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RESEARCH ARTICLE

On the radicals of linear algebraic monoids

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Abstract In this paper we consider the *Schwarz* radical of linear algebraic semigroups as defined in semigroup theory. We give some new characterizations of the complete regularity, regularity and solvability of irreducible linear algebraic monoids in terms of *Schwarz* radical data. Moreover, we give a generalization about the results of the kernel to the results of completely regular $\mathscr{J}$-classes.

Keywords Irreducible algebraic monoid · Kernel · Nilpotent · Radical · Regular · Solvable

1 Introduction

Throughout this paper, $\mathbb{Z}^+$ will denote the set of all positive integers. Let S be a semigroup and $I \subseteq S$ a (two-sided) ideal of S. Let $\sqrt{I}$ denote the set of all elements of S which satisfy that some power of them belongs to I, i.e.,

$$\sqrt{I} = \{a \in S \mid a^i \in I \text{ for some } i \in \mathbb{Z}^+\}.$$

There are five concepts of radical of S with respect to I, called the *Clifford* radical, *Luh* radical, *McCoy* radical, *Schwarz* radical, and *Ševrin* radical (see Definition 3.1), which are natural extensions of the concepts of radical of a ring. Denote the

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Clifford radical, *Luh* radical, *McCoy* radical, *Schwarz* radical, and *Ševrin* radical of S with respect to I by $\mathcal{R}^*(I)$, $\mathcal{C}(I)$, $\mathcal{M}(I)$, $\mathcal{R}(I)$, $\mathcal{L}(I)$ respectively. If a semigroup S has a kernel $\ker(S)$ (the minimal ideal of S), then the *Clifford* radical, *Luh* radical, *McCoy* radical, *Schwarz* radical, and *Ševrin* radical of S with respect to $\ker(S)$ are simply called the *Clifford* radical, *Luh* radical, *McCoy* radical, *Schwarz* radical, and *Ševrin* radical of S. Bosǎk [3] gives an example in abstract semigroups that all of the radicals mentioned above are distinct from one anther, and shows that for any semigroup S and any ideal I of S,

$$I \subseteq \mathcal{R}(I) \subseteq \mathcal{M}(I) \subseteq \mathcal{L}(I) \subseteq \mathcal{R}^*(I) \subseteq \sqrt{I} \subseteq \mathcal{C}(I) \subseteq S.$$

Kmeť [14] prove that $\mathcal{R}^*(I) = \sqrt{I} = \mathcal{C}(I)$ if and only if $\sqrt{I}$ is an ideal of S. And S is a semilattice of archimedean semigroups (i.e., $b \in S^1aS^1 \Rightarrow b^i \in S^1a^2S^1$ for some $i \in \mathbb{Z}^+$, for all $a, b \in S$), if and only if for any ideal I of S, $\sqrt{I}$ is an ideal of S (see [5] and [15]).

Semigroups which satisfy that the set $\sqrt{I}$ is a semigroup for any ideal I, are characterized in [1] by Bogdanović and Ćirić.

In this paper, we consider the above radicals in linear algebraic semigroups. A linear (or affine) algebraic semigroup S over an algebraically closed field K is both an affine variety over K and a semigroup for which the product map $S \times S \to S$ is a morphism of varieties. Then S has a kernel $\ker(S)$ (see [22, Theorem 3.28]). Moreover, by [7, II 2.3.3] S is isomorphic to a (Zariski) closed subsemigroup of total n by n matrix monoid $M_n(K)$ for some $n \in \mathbb{Z}^+$, and is strongly π-regular (i.e., some power of each element of S lies in a subgroup of S) by a theorem of Clark (see [22, Theorem 3.18]). Then we have that the *Clifford* radical, *McCoy* radical, *Schwarz* radical, and *Ševrin* radical of S are coincide, i.e.,

$$\mathcal{R}(\ker(S)) = \mathcal{M}(\ker(S)) = \mathcal{L}(\ker(S)) = \mathcal{R}^*(\ker(S)).$$

Hence we only need to consider the *Schwarz* radical of a linear algebraic semigroup. In particular, the radical-like property, introduced by J. Luh (see [17]), holds for the *Schwarz* (or *Clifford*) radical of a linear algebraic semigroup S, that is, the Rees factor semigroup $S/\mathcal{R}(\ker(S))$ of S modulo $\mathcal{R}(\ker(S))$ has zero *Schwarz* (or *Clifford*) radical.

The following theorems play very important role to study the theory of linear algebraic monoids. Note that if a linear algebraic semigroup S has a zero element, the *Schwarz* radical $\mathcal{R}(\ker(S))$ of S is the maximum nilpotent ideal of S and $\sqrt{\ker(S)}$ is the set of all nilpotent elements of S.

Theorem 1.1 [21, Theorem 2.1] *Let S be an irreducible linear algebraic semigroup with zero* 0. *Then the following conditions are equivalent:*

(i) *S is completely regular;*
(ii) *S has no non-zero nilpotent elements* (*i.e.*, $\sqrt{\ker(S)} = \{0\}$)*;*
(iii) *S is a monoid and the unit group of S is a torus.*

Theorem 1.2 [22, Theorem 7.3] *Let M be an irreducible linear algebraic monoid with zero* 0 *and unit group G. Then the following conditions are equivalent:*

(i) *G is reductive;*
(ii) *M is regular;*
(iii) *M has no non-zero nilpotent ideals* ($i.e.$, $\mathcal{R}(\ker(M)) = \{0\}$).

Theorem 1.3 [19, Theorem 23] *Let M be an irreducible linear algebraic monoid with zero* 0 *and unit group G. Then the following conditions are equivalent:*

(i) *G is solvable;*
(ii) *the nilpotents of M form an ideal of M ($i.e.$, $\mathcal{R}(\ker(M)) = \sqrt{\ker(M)}$);*
(iii) *$J^2 \subseteq J$ for all $J \in \mathcal{U}(M)$.*

These facts imply that the structure of irreducible linear algebraic monoids with zero can be characterized in terms of *Schwarz* radical (or nilpotency) data. In general, for an irreducible linear algebraic monoid M, it need not have a zero element, that is, its kernel $\ker(M)$ is nontrivial. Brion shows in [4, Corollary 3.1.5] that for any irreducible non-affine algebraic monoid (that is, it is non-affine as a variety), its kernel must be nontrivial. The kernel of the linear algebraic monoid M carries a lot of structural information about M as well as of the unit group G which are well studied by Huang (cf. [9,10,12]).

The purpose of this paper is to study the structure of linear algebraic monoids in terms of *Schwarz* radical data. We give generalizations of the above results (Theorems 1.1, 1.2, 1.3) for an irreducible linear algebraic monoid (without zero). Namely, we prove the following theorem.

Theorem 1.4 *Let M be an irreducible linear algebraic monoid with unit group G. Then*

(i) (*Theorem* 4.1) *M is completely regular if and only if $\sqrt{\ker(M)} = \ker(M)$;*
(ii) (*Theorem* 4.2) *M is regular if and only if the Schwarz radical of M is a completely simple semigroup* (*i.e.*, $\mathcal{R}(\ker(M)) = \ker(M)$)*;*
(iii) (*Corollary* 4.4) *G is solvable if and only if $\sqrt{\ker(M)}$ forms an ideal of M* (*i.e.*, $\mathcal{R}(\ker(M)) = \sqrt{\ker(M)}$) *and a maximal subgroup of* $\ker(M)$ *is solvable.*

Moreover, for any completely regular $\mathscr{J}$-class $J \in \mathcal{U}(M)$, we construct a submonoid M_J of M with kernel J, defined by

$$M_J = \{a \in M \mid aJ \subseteq J\}.$$

Then M_J is a linear algebraic monoid with kernel J. Moreover, the unit group of M_J is just the unit group of M. Hence we can generalize the known results about the kernel of a linear algebraic monoid (see [9,10,12]). For instance,

Theorem 1.5 (Corollary 5.5) *Let M be an irreducible linear algebraic monoid with unit group G. Let $J \in \mathcal{U}(M)$ be completely regular, and $e \in E(J)$. Then*

(i) $\dim R(G) = \dim E(J) + \dim R(G_e) + \dim R(eGe)$*;*
(ii) $\dim R_u(G) = \dim E(J) + \dim R_u(G_e) + \dim R_u(eGe)$*;*
(iii) *G is reductive if and only if G_e and J are both reductive groups.*

The article is organized as follows. Section 2 is for notions and notations. In Sect. 3, we work with various properties of the *Schwarz* radical of algebraic semigroups. In Sect. 4, we give characterizations of the completely regularity, regularity and solvability of irreducible linear algebraic monoids in terms of *Schwarz* radical data. In Sect. 5, we generalize the results of Sect. 4 to the case in terms of completely regular $\mathscr{J}$-classes of irreducible linear algebraic monoids.

2 Preliminaries

We now assemble some notions and notations. $\mathbb{Z}^+$ will denote the set of all positive integers. If X is a set, then $|X|$ denotes the cardinality of X. Let S be a semigroup. Let $S^1 := S \cup \{1\}$ be the natural monoid extension of S. The semigroup S is strongly π-regular (sπr) if for each $a \in S$, there exists $i \in \mathbb{Z}^+$ such that a^i lies in a subgroup of S. If $a, b \in S$, then $a|b$ (a divides b) if $xay = b$ for some $x, y \in S^1$. S is archimedean if for all $a, b \in S$, $a|b^i$ for some $i \in \mathbb{Z}^+$. Let S_α $(\alpha \in \Omega)$ denote a partition of S into subsemigroups. Then S is a semilattice (union) of S_α $(\alpha \in \Omega)$ if for all $\alpha, \beta \in \Omega$, there exists $\gamma \in \Omega$ such that $S_\alpha S_\beta \cup S_\beta S_\alpha \subseteq S_\gamma$. According to [22, Theorem 1.15], S is a semilattice of archimedean semigroups if and only if for all $a, b \in S$, $a \mid b$ implies $a^2 \mid b^i$ for some $i \in \mathbb{Z}^+$. Let $E(S)$ denote the set of all idempotents of S. Let $e \in E(S)$. We denote by J_e, L_e, R_e and H_e the $\mathcal{J}$-, $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{H}$-classes of e in S under Green's relations, respectively (see [22, Chap. 1]). Suppose S is an sπr-semigroup and J is a $\mathcal{J}$-class of S. Then J is regular if $E(J) \neq \varnothing$. Moreover, J is completely regular if J is regular and $J^2 \subseteq J$. Let $\mathcal{U}(S)$ be the set of all regular $\mathcal{J}$-classes of S. For any $J_1, J_2 \in \mathcal{U}(S)$, we denote $J_1 \leq J_2$ if $a_2 \mid a_1$ for some (all) $a_i \in J_i$, $i = 1, 2$. We write $\mathcal{U}(S)$ for the partially ordered set $(\mathcal{U}(S), \leq)$. Let $\varnothing \neq I \subseteq S$. Then I is a right ideal of S if $IS \subseteq I$; I is a left ideal of S if $SI \subseteq I$; I is an ideal of S if $S^1 I S^1 \subseteq I$. The minimum ideal of S, if it exists, is called the kernel of S, denoted by $\ker(S)$. A completely simple semigroup S is an sπr-semigroup with no ideals other than S.

Let K denote a fixed algebraically closed field. $M_n(K)$ will denote the algebra of all $n \times n$ matrices over K, and $GL_n(K)$ its unit group. Let S be a Zariski closed subsemigroup of $M_n(K)$. If $e \in E(S)$ and $a \in S$, then we let $\det_e(a) = det(eae + 1 - e)$. Thus $\det_e(a) \neq 0$ if and only if $eae \in H_e$ by [22, Remark 3.23].

Let M be an irreducible linear algebraic monoid over K with unit group G. M is regular (resp. completely regular) if it is so as a semigroup. We call M reductive (resp. semisimple, solvable, nilpotent, a d-monoid) if its unit group is reductive (resp. reductive with center 1-dimensional, solvable, nilpotent, a torus). We write $R(G)$ (resp. $R_u(G)$) for the radical (resp. unipotent radical) of G. The rank of G, denoted rank(G), is referred to as the dimension of a maximal torus of G. Let $W(G)$ denote the weyl group of G. Then by [13, Proposition 24.1A, Corollary 25.2C], $W(G)$ is finite, and G is solvable if and only if $|W(G)| = 1$. For a subset V of M, denote by $\overline{V}$ the Zariski closure of V in M. If N is a closed algebraic subsemigroup of M, let N^c be the identity component of N. If $e \in E(M)$, then we denote $M_e = \{a \in M \mid ea = ae = e\}^c$, $G^r(e) = \{x \in G \mid xe = e\}$, $G^l(e) = \{x \in G \mid ex = e\}$, and $G_e = (G^r(e) \cap G^l(e))^c$. For any subset X of M, $C^r_X(e) = \{a \in X \mid ae = eae\}$, $C^l_X(e) = \{a \in X \mid ea = eae\}$, $C_X(e) = \{a \in X \mid ae = ea\}$. Let T be a maximal torus of G. Then $\Lambda \subseteq E(\overline{T})$ is a cross-section lattice of M, if $|\Lambda \cap J| = 1$ for all $J \in \mathcal{U}(M)$ and $J_e \geq J_f$ implies $e \geq f$ for all $e, f \in \Lambda$. If $J \in \mathcal{U}(M)$, then the width of J, $\omega(J) = |J \cap E(\overline{T})|$. If $e \in E(J)$, $\omega(e) = \omega(J)$. For completely regular $\mathcal{J}$-classes, we have the following characterizations.

Theorem 2.1 [22, Remark 1.7(iii)] *Let S be a linear algebraic semigroup, $J \in \mathcal{U}(S)$. Then the following conditions are equivalent:*

(i) *J is completely regular;*
(ii) *J is completely simple;*
(iii) *$J^2 \subseteq J$;*
(iv) *$E(J)^2 \subseteq J$.*

Theorem 2.2 [22, Theorem 6.30, Corollary 6.34] *Let M be an irreducible linear algebraic monoid with unit group G, $J \in \mathcal{U}(M)$ and $e \in E(J)$. Then the following conditions are equivalent:*

(i) *J is completely regular;*
(ii) *$E(J) \subseteq \overline{B}$ for some Borel subgroup B of G;*
(iii) *$\omega(e) = 1$;*
(iv) *$e \in E(\overline{R(G)})$;*
(v) *$G = C^l_G(e)C^r_G(e)$;*
(vi) *eGe is the $\mathcal{H}$-class of e.*

[22,23] are our primary references for algebraic monoid theory, and [2,13,26] for algebraic group theory.

3 The radicals of linear algebraic semigroups

Let S be a linear algebraic semigroup and I a (two-sided) ideal of S. Let $\ker(S)$ be the kernel of S (the minimum ideal of S). The following concepts are used for abstract semigroups which are defined analogously to these concepts for rings.

An element a of S is termed a nilpotent element of S with respect to I if $a^i \in I$ for some $i \in \mathbb{Z}^+$. Let $\sqrt{I}$ denote the set of all nilpotent elements of S with respect to I, i.e.,

$$\sqrt{I} = \{a \in S \mid a^i \in I \ for\ some\ i \in \mathbb{Z}^+\}.$$

By [22, Corollary 3.30], the set $\sqrt{I}$ is closed in S. An ideal (left or right), or a subsemigroup A of S is nilpotent with respect to I if $A^i \subseteq I$ for some $i \in \mathbb{Z}^+$, and is nil with respect to I if every element of A is a nilpotent element of S with respect to I. For simplicity, an ideal A (left or right) is nilpotent (resp., nil) if A is nilpotent (resp., nil) with respect to $\ker(S)$. An ideal A of S is locally nilpotent with respect to I if every subsemigroup $S_1 \subseteq S$, generated by a finite number of elements of A, is nilpotent with respect to I. An ideal P of S is called a prime ideal of S if $I_1 I_2 \subseteq P$ implies that $I_1 \subseteq P$ or $I_2 \subseteq P$ where I_1 and I_2 are ideals of S. An ideal P of S is called a completely prime ideal of S if for any two elements $a, b \in S$, $ab \in I$ implies that $a \in I$ or $b \in I$. Evidently a completely prime ideal is a prime ideal. Now we define some radicals of S with respect to I.

Definition 3.1 Let S be a linear algebraic semigroup, and I an ideal of S.

- (i) The union $\mathcal{L}(I)$ of all locally nilpotent ideals of S with respect to I is called the $\check{S}evrin$ (or locally nilpotent) radical of S with respect to I.
- (ii) The union $\mathcal{R}^*(I)$ of all nil ideals of S with respect to I is called the $Clifford$ (or nil) radical of S with respect to I [6].
- (iii) The union $\mathcal{R}(I)$ of all nilpotent ideals of S with respect to I is called the $Schwarz$ (or nilpotent) radical of S with respect to I [24].
- (iv) The intersection $\mathcal{M}(I)$ of all prime ideals of S containing I is called the $McCoy$ (or prime) radical of S with respect to I [17].
- (v) The intersection $\mathcal{C}(I)$ of all completely prime ideals of S containing I is called the Luh (or completely prime) radical of S with respect to I [17].

Obviously, every nilpotent (left or right) ideal of S with respect to I is nil with respect to I. By [22, Corollary 3.16], there exists $n \in \mathbb{Z}^+$ such that S is isomorphic to a (Zariski) closed subsemigroup of $M_n(K)$, and thus for all $a \in S$, a^n lies in a subgroup of S following [22, Theorem 3.18]. Hence

$$\sqrt{I} = \{a \in S \mid a^n \in I\}.$$

Then every nil (left or right) ideal of S with respect to I is nilpotent with respect to I. Therefore, an ideal of S is nil with respect to I if and only if it is nilpotent with respect to I. Thus $\mathcal{R}(I) = \mathcal{R}^*(I)$. It is known that (see [3])

$$I \subseteq \mathcal{R}(I) \subseteq \mathcal{M}(I) \subseteq \mathcal{L}(I) \subseteq \mathcal{R}^*(I) \subseteq \sqrt{I} \subseteq \mathcal{C}(I) \subseteq S.$$

So the $Clifford$, $McCoy$, $Sevrin$, $Schwarz$ radicals of S with respect to I coincide, that is,

$$\mathcal{R}(I) = \mathcal{M}(I) = \mathcal{L}(I) = \mathcal{R}^*(I).$$

Hence we only consider the $Schwarz$ radical of a linear algebraic semigroup S with respect to an ideal I.

Throughout this paper, we use the notation $\mathcal{R}(I)$ to denote the $Schwarz$ radical of a linear algebraic semigroup S with respect to an ideal I of S. We write $\mathcal{R}_S(I)$, if we want to specify S. For simplicity, the $Schwarz$ radical of S with respect to $\ker(S)$ is called the $Schwarz$ radical of S, denoted by $\mathcal{R}\ker(S)$. Clearly, $\mathcal{R}(I)$ is the largest nilpotent (or nil) ideal of S with respect to I. Since every ideal of S contained in $\sqrt{I}$ is nil with respect to I, it is easy to see that $\mathcal{R}(I)$ is also the largest ideal of S contained in $\sqrt{I}$. The following lemma shows that $\mathcal{R}(I)$ is the largest nilpotent (or nil) left (or right) ideal of S with respect to I.

Lemma 3.2 *Let S be a linear algebraic semigroup and I an ideal of S. Then $\mathcal{R}(I)$ contains every nilpotent left (or right) ideal of S with respect to I.*

Proof Let A be a nilpotent left ideal of S with respect to I. Then there exists some $i \in \mathbb{Z}^+$ such that $A^i \subseteq I$. So $(AS^1)^i \subseteq A^i S^1 \subseteq IS^1 \subseteq I$. Hence AS^1 is a nilpotent ideal of S with respect to I. Therefore $A \subseteq AS^1 \subseteq \mathcal{R}(I)$. Similarly, we have that every nilpotent right ideal of S with respect to I is also contained in $\mathcal{R}(I)$. □

By Lemma 3.2, we have the following lemma directly.

Lemma 3.3 *Let S be a closed subsemigroup of $M_n(K)$, I an ideal of S. Then*

$$\begin{aligned}\mathcal{R}(I) &= \{a \in S \mid (xay)^n \in I \textit{ for all } x, y \in S^1\}\\ &= \{a \in S \mid (ay)^n \in I \textit{ for all } y \in S^1\}\\ &= \{a \in S \mid (xa)^n \in I \textit{ for all } x \in S^1\}\end{aligned}$$

Proposition 3.4 *Let M be a linear algebraic monoid with unit group G and let I be an ideal of M. Then*

$$\begin{aligned}\mathcal{R}(I) &= \{a \in M \mid GaG \subseteq \sqrt{I}\}\\ &= \{a \in M \mid aG \subseteq \sqrt{I}\}\\ &= \{a \in M \mid Ga \subseteq \sqrt{I}\}.\end{aligned}$$

Proof Without loss of generality, we may assume that M is a closed submonoid of $M_n(K)$ for some $n \in \mathbb{Z}^+$. By Lemma 3.3, we have

$$\mathcal{R}(I) = \{a \in M \mid MaM \subseteq \sqrt{I}\} = \{a \in M \mid aM \subseteq \sqrt{I}\} = \{a \in M \mid Ma \subseteq \sqrt{I}\}.$$

If $a \in \mathcal{R}(I)$, then $GaG \subseteq MaM \subseteq \sqrt{I}$. If $b \in M$ with $GbG \subseteq \sqrt{I}$, since $\sqrt{I}$ is closed in M, we have

$$MbM \subseteq \overline{GbG} = \overline{MbM} \subseteq \sqrt{I},$$

which implies $b \in \mathcal{R}(I)$. Therefore,

$$\mathcal{R}(I) = \{a \in M \mid GaG \subseteq \sqrt{I}\}.$$

Similarly, we can get

$$\mathcal{R}(I) = \{a \in M \mid aG \subseteq \sqrt{I}\} = \{a \in M \mid Ga \subseteq \sqrt{I}\}.$$

□

Proposition 3.5 *Let S be a linear algebraic semigroup, I an ideal of S and $e \in E(S)$. Then $\mathcal{R}(I)$ is closed in S.*

In particular,

$$\mathcal{R}(SeS) = \bigcap_{f \in E(S)\backslash E(SeS)} I(f),$$

where $I(f) = \{a \in S \mid a \nmid f\}$.

Proof Consider the product $\mu : S \times S \to S$ by $\mu(a, b) = ab$ for any $a, b \in S$. As $\mu(S \times \mathcal{R}(I)) = S\mathcal{R}(I) \subseteq \mathcal{R}(I)$, we deduce that

$$\mu(S \times \overline{\mathcal{R}(I)}) = \mu(\overline{S \times \mathcal{R}(I)}) \subseteq \overline{\mu(S \times \mathcal{R}(I))} \subseteq \overline{\mathcal{R}(I)}.$$

Similarly, we can get $\mu(\overline{\mathcal{R}(I)} \times S) \subseteq \overline{\mathcal{R}(I)}$. So $S\overline{\mathcal{R}(I)}S \subseteq \overline{\mathcal{R}(I)}$, which implies that $\overline{\mathcal{R}(I)}$ is an ideal of S. Since $\sqrt{I}$ is closed in S, $\overline{\mathcal{R}(I)} \subseteq \sqrt{I}$. By the maximality of $\mathcal{R}(I)$, we get $\overline{\mathcal{R}(I)} = \mathcal{R}(I)$, thus $\mathcal{R}(I)$ is closed in S.

Let $a \in \mathcal{R}(SeS)$ and $f \in E(S)\backslash E(SeS)$. If $a \mid f$, then $f \in SaS \subseteq \mathcal{R}(SeS)$, thus $f \in E(\mathcal{R}(SeS)) = E(SeS)$, a contradiction. So $a \nmid f$, and thus $a \in I(f)$. Hence

$$\mathcal{R}(SeS) \subseteq \bigcap_{f \in E(S)\backslash E(SeS)} I(f).$$

On the other hand, suppose $a \in S$ and $a \nmid f$ for every $f \in E(S)\backslash E(SeS)$. We claim that $a \in \mathcal{R}(SeS)$. In fact, if not, there exists $y \in S$, such that $(ay)^n \notin SeS$ for some $n \in \mathbb{Z}^+$ by Lemma 3.3. Thus there exits $f \in E(S)\backslash E(SeS)$ such that $(ay)^n \in H_f$. Then $a \mid f$, a contradiction. So $a \in \mathcal{R}(SeS)$. Therefore,

$$\mathcal{R}(SeS) = \bigcap_{f \in E(S)\backslash E(SeS)} I(f).$$

□

For a linear algebraic semigroup S with kernel $\ker(S)$, recall that an idempotent $e \in E(S)$ is called primitive if $E(eSe) \setminus E(\ker(S)) = \{e\}$, that is, for any $f \in E(S)$, $e \gneqq f$ implies $f \in E(\ker(S))$. The following corollary gives a characterization of the *Schwarz* radical of an irreducible linear algebraic monoid in terms of primitive idempotents as follows.

Corollary 3.6 *Let M be an irreducible linear algebraic monoid with unit group G. Let T be a maximal torus of G, $\Lambda \subseteq E(\overline{T})$ a cross-section lattice of M. Then*

$$\mathcal{R}\ker(M) = \bigcap_{e \in \Lambda_0} I(e),$$

where $I(e) = \{a \in M \mid a \nmid e\}$ *and* $\Lambda_0 = \{e \in \Lambda \mid e$ *is primitive*$\}$.

Proof First, we claim that if $e, f \in E(M)$ with $e \leq f$, then $I(e) \subseteq I(f)$. In fact, for any $a \in I(e)$, if $a \notin I(f)$, then $a \mid f$. Since $f \mid e$, we have $a \mid e$, a contradiction. Let $\Omega = \{e \in E(M) \mid e \text{ is primitive}\}$. By [22, Corollary 6.9, Theorem 6.20], for any $e \in E(M) \backslash E(\ker(M))$, there exists $e' \in \Omega$ such that $e \geq e'$. Hence, by Proposition 3.5,

$$\mathcal{R}\ker(M) = \bigcap_{e \in E(M) \backslash E(\ker(M))} I(e) = \bigcap_{e \in \Omega} I(e).$$

Next, we claim that if $e, f \in E(M)$ with $J_e = J_f$, then $I(e) = I(f)$. Since $J_e = J_f$, by [22, Corollary 6.8], there exists $x \in G$ such that $x^{-1}ex = f$. Let $a \in I(e)$. If $a \mid f$, then $f \in MaM$. Thus $e = xfx^{-1} \in MaM$, which implies $a \mid e$, a contradiction. So $I(e) \subseteq I(f)$. Similarly, we get $I(f) \subseteq I(e)$.Therefore, $I(e) = I(f)$. According to [22, Corollary 6.10], $E(M) = \bigcup_{x \in G} x^{-1}E(\overline{T})x$. So

$$\mathcal{R}\ker(M) = \bigcap_{e \in \Omega} I(e) = \bigcap_{e \in \Lambda_0} I(e),$$

where $\Lambda_0 = \{e \in \Lambda \mid e \text{ is primitive}\}$. □

Remark 3.7 (i) For an irreducible linear algebraic monoid M with unit group G, if $e \in E(M)$ such that the $\mathscr{J}$-class J_e of M is completely regular, then by the proof of Proposition 5.8,

$$I(e) = \{a \in M \mid a \nmid e\} = \{a \in M \mid det_e(a) = 0\},$$

and $I(e)$ is the union of some irreducible components of the non-units of M. In particular, if M is solvable with zero, then for any $e \in E(M)$, the $\mathscr{J}$-class J_e of M is completely regular by [22, Corollary 6.32]. Hence following Corollary 3.6, we obtain that

$$\mathcal{R}\ker(M) = \{a \in M \mid det_{e_i}(a) = 0,\ 1 \leq i \leq n\},$$

Where $\{e_i \mid 1 \leq i \leq n\}$ are the set of all primitive idempotents of $E(\overline{T})$ for a maximal torus T of G.

For example, let M be the set of all upper triangular matrices in $M_3(K)$, i.e.,

$$M = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix} := \left\{ \begin{pmatrix} a_1 & b_1 & c \\ 0 & a_2 & b_2 \\ 0 & 0 & a_3 \end{pmatrix} \mid a_i, b_j, c \in K, 1 \leq i \leq 3, 1 \leq j \leq 2 \right\}.$$

Then $\ker(M) = \{0\}$, and the *Schwarz* radical of M, $\mathcal{R}\ker(M) = \begin{pmatrix} 0 & K & K \\ 0 & 0 & K \\ 0 & 0 & 0 \end{pmatrix}$.

Choose

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus

$$I(e_1) = \begin{pmatrix} 0 & K & K \\ 0 & K & K \\ 0 & 0 & K \end{pmatrix},\ I(e_2) = \begin{pmatrix} K & K & K \\ 0 & 0 & K \\ 0 & 0 & K \end{pmatrix},\ I(e_3) = \begin{pmatrix} K & K & K \\ 0 & K & K \\ 0 & 0 & 0 \end{pmatrix}.$$

So

$$\mathcal{R}\ker(M) = \bigcap_{1\le i\le 3} I(e_i).$$

Note that the e_is are primitive idempotents of M, and the $I(e_i)$s are exactly the irreducible components of the set of all non-units of M.

(ii) Even thought the *Schwarz* radical of any linear algebraic semigroup is closed which is showed in Proposition 3.5, it may not be irreducible. Let

$$M = \left\{ a \otimes b \mid a, b \in \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \right\},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then the *Schwarz* radical of M, $\mathcal{R}\ker(M) = S_1 \cup S_2$, where

$$S_1 = \left\{ a \otimes E_{12} \mid a \in \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \right\},$$
$$S_2 = \left\{ E_{12} \otimes b \mid b \in \begin{pmatrix} K & K \\ 0 & K \end{pmatrix} \right\}.$$

So $\mathcal{R}\ker(M)$ is reducible.

Now we discuss the properties of the minimal non-nilpotent ideals of linear algebraic semigroups determined by primitive idempotents. We have the following proposition which is similar to the result of [16, Theorem 1] in compact semigroups with zero.

Proposition 3.8 *Let S be a linear algebraic semigroup, $e \in E(S)$. Then the following are equivalent:*

(i) *e is a primitive idempotent of S;*
(ii) *the set $eSe \setminus \sqrt{\ker(S)}$ is a group;*
(iii) *Se is a minimal non-nilpotent left ideal of S;*

(iv) *eS is a minimal non-nilpotent right ideal of S;*
(v) *SeS is a minimal non-nilpotent ideal of S;*
(vi) *each idempotent in $SeS \setminus \ker(S)$ is a primitive idempotent of S.*

Proof (i) $\Longrightarrow$ (ii): Assume that e is a primitive idempotent of S. Then we have $H_e \subseteq eSe\setminus\sqrt{\ker(S)}$. Let $a \in eSe\setminus\sqrt{\ker(S)}$. Then $a^m \in H_f$ for some $f \in E(eSe)$ and some $m \in \mathbb{Z}^+$. Since $a \notin \sqrt{\ker(S)}$, we have $f \notin E(\ker(S))$, which implies $f = e$. Hence $a^m \in H_e$, and then $a \in H_e$. Therefore, the set $eSe \setminus \sqrt{\ker(S)} = H_e$ is the maximal subgroup of S containing e.

(ii) $\Longrightarrow$ (iii): Assume (ii) and that I_l is a left non-nilpotent ideal of S contained in Se. We claim that there exists an idempotent $f \in E(I_l) \setminus E(\ker(S))$. Since S is a linear algebraic semigroup, I_l is a non-nil ideal of S. Thus there exists $x \in I_l \setminus \sqrt{\ker(S)}$. But by [22, Theorem 3.18], there exist $f \in E(S)$ and $m \in \mathbb{Z}^+$ such that $x^m \in H_f \cap I_l$. So there exists $y \in S$ such that $yx^m = f \in I_l$, since I_l is a left ideal of S. Since $x \notin \sqrt{\ker(S)}$ and $x^m \in H_f$, we have $f \notin E(\ker(S))$. Now $f \in Se$, implying $f = fe$. Then $ef = efef \in E(eSe)$. Since $fef = f \notin E(\ker(S))$, $ef \notin E(\ker(S))$. Since $eSe \setminus \sqrt{\ker(S)}$ is a group and $ef \in E(eSe \setminus \sqrt{\ker(S)})$, $e = ef \in I_l$. Hence $I_l = Se$.

(iii) $\Longrightarrow$ (v): Suppose I is a non-nilpotent ideal of S contained in SeS. Then I is a non-nil ideal of S, as S is a linear algebraic semigroup. Thus there exist $x \in I \setminus \sqrt{\ker(S)}$ and $m \in \mathbb{Z}^+$ such that $x^m \in H_f \cap I$ for some $f \in E(S) \setminus E(\ker(S))$. So there exists some $y \in S$ such that $f = yx^m \in I \subseteq SeS$. Hence there exist $a, b \in S$ such that $f = aeb$. Let $g = bfae \in I$. Then $g^2 = bfaebfae = bfae = g \in E(Se)$. Since $f \notin E(\ker(S))$ and $f = aegb$, $g \notin E(\ker(S))$. By the minimality of Se, we have $Se = Sg$. Thus $SeS = SgS \subseteq I$, implying $I = SeS$

(v) $\Longrightarrow$ (i): Assume (v). If e were not primitive, there would exist $f \in E(eSe) \setminus E(\ker(S))$ such that $e \neq f$. Thus SfS in a non-nilpotent ideal of S contained in SeS. By the minimality of SeS, $SfS = SeS$. Hence $f \in eSe \cap J_e$. By [22, Theorem 1.4(iii)], $eSe \cap J_e = H_e$. Then $f = e$, a contradiction. Therefore, e is a primitive idempotent of S;

The result (vi) $\Longleftrightarrow$ (i) follows from (v) $\Longleftrightarrow$ (i). And by symmetry, we finish the proof. □

In a similar way in [6, Theorem 1.1] and by Proposition 3.8, we have the following proposition.

Proposition 3.9 *Let S be a linear algebraic semigroup and let I be a minimal non-nilpotent ideal of S. Then any proper ideal of I is nilpotent. In particular, there exists a primitive idempotent $e \in E(S)$ such that $I = SeS$.*

Proof Note that the kernel of I is equal to $\ker(S)$ by [25]. Let A be a proper ideal of I. Suppose (by way of contradiction) A is not a nilpotent ideal of I with respect to $\ker(S)$. Then IAI is a two-sided ideal of S contained in I, and $IAI \subseteq A \subsetneq I$. By the minimality of I, IAI is a nilpotent ideal of S, and thus $IAI \subseteq \mathcal{R}\ker(S)$. Then $(IA)^2 = IAIA \subseteq \mathcal{R}\ker(S)A \subseteq \mathcal{R}\ker(S)$. So IA is a nilpotent left ideal of S. Similarly, AI is a nilpotent right ideal of S.

Now, SAS is a two-sided ideal of S contained in I. By the minimality of I, we have that SAS is a nilpotent ideal of S or $SAS = I$. In either event, since IA is nilpotent, we have that $(SA)^2 = SASA$ is nilpotent. Hence SA is a nilpotent left ideal of S. Therefore, $SA \subseteq \mathcal{R}\ker(S)$. Thus there exists some $m \in \mathbb{Z}^+$ such that $(SA)^m \subseteq \ker(S)$. So $A^{2m} = (A \cdot A)^m \subseteq (S \cdot A)^m \subseteq \ker(S)$. Hence A is a nilpotent ideal of I, a contradiction. Therefore, we have that any proper ideal of I is nilpotent.

Since I is non-nilpotent, I is non-nil, and thus there exists $a \in I$ such that $a \notin \sqrt{\ker(S)}$. By [22, Theorem 3.18], there exist some $n \in \mathbb{Z}^+$ and $e \in E(S)$ such that $a^n \in H_e$. Clearly, $e \notin E(\ker(S))$. Hence there exists $b \in S$ such that $e = a^n b$, which implies $e \in E(I) \setminus E(\ker(S))$. So SeS is a non-nilpotent ideal of S contained in I, which implies $SeS = I$. By Proposition 3.8, e is a primitive idempotent of S. □

4 The structure of linear algebraic monoids in terms of Schwarz radical data

For an irreducible linear monoid M, it is easy to see that

$$\ker(M) \subseteq \mathcal{R}\ker(M) \subseteq \sqrt{\ker(M)}.$$

In the Sect. 4.1, we give a characterization of the condition that $\ker(M) = \mathcal{R}\ker(M)$. And in the Sect. 4.2, we give a characterization of the condition that $\mathcal{R}\ker(M) = \sqrt{\ker(M)}$.

4.1 Completely regularity and regularity conditions

Theorem 4.1 *Let M be an irreducible linear algebraic monoid. Then the following are equivalent:*

(i) *M is completely regular;*
(ii) $\sqrt{\ker(M)} = \ker(M)$*;*
(iii) $\{a \in M \mid a^i = f$ *for some* $i \in \mathbb{Z}^+\} \subseteq fGf$ *for every* $f \in E(\ker(M))$.

Proof (i) $\Longrightarrow$ (ii). Suppose M is completely regular and $a \in \sqrt{\ker(M)}$. Then there exists $f \in E(\ker(M))$ such that $a \in H_f$. Hence $a \in \ker(M)$ as $H_f \subseteq \ker(M)$. So $\sqrt{\ker(M)} = \ker(M)$.

(ii) $\Longrightarrow$ (iii). Assume that $\sqrt{\ker(M)} = \ker(M)$. Let $f \in E(\ker(M)$. If $a^i = f$ for some $i \in \mathbb{Z}^+$, then $a \in \sqrt{\ker(M)}$, and thus $a \in \ker(M)$. So there exists $e \in E(M)$ such that $a \in H_e$. Then $a^i = f \in H_e$ which implies $e = f$. Therefore, $a \in H_f$. By Theorem 2.2, we have that $H_f = fGf$. Therefore, $a \in fGf$.

(iii) $\Longrightarrow$ (i). Suppose (iii) holds. For any $f \in E(\ker(M)$, if a is a nilpotent element of M_f, then $a^i = f$ for some $i \in \mathbb{Z}^+$. Thus $a \in fGf \cap M_f = \{f\}$, then $a = f$. This shows that M_f has no non-zero nilpotent elements, hence by [22, Theorem 5.12], M_f is completely regular. Following [22, Theorem 7.4], we obtain that M is regular. Moreover, by[18, Theorem 3.7], $E(M_f) = \{e \in E(M) \mid e \geq f\}$ is finite. Therefore, M is completely regular following [20, Theorem 3.10]. □

Theorem 4.2 *Let M be an irreducible linear algebraic monoid. Then M is regular if and only if the Schwarz radical of M is completely simple (i.e., $\mathcal{R}\ker(M) = \ker(M)$).*

Proof By [22, Theorem 3.15], M is isomorphic to a closed submonoid of $M_n(K)$ for some $n \in \mathbb{Z}^+$, thus $\sqrt{\ker(M)} = \{a \in M \mid a^n \in \ker(M)\}$. Suppose M is regular. For any $a \in \mathcal{R}\ker(M)$, there exists $e \in E(M)$ such that $MaM = MeM$. So

$$e \in MeM = MaM \subseteq \mathcal{R}\ker(M),$$

which implies $e \in \mathcal{R}\ker(M)$. Thus $e = e^n \in \ker(M)$. Therefore,

$$MaM = MeM = \ker(M),$$

and then $a \in \ker(M)$. This prove that $\mathcal{R}\ker(M) = \ker(M)$.

Suppose $\mathcal{R}\ker(M) = \ker(M)$. Let G be the unit group of M. If M is not regular, then $\overline{R(G)}$ is not completely regular following [22, Theorem 7.4]. Hence, from Theorem 4.1, there exist $a \in \overline{R(G)}$ and $f \in E(\ker(\overline{R(G)}))$, such that $a^n = f$ and $a \notin f\overline{R(G)}f$.

Let $x \in M$. Then $x \in \overline{B}$ for some Borel subgroup B of G. Now $a \in \overline{R(G)} \subseteq \overline{B}$. From [22, Corollaries 1.16, 1.17, 3.20], we have that for all $b_1, b_2 \in \overline{B}$, $b_1 \mid b_2$ implies $b_1^n \mid b_2^n$. So $a \mid ax$ implies $a^n \mid (ax)^n$, i.e. $f \mid (ax)^n$. But $(ax)^n \mid f$ following $f \in E(\ker(\overline{R(G)}))$ and $E(\ker(\overline{R(G)})) \subseteq E(\ker(M))$. So $f \mathscr{J} (ax)^n$. Thus $(ax)^n \in \ker(M)$, $ax \in \sqrt{\ker(M)}$. This proves that $aM \subseteq \sqrt{\ker(M)}$. Thus by Lemma 3.3, $a \in \mathcal{R}\ker(M) = \ker(M)$. Since $a^n = f$, we have $f \in aM$, $fM \subseteq aM$. But by [9, Lemma 3(2)], aM is a minimal right ideal of M since $a \in \ker(M)$. So $fM = aM$, $a \in fM$. Similarly, $a \in Mf$. Therefore, $a \in fMf$, and thus $a \in f\overline{R(G)}f$, a contradiction. So $\overline{R(G)}$ is completely regular, and hence M is regular.

Finally, by Proposition 3.5, $\mathcal{R}\ker(M)$ is closed. Therefore, $\mathcal{R}\ker(M)$ is a linear algebraic semigroup which implies that $\mathcal{R}\ker(M)$ is an $s\pi r$-semigroup. Note that the kernel of $\mathcal{R}\ker(M)$ is just $\ker(M)$. Hence we have that $\mathcal{R}\ker(M) = \ker(M)$ if and only if $\mathcal{R}\ker(M)$ is completely simple. □

4.2 The solvability condition

Theorem 4.3 *Let M be an irreducible linear monoid with unit group G and $e \in E(\ker(M))$. Then the following are equivalent:*

(i) *G_e is solvable;*
(ii) $|W(G)| = |W(H_e)|$;
(iii) *M is a semilattice of archimedean semigroups;*
(iv) *$\sqrt{\ker(M)}$ forms an ideal of M (i.e., $\mathcal{R}\ker(M) = \sqrt{\ker(M)}$);*
(v) *$\mathcal{U}(M)$ is a relatively complement lattice.*

Proof That (i) $\iff$ (iii) $\iff$ (v) follows from [20, Theorem 2.15]. Let T be a maximal torus of G. By [22, Proposition 6.25] and [22, Theorem 6.16], we know that $|W(G)| = \omega(e)|W(C_G(e)|$ where $\omega(e) = |J_e \cap E(\overline{T})|$, and $|W(C_G(e))| = |W(G_e)| \cdot |W(H_e)|$. Since $e \in E(\ker(M))$, we have $\omega(e) = 1$, thus $|W(G)| = |W(C_G(e)| = |W(G_e)| \cdot |W(H_e)|$. So (i) $\iff$ (ii).

By [22, Theorem 3.15], M is isomorphic to a closed submonoid of $M_n(K)$ for some $n \in \mathbb{Z}^+$, thus

$$\sqrt{\ker(M)} = \left\{a \in M \mid a^n \in \ker(M)\right\}.$$

If M is a semilattice of archimedean semigroups, by [22, Theorem 1.15], for every $a, b \in M$, $a \mid b$ implies $a^2 \mid b^i$ for some $i \in \mathbb{Z}^+$. Let $a \in \sqrt{\ker(M)}$, $b \in M$. since $a \mid ab$, we have $a^n \mid (ab)^n$. Moreover, $(ab)^n \mid a^n$ following $a^n \in \ker(M)$. So $a^n \mathscr{J} (ab)^n$, $ab \in \sqrt{\ker(M)}$. Similarly, $ba \in \sqrt{\ker(M)}$. Hence $MaM \in \sqrt{\ker(M)}$. This proves that $\sqrt{\ker(M)}$ is an ideal of M, that is $\mathcal{R}\ker(M) = \sqrt{\ker(M)}$. Thus (iii) $\implies$ (iv).

Now we clam that (iv) $\implies$ (iii). Suppose $\mathcal{R}\ker(M) = \sqrt{\ker(M)}$. Let $J \in \mathcal{U}(M)$ such that J covers $\ker(M)$. By [19, Theorem 23, Remark 24], we just need to show that $\omega(J) = |J \cap E(\overline{T})| = 1$. Suppose $\omega(J) > 1$, then there exist $e, f \in J \cap E(\overline{T})$, such that $e \neq f$. By [22, corollary 6.8], there exists $x \in G$ such that $f = xex^{-1}$. Since $e, f \in E(\overline{T})$, $ef = fe \in E(M)$. So $ef = fe \le e, f$. But $ef \neq e$, $ef \neq f$. Thus $ef = fe \in E(\ker(M))$ as J covers $\ker(M)$. Then $exex^{-1} = xex^{-1}e \in E(\ker(M))$. So $exe, ex^{-1}e \in \ker(M)$. Hence

$$(ex)(ex) = (exe)x \in \ker(M),\ (x^{-1}e)(x^{-1}e) = x^{-1}(ex^{-1}e) \in \ker(M).$$

Therefore, $ex, x^{-1}e \in \sqrt{\ker(M)}$. Since $\sqrt{\ker(M)}$ is an ideal of M, $e = (ex)(x^{-1}e) \in \sqrt{\ker(M)}$, which implies that $e \in E(\ker(M))$, a contradiction. This shows that $\omega(J) = 1$. Therefore, M is a semilattice of archimedean semigroups. □

Corollary 4.4 *Let M be an irreducible monoid with unit group G. Then G is solvable if and only if $\sqrt{\ker(M)}$ forms an ideal of M and a maximal subgroup of* $\ker(M)$ *is solvable.*

Proof It is known that G is solvable if and only if $|W(G)| = 1$. Let $e \in E(\ker(M))$. From the proof of Theorem 4.3, we have that $|W(G)| = |W(G_e)| \cdot |W(H_e)|$, and $|W(G_e)| = 1$ if and only if $\sqrt{\ker(M)}$ forms an ideal of M. So G is solvable if and only if $\sqrt{\ker(M)}$ forms an ideal of M and the subgroup H_e is solvable. □

5 On completely regular $\mathcal{J}$-classes

It is known that the kernel of a linear algebraic semigroup S is a completely regular $\mathcal{J}$-class of S, and every completely regular $\mathcal{J}$-class of S is completely simple (as a semigroup). In this section, we generalize the results about the kernel to completely regular $\mathcal{J}$-classes of linear algebraic semigroups.

The following construction plays an important role in this section.

Proposition 5.1 *Let S be a linear algebraic semigroup. Let $J \in \mathcal{U}(S)$ be completely regular and let $S_J = \{a \in S \mid aJ \subseteq J\}$. Then S_J is a linear algebraic semigroup with kernel J. If S is irreducible, S_J is irreducible.*

Proof By [22, Corollary 3.16], we can assume that S is a closed subsemigroup of some $M_n(K)$. For any $a, b \in S_J$, $abJ \subseteq aJ \subseteq J$, so $ab \in S_J$, which implies S_J is a semigroup. Since J is completely regular, $J^2 \subseteq J$, thus $J \subseteq S_J$. Let $e \in E(J)$. Now we claim that $S_J = \{a \in S \mid det_e(a) \neq 0\}$. Let $a \in S$ such that $eae \in J$. Then $e \mid ea \mid eae \mid e$, thus $ea \in J$. Then for any $b \in J$, we have $eab \in J$. So $ab \mid eab \mid b \mid ab$, thus $ab \in J$ which implies $aJ \subseteq J$. On the other hand, if $b \in S$ with $bJ \subseteq J$, then $ebe \in J$ since $J^2 \subseteq J$. Hence we get $eae \in J$ if and only if $aJ \subseteq J$, i.e., $S_J = \{a \in S \mid det_e(a) \neq 0\}$. Similarly, we have $eae \in J$ if and only if $Ja \subseteq J$. But by [22, Remark 3.23], we obtain that $eae \in J$ if and only if $\det_e(a) \neq 0$. So

$$S_J = \{a \in S \mid Ja \subseteq J\} = \{a \in S \mid det_e(a) \neq 0\}.$$

Note that J is an ideal of S_J and J is completely simple, thus J is the kernel of S. Since $S_J = \{a \in S \mid det_e(a) \neq 0\}$ which is open in S, S_J is a linear algebraic semigroup with kernel J, and if S is irreducible, S_J is irreducible. □

Throughout this section, we use the notation S_J defined by

$$S_J = \{a \in S \mid aJ \subseteq J\},$$

for a linear algebraic semigroup S and a completely regular $\mathcal{J}$-class J of S. Let $e \in E(S_J)$. By [22, Remark 1.3(iii)], the $\mathcal{H}$-class of e in S is equal to the $\mathcal{H}$-class of e in S_J. Moreover, if $f \in E(S)$ with $e\mathcal{L}f$ in S, then $ef = e$, $fe = f$. Thus $f \in S_J$ and $e\mathcal{L}f$ in S_J. Therefore, we have that the two sets of idempotents of the $\mathcal{L}$-class of e in S and in S_J coincide. Similarly, the two sets of idempotents of the $\mathcal{R}$-class of e in S and in S_J coincide. Then applying the results of the structure of the kernel of S described in [9] for S_J, it is easy to give the structure of completely regular $\mathcal{J}$-classes of S by the following two corollaries directly.

Corollary 5.2 *Let S be a linear algebraic semigroup. Let $J \in \mathcal{U}(S)$ be completely regular. Then*

(i) $$J = \bigcup_{h \in E(J)} H_h;$$

(ii) *for any $e, f \in E(J)$, H_e is isomorphic to H_f as an algebraic group under the morphism given by $x \mapsto gxf$ for some $g \in E(J)$.*

Corollary 5.3 *Let S be a linear algebraic semigroup. Let $J \in \mathcal{U}(S)$ be completely regular, and $e \in E(J)$. Then*

(i) *$J = E(L_e)H_eE(R_e)$;*

(ii) *under the Rees construction* $J = E(L_e) \times H_e \times E(R_e)$,

$$E(J) = \left\{(f, (gf)^{-1}, g) \mid f \in E(L_e),\ g \in E(R_e)\right\},$$

which is isomorphic to $E(L_e) \times E(R_e)$ *as an algebraic variety.*

Theorem 5.4 *Let* M *be an irreducible linear algebraic monoid with unit group* G, $J \in \mathcal{U}(M)$ *and* $e \in E(J)$. *Then*

$$\dim J = 2 \dim G + \dim H_e - \dim C_G^r(e) - \dim C_G^l(e).$$

In particular, if J *is completely regular,*

$$\dim J = \dim G - \dim G_e.$$

Proof The algebraic group $G \times G$ acts on M via left and right multiplication: $(g, h) \cdot x = gxh^{-1}$. From [22, proposition 6.1], we know that the orbit of the element $e \in E(M)$ under this action is just the $\mathcal{J}$-class of e, J_e. Consider $(G \times G)_e$, the isotropy subgroup of e under the action of $G \times G$. Then

$$\begin{aligned}(G \times G)_e &= \{(x, y) \in G \times G \mid xey^{-1} = e\} \\ &= \{(x, y) \in G \times G \mid xe = exe = eye = ey\}.\end{aligned}$$

Hence $(G \times G)_e$ is a closed subgroup of $G \times G$, which contained in $C_G^r(e) \times C_G^l(e)$. By [22, Theorem 6.16] and its proof, we have

$$\begin{aligned}C_G^r(e)e = eC_G^l(e) = eC_G(e) = H_e,\\ \dim C_G^r(e) = \dim G^r(e) + \dim H_e,\ \dim C_G^l(e) = \dim G^l(e) + \dim H_e,\end{aligned}$$

and $\dim C_G(e) = \dim G_e + \dim H_e$.

Let $\phi : (G \times G)_e \to H_e \times H_e$, defined by $(x, y) \mapsto (xe, ey)$. Obviously, ϕ is a homomorphism of algebraic groups. Then

$$\phi((G \times G)_e) = \{(xe, ey) \in H_e \times H_e \mid xe = ey\}$$

is isomorphic to H_e as algebraic groups. Hence $\dim \phi((G \times G)_e) = \dim H_e$. Note that $\ker(\phi) = G^r(e) \times G^l(e)$. So

$$\dim(G \times G)_e = \dim \phi((G \times G)_e) + \dim \ker(\phi) = \dim H_e + \dim G^r(e) + \dim G^l(e).$$

Since J is the $G \times G$-orbit of e in M and $(G \times G)_e$ is the isotropy subgroup of e, we have $\dim J = \dim(G \times G) - \dim(G \times G)_e = 2 \dim G - \dim H_e - \dim G^r(e) - \dim G^l(e) = 2 \dim G + \dim H_e - \dim C_G^r(e) - \dim C_G^l(e)$.

If J is completely regular and $e \in E(J)$, then by Theorem 2.2, $G = C_G^l(e)C_G^r(e)$. Hence $\dim G = \dim C_G^l(e) + \dim C_G^r(e) - \dim C_G(e)$. So

$$\dim J = \dim G - \dim C_G(e) + \dim H_e = \dim G - \dim G_e.$$

□

If M is a linear algebraic monoid with unit group G, then the unit group of M_J is exactly the unit group G of M. Hence we can study the structure of the algebraic group G in term of the data of completely regular $\mathcal{J}$-classes of M. By Theorem 2.2, $e \in E(\overline{R(G)})$ if only if J_e is a completely regular $\mathcal{J}$-class of M. Then using the results about the kernel [8, Theorem 2.1], [10, Theorem 5.5], [11, Proposition 2.3, Theorem 2.4], we obtain the followings directly.

Corollary 5.5 *Let M be an irreducible linear algebraic monoid with unit group G, and let $e \in E(\overline{R(G)})$. Then*

(i) $\dim R(G) = \dim E(J_e) + \dim R(G_e) + \dim R(eGe)$;
(ii) $\dim R_u(G) = \dim E(J_e) + \dim R_u(G_e) + \dim R_u(eGe)$.
(iii) *G is reductive if and only if G_e and J_e are both reductive groups.*

Corollary 5.6 *Let M be an irreducible linear algebraic monoid with unit group G, and let $e \in E(\overline{R(G)})$. Let P be a parabolic subgroup of G. Then*

(i) $P = C_P(e)R(G) = C_P(e)R_u(G)$;
(ii) *$P = C_P(e) \ltimes R_u(G)$ if and only if G_e and H_e are both reductive groups;*
(iii) *if G_e is a reductive group and $J_e = E(J_e)$, then $P = P_e \ltimes R_u(G)$.*

Now we want to generalize the results (Theorems 4.1, 4.2, 4.3) of the *Schwarz* radical to the case in terms of a completely regular $\mathcal{J}$-class J in an irreducible linear algebraic monoid M. First, we give some properties of M_J.

Lemma 5.7 *Let M be an irreducible linear algebraic monoid. Let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class and $e \in E(J)$. Then $a \mid e$ if and only if $eae \mid e$ for any $a \in M$.*

Proof Let G be the unit group of M, and $a \in M$. If $eae \mid e$, then $a \mid eae \mid e$. So we only need to prove that if $a \mid e$, then $eae \mid e$. Suppose $a \mid e$. It follows from [22, Corollary 6.13] that $a \in GM_eG$, and thus there exist $x, y \in G$ and $b \in M_e$ such that $a = xby$. Thus $eae = exbye$. And $exbe = exe\mathcal{H}e$ following Theorem 2.2. So $exb \mid e$ which implies $exb \in J$. Clearly, $ye \in J$. Since J is completely regular, $J^2 = J$. Hence $eae = exbye \in J$ and thus $eae \mid e$. □

Proposition 5.8 *Let M be an irreducible linear algebraic monoid with unit group G, and let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class, $e \in E(J)$. Let $S = M \setminus G = \cup_{i \in I} S_i$, where S_i are the irreducible components of S. Then*

$$M_J = \{a \mid a \in M, a \mid e\} = GM_eG = M \setminus (\cup_{i \in \Delta} S_i),$$

where $\Delta = \{i \in I \mid \text{for all } a \in S_i,\ a \nmid e\}$.

Proof Given an element $a \in M$, by Lemma 5.7, we have $a \mid e$ if and only if $eae \mid e$. By the proof of Proposition 5.1, $M_J = \{a \mid a \in M,\ eae \mid e\}$. Hence

$$M_J = \{a \mid a \in M,\ a \mid e\} = GM_eG$$

following [22, Corollary 6.13]. In particular,

$$a \nmid e \iff eae \nmid e \iff eae \notin H_e \iff det_e(a) = 0.$$

Let $I(e) = \{a \mid a \in M, a \nmid e\}$. Then $I(e) = \{a \mid a \in M, det_e(a) = 0\}$ and thus $M \setminus M_J = I(e)$.

If $M = G$, then $S = \varnothing$ and $M_J = G$. And if $J = \ker(M)$, then $M_J = M$. Suppose $M \neq G$ and $J \neq \ker(M)$. Now we claim that $I(e) = \cup_{i\in\Delta} S_i$, where $\Delta = \{i \in I \mid \text{for all } a \in S_i,\ a \nmid e\}$. By [22, Theorem 3.15], we can assume that M is a closed submonoid of some $M_n(K)$. Consider $\phi : M \to K$ given by $\phi(a) = det_e(a) = det(eae + 1 - e)$. Thus $I(e) = \phi^{-1}(0)$. By [22, Theorem 2.21], the dimension of every irreducible component of $I(e)$ is $p - 1$. Since J is completely regular, $\omega(e) = 1$. From Theorem 2.2, we have eGe is the $\mathcal{H}$-class of e. Thus if $a \in G$, then $eae \in H_e$, which implies $a \notin I(e)$. So $I(e) \subseteq M \setminus G = S$. Moreover, by [22, Proposition 6.2], the dimension of every irreducible component of S is $p - 1$. Hence every irreducible component of $I(e)$ is also the irreducible component of S. Let S_0 be an irreducible component of S which satisfy that for any element $a \in S_0$, $a \nmid e$. Then $S_0 \subseteq I(e)$. Hence $I(e) = \cup_{i\in\Delta} S_i$. Therefore, $M_J = M \setminus I(e) = M \setminus (\cup_{i\in\Delta} S_i)$. □

Corollary 5.9 *Let M be an irreducible submonoid of $M_n(K)$ with unit group G, and let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class. Then*

$$\left\{a \in M \mid a^n \in J\} = \{a \in M_J \mid a^n \in J\right\}.$$

Proof If $a \in M$ with $a^n \in J$. Let $e \in E(J)$. Then $a \mid a^n \mid e$. So $a \notin I(e)$. By Proposition 5.8, $a \in M_J$. Hence $\{a \in M \mid a^n \in J\} = \{a \in M_J \mid a^n \in J\}$. □

For an irreducible linear algebraic monoid M with unit group G. Let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class, we denote

$$\sqrt{J} = \{a \in M \mid a^i \in J \text{ for some } i \in \mathbb{Z}^+\}$$

and $\mathcal{R}(J) = \{a \in M \mid GaG \subseteq \sqrt{J}\}$. By Proposition 3.4 and Corollary 5.9, it is easy to see that the *Schwarz* radical $\mathcal{R}\ker(M_J)$ of M_J is equal to $\mathcal{R}(J)$. Moreover, J, $\sqrt{J}$ and $\mathcal{R}(J)$ are both affine variety. This is because M_J is an affine variety and J, $\sqrt{J}$, $\mathcal{R}(J)$ are both closed in M_J.

For an irreducible linear algebraic monoid M with unit group G, there is a natural algebraic group $G \times G$ acts on M via left and right multiplication:

$$(g, h) \cdot x = gxh^{-1} \text{ for } g, h \in G \text{ and } a \in M.$$

From [22, Proposition 6.1], we know that the orbit of an element $a \in M$ under this action is just the $\mathcal{J}$-class of a, J_a. Now we can give a generalization of the results in Sect. 4 in the language of linear algebraic group actions.

Corollary 5.10 *Let M be an irreducible linear algebraic monoid with unit group G, and let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class, $e \in E(J)$. Then*

- (i) *G_e is reductive if and only if J is the unique $G \times G$-stable affine subvariety of M contained in $\sqrt{J}$ (i.e., $\mathcal{R}(J) = J$);*
- (ii) *G_e is solvable if and only if $\sqrt{J}$ is a $G \times G$-stable affine subvariety of M (i.e., $\mathcal{R}(J) = \sqrt{J}$)*

Proof Let $M_J = \{a \in M \mid aJ \subseteq J\}$. By Proposition 5.1, M_J is an irreducible linear algebraic monoid with kernel J. Obviously, the unit group of M_J is G. Moreover, as we see above, the radical of the kernel of M_J is equal to $\sqrt{J}$, and $\mathcal{R}\ker(M_J) = \mathcal{R}_{M_J}(J) = \mathcal{R}(J)$. By [22, Theorem 7.4], M_J is regular if and only if G_e is reductive since $e \in E(J)$. Apply Theorems 4.2 and 4.3 for M_J, we have G_e is reductive if and only if $\mathcal{R}(J) = J$, and G_e is solvable if and only if $\mathcal{R}(J) = \sqrt{J}$. Since $\mathcal{R}(J) = \{a \in M \mid GaG \subseteq \sqrt{J}\}$ and J is the $G \times G$-orbit of e in M, we have that that $\mathcal{R}(J) = J$ is equivalent to that J is the unique $G \times G$-stable affine subvariety of M contained in $\sqrt{J}$, and $\mathcal{R}(J) = \sqrt{J}$ is equivalent to that $\sqrt{J}$ is a $G \times G$-stable affine subvariety of M. □

As an immediate consequence of Corollary 5.5(iii), 5.10 and [22, Proposition 6.25] we have,

Corollary 5.11 *Let M be an irreducible linear algebraic monoid with unit group G, let $J \in \mathcal{U}(M)$ be a completely regular $\mathcal{J}$-class. Then*

- (i) *G is reductive if and only if J is both a reductive group and the unique $G \times G$-stable affine subvariety of M contained in $\sqrt{J}$;*
- (ii) *G is solvable if and only if $\sqrt{J}$ is a $G \times G$-stable affine variety and a maximal subgroup of J is solvable.*

Remark 5.12 Let M be an irreducible solvable linear algebraic monoid. Since M is an $s\pi r$-semigroup, it is easy to verify that

$$M = \bigcup_{J \in \mathcal{U}(M)} \sqrt{J},$$

which is a semilattice of $\sqrt{J}$ ($J \in \mathcal{U}(M)$), and every $\sqrt{J}$ is an archimedean semigroup and a nil extension of the completely simple semigroup J.

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