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TRIBUTE

A tribute to John Meakin on the occasion of his 75th birthday

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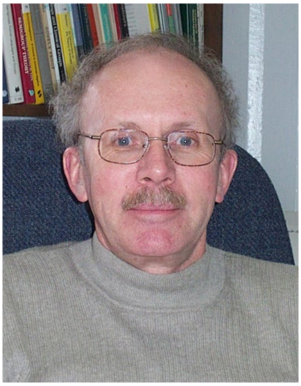
Abstract
John Meakin has had a distinguished career of over half a century in the theory of semigroups. This article gives a synopsis of his most important contributions. The author, a long time collaborator, gives his personal memories of working with John and the influence he has had on his career.



John Meakin

John Meakin was born on March 13, 1946 in Brisbane, Australia. He received his B. Sc. from the University of Queensland in 1968 and his Ph.D. at the age of 23 at Monash University under the direction of G.B. Preston. During Academic Year 1969–1970 he held a Post Doctoral position at the University of Florida. John thought of returning to Australia and because of the different academic calendars

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decided to take a one-semester Visiting Professorship at the University of Nebraska, Lincoln. Apparently Nebraska liked John and John liked Nebraska as he spent 89 additional semesters at UNL, retiring in 2015 as the Milton Mohr Distinguished Professor of Mathematics, a position that he held from 1987–2015. John was Chair of the Department of Mathematics from 2003–2011.

During his career, John was a visitor at a number of Universities and Institutions. Particularly important to his early research was a visit to the University of Kerala in 1977–1978, where he worked with Nambooripad on the then new theory of biordered sets and inductive groupoids and a visit to Rijksuniversiteit Gent in 1983–1984, where he worked with Pastijn on the four-spiral semigroup and related issues. I will describe these later in the paper. Among other institutions that John visited and where significant results followed were The Mathematical Sciences Research Institute, Berkeley, Université de Paris VI, Università di Siena, Politecnico di Milano and Xi'an Jiaotong University.

John mentored 17 Ph.D. students many of whom have had distinguished careers of their own. He has published 75 articles and given more than 200 invited seminars, colloquia and conference talks in some 30 different countries. He was an Editor for the International Journal of Algebra and Computation from 1990–2020, serving as a Managing Editor from 1990–2010. John served on a number of committees of the American Mathematical Society and was an organizer or member of the scientific advisory committee for some 20 international conferences.

John's prolific research career breaks into four distinct but related areas. I will now give a historical summary of each phase of John's work in the rest of the paper.

1 Thesis and the early 1970s

At the time of John's thesis, the school of semigroup theory in Monash was interested in generalizing results from the theory of inverse semigroups to more classes of regular semigroups. Chief among these was the class of orthodox semigroups. Indeed, John's thesis was entitled "Congruences on Orthodox Semigroups". Papers that reflect this period of John's career are [31–37]. These papers are still influential in this field of semigroup theory.

2 Structure mappings, locally inverse semigroups and the four-spiral semigroup, 1975–1984

This period in John's career started with the paper [38]. This introduced the structure mapping approach to inverse semigroups. In retrospect, it is equivalent to Schein's earlier work on inductive groupoids [22, 55], which was not well known in the West at that time. It was also a special case of Nambooripad's more general notion of inductive groupoids for regular semigroups and of biordered sets that was being developed and appeared a few years later [54]. This led to John's first Sabbatical at the University of Kerala in 1977–1978 to work with Nambooripad. This fruitful collaboration resulted in the papers [48–50].

Another paper that was written while visiting Nambooripad was [42]. John Rhodes had asked the natural question of which posets are isomorphic to the poset of $\mathcal{J}$-classes of some semigroup. It is clear that such a poset is down-directed. That is, any pair of elements has a lower bound. The main result of [42] is that any down-directed poset is the poset of $\mathcal{J}$-classes of some inverse semigroup. Note that the $\mathcal{R}$ and $\mathcal{L}$ orders of an inverse semigroup are semilattices easily seen to be isomorphic to the semilattice of idempotents of the inverse semigroup. So the result is surprising. The proof uses some subtle set theory. Chris Ash [1] independently proved this result at the same time.

During this period, John worked on extending the structure mapping approach to all regular semigroups. He also did work on co-extensions of inverse and regular semigroups, mainly by rectangular bands. This led naturally to the Rees construction over semigroups and to locally inverse semigroups and their biordered sets. Typical papers from this period are [39, 40, 43–45].

In 1983–1984, John had a Sabbatical visiting Francis Pastijn in Belgium. Pastijn was working on idempotent generated semigroups, another topic developed by Nambooripad, who constructed the free idempotent generated (regular) semigroup on a (regular) biordered set [54]. This led John to the discovery of the four-spiral semigroup. This was then extensively developed by John, Pastijn and John's then Ph.D. student Karl Byleen, first in [5], followed up by [6, 7, 41] among others.

3 Combinatorial and geometric inverse semigroup theory, 1984–2005

The above concluded John's work in what can be called the Clifford and Preston school of semigroups. That theory grew out of general semigroup theory and then inverse semigroup theory and its generalizations to orthodox and regular semigroups. Much of the rest of John's career has been involved with combinatorial and geometric inverse semigroup theory in the spirit of combinatorial and geometric group theory, and with applications of inverse semigroups in other fields of mathematics.

At this point, I made my way happily into collaborative life with John. On August 15, 1984, I moved to Lincoln, Nebraska where I was about to begin working at the University of Nebraska. I was ably helped by John to move furniture into the new house and we made an appointment to meet the next day in John's office. By the end of that day, we had constructed what is now known as the Margolis–Meakin Expansion of a group [23]. This is a functor from the category of groups to E-unitary inverse semigroups and in an appropriate setting the adjoint of the functor that sends an inverse semigroup to its maximal group image.

The paper was a harbinger of how John and I planned to work together, by combining the techniques of the Clifford–Preston school with that of the Rhodes school. Indeed, in the language of global semigroup theory, an inverse semigroup is E-unitary precisely when it has a homomorphism onto a group such that its derived category C [59] has the property that for each object c, the monoid of self-morphisms $Mor_C(c, c)$ is a semilattice. By taking the freest such category one is led to the Margolis–Meakin Expansion. That day, like for Rick and Captain Renault in the classic film Casablanca, was the beginning of a beautiful friendship. The reader will no doubt understand who is Bogey and who is Claude Rains in this analogy.

We decided to work on developing the up to then undeveloped geometric, combinatorial and topological theory of inverse semigroups by these methods. This naturally divides into two substrands, presentations of inverse semigroups and the theory of immersions and its relation to subgroups of free and related groups.

3.1 Presentation of inverse semigroups and inverse monoids

Inverse semigroups (monoids) form a variety of algebras of type $\langle 2, 1\rangle$ ($\langle 2, 1, 0\rangle$) where the unary operation is inversion (and the nullary operation is the identity element). Therefore there are free inverse semigroups (monoids) over any set X. We can thus talk about inverse semigroups (monoids) presented by generators and relations. In particular, there are a number of illuminating solutions to the word problem for free inverse semigroups, but the one most relevant from our point of view was Munn's solution via birooted finite subtrees (now called Munn trees) of the Cayley graph of the free group. This shows that the free inverse semigroup is the Margolis–Meakin Expansion of the free group and was a motivating example for us. Indeed, combinatorics and geometry of Munn trees play the same role in presentations of inverse semigroups that words play for semigroup presentations and reduced words play for group presentations.

An early and very important contribution was made by our Ph.D. student Joseph (Buck) Stephen [58]. Given a presentation of an inverse monoid, the strongly connected component of a word w of the corresponding (right) Cayley graph has vertices the elements of the $\mathcal{R}$-class of the image of w in that monoid and is the automaton of the (right) Schützenberger representation of the monoid. Stephen gave an iterative technique to build approximations to this graph that limits to it in an appropriate sense. This is essentially the Todd–Coxeter procedure adapted to inverse semigroups. Stephen emphasized the automata and language theoretic aspects of this construction. Indeed, the word problem is equivalent to deciding if a word labels a path in some finite Stephen approximate of the Schützenberger graph. Many papers thus use combinatorial and geometric arguments to assure that this latter membership problem is decidable. This had the desired effect of combining classical inverse semigroup theory with automaton theory, the theory of languages, geometry, in the sense of geometric group theory, combinatorics and topology. This method is still a major tool in the theory.

This proved to be a fruitful direction of our cooperation. The papers [24, 25, 27] give connections between inverse monoid presentations and context-free languages. Word problems for relatively free inverse semigroups, including certain Burnside varieties, are studied by these methods in [18, 29]. The methods are applied in [21] to free products of inverse semigroups, elucidating earlier work of Peter Jones.

Free products with amalgamation, HNN-extensions and analogues of Bass–Serre theory were studied. T.E. Hall proved that amalgams of inverse semigroups strongly embed into their free product with amalgamation [55]. The first paper using the methods discussed here was [16]. In [3], examples are given where the word problem for the factors and the embedding of the submonoid into the factors have decidable problems, but the free product with amalgamation has undecidable word problem. Cases where decidability can be proved were considered in [8, 9]. This continues to be a very active area with a big and growing literature.

It is natural to study one-relator inverse semigroup presentations. It is proved by topological methods in [19, 20] that if the one relator is of the form $w = 1$, where w is a cyclically reduced word over its alphabet, then the inverse monoid presented is E-unitary. Furthermore, the word problem for this presentation is reduced to the membership problem for the submonoid generated by the prefixes of w in the one-relator group with relation $w = 1$. Also, it is noted that if the word problem for one-relator inverse monoids with relation $w = 1$ is decidable, where w is a reduced word, then the one-relation problem for semigroups, a long standing open problem, is also decidable. Thus, this problem stands at the junction of semigroup theory, inverse semigroup theory and group theory.

A nice story came out of this research. John and I were waiting for a change of planes at O'Hare airport and working on an example. We wanted to find a positive word w, with no overlap such that the group of units of the inverse monoid with relator $w = 1$ was non-trivial. It is known that the monoid with this relator has a trivial group of units. We were working feverishly and getting close when I shouted out, "we have to blow up some idempotents". People around us, mistaking us for members of the Anti-Idempotent League, quickly moved to the other side of the airport.

We did find our word w at O'Hare. Recently, Igor Dolinka and Robert Gray called the inverse monoid with the presentation $w = 1$ the O'Hare Inverse Monoid. In [11] they develop methods to solve the word problem for this and related monoids. Gray also proved that the word problem for one-relator monoids with a non-reduced relator $w = 1$ can have an undecidable word problem [14], a spectacular result, considering the Magnus Theorem on one-relator groups and Adian's decidability result for one relator monoids with relator $w = 1$. Thus the problem for one-relator inverse monoids of the form $w = 1$ remains active and with deep results.

3.2 Subgroups of free groups, immersions and inverse automata

In a highly influential paper [57], John Stallings emphasized the use of immersions over a bouquet of circles to study finitely generated subgroups of free groups. From the eyes of an inverse semigroup theorist, one sees immediately that a connected immersion over a bouquet of $|X|$-circles is "the same thing" as a connected inverse automaton over a set X of generators. In turn, this is the same as a transitive representation of the free inverse monoid $FIM(X)$ by partial bijections. This observation is made precise in our paper [26], along with a number of other results. We also give a construction of all closed inverse submonoids of free inverse monoids of finite index. These are precisely the stabilizers of points in a representation of $FIM(X)$ by partial bijections on a finite set and thus play the role of finite index subgroups in free groups in the theory of free inverse monoids. Moreover, every finitely generated subgroup H of a free group is represented by a unique minimal finite inverse automaton $\mathcal{A}(H)$ and this is the connection to Stallings work. In this sense, the theory of finitely generated subgroups of free groups and immersions of graphs is a part of finite inverse semigroup theory.

One is led to consider an Eilenberg–Schützenberger type correspondence theorem [13] between varieties of finite inverse monoids and suitable collections of finitely generated subgroups of free groups. This was considered in the unpublished thesis of our joint Ph.D. student Robert Ruyle [56]. Very nice applications of these ideas appear in the paper [2]. A subgroup H of a group G is *pure* if whenever $g^n \in H$ for some $g \in G, n > 1$, then $g \in H$. Then one proves that a finitely generated subgroup of a free group is pure if and only if the transition inverse monoid of the minimal automaton $\mathcal{A}(H)$ of H is an aperiodic finite monoid, meaning that each of its subgroups is trivial. Furthermore the problem, given a finite set of generators of a subgroup of a free group to determine if the group it generates is pure, is a PSPACE-complete problem [2].

4 Applications of inverse semigroups to other fields, 2005–2020

In the last 15 years, John has applied inverse semigroup theory to topology and the theory of operator algebras. The papers [51, 52] consider immersions over more general classes of complexes than graphs. The paper [15] is also along these lines. They continue the thesis that inverse semigroup theory provides the correct algebraic tools to study immersions of complexes.

There has been an immense amount of work on C^*-algebras of inverse semigroups and related groupoids over the last few decades. John, Donsig and our late student Steven Haataja studied the relationship between amalgams of inverse semigroups and C^*-algebras in [12]. The papers [47, 53] are concerned with inverse semigroups and Leavitt path algebras and graph inverse semigroups respectively. Equations in free inverse semigroups were studied in [10].

John has continued his work on presentations of inverse semigroups. The paper [9] gives the structure of free products with amalgamations of finite inverse semigroups. Distortion functions are used in [30] to study membership problems for submonoids of groups and monoids. This has applications to the prefix monoid problem mentioned previously for one-relator groups and its relation to one-relator inverse monoid presentations. The notion of a sparse one-relator inverse semigroup and decision problems about them are studied in [17].

In a lovely summer visit to Nebraska in 2008, John, Mark Brittenham and I were led to an old question on the maximal subgroups of free idempotent generated semigroups over biordered sets [4]. We produced the first example of a biordered set whose free idempotent generated semigroup contained a non-free subgroup. This paper led to a very intensive sequence of papers by Ruskuc, Dolinka, Gray, Gould, Yang and others that have deeply elucidated the structure of free idempotent generated semigroups and its connections to group theory and other fields of mathematics.

In addition to his research work, John is an excellent lecturer and has written a number of survey articles that have been influential over the years [28, 46]. These are excellent sources for young mathematicians and researchers in related fields to get a foothold into the topics that John has pioneered.

5 Concluding remarks

Besides mathematics, John is an avid tennis player and bicyclist. He and his lovely wife Glory have traveled extensively all over the world. John has an excellent sense of humor (he had to put up with co-authors like myself) and a warm and kind manner to all his friends and colleagues. It was exciting and immensely influential in my mathematical and personal life to work closely with John over the last 36 years. We are both happy that the work we started in combinatorial and geometric aspects of inverse semigroup theory and idempotent generated semigroups has led to extensive work by many mathematicians. I am certain that all of John's many friends and colleagues wish John the best of health and happiness on the occasion of his 75th birthday, and wish him many more years of mathematics and enjoyment of life with Glory and the rest of their family.

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