})$ there is a permutation π of $\mathcal{P} \cap X$ such that

$$v_p(a) = v_{\pi(p)}(b)$$

for all $p \in \mathcal{P} \cap X$. We can think of "factorization type" as an equivalence class of the relation $\approx$. This definition is equivalent to that of a normalized type given by Geroldinger and Halter-Koch [2, Definition 3.5.7], cf. also Narkiewicz [6, (1) and 3.(a)]. We call a subset $A \subseteq \mathcal{F}(\mathcal{P})$ type-dependent if it is closed upon $\approx$. We say that A is trivial if $A = \emptyset$ or $A = \{1\}$.

3 Metrics of a type-dependent set

For $a \in \mathcal{F}(\mathcal{P})$ we call

$$\gamma(a) = \inf_{\substack{p \in \mathcal{P} \\ v_p(a) \geq 1}} v_p(a)$$

the (minimal) height of a. This is similar to [2, Definition 9.1.1], except $\gamma(1) = +\infty$. For $A \subseteq \mathcal{F}(\mathcal{P})$ we define the (minimal) height as

$$\gamma(A) = \inf_{a \in A} \gamma(a).$$

The height of A is therefore finite if and only if A is non-trivial. Height should be thought of in terms of sparsity, not size of a set. Indeed, the height of a subset is always less or equal to the height of a superset. We define a related measure

$$\mathrm{ed}(A) = \frac{1+\lambda}{\gamma(A)}$$

and call it the exponential density of A. Following [2, Definition 9.1.1] we put, for $a \in \mathcal{F}(\mathcal{P})$, $e \in \mathbf{N}$, $X \in \mathrm{Cl}(\mathcal{S})$,

$$\delta_{e,X}(a) = \left|\left\{p \in \mathcal{P} \cap X : v_p(a) = e\right\}\right|.$$

Narkiewicz [6] considered $\sum_{X \neq E} \delta_{1,X}(a)$ and called it the depth of a. For $e \in \mathbf{N}$, $U \subseteq \mathrm{Cl}(\mathcal{S})$, and $\alpha : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$ such that $\alpha(X) = 0$ for all $X \in U$, let

$$\mathbf{\Omega}_e(U, \alpha) = \left\{a \in \mathcal{F}(\mathcal{P}) : \gamma(a) \geq e \text{ and } \delta_{e,X}(a) = \alpha(X) \text{ for all } X \in \mathrm{Cl}(\mathcal{S}) \setminus U\right\}.$$

We call such sets cubes. In particular

$$\mathbf{\Omega}_e(\mathrm{Cl}(\mathcal{S}), 0) = \{a \in \mathcal{F}(\mathcal{P}) : \gamma(a) \geq e\}$$

and

$$\mathbf{\Omega}_e(\emptyset, 0) = \{a \in \mathcal{F}(\mathcal{P}) : \gamma(a) \geq e+1\} = \mathbf{\Omega}_{e+1}(\mathrm{Cl}(\mathcal{S}), 0). \tag{3}$$

It follows from the infinitude of primes in each class that $\gamma(\mathbf{\Omega}_e(U, \alpha)) = e$ whenever $U \neq \emptyset$ or $\alpha \neq 0$. If $\alpha \neq 0$, we have $\gamma(a) = e$ for all $a \in \mathbf{\Omega}_e(U, \alpha)$. If $\alpha = 0$, then $\mathbf{\Omega}_e(U, \alpha)$ contains all elements of height $\geq e+1$.

Let $A \subseteq \mathcal{F}(\mathcal{P})$ be a non-trivial, type-dependent set of height e. We define the rank of A, denoted $\mathrm{rk}(A)$, as the smallest r such that A is contained in a finite union

$$\bigcup_{i=1}^{m} \mathbf{\Omega}_e(U_i, \alpha_i), \tag{4}$$

where $|U_i| \leq r$ for $i = 1, \ldots, m$. We also define the degree of A, denoted $\deg(A)$, as the supremum of all $d \in \mathbf{N}_0$ such that the intersection $\mathbf{\Omega}_e(U, \alpha) \cap A$ is of the same height and rank as A for some U and α such that $\sum_X \alpha(X) = d$. For trivial sets we put:

$$\mathrm{rk}(\{1\}) = 0, \qquad \deg(\{1\}) = 0,$$
$$\mathrm{rk}(\emptyset) = -\infty, \qquad \deg(\emptyset) = +\infty.$$

In [7] the present author defined rank and degree for a subclass of type-dependent sets called Ω-sets. We show at the end of the section that the notions defined here extend the previous ones. Let

$$\mathrm{metrics}(A) = (\mathrm{ed}(A), \mathrm{rk}(A), \deg(A)).$$

We use the standard lexicographic ordering on $[0, +\infty) \times (\{-\infty\} \cup \mathbf{N}_0) \times (\mathbf{N}_0 \cup \{+\infty\})$, with the first coordinate most significant, to compare such triples.

Fact 4 *If $A \subseteq B \subseteq \mathcal{F}(\mathcal{P})$ and A and B are type-dependent, then*

$$\mathrm{metrics}(A) \leq \mathrm{metrics}(B).$$

Proof Follows from the definition of height, rank and degree. □

Lemma 5 *The intersection of two cubes*

$$A = \mathbf{\Omega}_e(U, \alpha) \cap \mathbf{\Omega}_{e'}(U', \alpha')$$

is either empty or is itself a cube. If $e > e'$ and $\alpha' \neq 0$, then $A = \emptyset$. If $e > e'$ and $\alpha' = 0$, then $A = \mathbf{\Omega}_e(U, \alpha)$. If $e = e'$, then A is non-empty if and only if $\alpha(X) = \alpha'(X)$ for every $X \in \mathrm{Cl}(\mathcal{S}) \setminus \big(U \cup U'\big)$. In that case $A = \mathbf{\Omega}_e(U \cap V, \beta)$, where $\beta(X) = \alpha(X)$ for $X \in \mathrm{Cl}(\mathcal{S}) \setminus U$ and $\beta(X) = \alpha'(X)$ for $X \in \mathrm{Cl}(\mathcal{S}) \setminus U'$.

Proof Follows from the definition of a cube. □

Fact 6 *If $A \subseteq \mathcal{F}(\mathcal{P})$ is a non-trivial type-dependent set of height e and rank r, then* $\deg(A)$ *equals the smallest d' such that A is contained in a finite union* (4) *with $|U_i| \leq r$ for all i and $\sum_{X \notin U} \alpha_i(X) \leq d'$ for all i with $|U_i| = r$. In particular* $\deg(A) < +\infty$.

Proof Let $d = \deg(A)$, d' be as in the assertion, and suppose A is contained in (4) with $|U_i| \leq r$ for all i and $d' = \max_{i:|U_i|=r} \sum_{X \notin U_i} \alpha_i(X)$. We may assume that the choice of pairs (U_i, α_i) is minimal, i.e. the number of i with $|U_i| = r$ is the smallest possible, and each intersection $A \cap \mathbf{\Omega}_e(U_i, \alpha_i)$ is non-empty. Moreover we can suppose that $|U_1| = r$ and $\sum_{X \notin U_1} \alpha_1(X) = d'$. First we show that

$$\mathrm{rk}\,(A \cap \mathbf{\Omega}_e(U_1, \alpha_1)) = r,$$

implying that $d \geq d'$. Indeed, if $\mathrm{rk}\,(A \cap \mathbf{\Omega}_e(U_1, \alpha_1)) < r$, then $A \cap \mathbf{\Omega}_e(U_1, \alpha_1) \subseteq \bigcup_{i=1}^{n} \mathbf{\Omega}_e(V_i, \beta_i)$ for some V_i, β_i such that $|V_i| \leq r - 1$ for all i. Then

$$A \subseteq \left(\bigcup_{i=1}^{n} \mathbf{\Omega}_e(V_i, \beta_i)\right) \cup \left(\bigcup_{i=2}^{m} \mathbf{\Omega}_e(U_i, \alpha_i)\right),$$

contrary to the minimality of the choice of pairs (U_i, α_i). To see that $d \leq d'$ let U, α be such that $\mathrm{rk}\,(\mathbf{\Omega}_e(U, \alpha) \cap A) = r$. We have

$$\mathbf{\Omega}_e(U, \alpha) \cap A \subseteq \bigcup_{i=1}^{m} (\mathbf{\Omega}_e(U, \alpha) \cap \mathbf{\Omega}_e(U_i, \alpha_i)).$$

Each non-empty intersection $\mathbf{\Omega}_e(U, \alpha) \cap \mathbf{\Omega}_e(U_i, \alpha_i)$ is of the form $\mathbf{\Omega}_e(U \cap U_i, \alpha_i')$, and we have $|U \cap U_i| < r$ unless

$$|U_i| = r, \qquad U_i \subseteq U, \qquad \text{and} \qquad \alpha(X) = \alpha_i(X) \text{ for all } X \in \mathrm{Cl}(\mathcal{S}) \setminus U. \tag{5}$$

Since rk $(\mathbf{\Omega}_e(U, \alpha) \cap A) = r$ we must have (5) for at least one i, impying that $\sum_{X \notin U} \alpha(X) \leq \sum_{X \notin U_i} \alpha_i(X) \leq d'$. □

We call

$$A \subseteq \bigcup_{i=1}^{m} \mathbf{\Omega}_e(U_i, \alpha_i) \tag{6}$$

a fair covering if $\max_i |U_i| = \mathrm{rk}(A)$, $\max_{i:|U_i|=\mathrm{rk}(A)} \sum_{X \notin U_i} \alpha_i(X) \leq \deg(A)$. It follows from Fact 6 that every non-trivial type-dependent set has a fair covering and that in fact $\max_{i:|U_i|=\mathrm{rk}(A)} \sum_{X \notin U_i} \alpha_i(X) = \deg(A)$.

Fact 7 *Let $A \subseteq \mathcal{F}(\mathcal{P})$ be non-trivial. We have $\mathrm{rk}(A) \in \{0, \ldots, h\}$ and $\deg(A) \in \mathbf{N}_0$. The rank and degree cannot both be equal to zero.*

Proof Let $e = \gamma(A)$. The assertion follows from the existence of a fair covering (6). If $U_i = \emptyset$ and $\alpha_i = 0$ for all i, then $\gamma(a) > e$ for all $a \in A$, contrary to the assumption. □

Fact 8 *Let $A, B \subseteq \mathcal{F}(\mathcal{P})$ be type-dependent. Then*

$$\mathrm{metrics}(A \cup B) = \max\left(\mathrm{metrics}(A), \mathrm{metrics}(B)\right).$$

Proof The fact that $\gamma(A \cup B) = \min(\gamma(A), \gamma(B))$ follows from the definition of height. Suppose, as we may, that $A \neq B$ and $A, B \neq \emptyset$, moreover

$$\mathrm{metrics}(A) \leq \mathrm{metrics}(B) = \max\left(\mathrm{metrics}(A), \mathrm{metrics}(B)\right).$$

This implies that B is non-trivial. Let $e = \gamma(B)$, $r = \mathrm{rk}(B)$ and $d = \deg(B)$. By Fact 4 we have $\mathrm{metrics}(A \cup B) \geq \mathrm{metrics}(B)$, so it is enough to show the converse. Suppose

$$B \subseteq \bigcup_{i=1}^{n} \mathbf{\Omega}_e(V_i, \beta_i)$$

is a fair covering. If $\gamma(A) > e$, then $A \cup B \subset \mathbf{\Omega}_e(\emptyset, 0) \cup \bigcup_{i=1}^{n} \mathbf{\Omega}_e(V_i, \beta_i)$, so $\mathrm{metrics}(A \cup B) \leq \mathrm{metrics}(B)$. Otherwise $\gamma(A) = e$ and $\mathrm{rk}(A) \leq r$. Moreover there exists a fair covering (6). The covering

$$A \cup B \subseteq \left(\bigcup_{i=1}^{m} \mathbf{\Omega}_e(U_i, \alpha_i)\right) \cup \left(\bigcup_{i=1}^{n} \mathbf{\Omega}_e(V_i, \beta_i)\right)$$

shows that $\mathrm{rk}(A \cup B) \le r$, so $\mathrm{rk}(A \cup B) = r$. If $\mathrm{rk}(A) = r$, then $\mathrm{metrics}(A) \le \mathrm{metrics}(B)$ implies that $\deg(A) \le \deg(B)$, so

$$\max\left(\max_{i:|U_i|=r} \sum_{X\notin U_i} \alpha_i(X), \max_{i:|V_i|=r} \sum_{X\notin V_i} \beta_i(X)\right) = d,$$

hence $\deg(A \cup B) \le d$. Otherwise $\mathrm{rk}(A) < r$, so $|U_i| < r$ for all i and $\max_{i:|V_i|=r} \sum_{X\notin V_i} \beta_i(X) = d$ implies $\deg(A \cup B) \le d$. □

Fact 9 *Let $A \subseteq \mathcal{F}(\mathcal{P})$ be a type-dependent set, $m \in \mathbf{N}$, and*

$$B = \left\{a^m : a \in A\right\}.$$

Then $\gamma(B) = m\gamma(A)$, $\mathrm{rk}(B) = \mathrm{rk}(A)$ *and* $\deg(B) = \deg(A)$.

Proof Let $e = \gamma(A)$. We may suppose that A is non-trivial. The equality $\gamma(B) = me$ follows from the definition. The other equalities follow from the fact that $a \in \mathbf{\Omega}_e(U, \alpha)$ is equivalent to $a^m \in \mathbf{\Omega}_{me}(U, \alpha)$, so (6) is a fair covering of A if and only if

$$B \subseteq \bigcup_{i=1}^{m} \mathbf{\Omega}_{me}(U_i, \alpha_i)$$

is a fair covering of B. □

Proposition 10 *Let $A, B \subseteq \mathcal{F}(\mathcal{P})$ be non-empty, type-dependent sets such that $\gamma(A) \ge \gamma(B)$. Then $\gamma(AB) = \gamma(B)$. If* $\gcd(a, b) = 1$ *for all $a \in A$, $b \in B$, then* $\mathrm{rk}(AB) = r + \mathrm{rk}(B)$*, and* $\deg(AB) = d + \deg(B)$*, where* $r = \mathrm{rk}(A)$ *and* $d = \deg(A)$ *if $\gamma(A) = \gamma(B)$, and $r = 0$ and $d = 0$ if $\gamma(A) > \gamma(B)$.*

Proof The assertion is easy to check if A is a trivial set, so we assume that A is non-trivial, hence so is B. The equality $\gamma(AB) = \gamma(B)$ follows by taking any $b \in B$ with $\gamma(b) = \gamma(B)$, and any $a \in A$ relatively prime to b (again, it exists by the infinitude of primes in each class and the fact that A is type-dependent). Then $\gamma(ab) = \gamma(b)$ and $ab \in AB$.

Suppose $\gcd(a, b) = 1$ for all $a \in A$, $b \in B$. If $p, q \in \mathcal{P}$ are such that $[p] = [q]$, $p \mid a$ for some $a \in A$, and $q \mid b$ for some $b \in B$, then there exists $a' \in \mathcal{F}(\mathcal{P})$, $a' \approx a$, such that $q \mid a$. Since A is type-dependent, we have $a' \in A$ and $q \mid \gcd(a', b)$, contrary to the assumptions. Therefore the sets $\widetilde{U}$ and $\widetilde{V}$, of classes of prime divisors of elements of A and B, respectively, are disjoint. Let $e = \gamma(B)$. Consider a fair covering

$$B \subseteq \bigcup_{j=1}^{n} \mathbf{\Omega}_e(V_j, \beta_j).$$

If $\gamma(A) = e$, let (6) be a fair covering, otherwise let $m = 1$, $U_1 = \emptyset$, and $\alpha_1 = 0$. In any case we have

$$r = \max_i |U_i|, \qquad \text{and} \qquad d = \max_{i:|U_i|=r} \sum_{X \notin U_i} \alpha_i(X).$$

Our goal is to show that $\gamma(AB) = e$, $\mathrm{rk}(AB) = r + \mathrm{rk}(B)$, and $\deg(AB) = d + \deg(B)$. We may assume that $U_i \subseteq \widetilde{U}$ and $V_j \subseteq \widetilde{V}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. Since $\mathrm{Supp}(\alpha_i) \subseteq \widetilde{U} \setminus U_i$ and $\mathrm{Supp}(\beta_j) \subseteq \widetilde{V} \setminus V_j$, we have

$$\mathbf{\Omega}_e(U_i, \alpha_i)\mathbf{\Omega}_e(V_j, \beta_j) \subseteq \bigcup_{\eta \le \alpha_i + \beta_j} \mathbf{\Omega}_e(U_i \cup V_j, \eta)$$

for all i, j. Hence

$$AB \subseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \bigcup_{\substack{\alpha' \le \alpha_i \\ \beta' \le \beta_i}} \mathbf{\Omega}_e(U_i \cup V_j, \alpha' + \beta'). \tag{7}$$

This implies $(\mathrm{rk}(AB), \deg(AB)) \le (r + \mathrm{rk}(B), d + \deg(B))$ in the lexicographic order, where the first coordinate is more significant. It also follows from (7) that if AB is covered with sets of the form $\mathbf{\Omega}_e(W, \eta)$, we can replace each $\mathbf{\Omega}_e(W, \eta)$ with the union of

$$\mathbf{\Omega}_e(W, \eta) \cap \mathbf{\Omega}_e(U_i \cup V_j, \alpha' + \beta'), \quad i = 1, \ldots m, j = 1, \ldots, n, \alpha' \le \alpha_i, \beta' \le \beta_i, \tag{8}$$

skipping the empty summands. If non-empty, the intersection (8) equals $\mathbf{\Omega}_e((W \cap U_i) \cup (W \cap V_j), \alpha'' + \beta'')$, where

$$\alpha''(X) = \begin{cases} \eta(X), & X \in \widetilde{U} \setminus W, \\ \alpha'(X), & X \in \widetilde{U} \setminus U_i, \\ 0, & X \in (W \cap U_i) \cup \left(\mathrm{Cl}(\mathcal{S}) \setminus \widetilde{U}\right), \end{cases}$$

$$\beta''(X) = \begin{cases} \eta(X), & X \in \widetilde{V} \setminus W, \\ \beta'(X), & X \in \widetilde{V} \setminus V_j, \\ 0, & X \in (W \cap V_i) \cup \left(\mathrm{Cl}(\mathcal{S}) \setminus \widetilde{V}\right). \end{cases}$$

Moreover $\sum_X \alpha''(X) \le \sum_X \alpha_i(X)$ unless $W \cap U_i \subsetneq U_i$, likewise $\sum_X \beta''(X) \le \sum_X \beta_j(X)$ unless $W \cap V_j \subsetneq V_j$, and finally $\sum_X \alpha''(X) + \beta''(X) \le \sum_X \eta(X)$ unless $(W \cap U_i) \cup (W \cap V_j) \subsetneq W$. Therefore there exists a fair covering of AB of the form

$$AB \subseteq \bigcup_{k=1}^{l} \mathbf{\Omega}_e(U'_k \cup V'_k, \alpha'_k + \beta'_k), \tag{9}$$

where, for all k, we have $U'_k \subseteq \widetilde{U}$, $V'_k \subseteq \widetilde{V}$, $\mathrm{Supp}(\alpha'_k) \subseteq \widetilde{U} \setminus U_k$, $\mathrm{Supp}(\beta'_k) \subseteq \widetilde{V} \setminus V_k$, and

$$\left(\left| U'_k \right|, \sum_X \alpha'_k(X) \right) \leq (r, d) \quad \text{and} \quad \left(\left| V'_k \right|, \sum_X \beta'(X) \right) \leq (\mathrm{rk}(B), \deg(B)) \tag{10}$$

in the lexicographic order. Let I and J be the sets of those k, for which the first, respectively the second, inequality in (10) is sharp. By Fact 6 (and by $I = \emptyset$ in the case $\gamma(A) > e$) there exist

$$a \in A \setminus \bigcup_{k \in I} \mathbf{\Omega}_e(U'_k, \alpha'_k)$$

and

$$b \in B \setminus \bigcup_{k \in J} \mathbf{\Omega}_e(V'_k, \beta'_k).$$

Fix k such that $ab \in \mathbf{\Omega}_e(U'_k \cup V'_k, \alpha'_k + \beta'_k)$. We have $a \in \mathbf{\Omega}_e(U'_k, \alpha'_k)$ and $b \in \mathbf{\Omega}_e(V'_k, \beta'_k)$ so $k \notin I \cup J$, and thus $\left| U'_k \cup V'_k \right| = r + \mathrm{rk}(B)$ and $\sum_X \alpha'_k(X) + \beta'_k(X) = d + \deg(B)$. The covering (9) is fair, so this implies the converse inequality $(\mathrm{rk}(AB), \deg(AB)) \geq (r + \mathrm{rk}(B), d + \deg(B))$. □

Next we show that when A is a special kind of type-dependent set called Ω-set, as defined in [7], the values of rank and degree introduced there agree with the ones defined here. We do that without re-introducing the language of [7], only a bare minimum. For $a \in \mathcal{F}(\mathcal{P})$ and $X \in \mathrm{Cl}(\mathcal{S})$ let

$$\Omega_X(a) = \sum_{p \in \mathcal{P} \cap X} v_p(x).$$

We call $A \subseteq \mathcal{F}(\mathcal{P})$ an Ω-set if for all $a \in A$ and all $b \in \mathcal{F}(\mathcal{P})$ such that

$$\Omega_X(a) = \Omega_X(b) \qquad \text{for all } X \in \mathrm{Cl}(S) \tag{11}$$

we have $b \in A$. Of course, every Ω-set is type-dependent. For $U \subseteq \mathrm{Cl}(\mathcal{S})$, and $\alpha : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$ such that $\alpha(X) = 0$ for all $X \in U$, let

$$\mathbf{\Omega}(U, \alpha) = \{a \in \mathcal{F}(\mathcal{P}) : \Omega_X(a) = \alpha(X) \text{ for all } X \in \mathrm{Cl}(\mathcal{S}) \setminus U\}.$$

Proposition 11 *Let A be a non-trivial Ω-set. Then $\gamma(A) = 1$. The rank of A equals the smallest r' such that A is contained in a finite union of the form*

$$\bigcup_{i=1}^{n} X_i \cap \mathbf{\Omega}(U_i, \alpha_i), \tag{12}$$

where $X_i \in \mathrm{Cl}(\mathcal{S})$ and $|U_i| \le r'$ for $i = 1, \ldots, n$. The degree of A equals the supremum d' of all $d \in \mathbf{N}_0$ such that

$$\mathrm{rk}\,(Y \cap \mathbf{\Omega}(U, \alpha) \cap A) = \mathrm{rk}(A)$$

for some $Y \in \mathrm{Cl}(\mathcal{S})$, $U \subseteq \mathrm{Cl}(\mathcal{S})$ and $\alpha : \mathrm{Cl}(S) \to \mathbf{N}_0$ such that $\alpha(X) = 0$ for $X \in U$ and $\sum_{X \notin U} \alpha(X) = d$.

Proof Let $r = \mathrm{rk}(A)$ and $d = \deg(A)$. Let r' and d' be as in the assertion and consider the smallest d'' such that A is contained in a finite union (12) with $|U_i| \le r$ for all i and $\sum_{X \notin U} \alpha_i(X) \le d''$ for all i with $|U_i| = r$. If $a \in A \setminus \{1\}$, then, by the infinitude of primes in each class, there exists $b \in \mathcal{F}(\mathcal{P})$ which is a product of distinct primes and satisfies (11), hence $b \in A$. Therefore $\gamma(A) = 1$. For every $a \in \mathcal{F}(\mathcal{P})$, $X \in \mathrm{Cl}(\mathcal{S})$ we have $0 \le \delta_{1,X}(a) \le \Omega_X(a)$, therefore if A is contained in (12), it is also contained in

$$\bigcup_{i=1}^{m} \bigcup_{\substack{\beta:\mathrm{Cl}(S)\to\mathbf{N}_0 \\ \beta \le \alpha_i}} \mathbf{\Omega}_1(U_i, \beta).$$

Hence $r \le r'$ and, by Fact 6, also $(r, d) \le (r', d'')$ in the lexicographic order. On the other hand, if A is contained in (4) with $e = 1$, and $a \in A$, we can find (again) $b \in A$ which is a product of distinct primes and satisfies (11). Let i be such that $b \in \mathbf{\Omega}_1(U_i, \alpha_i)$. Then $a \in \mathbf{\Omega}(U_i, \alpha_i)$. The choice of a is arbitrary, so A is contained in

$$\bigcup_{i=1}^{m} \bigcup_{X \in \mathrm{Cl}(\mathcal{S})} X \cap \mathbf{\Omega}(U_i, \alpha_i).$$

This implies $(r', d'') \le (r, d)$ in the lexicographic order. The equality $d'' = d'$ follows from [7, Lemma 4.4] and the basic properties of the "old" rank and degree proved there. □

4 Regular type-dependent sets

We call a non-trivial, type-dependent subset A of the divisor monoid $\mathcal{F}(\mathcal{P})$ (of a shifted formation $\mathcal{S}$) *regular* if $Z_A(s)$ has an extension of the form

$$Z_A(s) = \sum_{i=1}^{n} H_i(s) \prod_{\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}} L(es, \mathcal{S}, \chi)^{w_{i,\chi}} \left(\log L(es, \mathcal{S}, \chi)\right)^{k_{i,\chi}}, \tag{13}$$

where $e = \gamma(A)$, $H_i \in \mathcal{A}_{e+1}$, $w_{i,\chi} \in \mathbf{C}$, and $k_{i,\chi} \in \mathbf{N}_0$ for all i, χ, moreover the limit

$$\lim_{s \to \sigma_0^+} Z_A(s)(s - \sigma_0)^{\mathrm{rk}(A)/h} (\log((s - \sigma_0)^{-1}))^{-\deg(A)}, \tag{14}$$

where $\sigma_0 = \mathrm{ed}(A)$, is finite and non-zero. In addition the trivial sets are also considered regular. We call A *regular across classes* if for every $X \in \mathrm{Cl}(\mathcal{S})$ such that $X \cap A \neq \emptyset$ the intersection $X \cap A$ is regular with $\mathrm{metrics}(X \cap A) = \mathrm{metrics}(A)$, and *completely regular across classes* if, in addition, $X \cap A \neq \emptyset$ for all X. Further, A is *almost regular* if there exists a type-dependent set B such that metrics $(B) < \mathrm{metrics}(A)$ and $A \triangle B$ is regular. Finally, A is *semi-regular* if there are almost regular, type-dependent sets B and B' such that $B \subseteq A \subseteq B'$ and $\mathrm{metrics}(B) = \mathrm{metrics}(B') = \mathrm{metrics}(A)$.

Lemma 12 *Let $e \in \mathbf{N}$ and*

$$A \subseteq \mathbf{\Omega}_e(\mathrm{Cl}(\mathcal{S}), 0) = \{a \in \mathcal{F}(\mathcal{P}) : \gamma(a) \geq e\}.$$

Then $Z_A(s, f) \in \mathcal{A}_e$ for every $f : A \to \mathbf{C}$ such that $\forall_{\varepsilon > 0} f(a) \ll \|a\|^{\varepsilon}$ for $a \in A$.

Proof Let $\sigma > (1 + \lambda)/e$, $\varepsilon = (\sigma - (1 + \lambda)/e)/2$, $\sigma' = \sigma - \varepsilon = (1 + \lambda)/e + \varepsilon$. We have

$$\begin{aligned}
\sum_{a \in A} |f(a)| \, \|a\|^{-s} &\ll \sum_{\substack{a \in \mathcal{F}(\mathcal{P}) \\ \gamma(a) \geq e}} \|a\|^{-\sigma'} \\
&< \prod_{p \in \mathcal{P}} \left(1 + \|p\|^{-e\sigma'} + \|p\|^{-(e+1)\sigma'} + \ldots\right) \\
&= \prod_{p \in \mathcal{P}} \left(1 + \|p\|^{-e\sigma'} \left(1 - \|p\|^{-\sigma'}\right)^{-1}\right) \\
&< +\infty.
\end{aligned}$$

□

Lemma 13 *Suppose A is a type-dependent set and*

$$\mathrm{metrics}(A) < (\sigma_0, r, d)$$

for some $\sigma_0 = (1 + \lambda)/e$, $e \in \mathbf{N}$, and $r, d \in \mathbf{N}_0$ not both zero. If $\gamma(A) > e$ or A is regular, then

$$\lim_{s \to \sigma_0^+} Z_A(s)(s - \sigma_0)^{r/h} (\log((s - \sigma_0)^{-1}))^{-d} = 0.$$

Proof The assertion is obvious when $A = \emptyset$, so we assume that $A \neq \emptyset$. If $\gamma(A) > e$, then $\lim_{s\to\sigma_0^+} Z_A(s)$ is finite by Lemma 12. This implies the assertion. Suppose $\gamma(A) = e$. Then the limit (14) is finite and either $r > \mathrm{rk}(A)$ or $r = \mathrm{rk}(A)$ and $d > \deg(A)$. Therefore

$$\lim_{s\to\sigma_0^+}(s-\sigma_0)^{(r-\mathrm{rk}(A))/h}(\log((s-\sigma_0)^{-1}))^{\deg(A)-d} = 0,$$

so the assertion holds again. □

Fact 14 *Suppose A is a type-dependent set. If A is regular (respectively completely regular across classes), then so is $A \triangle B$ for every type-dependent set B satisfying $\gamma(B) > \gamma(A)$.*

Proof Suppose A is regular and $B \neq \emptyset$. We have $\mathrm{metrics}(A \triangle B) = \mathrm{metrics}(A)$ by Fact 8. Both $B \setminus A$ and $A \cap B$ are of greater height than A by Fact 4. Lemma 12 and

$$Z_{A\triangle B}(s) = Z_A(s) + Z_{B\setminus A}(s) - Z_{A\cap B}(s), \qquad \sigma > 1+\lambda,$$

imply that $Z_{A\triangle B}(s)$ is of the form (13). Fact 7 and Lemma 13 show that the limit (14) for $Z_{A\triangle B}(s)$ is the same as for $Z_A(s)$. By replacing A and B with $X \cap A$ and $X \cap B$ and using Fact 4 we obtain the assertion for sets completely regular across classes. □

Fact 15 *Suppose A is a type-dependent set. If A is almost regular (respectively semi-regular), then so is $A \triangle B$ for every type-dependent set B satisfying* $\mathrm{metrics}(B) < \mathrm{metrics}(A)$.

Proof We have $\mathrm{metrics}(A \triangle B) = \mathrm{metrics}(A)$ by Fact 8. If A is almost regular and $A \triangle B'$ is regular for some B' satisfying $\mathrm{metrics}(B') < \mathrm{metrics}(A)$, then

$$(A \triangle B) \triangle \left(B \triangle B'\right) = A \triangle B'.$$

Facts 4 and 8 imply that $\mathrm{metrics}\left(B \triangle B'\right) < \mathrm{metrics}(A)$, hence $A \triangle B$ is almost regular. If A is semi-regular and $B' \subseteq A \subseteq B''$ where B', B'' are almost regular with the same metrics as A, then

$$B' \setminus B \subseteq A \subseteq B'' \cup B.$$

Again, Fact 8 implies that $\mathrm{metrics}(B' \setminus B) = \mathrm{metrics}(B'' \cup B) = \mathrm{metrics}(A)$, and the first assertion shows that $B' \setminus B$ and $B'' \cup B$ are almost regular. This implies the assertion. □

Fact 16 *Suppose $A, B \subseteq \mathcal{F}(\mathcal{P})$ are disjoint, type-dependent sets. If A and B are regular, then so is $A \cup B$. If A and B are regular across classes and* $\mathrm{metrics}(A) = \mathrm{metrics}(B)$*, then $A \cup B$ is regular across classes. If A is completely regular across classes, B is regular across classes and* $\mathrm{metrics}(B) \leq \mathrm{metrics}(A)$*, then $A \cup B$ is completely regular across classes.*

Proof Suppose A and B are regular. It follows from

$$Z_{A\cup B}(s) = Z_A(s) + Z_B(s), \qquad \sigma > 1 + \lambda,$$

that $Z_{A\cup B}(s)$ is of the form (13). Facts 7 and 8, and Lemma 13, imply that $A \cup B$ is regular when $\mathrm{metrics}(A) \neq \mathrm{metrics}(B)$, otherwise it follows from the additivity of limits. The other assertions follow from the first one and the fact that for every $X \in \mathrm{Cl}(\mathcal{S})$ the sets $X \cap A$ and $X \cap B$ are empty or regular, so their disjoint sum is regular. Moreover, for every $X \in \mathrm{Cl}(\mathcal{S})$ such that $X \cap (A \cup B) \neq \emptyset$ the value

$$\mathrm{metrics}\left(X \cap (A \cup B)\right) = \max\left(\mathrm{metrics}\left(X \cap A\right), \mathrm{metrics}\left(X \cap B\right)\right)$$

equals $\mathrm{metrics}(A \cup B)$, because either

$$\mathrm{metrics}\left(X \cap A\right) = \mathrm{metrics}(A) \geq \mathrm{metrics}(B)$$

or

$$\mathrm{metrics}\left(X \cap B\right) = \mathrm{metrics}(B) = \mathrm{metrics}(A).$$

□

Fact 17 *Suppose A is a type-dependent set. If A is completely regular across classes, it is regular across classes. If it is regular across classes, it is regular. If it is regular, it is almost regular. If it is almost regular, it is semi-regular.*

Proof By Fact 16 if $X \cap A$ is regular for every $X \in \mathrm{Cl}(\mathcal{S})$, then so is the disjoint union

$$A = \bigcup_{X \in \mathrm{Cl}(\mathcal{S})} X \cap A.$$

This implies the second assertion. The others are obvious. □

Fact 18 *Suppose $A, B \subseteq \mathcal{F}(\mathcal{P})$ are type-dependent sets such that $B \subsetneq A$ and* $\mathrm{metrics}(B) < \mathrm{metrics}(A)$. *If A and B are regular (respectively regular across classes, completely regular across classes), then so is $A \setminus B$.*

Proof Suppose A and B are regular. We have $\mathrm{metrics}(A \setminus B) = \mathrm{metrics}(A)$ by Fact 8. The identity

$$Z_{A\setminus B}(s) = Z_A(s) - Z_B(s), \qquad \sigma > 1 + \lambda,$$

Fact 7 and Lemma 13 imply the first assertion. The assertions across classes follow from the first one upon observing that $X \cap B \neq \emptyset$ implies $X \cap A \neq \emptyset$. □

Fact 19 *Let $A \subseteq \mathcal{F}(\mathcal{P})$ be a type-dependent set, $m \in \mathbf{N}$, and $B = \{a^m : a \in A\}$. If A is regular, then so is B. If A is regular across classes, then so is B.*

Proof Let $e = \gamma(A)$, $\sigma_A = \mathrm{ed}(A)$, $r = \mathrm{rk}(A)$, $d = \deg(A)$ and $\sigma_B = \mathrm{ed}(B)$. We have $\gamma(B) = me$, $\sigma_B = \sigma_A/m$, $\mathrm{rk}(B) = r$ and $\deg(B) = d$ by Fact 9. Suppose A is regular. It follows from

$$Z_B(s) = Z_A(ms), \qquad \sigma > 1 + \lambda,$$

that $Z_B(s)$ has the required form. Moreover

$$\begin{aligned}
&\lim_{s\to\sigma_B^+} Z_B(s)(s-\sigma_B)^{r/h}(\log((s-\sigma_B)^{-1}))^{-d} \\
&\quad = \lim_{s\to\sigma_A^+} Z_A(s)(s/m-\sigma_B)^{r/h}(\log((s/m-\sigma_B)^{-1}))^{-d} \\
&\quad = m^{-r/h}\lim_{s\to\sigma_A^+} Z_A(s)(s-\sigma_A)^{r/h}(\log(m)+\log((s-\sigma_A)^{-1}))^{-d} > 0.
\end{aligned}$$

Hence B is regular. Let $Y \in \mathrm{Cl}(\mathcal{S})$ and let $X_1, \ldots, X_n$ be all the solutions of $X^m = Y$. If A is regular across classes, then $\bigcup_{j=1}^n \left(X_j \cap A\right)$ is regular by Fact 16 and

$$Y \cap B = \left\{a^m : a \in \bigcup_{j=1}^n \left(X_j \cap A\right)\right\},$$

so the second part of the assertion follows from the first one. □

Proposition 20 *Let $A, B \subseteq \mathcal{F}(\mathcal{P})$ be type-dependent sets such that $\gcd(a, b) = 1$ for all $a \in A$, $b \in B$. If A and B are regular (respectively regular across classes, completely regular across classes), then so is AB. If $\gamma(B) > \gamma(A)$ and A is regular (respectively regular across classes, completely regular across classes), then so is AB.*

Proof Suppose $\mathrm{metrics}(B) \leq \mathrm{metrics}(A)$ and let $e = \gamma(A)$ and $\sigma_0 = \mathrm{ed}(A)$.

Case 1. A and B are regular and $\gamma(B) = e$.
We have $\gamma(AB) = e$, $\mathrm{rk}(AB) = \mathrm{rk}(A) + \mathrm{rk}(B)$ and $\deg(AB) = \deg(A) + \deg(B)$ by Proposition 10. We also have

$$Z_{AB}(s) = Z_A(s)Z_B(s), \qquad \sigma > 1 + \lambda,$$

so $Z_{AB}(s)$ has the required form, because both $Z_A(s)$ and $Z_B(s)$ do. Moreover

$$\begin{aligned}
&\lim_{s\to\sigma_0^+} Z_{AB}(s)(s-\sigma_0)^{(\mathrm{rk}(A)+\mathrm{rk}(B))/h}(\log((s-\sigma_0)^{-1}))^{-(\deg(A)+\deg(B))} \\
&\quad = \lim_{s\to\sigma_0^+} Z_A(s)(s-\sigma_0)^{\mathrm{rk}(A)/h}(\log((s-\sigma_0)^{-1}))^{-\deg(A)} \\
&\quad\quad \cdot \lim_{s\to\sigma_0^+} Z_B(s)(s-\sigma_0)^{\mathrm{rk}(B)/h}(\log((s-\sigma_0)^{-1}))^{-\deg(B)}.
\end{aligned}$$

Case 2. A is regular and $\gamma(B) > e$.
We have $\gamma(AB) = e$, $\mathrm{rk}(AB) = \mathrm{rk}(A)$ and $\deg(AB) = \deg(A)$ by Proposition 10. Moreover

$$Z_{AB}(s) = Z_A(s) Z_B(s), \qquad \sigma > 1 + \lambda,$$

and $Z_B(s) \in \mathcal{A}_{e+1}$, so $Z_{AB}(s)$ has the required form. We also have

$$\lim_{s \to \sigma_0^+} Z_{AB}(s)(s - \sigma_0)^{\mathrm{rk}(A)/h} (\log((s - \sigma_0)^{-1}))^{-\deg(A)}$$
$$= \lim_{s \to \sigma_0^+} Z_A(s)(s - \sigma_0)^{\mathrm{rk}(A)/h} (\log((s - \sigma_0)^{-1}))^{-\deg(A)} \cdot \lim_{s \to \sigma_0^+} Z_B(s)$$

and $\lim_{s \to \sigma_0^+} Z_B(s) > 0$ follows from the definition and absolute convergence of the series.

Case 3. A and B are regular across classes and $\gamma(B) = e$.
The non-empty among the sets $X \cap A$, $Y \cap B$, where $X, Y \in \mathrm{Cl}(\mathcal{S})$, are regular with metrics equal to metrics(A), respectively metrics(B). By Fact 8, Proposition 10, and the first case, each of

$$Z \cap AB = \bigcup_{XY = Z} (X \cap A) \cdot (Y \cap B), \qquad Z \in \mathrm{Cl}(\mathcal{S}), \tag{15}$$

is either empty or regular with metrics equal to metrics(AB). If, in addition, A is completely regular across classes and B is non-empty, we have $Y \cap B \neq \emptyset$ for at least one $Y = Y_0$, so the right-hand side of (15) always has a non-empty summand for $X = ZY_0^{-1}$. Therefore AB is completely regular across classes.

Case 4. A is regular across classes and $\gamma(B) > e$.
This is analogous to Case 3. □

5 Sets of rank 0

Fact 21 *Let A be a non-trivial, type-dependent set with $e = \gamma(A)$. We have* $\mathrm{rk}(A) = 0$ *if and only if the quantity*

$$d = \sup_{a \in A} \sum_{X \in \mathrm{Cl}(\mathcal{S})} \delta_{e,X}(a)$$

is finite. In that case $\deg(A) = d$ *and* $d \in \mathbf{N}$.

Proof If $A \subseteq \bigcup_{i=1}^{m} \mathbf{\Omega}_e(\emptyset, \alpha_i)$, then $\delta_{e,X}(a) < \alpha_i(X)$ for every $a \in \mathbf{\Omega}_e(\emptyset, \alpha_i)$, $X \in \mathrm{Cl}(\mathcal{S})$, implying $d < +\infty$. Conversely, if $d < +\infty$ holds, we let $\mathcal{D}$ denote the set of all $\alpha : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$, satisfying $\sum_X \alpha(X) \leq d$. We have

$$A \subseteq \bigcup_{\alpha \in \mathcal{D}} \mathbf{\Omega}_e(\emptyset, \alpha),$$

so $\mathrm{rk}(A) = 0$ and $\deg(A) \leq d$ by Fact 6. Let us pick an element $a_{\max} \in A$ satisfying $\sum_{X \in \mathrm{Cl}(\mathcal{S})} \delta_{e,X}(a) = d$ and let $\alpha_{\max} : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$, $\alpha_{\max}(X) = \delta_{e,X}(a_{\max})$. The set

$$\mathbf{\Omega}_e(\emptyset, \alpha_{\max}) \cap A$$

is a subset of A and contains $[a_{\max}]_{\approx}$, so it is of height e and rank 0 by Fact 4. Hence $\deg(A) = d$. Positivity of d follows from the fact that $\gamma(A) = e$, so there is at least one element $a \in A$ satisfying $\gamma(a) = e$, and therefore $\sum_{X \in \mathrm{Cl}(\mathcal{S})} \delta_{e,X}(a) > 0$. □

Lemma 22 *Let $d \in \mathbf{N}_0$, $e \in \mathbf{N}$, $b = \prod_{i=1}^{d} p_i^e$ for some distinct $p_i \in \mathcal{P}$, $C \subseteq \mathcal{F}(\mathcal{P})$ a non-empty, type-dependent set satisfying $\gamma(C) > e$,*

$$A = \left\{ b'c : b' \approx b, c \in C, \gcd(b', c) = 1 \right\}, \tag{16}$$

and let $f : C \to \mathbf{C}$ be such that $\forall_{\varepsilon > 0} f(c) \ll \|c\|^{\varepsilon}$ for $c \in C$. Then f has a unique extension to $A \cup C$ satisfying

$$f(b'c) = f(c), \qquad \text{for all } b' \approx b, c \in C, \gcd(b', c) = 1, \tag{17}$$

and $Z_A(s, f)$ is a polynomial in $(P_X(es))_{X \in \mathrm{Cl}(\mathcal{S})}$ with coefficients in $\mathcal{A}_{e+1}$, of degree $\leq d$, and no other terms of degree d than

$$Z_C(s, f) \prod_{X \in \mathrm{Cl}(\mathcal{S})} P_X(es)^{\delta_{e,X}(b)}.$$

Proof Suppose $d = 0$. We have $b = 1$ and $A = C$. The only representation of $a = b'c$ satisfying the conditions in (16) is with $b' = 1$, so (17) holds trivially. We have $Z_A(s, f) = Z_C(s, f) \in \mathcal{A}_{e+1}$ by Lemma 12.

Suppose $d > 0$ is the smallest possible such that the assertion fails. We have $\gamma(a) = e$ for every $a \in A$, so $A \cap C = \emptyset$. The only representation of $a = b'c$ satisfying the conditions in (16) is with $b' = \prod_{\substack{p \in \mathcal{P} \\ v_p(a) = e}} p^e$, so the extension of f is unique. When $b' \approx b$ and $c \in C$ we have $\gamma(c) > e$, so $\gcd(b', c)$ is always of the same type as

$$\prod_{p \in U} p^e \tag{18}$$

for some $U \subseteq \{p_1, \ldots, p_d\}$. From among the divisors of b of the form (18) with $U \neq \emptyset$ we choose a set D containing one representative of each type. By the uniqueness of representation of elements $a \in A$ in the form $a = b'c$ in (16) we have

$$\begin{aligned} Z_A(s, f) &= Z_{[b]_{\approx}}(s) Z_C(s, f) - \sum_{u \in D} \sum_{\substack{b' \approx b, c \in C \\ (b', c) \approx u}} f(c) \left\| b'c \right\|^{-s} \\ &= Z_{[b]_{\approx}}(s) Z_C(s, f) - \sum_{u \in D} Z_{A_u}(s, f_u), \end{aligned} \tag{19}$$

where

$$A_u = \left\{ b'c : b' \approx b, c \in C, (b', c) \approx u \right\},$$
$$f_u(a) = \sum_{\substack{b' \approx b, c \in C \\ (b', c) \approx u \\ b'c = a}} f(c), \qquad a \in A_u, \tag{20}$$

provided that each $Z_{A_u}(s, f_u)$ is well defined, i.e. $\forall_{\varepsilon > 0} f_u(a) \ll \|a\|^{\varepsilon}$ on A_u, which we show below. For each $u \in D$ let $b_u = b/u$,

$$C_u = \left\{ u'c : u' \approx u, c \in C, u' \mid c \right\}.$$

Then C_u is type-dependent. We have $\gcd(b_u, u) = 1$, so whenever $b' \approx b$, $c \in C$ and $(b', c) = u' \approx u$ we have $\gcd(b'/u', u'c) = 1$. Therefore A_u is contained in

$$\left\{ b'_u c_u : b'_u \approx b_u, c_u \in C_u, \gcd(b'_u, c_u) = 1 \right\}. \tag{21}$$

To see that (21) equals A_u consider any $b'_u \approx b_u$ and $c_u = u'c$ where $u' \approx u$, $c \in C$, $u' \mid c$ and $\gcd(b'_u, u'c) = 1$. We have $b'_u c_u = b'_u u' \cdot c$, $b'_u u' \approx b$ and $\gcd(b'_u u', c) = u'$, so $b'_u c_u \in A_u$, and (21) is contained in A_u. We have $\gamma(C_u) > e$, so the representation of $a \in A_u$ as $a = b'_u c_u$ in (21) is unique. The representation $a = b'c$ in (20) need not be unique, but it implies that $b'/u' = b'_u$ and $u'c = c_u$ for some $u' \approx u$. and Hence

$$f_u(b'_u c_u) = \sum_{\substack{u' \approx u \\ u' | c_u}} f\left(\frac{c_u}{u'}\right),$$

in particular $f_u(b'_u c_u)$ depends on c_u alone. For every $\varepsilon > 0$ we have $f(c_u/u') \ll \left\|c_u/u'\right\|^{\varepsilon/2} \ll \|c_u\|^{\varepsilon/2}$, $u' \approx u$, and

$$\sum_{\substack{u' \approx u \\ u' | c_u}} 1 \ll \omega(c_u)^d \le \log_2 (\|c_u\|)^d \ll \|c_u\|^{\varepsilon/2}.$$

We obtain $f_u(a) = f_u(c_u) \ll \|c_u\|^{\varepsilon} \ll \|a\|^{\varepsilon}$. By the inductive hypothesis each function $\sum_{u \in D} Z_{A_u}(s, f_u)$, $u \in D$, satisfies the assertion with $d - \omega(u)$ in place of d.

Finally, note that for each $X \in \mathrm{Cl}(\mathcal{S})$, $\delta = \delta_{e,X}(b)$, we have

$$P_X(es)^{\delta} = \sum_{r=1}^{\delta} \sum_{\substack{1 \le m_1 \le \ldots \le m_r \le \delta \\ m_1 + \ldots + m_r = \delta}} \kappa(m_1, \ldots, m_r) Z_{A(m_1, \ldots, m_r)}(s), \tag{22}$$

where $\kappa(m_1, \ldots, m_r) \in \mathbf{N}$ and $A(m_1, \ldots, m_r)$ denotes the set of all elements of type $q_1^{m_1 e} \ldots q_r^{m_r e}$ with $q_1, \ldots, q_r \in X \cap \mathcal{P}$ distinct. When $m_1 = \ldots = m_k = 1 < m_{k+1} \le \ldots \le m_r$ for $k < r$, the set $A' = A(m_1, \ldots, m_r)$ satisfies the assumptions of the lemma with $b' = q_1^e \dots q_k^e, C' = A(m_{k+1}, \dots, m_r)$, so by the inductive hypothesis $Z_{A(m_1,\dots,m_r)}(s)$ is a polynomial of degree $k \le \delta - 1 < d$. Otherwise we have $r = \delta$ and $m_1 = \dots = m_\delta = 1$. Comparing the products of both sides of (22) over all $X \in \mathrm{Cl}(\mathcal{S})$ we conclude that

$$Z_{[b]_\approx}(s) - \prod_{X \in \mathrm{Cl}(\mathcal{S})} P_X(es)^{\delta_{e,X}(b)}$$

is a polynomial of degree $\le d - 1$. This, in view of (19), implies our assertion. □

Theorem 23 *Every type-dependent set of rank* 0 *is regular.*

Proof Trivial sets are regular, so suppose A is a non-trivial type-dependent set of rank 0. Let $e = \gamma(A)$. It follows from Fact 21 that $\deg(A) = d$, where

$$d = \sup_{a \in A} \sum_{X \in \mathrm{Cl}(\mathcal{S})} \delta_{e,X}(a)$$

and $d \in \mathbf{N}$. Consider a fair covering

$$A \subseteq \bigcup_{i=1}^{m} \mathbf{\Omega}_e(\emptyset, \alpha_i),$$

such that $A_i := \mathbf{\Omega}_e(\emptyset, \alpha_i) \cap A \neq \emptyset$ for every i. Fix $\alpha = \alpha_i$. Let $\delta = \delta_i = \sum_X \alpha(X)$, and let $b = \prod_{j=1}^{\delta} p_j^e$ be such that $b \in \mathbf{\Omega}_e(\emptyset, \alpha)$. Obviously for every $a \in A_i$ there exists a unique $b' \approx b$ such that $b' \mid a$, namely

$$b' = \prod_{\substack{p \in \mathcal{P} \\ v_p(a) = e}} p^e.$$

Of course $\gcd(b', a/b') = 1$. Let

$$C_i = \left\{ c \in \mathcal{F}(\mathcal{P}) : b'c \in A_i \text{ for some } b' \approx b \right\}. \tag{23}$$

We have $\gamma(C_i) > e$. If $c \in C_i$, $b' \approx b$, and $\gcd(b', c) = 1$, then by (23) there exists a $b'' \approx b$ such that $b''c \in A_i$. We have $\gcd(b'', c) = 1$, so $b''c \approx b'c$, implying $b'c \in A_i$. We conclude that

$$A_i = \left\{ b'c : b' \approx b, c \in C_i, \gcd(b', c) = 1 \right\}.$$

By Lemma 22 with $f = 1$, and Lemma 3, the function $Z_{A_i}(s)$ is a polynomial of degree $\le d$ in $\log L(es, \mathcal{S}, \chi)$, $\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}$. Hence

$$Z_A(s) = \sum_{i=1}^{m} Z_{A_i}(s) = \sum_{j=1}^{n} H_j(s) \prod_{\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}} (\log L(es, \mathcal{S}, \chi))^{k_{j,\chi}}, \tag{24}$$

where $H_i \in \mathcal{A}_{e+1}$ and $k_{j,\chi} \in \mathbf{N}_0$, as in (13), moreover $\sum_\chi k_{j,\chi} \leq d$ for all j. It also follows from Lemmas 22 and 3 that the sum of terms of (24) with $k_{j,\chi_0} = d$ equals

$$h^{-d} Z_C(s) \left(\log L(es, \mathcal{S}, \chi_0)\right)^d ,$$

where $C = \bigcup_{\delta_i = d} C_i$. Each function $\log L(es, \mathcal{S}, \chi)$, $\chi \neq \chi_0$, is regular at $\sigma_0 = \mathrm{ed}(A)$. Therefore the limit (14) equals $h^{-d} Z_C(\sigma_0) > 0$. □

6 Sets generated by cubes

Lemma 24 *Let $e, l \in \mathbf{N}$, $X \in \mathrm{Cl}(\mathcal{S})$,*

$$A = A_{e,l,X} = \left\{a \in \mathcal{F}(\mathcal{P}) : v_p(a) \in e + l\mathbf{N}_0 \text{ and } p \in X \text{ for every } p \mid a,\, p \in \mathcal{P}\right\} \tag{25}$$

and

$$B = B_{e,X} = \left\{b \in A : v_p(b) = e \text{ for every } p \mid b,\, p \in \mathcal{P}\right\}. \tag{26}$$

We have $\gamma(A) = \gamma(B) = e$, $\mathrm{rk}(A) = \mathrm{rk}(B) = 1$ and $\deg(A) = \deg(B) = 0$. For $Y \in \mathrm{Cl}(\mathcal{S})$ we have $Y \cap A \neq \emptyset$ if and only if $Y \in \left\langle X^{\gcd(e,l)}\right\rangle$. In that case $\mathrm{metrics}(Y \cap A) = \mathrm{metrics}(A)$. *We have $Y \cap B \neq \emptyset$ if and only if $Y \in \langle X^e \rangle$. In that case* $\mathrm{metrics}(Y \cap B) = \mathrm{metrics}(B)$.

Proof We have $B \subseteq A \subseteq \mathbf{\Omega}_e(\{X\}, 0)$, so $\gamma(B) \geq \gamma(A) \geq e$. If $\gamma(B) = \gamma(A) = e$, then $\mathrm{rk}(B) \leq \mathrm{rk}(A) \leq 1$ and, if $\mathrm{rk}(B) = \mathrm{rk}(A) = 1$ as well, then $\deg(A) = \deg(B) = 0$ by Fact 6. For each $a \in A$ we have $[a] \in X^{me+nl}$ for some $m, n \in \mathbf{N}_0$ and $[a] \in X^{me}$ when $a \in B$. Conversely, let $Y = X^{me+nl}$ and let $k \equiv m \pmod{\mathrm{ord}(X)}$ be arbitrarily large. We have $\mathrm{metrics}(Y \cap A) \leq \mathrm{metrics}(A)$ by Fact 4. For distinct primes $p_1, \ldots, p_k \in X$ we have $p_1^{e+nl} p_2^e \ldots p_k^e \in Y \cap A$, so $\gamma(Y \cap A) = \gamma(A) = e$ and $\sup_{a \in Y \cap A} \delta_{e,X}(a) = +\infty$. Hence $\mathrm{rk}(Y \cap A) = \mathrm{rk}(a) = 1$ by Fact 21. Consequently $\deg(Y \cap A) = \deg(A) = 0$. Likewise, when $n = 0$, we have $p_1^e p_2^e \ldots p_k^e \in Y \cap B$ and the same argument shows that $\mathrm{metrics}(Y \cap B) = \mathrm{metrics}(B) = \mathrm{metrics}(A)$. □

Lemma 25 *Let $e \in \mathbf{N}$ and $X \in \mathrm{Cl}(\mathcal{S})$. The set B defined by (26) is regular across classes.*

Proof We have $\gamma(B) = e$, $\mathrm{rk}(B) = 1$ and $\deg(B) = 0$ by Lemma 24. Moreover, for every $\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}$ we have

$$Z_B(s, \chi) = \prod_{p \in X \cap \mathcal{P}} \left(1 + \chi(p^e)\, \|p\|^{-es}\right), \qquad \sigma > 1 + \lambda,$$

so

$$\begin{aligned}\log Z_B(s,\chi) &= \sum_{p\in X\cap\mathcal{P}} \left(\chi(p^e)\,\|p\|^{-es} + O\left(\|p\|^{-(e+1)\sigma}\right)\right)\\ &= \chi(X)^e \frac{1}{h} \sum_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\psi(X)} \sum_{p\in\mathcal{P}} \left(\psi(p)\,\|p\|^{-es} + O\left(\|p\|^{-(e+1)\sigma}\right)\right)\\ &= \chi(X)^e \frac{1}{h} \sum_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\psi(X)} \log L(es,\mathcal{S},\psi) + R_\chi(s)\\ &= \sum_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})}} \left\langle \chi^e\overline{\psi} \mid \{X\}\right\rangle \log L(es,\mathcal{S},\psi) + R_\chi(s),\end{aligned}$$

where $R_\chi(s) \in \mathcal{A}_{e+1}$. The coefficient at $\log L(es,\mathcal{S},\chi_0)$, having a singularity at $s = (1+\lambda)/e$, is $\langle \chi^e \mid \{X\}\rangle$. We have $\langle \chi^e \mid \{X\}\rangle = 1/h$ if and only if

$$\chi^e(X) = 1. \tag{27}$$

Otherwise $\Re\,\langle \chi^e \mid \{X\}\rangle < 1/h$. Let $Y \in \langle X^e\rangle$. We have

$$\begin{aligned}Z_{Y\cap B}(s) &= \frac{1}{h} \sum_{\chi\in\widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\chi(Y)} Z_B(s,\chi)\\ &= \frac{1}{h} \sum_{\chi\in\widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\chi(Y)} \prod_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})}} L(es,\mathcal{S},\psi)^{\langle \chi^e\overline{\psi}\mid\{X\}\rangle} \exp\left(R_\chi(s)\right).\end{aligned}$$

Let G denote the set of those χ that satisfy (27) and $\sigma_0 = (1+\lambda)/e$. We have $Z_B(s,\chi) = Z_B(s)$ and $\chi(Y) = 1$ for $\chi \in G$, so

$$\begin{aligned}\lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} Z_{Y\cap B}(s) &= \lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} \frac{1}{h} \sum_{\chi\in G} \overline{\chi(Y)} Z_B(s,\chi)\\ &= \frac{|G|}{h} \lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} Z_B(s).\end{aligned}$$

Moreover

$$Z_B(s) = \exp\left(R_{\chi_0}(s)\right) \prod_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})}} L(es,\mathcal{S},\psi)^{\langle\overline{\psi}\mid\{X\}\rangle}.$$

The factor $\exp\left(R_{\chi_0}(s)\right)$ is regular and non-zero at σ_0 and so is

$$\prod_{\psi\in\widehat{\mathrm{Cl}(\mathcal{S})},\,\psi\neq\chi_0} L(es,\mathcal{S},\psi)^{\langle\overline{\psi}\mid\{X\}\rangle}$$

by definition of a shifted formation. It remains to note that

$$\lim_{s\to\sigma_0^+}(s-\sigma_0)^{1/h}L(es,\mathcal{S},\chi_0)^{1/h}$$

is finite and non-zero, again, by definition of a shifted formation. □

Lemma 26 *Let* $e, l \in \mathbf{N}$ *and* $X \in \mathrm{Cl}(\mathcal{S})$. *The set* A *defined by* (25) *is regular across classes.*

Proof We recall that $\gamma(A) = e$, $\mathrm{rk}(A) = 1$ and $\deg(A) = 0$ by Lemma 24. Let B be as defined by (26). If $m = \gcd(e, l)$ it is sufficient to prove the regularity of $A_{e/m,l/m,X}$ across classes and apply Fact 19. Therefore we assume that $\gcd(e, l) = 1$. We have, for every $\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}$ and $\sigma > 1 + \lambda$,

$$\begin{aligned}Z_A(s,\chi) &= \prod_{p\in X\cap\mathcal{P}}\left(1+\sum_{n=0}^{\infty}\chi(p^{e+nl})\,\|p\|^{-(e+nl)s}\right)\\ &= Z_B(s,\chi)\prod_{p\in X\cap\mathcal{P}}\left(1+\left(\sum_{n=1}^{\infty}\chi(p^{nl})\,\|p\|^{-nls}\right)\right.\\ &\qquad\left.\times\left(\sum_{m=1}^{\infty}(-1)^{m+1}\chi(p^{me})\,\|p\|^{-mes}\right)\right),\end{aligned}$$

where the last equality follows from the power series identity

$$1+\sum_{n=0}^{\infty}z^{e+nl} = \left(1+z^e\right)\left(1+\left(\sum_{n=1}^{\infty}z^{nl}\right)\left(\sum_{m=1}^{\infty}(-1)^{m+1}z^{me}\right)\right),\qquad |z|<1.$$

Hence

$$\begin{aligned}Z_A(s,\chi) &= Z_B(s,\chi)\prod_{p\in X\cap\mathcal{P}}\left(1+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+1}\chi(p^{me+nl})\,\|p\|^{-(me+nl)s}\right)\\ &= Z_B(s,\chi)Z_C(s,\chi f),\end{aligned}$$

where

$$C = \left\{a\in\mathcal{F}(\mathcal{P}) : p\in X \text{ and } v_p(a)\geq e+l \text{ for every } p\mid a,\ p\in\mathcal{P}\right\}$$

and $f : C \to \mathbf{R}$ is a multiplicative function defined by

$$\begin{aligned}f(p^j) = \#&\left\{(m,n)\in\mathbf{N}^2 : j = me+nl,\ 2\nmid m\right\}\\ &-\#\left\{(m,n)\in\mathbf{N}^2 : j = me+nl,\ 2\mid m\right\}\end{aligned}$$

for $p \in X \cap \mathcal{P}$. We have $\gamma(C) = e + l$ and $\left|f(p^j)\right| < j \leq \log_2\left(\left\|p^j\right\|\right)$, hence $Z_C(s, \chi f) \in \mathcal{A}_{e+1}$ by Lemma 12. Let $Y \in \mathrm{Cl}(\mathcal{S})$ be such that $Y \cap A \neq \emptyset$. We have $Y \in \langle X \rangle$. Similarly to the proof of Lemma 25 we have

$$Z_{Y\cap A}(s) = \frac{1}{h} \sum_{\chi \in \widehat{\mathrm{Cl}(\mathcal{S})}} \overline{\chi(Y)} Z_B(s, \chi) Z_C(s, \chi f)$$

and

$$Z_B(s, \chi) = \prod_{\psi \in \widehat{\mathrm{Cl}(\mathcal{S})}} L(es, \mathcal{S}, \psi)^{\langle \chi^e \overline{\psi} | \{X\} \rangle} \exp\left(R_\chi(s)\right), \qquad \chi \in \widehat{\mathrm{Cl}(\mathcal{S})},$$

where $R_\chi(s) \in \mathcal{A}_{e+1}$ and $\Re \langle \chi^e \mid \{X\} \rangle < 1/h$ unless χ satisfies (27). It follows that $Z_{Y\cap A}(s)$ is of the required form. Again, let G denote the set of χ satisfying (27). We have

$$\begin{aligned}
\lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} Z_{Y\cap A}(s) &= \lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} \frac{1}{h} \sum_{\chi \in G} \overline{\chi(Y)} Z_B(s, \chi) Z_C(s, \chi f) \\
&= \lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} Z_B(s) \frac{1}{h} \sum_{\chi \in G} \overline{\chi(Y)} Z_C(s, \chi f) \\
&= \frac{|G|}{h} \lim_{s\to\sigma_0^+} (s-\sigma_0)^{1/h} Z_B(s) Z_{Y'\cap C}(s, f),
\end{aligned}$$

where

$$Y' = \left\{a \in \mathcal{F}(\mathcal{P}) : [a] = Y X^{me} \text{ for some } m \in \mathbf{N}_0\right\}.$$

It follows from $Z_C(s, f) \in \mathcal{A}_{e+1}$ that $\lim_{s\to\sigma_0^+} Z_{Y'\cap C}(s, f) = Z_{Y'\cap C}(\sigma_0, f)$, which is finite and real. By Lemma 25 the limit $\lim_{s\to\sigma_0^+}(s-\sigma_0)^{1/h} Z_B(s)$ is positive. It remains to show that $Z_{Y'\cap C}(\sigma_0, f) > 0$. For every $\sigma \geq \sigma_0$ we have

$$Z_{Y'\cap C}(s, f) = \sum_{r=0}^{\infty} \sum_{\substack{a \in Y'\cap C \\ \omega(a)=r}} f(a) \left\|a\right\|^{-\sigma}.$$

The summand with $r = 0$ is either 1 or 0, depending on whether $Y \in \langle X^e \rangle$ or not. For $r \geq 1$, $a \in C$ and $\omega(a) = r$ we have $a = p_1^{k_1} \ldots p_r^{k_r}$ for some $k_1, \ldots, k_r \in \mathbf{N}$ and distinct $p_1, \ldots, p_r \in X \cap \mathcal{P}$. Moreover, $a \in Y'$ is equivalent to $k_1 + \ldots + k_r \equiv k \pmod{e}$, where k is such that $Y = X^k$. Therefore

$$\sum_{\substack{a \in Y'\cap C \\ \omega(a)=r}} f(a) \left\|a\right\|^{-\sigma} = \sum_{\substack{k_1,\ldots,k_r \in \{0,\ldots,e-1\} \\ k_1+\ldots+k_r \equiv k \pmod{e}}} \sum_{p_1,\ldots,p_r} \prod_{j=1}^{r} \sum_{\substack{m \geq e+l \\ m \equiv k_j \pmod{e}}} f(p_j^m) \left\|p_j\right\|^{-m\sigma},$$

where $\sum_{p_1,\ldots,p_r}$ indicates summation over all sets of r distinct primes in X. Given j, k_j and p_j we have

$$\begin{aligned}
&\sum_{\substack{m\geq e+l\\ m\equiv k_j \pmod e}} f(p_j^m)\,\|p_j\|^{-m\sigma}\\
&= \sum_{\substack{m,n\in\mathbf{N}\\ me+nl\equiv k_j \pmod e\\ 2\nmid m}} \|p_j\|^{-(me+nl)\sigma} - \sum_{\substack{m,n\in\mathbf{N}\\ me+nl\equiv k_j \pmod e\\ 2\mid m}} \|p_j\|^{-(me+nl)\sigma}\\
&= \sum_{\substack{m,n\in\mathbf{N}\\ me+nl\equiv k_j \pmod e\\ 2\nmid m}} \left(\|p_j\|^{-(me+nl)\sigma} - \|p_j\|^{-((m+1)e+nl)\sigma}\right).
\end{aligned}$$

Every term in the final sum is positive and the sum itself is non-empty by $\gcd(e,l) = 1$. Solutions to $k_1 + \ldots + k_r \equiv k \pmod e$ do exist whenever $r \geq 1$. Therefore $Z_{Y'\cap C}(\sigma, f) > 0$ for all $\sigma \geq \sigma_0$. □

Lemma 27 *Let $e \in \mathbf{N}$, $V \subseteq \mathrm{Cl}(\mathcal{S})$, $\alpha : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$, $\alpha \neq 0$, $\mathrm{Supp}(\alpha) \subseteq V$, and*

$$B = \{a \in \mathbf{\Omega}_e(\emptyset, \alpha) : [p] \in V \text{ for every } p \mid a,\ p \in \mathcal{P}\}. \tag{28}$$

The set B is regular across classes with $\gamma(B) = e$, $\mathrm{rk}(B) = 0$ and $\deg(B) = \sum_X \alpha(X)$. For $Y \in \mathrm{Cl}(\mathcal{S})$ we have $Y \cap B \neq \emptyset$ if and only if $Y \in \langle V\rangle$.

Proof We have $[a] \in \langle V\rangle$ for every $a \in B$, so suppose $Y \in \langle V\rangle$. Let b be any element of B and suppose, as we may, that $Y[b]^{-1} = [p_1]^{k_1}\ldots[p_r]^{k_r}$ for some primes $p_1,\ldots,p_r$ with classes in V, $p_i \nmid b$ for all i. The element

$$a = bp_1^{k_1+eh}p_2^{k_2+eh}\ldots p_r^{k_r+eh}$$

satisfies $a \in Y \cap B$, so $\gamma(Y \cap B) = e$. We have $\mathrm{rk}(Y \cap B) = 0$ and $\deg(Y \cap B) \leq d := \sum_X \alpha(X)$ by $Y \cap B \subseteq \mathbf{\Omega}_e(\emptyset, \alpha)$ and Fact 6. To see that $\deg(Y \cap B) = d$ we note that $Y \cap B \subseteq \mathbf{\Omega}_e(\emptyset, \alpha)$, so $\mathrm{rk}\,(\mathbf{\Omega}_e(\emptyset, \alpha) \cap Y \cap B) = \mathrm{rk}(Y \cap B)$. Regularity follows from Theorem 23. □

Lemma 28 *Every cube $\mathbf{\Omega}_e(U, \alpha)$ is completely regular across classes. If $U \neq \emptyset$ or $\alpha \neq 0$, then $\gamma(\mathbf{\Omega}_e(U,\alpha)) = e$, $\mathrm{rk}(\mathbf{\Omega}_e(U,\alpha)) = |U|$ and $\deg(\mathbf{\Omega}_e(U,\alpha)) = \sum_X \alpha(X)$.*

Proof Let $V = \mathrm{Cl}(\mathcal{S}) \setminus U$. We have

$$\mathbf{\Omega}_e(U,\alpha) = \left(\prod_{X\in U} A_{e,1,X}\right) B,$$

where $A_{e,1,X}$ is defined by (25), and B is as in (28). Let $A = \prod_{X\in U} A_{e,1,X}$. If $U \neq \emptyset$, then it follows from Lemmas 24 and 26 and Propositions 10 and 20 that $\gamma(A) = e$,

$\mathrm{rk}(A) = |U|$, $\deg(A) = 0$ and A is regular across classes. If $\alpha \neq 0$, then B is regular across classes, with $\gamma(B) = e$, $\mathrm{rk}(B) = 0$, and $\deg(B) = \sum_X \alpha(X)$, by Lemma 27 and Theorem 23. Therefore, when both $U \neq \emptyset$ and $\alpha \neq 0$, we obtain the metrics asserted from Proposition 10, moreover the set AB is regular across classes by Proposition 20. When $U = \emptyset$ and $\alpha \neq 0$, we have $A = \{1\}$, so $AB = B$ is, again, regular across classes with the asserted metrics. When $U \neq \emptyset$ and $\alpha = 0$, we have $\gamma(B) > e$, so AB is again regular across classes by Proposition 20 and the metrics follow from Proposition 10. We reduce the case $U = \emptyset$ and $\alpha = 0$ to $U = \mathrm{Cl}(\mathcal{S})$ and $\alpha = 0$ using (3). Finally $Y \cap AB \neq \emptyset$ for each $Y \in \mathrm{Cl}(\mathcal{S})$ thanks to Lemmas 24 and 27 and the fact that $\langle U \rangle \langle V \rangle = \mathrm{Cl}(\mathcal{S})$. □

Corollary 29 *If a cube $\mathbf{\Omega}_e(U, \alpha)$ is a proper subset of another cube $\mathbf{\Omega}_{e'}(U', \alpha')$, then*

$$\mathrm{metrics}(\mathbf{\Omega}_e(U, \alpha)) < \mathrm{metrics}(\mathbf{\Omega}_{e'}(U', \alpha')).$$

Theorem 30 *Let $\mathfrak{A}$ be the algebra of subsets of $\mathcal{F}(\mathcal{P})$ generated by all cubes with the binary operations of set union, intersection and difference. Every non-empty set $A \in \mathfrak{A}$ is completely regular across classes.*

Proof First we show by induction that every set of the form

$$\mathbf{\Omega}_e(U, \alpha) \setminus \bigcup_{i=1}^{m} \mathbf{\Omega}_{e_i}(U_i, \alpha_i). \tag{29}$$

is completely regular across classes. Let D be given by (29). For $m = 0$ the assertion follows from Lemma 28. Suppose $m \geq 1$ and the assertion holds for $m - 1$. Suppose $D \neq \emptyset$. Note that

$$D = \mathbf{\Omega}_e(U, \alpha) \setminus \bigcup_{i=1}^{m} \big(\mathbf{\Omega}_e(U, \alpha) \cap \mathbf{\Omega}_{e_i}(U_i, \alpha_i)\big),$$

so, by Lemma 5 and $D \neq \emptyset$ we may assume

$$\mathbf{\Omega}_{e_i}(U_i, \alpha_i) \subsetneq \mathbf{\Omega}_e(U, \alpha)$$

and, by Corollary 29,

$$\mathrm{metrics}(\mathbf{\Omega}_{e_i}(U_i, \alpha_i)) < \mathrm{metrics}(\mathbf{\Omega}_e(U, \alpha))$$

for each i. It follows from Fact 8 that $\mathrm{metrics}(D) = \mathrm{metrics}(\mathbf{\Omega}_e(U, \alpha))$. Let

$$A = \mathbf{\Omega}_e(U, \alpha) \setminus \bigcup_{i=1}^{m-1} \mathbf{\Omega}_{e_i}(U_i, \alpha_i)$$

and

$$B = \mathbf{\Omega}_{e_m}(U_m, \alpha_m) \setminus \bigcup_{i=1}^{m-1} \mathbf{\Omega}_{e_i}(U_i, \alpha_i).$$

We have $D = A \setminus B$. By the inductive hypothesis A is completely regular across classes, and so is B, unless $B = \emptyset$. In the latter case we are done, so suppose $B \neq \emptyset$. We have

$$\operatorname{metrics}(B) \leq \operatorname{metrics}(\mathbf{\Omega}_{e_m}(U_m, \alpha_m)) < \operatorname{metrics}(D)$$

and $\operatorname{metrics}(D) \leq \operatorname{metrics}(A)$, so indeed $D = A \setminus B$ is regular by Fact 18. Every element of $\mathfrak{A}$ is a finite union of disjoint sets of the form (29). The theorem's assertion follows from Fact 16. □

Corollary 31 *Every non-trivial type-dependent set A is contained in a type-dependent set B, completely regular across classes, with* $\operatorname{metrics}(B) = \operatorname{metrics}(A)$.

Proof Let (6) be a fair covering of A and let B be the right-hand side of (6). We have $\operatorname{metrics}(B) = \operatorname{metrics}(A)$ and the regularity of B follows from Theorem 30. □

Corollary 32 *If a type-dependent set A contains an almost regular set B with* $\operatorname{metrics}(B) = \operatorname{metrics}(A)$*, then A is semi-regular.*

Corollary 33 *The assumption "if $\gamma(A) > e$ or A is regular" in Lemma 13 may be dropped.*

Proof It is enough to apply Lemma 13 to a regular superset A' of A such that $\operatorname{metrics}(A') = \operatorname{metrics}(A)$. The assertion for A follows from $|Z_A(\sigma)| \leq |Z_{A'}(\sigma)|$. □

Corollary 34 *If A is an almost regular set satisfying (13), then A is regular.*

Proof Let $B \subseteq \mathcal{F}(\mathcal{P})$ be such that $\operatorname{metrics}(B) < \operatorname{metrics}(A)$ and $A \triangle B$ is regular. The assertion follows from:

$$\operatorname{metrics}(A) = \operatorname{metrics}(A \triangle B),$$
$$\left|Z_A(s) - Z_{A \triangle B}(s)\right| \leq |Z_B(\sigma)|, \qquad \sigma > \operatorname{ed}(A),$$

and Corollary 33. □

7 Counting functions of regular sets

Theorem 35 *Let A be a non-trivial type-dependent set. If A is almost regular, then*

$$A(x) \sim C x^{(1+\lambda)/\gamma(A)} (\log x)^{-1+\operatorname{rk}(A)/h} (\log\log x)^d, \qquad x \to \infty, \tag{30}$$

for some $C > 0$*, where* $d = \deg(A)$ *if* $\mathrm{rk}(A) > 0$ *and* $d = \deg(A) - 1$ *if* $\mathrm{rk}(A) = 0$. *If A is semi-regular, then*

$$A(x) \asymp x^{(1+\lambda)/\gamma(A)} (\log x)^{-1+\mathrm{rk}(A)/h} (\log\log x)^d, \qquad x \to \infty. \tag{31}$$

Otherwise

$$A(x) \ll x^{(1+\lambda)/\gamma(A)} (\log x)^{-1+\mathrm{rk}(A)/h} (\log\log x)^d, \qquad x \to \infty \tag{32}$$

and

$$A(x) \gg x^{(1+\lambda)/\gamma(A)} (\log x)^{-1} (\log\log x)^M, \qquad x \to \infty \tag{33}$$

for every $M > 0$.

Proof When A is regular with $\sigma_0 = \mathrm{ed}(A)$, it follows from (13) that we have

$$Z_A(s) = \sum_{i=1}^{m} \widetilde{H}_i(s)(s-\sigma_0)^{-w_i} \left(\log((s-\sigma_0)^{-1})\right)^{k_i}, \qquad \sigma > \sigma_0 \tag{34}$$

for some $w_i \in \mathbf{C}$, $k_i \in \mathbf{N}_0$, and $\widetilde{H}_i(s)$ holomorphic in $\sigma \geq \sigma_0$, satisfying $\widetilde{H}_i(\sigma_0) \neq 0$. We may assume that m is the smallest possible, hence the pairs (w_i, k_i) are all distinct. Let $w = \max_i \Re w_i$ and $k = \max_{\Re w_i = w} k_i$. Suppose $\Re w_i = w$ and $k_i = k$ for $i \leq m'$, moreover $\Re w_i < w$ or $k_i < k$ for each $i > m'$. We have

$$Z_A(\sigma)(\sigma-\sigma_0)^w (\log((s-\sigma_0)^{-1}))^{-k} = \sum_{i=1}^{m'} \widetilde{H}_i(\sigma_0)(\sigma-\sigma_0)^{-i\Im w_i} + o(1), \quad \sigma \to \sigma_0^+,$$

and the trigonometric polynomial $\sum_{i=1}^{m'} \widetilde{H}_i(\sigma_0)(\sigma-\sigma_0)^{-i\Im w_i}$ is not identically zero, the exponents $-i\Im w_i$ being all distinct. By (14) we have $w = \mathrm{rk}(A)/h$, $k = \deg(A)$, $m' = 1$ and $w_1 = w$. The first assertion now follows from the Tauberian theorem of Delange and Ikehara, e.g., in the form given in [2, Theorem 8.2.5] that we can apply to $Z_A(\sigma_0 s)$, because $\sigma_0 > 0$. Suppose A is almost regular and $A \bigtriangleup B$ is regular for some $B \subseteq \mathcal{F}(\mathcal{P})$ such that metrics$(B) <$ metrics(A), then let B' be a regular superset of B satisfying metrics$(B') =$ metrics(B), that exists by Corollary 31. We have metrics$(A \bigtriangleup B) =$ metrics(A) by Fact 8, so

$$(A \bigtriangleup B)(x) \sim C x^{(1+\lambda)/\gamma(A)} (\log x)^{-1+\mathrm{rk}(A)/h} (\log\log x)^d, \qquad x \to \infty,$$

for some $C > 0$ and

$$B'(x) = o\left(x^{(1+\lambda)/\gamma(A)} (\log x)^{-1+\mathrm{rk}(A)/h} (\log\log x)^d\right), \qquad x \to \infty.$$

Since $A(x) = (A \bigtriangleup B)(x) + (A \cap B)(x) - (B \setminus A)(x)$ and $|(A \cap B)(x) - (B \setminus A)(x)| \leq B'(x)$ for every x, we obtain (30). When A is semi-regular, (31) follows from the definition and (30). Suppose A is not regular and let A' be its regular superset, as implied by Corollary 31. Then (30) for A' in place of A implies (32) for A. Similarly (33) follows from the fact that for arbitrarily large $d \in \mathbf{N}$ the set A contains a regular subset of height $e = \gamma(A)$, rank 0 and degree d. Indeed, let

$$A_d = \left\{ a \in A : \sum_{X \in \mathrm{Cl}(\mathcal{S})} \delta_{e,X}(a) = d \right\}.$$

We have $\mathrm{rk}(A) > 0$ by Theorem 23. Therefore by Fact 21 there are arbitrarily large d such that $A_d \neq \emptyset$. For such a d we conclude from Fact 21 that $\mathrm{rk}(A_d) = 0$ and $\deg(A_d) = d$, and from Theorem 23 that A_d is regular. □

The next theorem shows that (33) is sharp.

Theorem 36 *Let $e \in \mathbf{N}$ and let $f(x)$ be a real function such that $f(x) \gg x^{(1+\lambda)/e}(\log x)^{-1}(\log\log x)^M$ when $x \to +\infty$, for every $M > 0$. There exists a type-dependent set A of height e and rank h such that*

$$\liminf_{x \to +\infty} \frac{A(x)}{f(x)} = 0.$$

Proof We construct an ascending sequence of type-dependent sets (A_n) of height e and rank 0 and an ascending sequence of positive numbers (x_n). Let $A_1 = \{p^e : p \in \mathcal{P}\}$. When A_n is constructed we conclude from Theorems 23 and 35 that $A_n(x) = o(f(x))$, $x \to +\infty$, so we can find $x_n > n$ such that $A_n(x) \leq f(x)/n$ for all $x \geq x_n$. Let $m = m_n = \lceil \log_2(x_n) \rceil$ and

$$A_{n+1} = A_n \cup \left\{ p_1^e \dots p_m^e : p_1, \dots, p_m \in \mathcal{P} \right\}$$

We have $\|a\| \geq 2^m$ for all $a \in A_{n+1} \setminus A_n$, so $A_{n+1}(x) = A_n(x)$ for all $x \leq x_n$. The fact that A_{n+1} is also of rank 0 follows from Facts 8 and 21. Then $A = \bigcup_{n=1}^{\infty} A_n$ satisfies

$$\lim_{n \to \infty} \frac{A(x_n)}{f(x_n)} = \lim_{n \to \infty} \frac{A_n(x_n)}{f(x_n)} = 0$$

and $\lim_{n\to\infty} x_n + \infty$. It remains to show that $\mathrm{rk}(A) = h$. Suppose the contrary and let (6) be a fair covering with $\max_i |U_i| < h$. Let $M = \max_i \sum_X \alpha_i(X)$ and let n be such that $x_n > 2^{h(M+1)}$. Then $m_n > h(M+1)$ and we can find $a \in A_{n+1}$ such that $\delta_{e,X}(a) > M + 1$ for every $X \in \mathrm{Cl}(\mathcal{S})$. Let i be such that $a \in \mathbf{\Omega}_e(U_i, \alpha_i)$ and let X be such that $X \notin U_i$. Then $\delta_{e,X}(a) = \alpha_i(X) \leq M$, a contradiction. □

Proposition 37 *Suppose $U \subseteq \mathrm{Cl}(\mathcal{S})$, $V = \mathrm{Cl}(\mathcal{S}) \setminus U$, and $\alpha, \beta : \mathrm{Cl}(\mathcal{S}) \to \mathbf{N}_0$, $\mathrm{Supp}(\beta) \subseteq \mathrm{Supp}(\alpha) = U$. Let*

$$B = \left\{ b \in \mathcal{F}(\mathcal{P}) : [p] \in U \text{ and } v_p(b) \in \alpha([p]) + \beta([p])\mathbf{N}_0 \text{ for all } p \mid b,\, p \in \mathcal{P} \right\}.$$

Suppose C is a (non-empty) type-dependent set such that $[p] \in V$ for every $p \mid c$, $c \in C$, $p \in \mathcal{P}$, moreover $\mathrm{rk}(C) = 0$ or $\gamma(C) > \inf_{X \in U} \alpha(X)$. *Then the set $A = BC$ is regular with*

$$\gamma(A) = \min(\inf_{X \in U} \alpha(X), \gamma(C)),$$
$$\mathrm{rk}(A) = \#\{X \in U : \alpha(X) = \gamma(A)\}$$

and

$$\deg(A) = \sup_{c \in C} \sum_{X \in V} \delta_{\gamma(A),X}(c),$$

where we understand $\inf_{X \in U} \alpha(X)$ *as* $+\infty$ *when* $U = \emptyset$. *If* $\gamma(C) > \inf_{X \in U} \alpha(X)$ *or* $\mathrm{metrics}(X \cap C) = \mathrm{metrics}(C)$ *for every* $X \in \mathrm{Cl}(\mathcal{S})$ *such that* $X \cap C \neq \emptyset$, *then A is regular across classes.*

Proof When $U = \emptyset$ we have $B = \{1\}$, so B is regular across classes with $\gamma(B) = +\infty$. When $U \neq \emptyset$, then $B = \prod_{X \in U} B_X$ where

$$B_X = \left\{b \in \mathcal{F}(\mathcal{P}) : p \in X \text{ and } v_p(b) \in \alpha(X) + \beta(X)\mathbf{N}_0 \text{ for all } p \mid b,\, p \in \mathcal{P}\right\}$$

for all $X \in U$. If $\beta(X) = 0$ then B_X coincides with the set $B_{\alpha(X),X}$ defined in (26). Otherwise $B_X = A_{\alpha(X),\beta(X),X}$, as defined by (25). It follows from Lemmas 24, 25 and 26, that B_X is regular across classes with $\gamma(B_X) = \alpha(X)$, $\mathrm{rk}(B_X) = 1$ and $\deg(B_X) = 0$. It follows from Propositions 10 and 20 that B is regular across classes with

$$\gamma(B) = \inf_{X \in U} \alpha(X),$$
$$\mathrm{rk}(B) = \#\{X \in U : \alpha(X) = \gamma(B)\}$$

and

$$\deg(B) = 0.$$

If $\gamma(C) > \gamma(B)$, the assertions follow from Propositions 20 and 10. Otherwise $\mathrm{rk}(C) = 0$, so C is regular by Theorem 23 and so is $Y \cap C$ for every $Y \in \mathrm{Cl}(\mathcal{S})$ such that $\gamma(Y \cap C) = \gamma(C)$. If $\mathrm{metrics}(X \cap C) = \mathrm{metrics}(C)$ whenever $X \cap C \neq \emptyset$, then C is regular across classes. By Proposition 20 the set A is regular, and if C is regular across classes, then A is regular across classes. The assertions about $\mathrm{metrics}(A)$ follow from Lemma 21 and Proposition 10. □

Proof of Theorem 1 We have

$$S = \left\{b \in \mathcal{F}(\mathcal{P}) : \mathrm{ord}([p]) \mid v_p(b) \text{ for all } p \mid b,\, p \in \mathcal{P}\right\},$$
$$S_1 = \left\{b \in \mathcal{F}(\mathcal{P}) : v_p(b) = \mathrm{ord}([p]) \text{ for all } p \mid b,\, p \in \mathcal{P}\right\},$$

and

$$E_k = \left\{ b \in \mathcal{F}(\mathcal{P}) : v_p(b) \geq k \text{ for all } p \mid b,\, p \in \mathcal{P} \right\}.$$

Proposition 37 applies to these sets with $U = \mathrm{Cl}(\mathcal{S})$ and $C = \{1\}$, so $A = B$. The functions α and β should be defined as $\alpha(X) = \beta(X) = \mathrm{ord}(X)$ in the case of S, $\alpha(X) = \mathrm{ord}(X)$ and $\beta(X) = 0$ in the case of S_1, and $\alpha(X) = k$ and $\beta(X) = 1$ in the case of E_k. The assertions follow from Theorem 35. □

Proof of Theorem 2 We have $F_k = BC$, where

$$B = \{b \in \mathcal{F}(\mathcal{P}) : [p] = E \text{ for all } p \mid b,\, p \in \mathcal{P}\}$$

and

$$C = \{a \in F_k : p \notin E \text{ for every } p \mid a,\, p \in \mathcal{P}\}.$$

Suppose $h > 1$. Then for every $X \in \mathrm{Cl}(\mathcal{S}) \setminus \{E\}$ and $p_1, \ldots, p_{\mathrm{ord}(X)} \in X \cap \mathcal{P}$ we have $p_1 \ldots p_{\mathrm{ord}(X)} \in C$, so $\gamma(C) = 1$. We note, following Narkiewicz, that if $p_1, \ldots, p_{(k+1)\,\mathrm{ord}(X)} \in X \cap \mathcal{P}$ are distinct primes dividing $a \in \mathcal{S}$ and $X \in \mathrm{Cl}(\mathcal{S}) \setminus \{E\}$, then $p_1, \ldots, p_{(k+1)\,\mathrm{ord}(X)}$ can be divided to $k+1$ groups of $\mathrm{ord}(X)$ elements in

$$\frac{((k+1)\,\mathrm{ord}(X))!}{(\mathrm{ord}(X)!)^{k+1}} > k$$

ways, giving rise to more than k distinct factorizations of a. Hence for every $a \in C$ we have $\sum_X \delta_{1,X}(a) \leq \omega(a) < h(h-1)(k+1)$, so $\mathrm{rk}(C) = 0$ and $N_k := \deg(C) < h(h-1)(k+1)$. Otherwise, if $h = 1$, we have $C = \{1\}$ and $F_k = B$, and we put $N_k := \deg(C) = 0$. In either case the assumptions of Proposition 37 are therefore satisfied with $U = \{E\}$ and $\alpha(E) = \beta(E) = 1$. The assertion follows from Theorem 35. □

Acknowledgements This work has been supported by the Polish National Science Centre Grant No. 2017/25/B/ST1/00208. The author is grateful for the remarks of Professor Jerzy Kaczorowski, that helped improve the paper, and for the referee's suggestion regarding the more specific title.

References

1. Fogels, E.: Zur Arithmetik quadratischer Zahlenkörper. Univ. Riga. Wiss. Abh. Kl. Math. Abt. **1**, 23–47 (1943)

2. Geroldinger, A., Halter-Koch, F.: Non-unique Factorizations. Chapman and Hall, Boca Raton (2006). https://doi.org/10.1201/9781420003208
3. Kaczorowski, J.: A pure arithmetical definition of the class group. Colloq. Math. **48**, 265–267 (1984). https://doi.org/10.4064/cm-48-2-265-267
4. Kaczorowski, J.: On completely irreducible algebraic integers. Colloq. Math. Soc. János Bolyai **34**, 831–853 (1984)
5. Kaczorowski, J.: Analytic monoids and factorization problems. Semigroup Forum **94**, 532–555 (2017). https://doi.org/10.1007/s00233-016-9778-9
6. Narkiewicz, W.: Numbers with unique factorization in an algebraic number field. Acta Arith. **21**, 313–322 (1972). https://doi.org/10.4064/aa-21-1-313-322
7. Radziejewski, M.: Structural results for semigroup subsets defined by factorization properties dependent on Ω functions. Semigroup Forum **89**, 609–626 (2014). https://doi.org/10.1007/s00233-014-9593-0
8. Radziejewski, M.: The asymptotic behaviour of the counting functions of Ω-sets in arithmetical semigroups. Acta Arith. **163**, 179–198 (2014). https://doi.org/10.4064/aa163-2-7