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RESEARCH ARTICLE

Infinite Banach direct sums and diagonal C_0-semigroups with applications to a stochastic particle system

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Abstract We show an alternative—axiomatic approach to the direct sums of countably many Banach spaces, called here *B-direct sums* (BDS). We develop "direct summing" of linear operators and the problems of generating the C_0-semigroups in such "Banach direct sums of Banach spaces" by "the direct sums of the generators". We study special types of BDS, including M-BDS—the one closely related to Day's construction of a direct sum (M.M. Day, Normed linear spaces, 1973). The main abstract results of the paper concern C_0-semigroup generation properties for diagonal operators in M-BDS type direct sums. The paper is motivated by stochastic particle systems and the problem of existing C_0-semigroups defined by the corresponding hierarchies for marginal probability densities. We use our abstract M-BDS results to get a solution of the respective differential equation in the C_0-semigroup sense.

Keywords Direct sum of Banach spaces · C_0-semigroup of operators · Stochastic particle system

1 Introduction

The notion of a direct sum (or external product) of normed spaces, being unique up to an isometry in the Hilbert space case, is far from being unique in Banach space case,

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when we impose only Banach space requirement for the appropriate direct sum. This non-uniqueness is especially visible when we consider an infinite family of spaces. The existing literature of such infinite case seems to be not very rich. The best known constructions are the " ℓ^p type direct sums" (see e.g. [7]) which are natural generalizations of the standard Hilbert space construction corresponding to $p = 2$. Much more general is the construction introduced by Day (see [3,4]), based on the notion of *full function space*.

One of the main goals (an abstract one) of the present paper is to show a different approach to the direct sums of Banach spaces. Contrary to the above constructions, we show an axiomatic way of thinking on various direct sums. Using this approach we mainly develop "direct summing" of linear operators and problems of generating the C_0-semigroups in such "Banach direct sums of Banach spaces" by "the direct sums of the generators". We study here only the case of countably mamy spaces, however some parts of our considerations could be also generalized onto an arbitrary index set of the Banach spaces.

Our second main goal is to use the abstract results of this theory to show the existence of solutions with certain strong differentiability properties for an important example of stochastic particle system.

Section 2 starts with some preliminaries, including notation. We consider there *the formal direct sum* $\bigoplus_{j\in\mathbb{N}} X_j$ of the Banach spaces X_j, $j \in \mathbb{N}$, being just a linear space of all the sequences $f = \{f_j\}_{j\in\mathbb{N}}$ with $f_j \in X_j$, and with the standard pointwise linear structure. Then we distinguish in an natural axiomatic way some normed spaces $\{(X, \|\cdot\|)\}_{j\in\mathbb{N}}$ with X being linear subspaces of $\bigoplus_{j\in\mathbb{N}} X_j$, called here *B-direct sums* (abbreviated to *BDS*). Surely, one of the conditions imposed onto this normed space is the completness. We also study here several further abstract conditions for such normed spaces, and some relations between them. This leads us to the definitions of some special kinds of S-BDS: S-BDS, S$^+$-BDS and M-BDS.

In practice the most difficult part of the proof that a given $\{(X, \|\cdot\|)\}_{j\in\mathbb{N}}$ is a BDS of the spaces X_j is often checking the completness. Thus we formulate a result on sufficient conditions for S-BDS and for M-BDS, which allow to omit the necessity of a direct proof of the completness We also study close relations between the Day's construction and our M-BDS notion.

In Sect. 3 we consider *the diagonal operators* $\operatorname*{diag}_{j\in\mathbb{N}} A_j$, being "maximal domain"*direct sums of operators* A_j in X_j, and we collect several results on such sums. Note, that the most important case for us is the M-BDS case, where we get the naturally expected formula for the operator norms:

$$\|A\|_{\mathrm{Op}} = \sup_{j\in\mathbb{N}} \|A_j\|_{\mathrm{Op}}.$$

We also study some elementary spectral properties for such direct sums.

The most important abstract results are contained in Sect. 4. We prove that having C_0-semigroups in all the spaces X_j, $j \in \mathbb{N}$ with growth bounds uniform in j, we obtain a C_0-semigroup on any M-BDS of X_j by taking direct sums of the semigroup operators from X_j. Moreover we prove that the generator of this semigroup is a direct sum of the appropriate generators of the semigroups on X_j. (see Theorems 4.3 and 4.4).

The results of the present paper are motivated by the general theory of stochastic particle systems. In such a system usually in the limit of $N \to \infty$, where N is a number of interacting agents, the infinity hierarchy of equations is obtained for the marginal probability densities (see [1, Chapter 8], [8] and references therein). The main problem is the existence and uniqueness results in the appropriate Banach space setting. The corresponding results in the mentioned references are of a weak type. Although there are some semigroup approaches in similar contexts (see e.g. [6,11]) there is no general semigroup approach referred to the infinite hierarchies mentioned before. This is why we decided to develop the general theory of infinite Banach direct sums and diagonal C_0-semigroups contained in Sects. 2–4.

An application to a simplified version of the infinite hierarchy of equations appearing in (see [1, Chapter 8], [8] and references therein) is showed in Sect. 5. The simplification is in that we assume the hierarchy in a case when the equations of the hierarchy are decoupled. It follows from the assumption that during the interaction between two agents the new state of the first agent is chosen with probability that is independent of the current state of the second agent. From the mathematical point of view the assumption leads to the diagonal operator.

The main goal is there to show that thanks to the abstract results of the previous sections, the appropriate system can be solved in a „strong" way for some particular initial conditions.

To make the presentation more convenient some more technical proofs and additional facts are collected in the Appendix.

2 Infinite Banach direct sums

2.1 General notation

Having a normed space $(Y, \|\cdot\|)$ we shall often use here the common shorter notation Y for it, if the choice of the norm $\|\cdot\|$ is fixed or standard. The symbols

$$\xrightarrow{Y}, \quad \xrightarrow[n\to+\infty]{Y},$$

are then used for the convergence of sequences in this space. *C-sequence* is here the abbreviation for Cauchy sequence. We shall also use the abbreviation: *Y is a norm subspace of X*, which means here that Y is the linear subspace of the normed space X and the norm in Y is just the restriction to Y of the norm in X (i.e.: Y is a subspace of X in the normed spaces sense).

The linear space of the all complex functions (sequences) on $\mathbb{N}$ is denoted here by

$$\ell(\mathbb{N})$$

and $\ell^p(\mathbb{N})$ is the standard p-summable complex sequence space with $\|f\| := \left(\sum_{j\in\mathbb{N}} |f_j|^p\right)^{\frac{1}{p}}$ for $1 \leq p < +\infty$. $\ell^\infty(\mathbb{N})$ is the bounded complex sequence space with $\|f\| := \sup_{j\in\mathbb{N}} |f_j|$ and $c_0(\mathbb{N})$ is the norm subspace of $\ell^\infty(\mathbb{N})$, consisting of all the sequences converging to 0.

The operator norm of linear map A between normed spaces $Y_1 \neq \{0\}$ and Y_2 is denoted by

$$\|A\|_{\mathrm{Op}}.$$

The symbols

$$\mathcal{L}(Y),\ \mathcal{C}(Y),\ \mathcal{B}(Y)$$

denote here the spaces/sets of linear, closed, and bounded operators on the normed space Y, respectively, in the usual meaning, i.e.: $A \in \mathcal{L}(Y)$ means that the domain $D(A)$ of A is a linear subspace of Y and $A : D(A) \longrightarrow Y$ is a linear map; $A \in \mathcal{C}(Y)$ means that $A \in \mathcal{L}(Y)$ and A is closed in the graph sense, and $A \in \mathcal{B}(Y)$ means that $A \in \mathcal{L}(Y)$, $D(A) = Y$ and $\|A\|_{\mathrm{Op}} < +\infty$ (so A is continuous).

Let us adopt here the following notation for the "generalized inverse operator" of $A \in \mathcal{L}(Y)$. For A such that $\operatorname{Ker} A = \{0\}$ the symbol $A^{-1\bullet}$ denotes the operator in $\mathcal{L}(Y)$ being the inverse map to A treated as the map $A : D(A) \longrightarrow \operatorname{Ran} A$, i.e.,

$$\forall_{x,y\in Y}\ \left(A^{-1\bullet}x = y \iff Ay = x\right). \tag{2.1}$$

In particular $D(A^{-1\bullet}) = \operatorname{Ran} A$ and $\operatorname{Ran}(A^{-1\bullet}) = D(A)$. When $\operatorname{Ran} A = Y$ then obviously $A^{-1\bullet} = A^{-1}$, with the common meaning of $^{-1}$. Using the notions Ker and $^{-1\bullet}$ we can easily express the standard notions of the point spectrum $\sigma_{\mathrm{p}}(A)$ of A, of the resolvent $\rho(A)$ of A and of the spectrum $\sigma(A)$ of A for an arbitrary $A \in \mathcal{L}(Y)$, $Y \neq \{0\}$, because for $\lambda \in \mathbb{C}$, we have:

$$\lambda \in \sigma_{\mathrm{p}}(A) \iff \operatorname{Ker}(A-\lambda) \neq \{0\}, \tag{2.2}$$

$$\lambda \in \rho(A) \iff \operatorname{Ker}(A-\lambda) = \{0\} \text{ and } (A-\lambda)^{-1\bullet} \in \mathcal{B}(Y), \tag{2.3}$$

$$\lambda \in \sigma(A) \iff \lambda \notin \rho(A); \tag{2.4}$$

by $\|A\|_{\mathrm{sp}}$ we denote the spectral norm for bounded operator A

We shall use here also the following notions concerning maps $F : Y_1 \longrightarrow Y_2$ between two normed spaces $(Y_1, \|\cdot\|_1)$ and $(Y_2, \|\cdot\|_2)$:

- F is *norm-monotonic* iff for any $y, y' \in Y_1$ $\|y\|_1 \leq \|y'\|_1 \implies \|F(y)\|_2 \leq \|F(y')\|_2$.
- F is a *similarity* iff there exists a constant $C \geq 0$ such that for any $y \in Y_1$ $\|F(y)\|_2 = C\|y\|_1$.

Some notation is also placed in the proper subsections.

2.2 Formal direct sums and its base subspaces

Consider Banach spaces $(X_j, \|\cdot\|_j)$, $j \in \mathbb{N}$. Denote the linear space being the direct sum[1] of all the linear spaces X_j by

[1] The name *product* instead of "direct sum" is also very popular, especially in set theory and topology context. Then also the „multiplicative" notation $\prod$ or $\times$ instead of $\oplus$ is used.

$$\bigoplus_{j\in\mathbb{N}} X_j,$$

i.e., $\bigoplus_{j\in\mathbb{N}} X_j$ is the set of all the sequences $f = \{f_j\}_{j\in\mathbb{N}}$ such that $\forall_{j\in\mathbb{N}}\ f_j \in X_j$, with the standard pointwise linear structure. We call here $\bigoplus_{j\in\mathbb{N}} X_j$ *the formal direct sum* of the above spaces.

Fix $r \in \mathbb{N}$. The "copy of X_r in $\bigoplus_{j\in\mathbb{N}} X_j$" is denoted by $\widetilde{X_r}$, i.e.

$$\widetilde{X_r} := \left\{ f \in \bigoplus_{j\in\mathbb{N}} X_j : \forall_{j\neq r}\ f_j = 0 \right\}.$$

For $u \in X_r$ the symbol $\widetilde{u}^r$ denotes the appropriate "copy of u in $\bigoplus_{j\in\mathbb{N}} X_j$":

$$\left(\widetilde{u}^r\right)_j := \begin{cases} u \text{ for } j = r \\ 0 \text{ for } j \neq r, \end{cases}$$

and $\mathcal{I}_r : X_r \longrightarrow \bigoplus_{j\in\mathbb{N}} X_j$ is the map given by

$$\mathcal{I}_r(u) := \widetilde{u}^r, \qquad u \in X_r. \tag{2.5}$$

Consider also "the r-th coordinate map" $p_r : \bigoplus_{j\in\mathbb{N}} X_j \longrightarrow X_r$:

$$p_r f := f_r, \qquad f \in \bigoplus_{j\in\mathbb{N}} X_j, \tag{2.6}$$

and "the natural projection onto $\widetilde{X_r}$", i.e. the map $\pi_r : \bigoplus_{j\in\mathbb{N}} X_j \longrightarrow \bigoplus_{j\in\mathbb{N}} X_j$ given by:

$$\pi_r(f) := \begin{cases} f_r \text{ for } j = r \\ 0 \quad \text{for } j \neq r, \end{cases} \qquad f \in \bigoplus_{j\in\mathbb{N}} X_j, \tag{2.7}$$

i.e., $\pi_r := \mathcal{I}_r \circ p_r$, and we have $\widetilde{X_r} = \operatorname{Ran} \mathcal{I}_r = \operatorname{Ran} \pi_r$.

Define also "the cut-off projection" $\mathcal{P}_r : \bigoplus_{j\in\mathbb{N}} X_j \longrightarrow \bigoplus_{j\in\mathbb{N}} X_j$:

$$\mathcal{P}_r f := \begin{cases} f_j \text{ for } j \leq r \\ 0 \quad \text{for } j > r, \end{cases} \qquad f \in \bigoplus_{j\in\mathbb{N}} X_j. \tag{2.8}$$

Denote

$$X_{\text{fin}} := \left\{ f \in \bigoplus_{j\in\mathbb{N}} X_j :\ \exists_{r\in\mathbb{N}} \forall_{j>r}\ f_j = 0 \right\}.$$

In particular $\widetilde{X_r} \subset X_{\text{fin}}$ and X_{fin} is the linear span of all the $\widetilde{X_j}$-s.

Consider the following linear subspaces of $X_{\text{fin}} \subset \bigoplus_{j\in\mathbb{N}} X_j$:

$$X^{\leq r} := \left\{ f \in \bigoplus_{j\in\mathbb{N}} X_j : \forall_{j>r}\ f_j = 0 \right\}, \quad r \in \mathbb{N}.$$

We have thus $X^{\leq r} = \operatorname{Ran} \mathcal{P}_r$ and $X_{\text{fin}} = \bigcup_{r\in\mathbb{N}} X^{\leq r}$.

For $f, g \in \bigoplus_{j\in\mathbb{N}} X_j$ denote also

$$f \preceq g \iff \left(\forall_{j\in\mathbb{N}}\ \|f_j\|_j \leq \|g_j\|_j\right).$$

We shall be interested only in such linear subspaces X of $\bigoplus_{j\in\mathbb{N}} X_j$ which satisfy

$$X_{\text{fin}} \subset X. \tag{2.9}$$

Each such X is called here *a base subspace of* $\bigoplus_{j\in\mathbb{N}} X_j$, and we use the notation $X \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$ for such a case. If, moreover, X is a normed space with a norm $\|\cdot\|$ then we call it *a normed base subspace of* $\bigoplus_{j\in\mathbb{N}} X_j$, and we use the similar notation $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$.

2.3 BDS and S-BDS

We distinguish some normed base subspaces called here *B-direct sums*.

Definition 2.1 A normed base subspace $(X, \|\cdot\|)$ of $\bigoplus_{j\in\mathbb{N}} X_j$ is a *B-direct sum* (abbreviated to *BDS*) of the sequence $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ of Banach spaces iff

(i) X is a Banach space;
(ii) $\overline{X_{\text{fin}}} = X$;
(iii) $\mathcal{I}_j : X_j \longrightarrow X$ is continuous (as the map between the normed space X_j and X) for any $j \in \mathbb{N}$.
(iv) $p_j|_X$ is continuous (as the map between normed spaces X and X_j) for any $j \in \mathbb{N}$.

We shall often simplify the use of the above BDS notion by writing something like: *X is a BDS of the spaces X_j*, and similarly for some stronger versions of BDS, which we introduce later.

Example 2.2 Let $X_j = \mathbb{C}$ for any $j \in \mathbb{N}$ (with the absolute value as the norms). So we have $\bigoplus_{j\in\mathbb{N}} X_j = \mathbb{C}^{\mathbb{N}} = \ell(\mathbb{N})$. Let us choose $X := \ell^p(\mathbb{N})$ for $p \in [1; +\infty)$. Obviously, $\ell^p(\mathbb{N})$ is a BDS of spaces $\mathbb{C}$ for all $p \in [1; +\infty)$. But for "$p = +\infty$" the similar fact is not true, because the above density condition **(ii)** is not satisfied. However, we can consider the norm subspace $c_0(\mathbb{N})$ of $\ell^\infty(\mathbb{N})$, which is obviously also a BDS of spaces $\mathbb{C}$.

There exists a well known generalization of the above construction (see e.g. [7]).

Example 2.3 Now the above choice of constant sequence of the spaces with one-dimensional space $\mathbb{C}$ is replaced by an arbitrary sequence $\mathcal{X} = \{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ of Banach spaces. With the previous "scalar context" it is natural to denote here $\ell(\mathbb{N}, \mathcal{X}) := \bigoplus_{j\in\mathbb{N}} X_j$ and

$$\ell^p(\mathbb{N}, \mathcal{X}) := \{f \in \ell(\mathbb{N}, \mathcal{X}) : \|f\|_{p,\mathcal{X}} < +\infty\}, \qquad 1 \le p \le +\infty,$$
$$c_0(\mathbb{N}, \mathcal{X}) := \{f \in \ell(\mathbb{N}, \mathcal{X}) : \lim_{j\to+\infty} \|f_j\|_j = 0\},$$

where $\|f\|_{p,\mathcal{X}} := \left(\sum_{j\in\mathbb{N}} \|f_j\|_j^p\right)^{\frac{1}{p}}$ when $p < +\infty$ and $\|f\|_{\infty,\mathcal{X}} := \sup_{j\in\mathbb{N}} \|f_j\|_j$ for any $f \in \ell(\mathbb{N}, \mathcal{X})$. The restrictions of $\|\cdot\|_{p,\mathcal{X}}$ and $\|\cdot\|_{\infty,\mathcal{X}}$ to $\ell^p(\mathbb{N}, \mathcal{X})$ $(1 \le p \le +\infty)$ and to $c_0(\mathbb{N}, \mathcal{X})$, respectively, are choosen for the norms here. Then it is known [7] that $\ell^p(\mathbb{N}, \mathcal{X})$ for $1 \le p \le +\infty$ and $c_0(\mathbb{N}, \mathcal{X})$ are Banach spaces. Knowing this we can immediately check that $\ell^p(\mathbb{N}, \mathcal{X})$ for $1 \le p < +\infty$ and $c_0(\mathbb{N}, \mathcal{X})$ are BDS of the spaces X_j. Note, that condition **(iii)** of the definition of BDS is satisfied "with the excess" in those cases—here $\mathcal{I}_j : X_j \longrightarrow \widetilde{X_j}$ are even **isometries** between the normed space X_j and the normed subspace $\widetilde{X_j}$ of X. Hovever this is not a necessary property of BDS (see e.g. Example 2.18, where similar but weighted classes are considered) and we do not need it for purposes of this paper.

BDS is the "weakest" variant of the notion of "infinite direct sum of Banach spaces" considered here. This weakness is related to the condition **(ii)**, which does not say anything on a concrete way of approximating the vectors of X by the vectors of X_{fin}. In the next notion the "approximation by cut-off" is postulated.

Definition 2.4 A normed base subspace $(X, \|\cdot\|)$ of $\bigoplus_{j\in\mathbb{N}} X_j$ is an *S-B-direct sum* (abbreviated to *S-BDS*) of the sequence $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ of Banach spaces iff it is a BDS of $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ and

(ii') $\forall_{f\in X}\ \|f - \mathcal{P}_r f\| \xrightarrow[r\to+\infty]{} 0.$

One can easily see that all the examples of BDS from Example 2.3 (and thus also from Example 2.2) are also the examples of S-BDS.

Open Question 2.5 *How to construct a BDS which is not an S-BDS? Is it possible?*

2.4 Some abstract properties and special kinds of S-BDS

We study here some abstract conditions that could be satisfied by a normed base subspace $(X, \|\cdot\|)$ of $\bigoplus_{j\in\mathbb{N}} X_j$. Most of them are typical conditions satisfied in the cases of $\ell^p(\mathbb{N}, \mathcal{X})$ (with $1 \le p < +\infty$) and of $c_0(\mathbb{N}, \mathcal{X})$. Some of them are often very simple to check in concrete cases. They include also all the conditions which already appeared in the definitions of BDS and S-BDS.

Definition 2.6 Let $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$. We say that X (more precisely: $(X, \|\cdot\|)$) satisfies[2]:

(coor) if $\forall_{r\in\mathbb{N}}\ p_r|_X$ is continuous, i.e.

$$\forall_{r\in\mathbb{N}}\exists_{C_r>0}\forall_{f\in X}\ \|f_r\|_r \le C_r\|f\|;$$

(inj) if $\forall_{r\in\mathbb{N}}\ \mathcal{I}_r : X_r \longrightarrow X$ is continuous, i.e.

$$\forall_{r\in\mathbb{N}}\exists_{D_r>0}\forall_{u\in X_r}\ \|\widetilde{u}^r\| \le D_r\|u\|_r;$$

(iso) if $\forall_{r\in\mathbb{N}}\ \mathcal{I}_r : X_r \longrightarrow \widetilde{X_r}$ is an isomorphism, i.e.

$$\forall_{r\in\mathbb{N}}\exists_{D_r,d_r>0}\forall_{u\in X_r}\ d_r\|u\|_r \le \|\widetilde{u}^r\| \le D_r\|u\|_r;$$

(sim) if $\forall_{r\in\mathbb{N}}\ \mathcal{I}_r : X_r \longrightarrow X$ is a similarity, i.e.

$$\forall_{r\in\mathbb{N}}\exists_{C_r\ge 0}\forall_{u\in X_r}\ \|\widetilde{u}^r\| = C_r\|u\|_r;$$

(pointbound) if

$$\forall_{f\in X}\ \sup_{r\in\mathbb{N}}\|\mathcal{P}_r f\| < +\infty;$$

(proj −−) if $\forall_{r\in\mathbb{N}}\ \mathcal{P}_r|_X : X \longrightarrow X$ is continuous , i.e.

$$\forall_{r\in\mathbb{N}}\exists_{K_r>0}\forall_{f\in X}\ \|\mathcal{P}_r f\| \le K_r\|f\|;$$

(proj−) if

$$\exists_{C>0}\forall_{f\in X}\ \sup_{r\in\mathbb{N}}\|\mathcal{P}_r f\| \le C\|f\|;$$

(proj) if

$$\forall_{f\in X}\ \sup_{r\in\mathbb{N}}\|\mathcal{P}_r f\| \le \|f\|;$$

(proj+) if

$$\forall_{f\in X}\ \sup_{r\in\mathbb{N}}\|\mathcal{P}_r f\| = \|f\|;$$

[2] In some cases it would be somewhat more correctly to say "$(X, \|\cdot\|)$ satisfies ...with respect to $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$".

(proj + −) if

$$\exists_{C,c>0}\forall_{f\in X}\ c\|f\| \le \sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\| \le C\|f\|;$$

(ban) if X is a Banach space;
(bel) if

$$\forall_{f\in\bigoplus_{j\in\mathbb{N}} X_j} \left(\sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\| < +\infty \implies f \in X\right); \tag{2.10}$$

(bel−) if

$$\forall_{f\in\bigoplus_{j\in\mathbb{N}} X_j} \left(\{\mathcal{P}_r f\}_{r\ge 1} \text{ is a C-sequence} \implies f \in X\right);$$

(bel − +) if

$$\forall_{f\in\bigoplus_{j\in\mathbb{N}} X_j} \left(\{\mathcal{P}_r f\}_{r\ge 1} \text{ is a C-sequence} \implies \left(f \in X \text{ and } \mathcal{P}_r f \xrightarrow[r\to+\infty]{X} f\right)\right);$$

(den) if $\overline{X_{\text{fin}}} = X$;
(appr) if

$$\forall_{f\in X}\ \|f - \mathcal{P}_r f\| \xrightarrow[r\to+\infty]{} 0; \tag{2.11}$$

(incr) if

$$\forall_{r\ge s\ge 1}\forall_{f\in\bigoplus_{j\in\mathbb{N}} X_j}\ \|\mathcal{P}_r f\| \ge \|\mathcal{P}_s f\|;$$

(mono) if

$$\forall_{f,g\in X_{\text{fin}}} (f \preceq g \implies \|f\| \le \|g\|);$$

(major) if

$$\forall_{f,g\in\bigoplus_{j\in\mathbb{N}} X_j} ((f \preceq g \text{ and } g \in X) \implies (f \in X \text{ and } \|f\| \le \|g\|)).$$

Example 2.7 The spaces $\ell^p(\mathbb{N}, \mathcal{X})$ for $1 \le p < +\infty$ satisfy all the above conditions. Typically (even for the scalar case of the Example 2.2) $c_0(\mathbb{N}, \mathcal{X})$ do not satisfy the condition **(bel)** and $\ell^\infty(\mathbb{N}, \mathcal{X})$ do not satisfy the condition **(appr)** nor **(den)**, but both spaces always satisfy the remaining conditions.

Remark 2.8 Replacing the norm in X or the norms in X_j by any equivalent norms we do not change the BDS nor the S-BDS property. More precisely, if $(X, \|\cdot\|)$ is a BDS (resp. S-BDS) of the sequence $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ of Banach spaces, the norm $\|\cdot\|'$ in X is equivalent to $\|\cdot\|$ and the norms $\|\cdot\|'_j$ in X_j are equivalent to $\|\cdot\|_j$ for $j = 1, \ldots,$

then $(X, \|\cdot\|')$ is a BDS (resp. S-BDS) of the sequence $\left\{(X_j, \|\cdot\|'_j)\right\}_{j\in\mathbb{N}}$. Also the properties **(coor)**, **(inj)**, **(iso)**, **(pointbound)**, (**proj** $--$), (**proj**$-$), (**proj** $+-$), **(ban)**, **(bel)**, (**bel**$-$), (**bel** $-+$), **(den)**, **(appr)** do not change under replacing the norm in X or the norms in X_j by any equivalent norms.

The proof of this remark is obvious directly by the definitions of all the properties mentioned there.

Using Remark 2.8 one can easily construct examples of such $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$ that satisfy all the properties mentioned in the remark, but do not satisfy: **(proj)**, (**proj**$+$), **(incr)**, **(mono)**. It suffices to "deform equivalently" (in an appropriate manner) the norm in one of BDS-s from Examples 2.2 and 2.3. In particular, (**proj**$+$) does not have to hold in each S-BDS. So we distinguish here two special types of S-BDS.

Definition 2.9 A normed base subspace $(X, \|\cdot\|)$ of $\bigoplus_{j\in\mathbb{N}} X_j$ is

- a S^+-*B-direct sum* (abbreviated to S^+-*BDS*) of the sequence $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ of Banach spaces if it is a S-BDS of $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ and X satisfies (**proj**$+$);
- a *M-B-direct sum* (abbreviated to *M-BDS*) of the sequence $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ of Banach spaces if it is a S^+-BDS of $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ and X satisfies **(mono)**.

Let us summarize now the definitions of all the kinds of BDS, assuming that we consider here only normed base subspaces:

$$\begin{aligned}
\textbf{BDS} &\Leftrightarrow (\textbf{ban})\ \&\ (\textbf{den})\ \&\ (\textbf{inj})\ \&\ (\textbf{coor}),\\
\textbf{S-BDS} &\Leftrightarrow (\textbf{ban})\ \&\ (\textbf{appr})\ \&\ (\textbf{inj})\ \&\ (\textbf{coor}),\\
\textbf{S}^+\textbf{-BDS} &\Leftrightarrow (\textbf{ban})\ \&\ (\textbf{appr})\ \&\ (\textbf{inj})\ \&\ (\textbf{coor})\ \&\ (\textbf{proj}+),\\
\textbf{M-BDS} &\Leftrightarrow (\textbf{ban})\ \&\ (\textbf{appr})\ \&\ (\textbf{inj})\ \&\ (\textbf{coor})\ \&\ (\textbf{proj}+)\ \&\ (\textbf{mono}).
\end{aligned}$$

However, it will be clear soon by Proposition 2.11, that the last definition can be much simplified—see Corollary 2.12. In Theorem 2.14 some convenient sufficient conditions for S-BDS and M-BDS are formulated.

2.5 "The distance" between BDS, S-BDS and S$^+$-BDS, and relations between the abstract properties

On the other hand each of our particular types of BDS (S-BDS, S$^+$-BDS and M-BDS) automatically satisfies more conditions from Definition 2.6.

Theorem 2.10 *If X is a BDS of spaces X_j, then it satisfies* (**bel** $-+$), (**proj** $--$) *and* **(iso)**. *If it is an S-BDS of spaces X_j, then it satisfies also* **(pointbound)** *and* (**proj** $+-$). *If it is an S^+-BDS of spaces X_j, then it satisfies also* **(incr)**. *If it is an M-BDS of spaces X_j, then it satisfies also* **(sim)** *and* **(major)**.

Moreover:

(a) *If X is an BDS of spaces X_j, then the following conditions are equivalent:*
  (i) *X is an S-BDS of spaces X_j;*

(ii) *X satisfies* **(pointbound)***;*
(iii) *X satisfies* (**proj**$-$)*;*
(iv) *X satisfies* (**proj** $+-$).

(b) *If X is an S-BDS of spaces X_j, then the formula*

$$\|f\|_+ := \sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\|, \qquad f \in X \tag{2.12}$$

defines a norm in X which is equivalent to $\|\cdot\|$ and $(X, \|\cdot\|_+)$ is an S^+-BDS of spaces X_j.

We shall prove this result later, after discussing some basic relations between the considered properties.

Proposition 2.11 *Suppose $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$.*

1. *If X satisfies* **(mono)***, then it satisfies* **(incr)**.
2. *If X satisfies* **(incr)***, then*

$$\forall_{f\in\bigoplus_{j\in\mathbb{N}} X_j} \quad \sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\| = \lim_{r\to+\infty} \|\mathcal{P}_r f\|. \tag{2.13}$$

3. *If X satisfies* **(appr)***, then it satisfies* **(pointbound)** *and*

$$\forall_{f\in X} \quad \|f\| = \lim_{r\to+\infty} \|\mathcal{P}_r f\|. \tag{2.14}$$

4. *If X satisfies* **(coor)** *and* **(inj)***, then it satisfies* (**proj** $--$).
5. *If X satisfies* **(ban)***,* **(pointbound)** *and* (**proj** $--$)*, then it satisfies* (**proj**$-$).
6. *If X satisfies* **(appr)** *and* (**proj**$-$)*, then it satisfies* (**proj** $+-$).
7. *If X satisfies* **(den)** *and* (**proj**$-$)*, then it satisfies* **(appr)**.
8. *If* **(appr)** *and* **(proj)***, then* (**proj**$+$) *and moreover*

$$\forall_{f\in X} \quad \|f\| = \lim_{r\to+\infty} \|\mathcal{P}_r f\| = \sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\|. \tag{2.15}$$

9. *If X satisfies* **(appr)** *and* **(mono)***, then it satisfies* (**proj**$+$) *and moreover (2.15).*
10. *Suppose that X satisfies* **(inj)** *and let $r \in \mathbb{N}$. If $f, f^{(n)} \in X^{\le r}$ for any $n \in \mathbb{N}$ and $f_j^{(n)} \xrightarrow[n\to+\infty]{X_j} f_j$ for any $j = 1, \ldots, r$, then $f^{(n)} \xrightarrow[n\to+\infty]{X} f$.*
11. *If X satisfies* **(coor)***, then $X^{\le r}$ and $\widetilde{X_r}$ are closed subspaces of X for any $r \in \mathbb{N}$.*
12. *Suppose that X satisfies* **(inj)***,* **(coor)** *and* **(ban)***, then it satisfies* **(iso)**.
13. *If X satisfies* **(mono)***, then it satisfies* **(iso)** *and* **(sim)***, and in particular* **(inj)** *holds.*
14. *If X satisfies* **(mono)** *and* (**proj** $--$)*, then it satisfies* **(coor)**.
15. *If X satisfies* **(proj)***, then it satisfies* **(incr)**.
16. *If X satisfies* **(bel)***, then it satisfies* (**bel**$-$).
17. *If X satisfies* **(ban)** *and* **(coor)***, then it satisfies* (**bel** $-+$).
18. *If X satisfies* **(mono)***,* (**bel**$-$) *and* **(appr)***, then it satisfies* **(major)**.

For the proof see Appendix "The proof of Proposition 2.11" section.

Proof of Theorem 2.10 The proof is based on the appropriate parts of Proposition 2.11—the numbers of the parts refer to this proposition. When X is BDS, then we get (**bel** $-+$) from part 17., (**proj** $--$) from part 4. and **(iso)** from part 11. When X is S-BDS, then we get **(pointbound)** from part 3. and (**proj** $+-$) from parts 5. and 6. If X is S^+-BDS, then we get **(incr)** from part 15. For X being M-BDS we get **(sim)** from part 13 and **(major)** from part 18.

Let us prove now the following implications of part a) of the theorem:

(**i**) $\Longrightarrow$ (**ii**) we get from part 3;
(**ii**) $\Longrightarrow$ (**iii**) we get from the first part (just proved) of the theorem and part 5;
(**iii**) $\Longrightarrow$ (**i**) It suffices to use part 7;
(**iii**) $\Longrightarrow$ (**iv**) We use the previuos implication and part 6;
(**iv**) $\Longrightarrow$ (**iii**) — it is obvious by the definitions of (**proj**$-$) and (**proj** $+-$).

Having all these implications we get a).

To obtain b) note first that the formula (2.12) for $\|f\|_+$ makes sense for any $f \in X$, because **(pointbound)** is satisfied by the part a). And to prove that $\|\cdot\|_+$ satisfies all the conditions for norm in X we just need to use that $\|\cdot\|$ satisfied them and that $\mathcal{P}_r$ are linear operators. These two norms are equivalent—this is just exactly the meaning of (**proj** $+-$) which holds by part a). To finish the proof of b) we have only to prove that for any $f \in X$

$$\sup_{n\in\mathbb{N}} \|\mathcal{P}_n f\|_+ = \|f\|_+. \tag{2.16}$$

Denote $A_n := \{\|\mathcal{P}_r f\| : r \leq n\}$. For $f \in X$ and $n \geq 1$ we have by definition

$$\|\mathcal{P}_n f\|_+ = \sup_{r\in\mathbb{N}} \|\mathcal{P}_r(\mathcal{P}_n f)\| = \sup_{r\leq n} \|\mathcal{P}_r f\| = \sup A_n.$$

Thus using the formula for supremum of a general sum of subsets of $\mathbb{R}$, we get

$$\sup_{n\in\mathbb{N}} \|\mathcal{P}_n f\|_+ = \sup\{\sup A_n : n \in \mathbb{N}\} = \sup\left(\bigcup_{n\in\mathbb{N}} A_n\right).$$

But $\bigcup_{n\in\mathbb{N}} A_n = \{\|\mathcal{P}_n f\| : n \in \mathbb{N}\}$, hence we finally get

$$\sup_{n\in\mathbb{N}} \|\mathcal{P}_n f\|_+ = \sup_{n\in\mathbb{N}} \|\mathcal{P}_n f\| = \|f\|_+.$$

□

Corollary 2.12 *For normed base subspaces*

M-BDS $\Leftrightarrow$ (**ban**) & (**appr**) & (**mono**).

Proof See Proposition 2.11 parts 9., 13. and 14. □

2.6 Some sufficient conditions for S-BDS and for M-BDS. Completness almost for free...

Usually the most complex part of the proof that a normed base subspace is a BDS of the spaces X_j is checking the completness (condition **(i)** of the definition, denoted also by **(ban)**). We shall formulate here some general and easy to check sufficient conditions for S-BDS and for M-BDS, which allow us to omit the necessity of a direct proof of completness. As we shall see—the key property allowing to omit it is the one we called **(bel)** (see (2.10)), which is in fact usually much easier to check than **(ban)**. In some sense **(bel)** helps us "even when it is not satisfied" (as in the case of the space $c_0(\mathbb{N}, \mathcal{X})$, where it does not hold, but it holds in the larger space $\ell^\infty(\mathbb{N}, \mathcal{X})$). All the remaining sufficient conditions we shall use are some necessary conditions. The results presented here generalize the standard methods used in the proofs of completness for $\ell^p(\mathbb{N}, \mathcal{X})$ (with $1 \leq p < +\infty$) and for $c_0(\mathbb{N}, \mathcal{X})$.

For $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$ let us denote:

$$X_{\text{appr}} := \left\{ f \in X : \ \ \|f - \mathcal{P}_n f\| \xrightarrow[n\to+\infty]{} 0 \right\}.$$

Remark 2.13 X_{appr} is a linear subspace of X satisfying $X_{\text{fin}} \subset X_{\text{appr}} \subset \overline{X_{\text{fin}}}$. If X satisfies (**proj**$-$), then $X_{\text{appr}} = \overline{X_{\text{fin}}}$.

Proof One obtains immediately the first part by the linearity of the projections $\mathcal{P}_n$ and by the fact that $X_{\text{fin}} = \bigcup_{r\in\mathbb{N}} X^{\leq r} = \bigcup_{r\in\mathbb{N}} \operatorname{Ran} \mathcal{P}_r$ and that $\mathcal{P}_n\mathcal{P}_r = \mathcal{P}_r$ for $n \geq r$. If X satisfies (**proj**$-$), then $\overline{X_{\text{fin}}}$, treated as a norm subspace of X, satisfies **(den)** and (**proj**$-$). Hence by Proposition 2.11 part 7. X satisfies **(appr)**, i.e. $\overline{X_{\text{fin}}} \subset X_{\text{appr}}$. □

Theorem 2.14 *Suppose that* $\{X_j\}_{j\in\mathbb{N}}$ *is a sequence of Banach spaces and* $(X, \|\cdot\|)$ *is a normed base subspace of* $\bigoplus_{j\in\mathbb{N}} X_j$.

1. *If X satisfies* **(coor)**, **(inj)**, (**proj** $+ -$) *and* **(bel)**, *then X is a Banach space, $X_{\text{appr}} = \overline{X_{\text{fin}}}$ and $\overline{X_{\text{fin}}}$ is an S-BDS of spaces X_j. In particular, if X satisfies* **(coor)**, **(inj)**, (**proj** $+ -$), **(bel)** *and* **(den)**, *then X is an S-BDS of spaces X_j.*
2. *If X satisfies* **(mono)**, (**proj**$+-$) *and* **(bel)**, *then X is a Banach space, $X_{\text{appr}} = \overline{X_{\text{fin}}}$ and $\overline{X_{\text{fin}}}$ is an M-BDS of spaces X_j. If X satisfies* **(mono)**, **(appr)** *and* **(bel)**, *then X is an M-BDS of spaces X_j.*

Proof 1. Let us prove the completness of X, assuming that **(coor)**, **(inj)**, (**proj** $+ -$) and **(bel)** hold. First, using (**proj** $+ -$), choose real positive c and C such that

$$\forall_{f\in X}\ c\|f\| \leq \sup_{r\in\mathbb{N}} \|\mathcal{P}_r f\| \leq C\|f\|. \tag{2.17}$$

Consider a Cauchy sequence $\{f^{(n)}\}_{n\geq n_0}$ in X. We shall prove its convergence, so let $\epsilon > 0$. Choose first $N \geq n_0$ such that

$$\forall_{n,m\geq N}\ \|f^{(n)} - f^{(m)}\| < \epsilon. \tag{2.18}$$

For any $j \in \mathbb{N}$, by **(coor)**, $\left\{f_j^{(n)}\right\}_{n \geq n_0}$ is a Cauchy sequence in X_j, so it is convergent in X_j. Let g be such the element of $\bigoplus_{j\in\mathbb{N}} X_j$ that $f_j^{(n)} \xrightarrow[n\to+\infty]{X_j} g_j$ for any $j \in \mathbb{N}$.
Now, consider an arbitrary $r \in \mathbb{N}$ and let $n \geq N$. For any $j = 1, \ldots, r$ we have

$$\left(\mathcal{P}_r\left(f^{(n)} - f^{(m)}\right)\right)_j = f_j^{(n)} - f_j^{(m)} \xrightarrow[m\to+\infty]{X_j} f_j^{(n)} - g_j = \left(\mathcal{P}_r\left(f^{(n)} - g\right)\right)_j$$

and by **(inj)** and by Proposition 2.11 part 10. we get

$$\mathcal{P}_r(f^{(n)} - f^{(m)}) \xrightarrow[m\to+\infty]{X} \mathcal{P}_r(f^{(n)} - g).$$

Thus, using (2.17) and (2.18), we obtain

$$\|\mathcal{P}_r(f^{(n)} - g)\| = \lim_{m\to+\infty} \|P_r(f^{(n)} - f^{(m)})\| \leq C\epsilon,$$

and hence $\sup_{r\in\mathbb{N}} \|\mathcal{P}_r(f^{(n)} - g)\| \leq C\epsilon$ (note that c, C are "r - independent"). Therefore by **(bel)** $f^{(n)} - g \in X$ for any $n \geq N$. So, in particular, $g \in X$ and the LHS inequality from (2.17) can be used also for $f^{(n)} - g$, which gives

$$c\|f^{(n)} - g\| \leq \sup_{r\in\mathbb{N}} \|\mathcal{P}_r(f^{(n)} - g)\| \leq C\epsilon.$$

We get $\|f^{(n)} - g\| \leq \frac{C}{c}\epsilon$ for any $n \geq N$, so $f^{(n)} \xrightarrow[n\to+\infty]{X} g$ is proved, and **(ban)** is satisfied for X. Thus $\overline{X_{\text{fin}}}$ is also a Banach space and by Remark 2.13 we get $X_{\text{appr}} = \overline{X_{\text{fin}}}$. This means that $\overline{X_{\text{fin}}}$ satisfies **(ban)**, **(appr)**, **(inj)** and **(coor)**, i.e. it is an S-BDS of spaces X_j.

2. Assume **(mono)**, (**proj** $+ -$) and **(bel)** for X. In particular we have (**proj** $- -$). We get **(inj)** and **(coor)** for X using parts 13. and 14. of Proposition 2.11. So using the first part of the theorem we get **(ban)** for X. Moreover $X_{\text{appr}} = \overline{X_{\text{fin}}}$ and $\overline{X_{\text{fin}}}$ is an S-BDS of spaces X_j. To prove that it is also an M-BDS it suffices to prove that $\overline{X_{\text{fin}}}$ satisfies (proj+), but this property follows from part 9. of Proposition 2.11 used to the space $\overline{X_{\text{fin}}}$.
   Now assume **(mono)**, **(appr)** and **(bel)** for X. From **(mono)** and **(appr)** we get (proj+), and thus also (**proj** $+ -$) for X, by part 9. of Proposition 2.11. Now the above proved part of the theorem gives that X_{appr} is an M-BDS of spaces X_j, but by **(appr)** we have $X_{\text{appr}} = X$. Therefore X is an M-BDS. □

Note that both parts of this theorem generalize e. g. the case of $\ell^p(\mathbb{N}, \mathcal{X})$ with $1 \leq p < +\infty$ (see Examples 2.3, 2.7). Note also that both assertions concerning the **(ban)** property of X works also for $\ell^\infty(\mathbb{N}, \mathcal{X})$ (see Examples 2.3, 2.7 again) and the assertions on $\overline{X_{\text{fin}}}$ work for $c_0(\mathbb{N}, \mathcal{X})$.

2.7 Some more general examples of M-BDS: Day's construction

There exists a generalization of such constructions as $\ell^p(\mathbb{N}, \mathcal{X})$ and $c_0(\mathbb{N}, \mathcal{X})$ from Example 2.3. It is given by Day's construction[3] - see [4, Def. 2, Sect. 2, p. 35]. Let us recall it for the case of the index set equal $\mathbb{N}$. A *full function space (on* $\mathbb{N}$*)* is any Banach space $(F, \|\cdot\|_F)$ such that F is a linear subspace of the scalar sequence space $\ell(\mathbb{N})$ (real or complex case) and the following condition holds

$$\forall_{h',h\in\ell(\mathbb{N})} \left(h \in F \text{ and } \forall_{n\in\mathbb{N}}\, |h'_n| \le |h_n|\right) \Longrightarrow \left(h' \in F \text{ and } \|h'\|_F \le \|h\|_F\right). \quad (2.19)$$

Now, for a sequence $\mathcal{X} = \left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ of Banach spaces we consider *the substitution space*[4] (of $\mathcal{X}$ in F) denoted by ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ and defined as follows:

$${}^F\bigoplus_{j\in\mathbb{N}} X_j := \left\{ f \in \bigoplus_{j\in\mathbb{N}} X_j : \left\{\|f_j\|_j\right\}_{j\in\mathbb{N}} \in F \right\},$$

and for $f \in {}^F\bigoplus_{j\in\mathbb{N}} X_j$ the norm is given by

$$\|f\|_{F,\mathcal{X}} := \|\left\{\|f_j\|_j\right\}_{j\in\mathbb{N}}\|_F.$$

Proposition 2.15 ([4, (10), Sect. 2,p. 35][5])
If F is a full function space and X_j are Banach spaces for any $j \in \mathbb{N}$, then ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ is a Banach space.

The natural question is:

What should be extra assumed on the space F to get ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ being one of the BDS types?

It turns out that adding only the natural assumption on the density of $\ell_{\text{fin}}(\mathbb{N})$ we immediately get the M-BDS.

Theorem 2.16 *Suppose that F is a full function space, $\ell_{\text{fin}}(\mathbb{N}) \subset F$ and $\ell_{\text{fin}}(\mathbb{N})$ is dense in F. If $\mathcal{X} = \left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ is a sequence of Banach spaces, then ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ is an M-BDS of $\mathcal{X}$.*

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[3] Most probably, the construction was introduced by Day - see [3,4].

[4] Note some mishmash in the existing terminology: some mathematicians use the notion *substitution space* instead of the notion *full function space* formulated above (or instead of some notion „equivalent" to f.f. space)—see e.g [9]. This change should be treated as a mistake, since it is not consistent with the Day terminology of [4]. Unfortunately, [4] contains some misprints slightly complicating the decision to choose his original terminology…

[5] The result in Day's book is formulated as: "all X_j are complete $\Longleftrightarrow$ ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ is complete", but there is a small mistake. One can easily see that the part "$\Longleftarrow$" is not true without some extra asumptions on the space F. E.g. F cannot be a subspace of functions vanishing in a certain point.

For the proof see Appendix "The proof of Theorem 2.16" section.

Remark 2.17 The converse of the above result is also true, namely: If X is an M-BDS of a sequence of Banach spaces $\mathcal{X}$, then there exists such a full function space F that $\ell_{\text{fin}}(\mathbb{N}) \subset F$, $\ell_{\text{fin}}(\mathbb{N})$ is dense in F and $X = {}^{F}\bigoplus_{j\in\mathbb{N}} X_j$.

We shall not give here the details of the proof—we shall only give the construction of the appropriate full function space F for the M-BDS X. We define:

$$F := \{h \in \ell(\mathbb{N}) : \exists_{f\in X} \forall_{j\in\mathbb{N}} \|f_j\|_j = |h_j|\},$$

and for $h \in F$

$$\|h\|_F := \|f\|,$$

where f is such an element of X that $\forall_{j\in\mathbb{N}} \|f_j\|_j = |h_j|$ (one may check, that the choice of particular f satisfying this condition does not change the value of $\|f\|$).

Example 2.18 Let $w := \{w_j\}_{j\in\mathbb{N}}$ be "a weight sequence"—a sequence of positive (> 0) numbers and let $\mathcal{X} = \{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ be a sequence of Banach spaces. As in Example 2.3 we denote here $\ell(\mathbb{N}, \mathcal{X}) := \bigoplus_{j\in\mathbb{N}} X_j$ and

$$\ell^p_w(\mathbb{N}, \mathcal{X}) := \{f \in \ell(\mathbb{N}, \mathcal{X}) : \|f\|_{p,w,\mathcal{X}} < +\infty\}, \qquad 1 \le p \le +\infty,$$
$$c_{0,w}(\mathbb{N}, \mathcal{X}) := \{f \in \ell(\mathbb{N}, \mathcal{X}) : \lim_{j\to+\infty} w_j \|f_j\|_j = 0\},$$

where $\|f\|_{p,w,\mathcal{X}} := \left(\sum_{j\in\mathbb{N}} w_j \|f_j\|_j^p\right)^{\frac{1}{p}}$ when $p < +\infty$ and $\|f\|_{\infty,w,\mathcal{X}} := \sup_{j\in\mathbb{N}} w_j \|f_j\|_j$ for any $f \in \ell(\mathbb{N}, \mathcal{X})$. The restrictions of $\|\cdot\|_{p,w,\mathcal{X}}$ and $\|\cdot\|_{\infty,w,\mathcal{X}}$ to $\ell^p_w(\mathbb{N}, \mathcal{X})$ $(1 \le p \le +\infty)$ and to $c_{0,w}(\mathbb{N}, \mathcal{X})$, respectively, are choosen for the norms here. In particular, for the constant sequence of spaces $\mathbb{C}$ (i.e. $X_j = \mathbb{C}$) we get a standard "weighted" generalization of scalar sequences spaces from Example 2.2 and we use a simpler notation $\ell^p_w(\mathbb{N})$, $c_{0,w}(\mathbb{N})$, $\|f\|_{p,w}$ in these cases. In the general case (for arbitrary Banach space sequence $\mathcal{X}$) we get generalization of spaces from Example 2.3. Moreover, this is a special case of ${}^{F}\bigoplus_{j\in\mathbb{N}} X_j$—the substitution space (of $\mathcal{X}$ in F) defined above with the full function space $(F, \|\cdot\|_F)$ equall to $(\ell^p_w(\mathbb{N}), \|\cdot\|_{p,w})$ or $(c_0(\mathbb{N}), \|\cdot\|_{\infty,w})$, respectively. Thus, by Theorem 2.16, $\ell^p_w(\mathbb{N})$ for $1 \le p < +\infty$ and $c_{0,w}(\mathbb{N})$ is an M-BDS of $\mathcal{X}$.

3 Direct sums of operators (diagonal operators)

3.1 Diagonal operators: "the maximal" choice

Suppose that $(X_j, \|\cdot\|_j)$ are Banach spaces and that $A_j \in \mathcal{L}(X_j)$ for any $j \in \mathbb{N}$. Let $(X, \|\cdot\|)$ be a normed base subspace of $\bigoplus_{j\in\mathbb{N}} X_j$. We define "the direct sum in X" for the above operators choosing the maximal reasonable domain in X for it.

Definition 3.1 *The direct sum of* $\{A_j\}_{j\in\mathbb{N}}$ *on* X is the operator $A \in \mathcal{L}(X)$ denoted by $\operatorname*{diag}_{j\in\mathbb{N}} A_j$[6] and given by

$$D(A) := \left\{ f \in X : \forall_{j\in\mathbb{N}}\ f_j \in D(A_j),\ \{A_j f_j\}_{j\in\mathbb{N}} \in X \right\}, \tag{3.1}$$

$$Af := \{A_j f_j\}_{j\in\mathbb{N}} \qquad \text{for} \quad f \in D(A). \tag{3.2}$$

An operator $T \in \mathcal{L}(X)$ is called *diagonal* iff $T = \operatorname*{diag}_{j\in\mathbb{N}} A_j$ for some sequence $\{A_j\}_{j\in\mathbb{N}}$ of operators.

It is easily seen that the above definition can be also expressed as follows:

$$\forall_{f,g\in X}\ \left(\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j \right) f = g \iff \forall_{j\in\mathbb{N}}\ A_j f_j = g_j \right). \tag{3.3}$$

3.2 The closedness, dense definitness and the boundedness of diagonal operators

We collect here several basic facts on direct sums of operators. Note that extra assumptions on the base space X (e.g., assumptions of a BDS kind) differ here for various results. We start from the closedness problem.

Proposition 3.2 *Suppose that* $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$ *and* X *satisfies* **(coor)**. *If* $A_j \in \mathcal{C}(X_j)$ *for any* $j \in \mathbb{N}$, *then* $\operatorname*{diag}_{j\in\mathbb{N}} A_j \in \mathcal{C}(X)$.

Proof Let $A = \operatorname*{diag}_{j\in\mathbb{N}} A_j$ and suppose that $f, g \in X$, $D(A) \ni f^{(n)} \xrightarrow[n\to+\infty]{X} f$ and $Af^{(n)} \xrightarrow[n\to+\infty]{X} g$. Then by **(coor)**, for any $j \in \mathbb{N}$, we have $D(A_j) \ni f_j^{(n)} \xrightarrow[n\to+\infty]{X_j} f_j$ and $\left(Af^{(n)}\right)_j \xrightarrow[n\to+\infty]{X_j} g_j$, so by the closedness of A_j we have $f_j \in D(A_j)$ and $A_j f_j = g_j$. Hence $\{A_j f_j\}_{j\in\mathbb{N}} = g \in X$, which means that $f \in D(A)$ and $Af = g$. □

Let us study now the problem of density of the domain. Denote

$$D_{\text{fin}}\left(\{A_j\}_{j\in\mathbb{N}}\right) := \{f \in X_{\text{fin}} : \forall_{j\in\mathbb{N}}\ f_j \in D(A_j)\}.$$

If $f \in D_{\text{fin}}\left(\{A_j\}_{j\in\mathbb{N}}\right)$, then $\{A_j f_j\}_{j\in\mathbb{N}} \in X_{\text{fin}} \subset X$. Hence we always have

$$D_{\text{fin}}\left(\{A_j\}_{j\in\mathbb{N}}\right) \subset D(\operatorname*{diag}_{j\in\mathbb{N}} A_j). \tag{3.4}$$

[6] Note that $\operatorname*{diag}_{j\in\mathbb{N}} A_j$ is determined not only by the choice of $\{A_j\}_{j\in\mathbb{N}}$ but also by the choice of X, however we omit here 'X' in the notation, assuming that the choice of X "is fixed".

Observe also, that

$$\forall_{j\in\mathbb{N}}\; D(A_j) = X_j \quad\Longrightarrow\quad X_{\mathrm{fin}} = D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right) \subset D(A). \tag{3.5}$$

Proposition 3.3 *Suppose that* $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$ *and* X *satisfies* **(inj)** *and* **(den)**. *If* $A_j \in \mathcal{L}(X_j)$ *and* A_j *is densely defined for any* $j \in \mathbb{N}$, *then* $D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right)$ *is dense in* X, *and in particular* $\operatorname*{diag}_{j\in\mathbb{N}} A_j$ *is densely defined.*

Proof We shall prove that $X_{\mathrm{fin}} \subset \overline{D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right)}$, which gives the assertion by **(den)**. Let $f \in X_{\mathrm{fin}}$; choose $r \in \mathbb{N}$ such that $f \in X^{\le r}$ and for any $j = 1, \dots, r$ choose a sequence $u^{(j)} = \left\{u^{(j,n)}\right\}_{n\ge 1}$ such that $D(A_j) \ni u^{(j,n)} \xrightarrow[n\to+\infty]{X_j} f_j$. For $n \ge 1$ define

$$f^{(j,n)} := \begin{cases} u^{(j,n)} & \text{for } j \le r \\ 0 & \text{for } j > r, \end{cases}$$

and $f^{(n)} := \left\{f^{(j,n)}\right\}_{j\in\mathbb{N}} \in \bigoplus_{j\in\mathbb{N}} X_j$. For any $n \ge 1$ we have $f^{(n)} \in D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right) \cap X^{\le r}$, and for any $j = 1, \dots, r$

$$f_j^{(n)} = f^{(j,n)} = u^{(j,n)} \xrightarrow[n\to+\infty]{X_j} f_j.$$

By Proposition 2.11 part 10. we get $f^{(n)} \xrightarrow[n\to+\infty]{X} f$, i.e. $f \in \overline{D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right)}$. □

In particular we immediately obtain the following result in BDS case.

Corollary 3.4 *Suppose that* $(X, \|\cdot\|)$ *is a BDS of a sequence* $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ *of Banach spaces, and* $A_j \in \mathcal{L}(X_j)$ *for any* $j \in \mathbb{N}$.

(i) *If* A_j *is densely defined for any* $j \in \mathbb{N}$, *then* $\operatorname*{diag}_{j\in\mathbb{N}} A_j$ *is densely defined with* $D_{\mathrm{fin}}\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right)$ *being a dense subdomain.*
(ii) *If* $A_j \in \mathcal{C}(X_j)$ *for any* $j \in \mathbb{N}$, *then* $\operatorname*{diag}_{j\in\mathbb{N}} A_j \in \mathcal{C}(X)$.

Now we come to the boudedness of operators.

Proposition 3.5 *Suppose that* $(X, \|\cdot\|)$ *is an M-BDS of a sequence* $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ *of Banach spaces and* $A = \operatorname*{diag}_{j\in\mathbb{N}} A_j$, *where* $X_j \ne \{0\}$, $A_j \in \mathcal{L}(X_j)$ *for any* $j \in \mathbb{N}$. *For any* $M \ge 0$ *the following conditions are equivalent:*

(i) $A \in \mathcal{B}(X)$ *and* $\|A\|_{\mathrm{Op}} \le M$,
(ii) $\forall_{j\in\mathbb{N}}\; A_j \in \mathcal{B}(X_j)$ *and* $\|A_j\|_{\mathrm{Op}} \le M$.

In particular, if $A \in \mathcal{B}(X)$, *then* $\|A\|_{\mathrm{Op}} = \sup_{j\in\mathbb{N}} \|A_j\|_{\mathrm{Op}}$.

Proof Suppose **(ii)**. Let $f \in X$ and let $h := \{A_j f_j\}_{j \in \mathbb{N}} \in \bigoplus_{j \in \mathbb{N}} X_j$. We have

$$\|h_j\|_j \leq M\|f_j\|_j = \|(Mf)_j\|_j, \quad j \in \mathbb{N},$$

i. e., $h \preceq Mf$. By Theorem 2.10 we know that **(major)** holds for X and thus $h \in X$ and $\|h\| \leq \|Mf\| = M\|f\|$. Hence $f \in D(A)$, $h = Af$ and $\|Af\| \leq M\|f\|$, i. e., **(i)** holds.

To get the opposite implication, assume **(i)**, consider arbitrary $j \in \mathbb{N}$ and $u \in X_j$. Since $D(A) = X \supset X_{\text{fin}}$ we have $\mathcal{I}_j u \in D(A)$ and thus, by $A = \operatorname*{diag}_{k \in \mathbb{N}} A_k$, $u = (\mathcal{I}_j u)_j \in D(A_j)$. Hence $D(A_j) = X_j$, and $A \circ \mathcal{I}_j = \mathcal{I}_j \circ A_j$. But by Theorem 2.10 **(iso)** and **(sim)** hold for X, which means that for some $C_j > 0$ we have

$$\|\mathcal{I}_j v\| = C_j \|v\|_j \quad \text{for any} \quad v \in X_j.$$

Thus

$$\|A_j u\|_j = C_j^{-1}\|\mathcal{I}_j A_j u\| = C_j^{-1}\|A\mathcal{I}_j u\| \leq MC_j^{-1}\|\mathcal{I}_j u\| = M\|u\|_j,$$

and **(ii)** holds. Note that $\|A\|_{\text{Op}}$ as well as $\|A_j\|_{\text{Op}}$ have been well-defined, because it was assumed, that $X_j \neq \{0\}$. We have proved that **(i)** $\Longleftrightarrow$ **(ii)** for any $M \geq 0$, i. e. the set of the upper bounds of the set $\{\|A_j\|_{\text{Op}} : j \in \mathbb{N}\}$ is equal to the set of the upper bounds of the one point set $\{\|A\|_{\text{Op}}\}$, hence the suprema are also equal. □

3.3 Basic spectral properties of diagonal operators

The last results of this section concern some elementary spectral properties of general linear operators.

Assume that $(X, \|\cdot\|) \sqsubset \bigoplus_{j \in \mathbb{N}} X_j$ and $A_j \in \mathcal{L}(X_j)$ for any $j \in \mathbb{N}$. We shall use the following notation:

$$\rho_\infty\left(\{A_j\}_{j \in \mathbb{N}}\right) := \left\{\lambda \in \bigcap_{j \in \mathbb{N}} \rho(A_j) : \sup_{j \in \mathbb{N}} \|(A_j - \lambda)^{-1}\|_{\text{Op}} = +\infty\right\}.$$

Proposition 3.6 *Assume that* $(X, \|\cdot\|)$ *is an M-BDS of a sequence* $\{(X_j, \|\cdot\|_j)\}_{j \in \mathbb{N}}$ *of Banach spaces and* $X_j \neq \{0\}$, $A_j \in \mathcal{L}(X_j)$ *for any* $j \in \mathbb{N}$. *Then:*

(i) $\sigma_{\text{p}}\left(\operatorname*{diag}_{j \in \mathbb{N}} A_j\right) = \bigcup_{j \in \mathbb{N}} \sigma_{\text{p}}(A_j)$;

(ii) $\rho\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j\right) = \left\{\lambda \in \bigcap_{j\in\mathbb{N}} \rho(A_j) : \sup_{j\in\mathbb{N}} \|(A_j-\lambda)^{-1}\|_{\mathrm{Op}} < +\infty\right\}$ *and*

$$\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j - \lambda\right)^{-1} = \operatorname*{diag}_{j\in\mathbb{N}}(A_j-\lambda)^{-1} \quad \textit{for } \lambda \in \rho\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j\right); \tag{3.6}$$

(iii) $\sigma\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j\right) = \bigcup_{j\in\mathbb{N}} \sigma(A_j) \cup \rho_\infty\left(\{A_j\}_{j\in\mathbb{N}}\right)$;

Proof Let $A := \operatorname*{diag}_{j\in\mathbb{N}} A_j$. Obviously, for any $\lambda \in \mathbb{C}$ we have $A - \lambda = \operatorname*{diag}_{j\in\mathbb{N}}(A_j - \lambda)$. Thus, by (2.2), we obtain **(i)** using Lemma 6.3 **(i)** to the sequence of operators $\{A_j-\lambda\}_{j\in\mathbb{N}}$. Similarly, we get **(ii)** by (2.3), using Lemma 6.3 **(ii)** and Proposition 3.5. And now **(iii)** follows from **(ii)**. □

Corollary 3.7 *With the assumptions of Proposition 3.6, if moreover $A_j \in \mathcal{C}(X_j)$ for any $j \in \mathbb{N}$ and there exists $\delta > 0$ such that*

$$\|(A_j-\lambda)^{-1}\|_{\mathrm{sp}} \geq \delta \|(A_j-\lambda)^{-1}\|_{\mathrm{Op}}, \quad j \in \mathbb{N}, \tag{3.7}$$

then $\sigma\left(\operatorname*{diag}_{j\in\mathbb{N}} A_j\right) = \overline{\bigcup_{j\in\mathbb{N}} \sigma(A_j)}$.

See Appendix "Generalized inversion of diagonal operators and the proof of Corollary 3.7" section for the proof.

4 Diagonal C_0-semigroups and their generators

We prove here that having C_0-semigroups[7] in all the spaces X_j, $j \in \mathbb{N}$ with a growth bound uniform in j we can obtain in a natural way a C_0-semigroup on an M-BDS of X_j. Moreover we shall see, that the generator of this semigroup is a direct sum of the appropriate generators of the semigroups on X_j.

4.1 The direct sum of operator functions and operator semigroups

First let us define "the direct sum of operator functions".

Suppose that $(X_j, \|\cdot\|_j)$ are Banach spaces, that $\mathcal{T}_j : [0;+\infty) \longrightarrow \mathcal{L}(X_j)$ for any $j \in \mathbb{N}$, and that $(X, \|\cdot\|)$ is a normed base subspace of $\bigoplus_{j\in\mathbb{N}} X_j$.

[7] We use here the following terminology—see e.g. [5]: $\mathcal{T}$ *is a semigroup of bounded operators on* Y (Y—a Banach space) iff $\mathcal{T} : [0;+\infty) \longrightarrow \mathcal{B}(Y)$ satisfies $\mathcal{T}(s+t) = \mathcal{T}(s)\mathcal{T}(t)$ for any $s, t \geq 0$ and $\mathcal{T}(0) = I$; such a semigroup $\mathcal{T}$ on Y is *a C_0-semigroup* iff $\lim_{t\to 0+} \mathcal{T}(t)y = y$ for any $y \in Y$.

Definition 4.1 *The direct sum of* $\{\mathcal{T}_j\}_{j\in\mathbb{N}}$ *on* X is the operator function $\mathcal{T}$: $[0;+\infty) \longrightarrow \mathcal{L}(X)$ denoted by $\operatorname*{diag}_{j\in\mathbb{N}} \mathcal{T}_j$ and given by

$$\mathcal{T}(t) := \operatorname*{diag}_{j\in\mathbb{N}} \mathcal{T}_j(t), \qquad t \geq 0.$$

Let us start from families being semigroups of bounded operators, without assuming yet the C_0 (the strong continuity) property. Note, that the assumption, that $\mathcal{T}(t)$ are in $\mathcal{B}(X)$, and not only in $\mathcal{L}(X)$, enforces by Proposition 3.5 the necessity of the uniform in j estimates for $\|\mathcal{T}_j(t)\|_{\mathrm{Op}}$.

Remark 4.2 Assume that $(X, \|\cdot\|)$ is an M-BDS of a sequence $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ of Banach spaces, and for any $j \in \mathbb{N}$ let $\mathcal{T}_j$ be a semigroup of bounded operators on $X_j \neq \{0\}$. If $M : [0;+\infty) \longrightarrow [0;+\infty)$ and

$$\forall_{\substack{j\in\mathbb{N}\\ t\geq 0}} \quad \|\mathcal{T}_j(t)\|_{\mathrm{Op}} \leq M(t),$$

then $\mathcal{T} := \operatorname*{diag}_{j\in\mathbb{N}} \mathcal{T}_j$ is a semigroup of bounded operators on X satisfying

$$\forall_{t\geq 0} \quad \|\mathcal{T}(t)\|_{\mathrm{Op}} \leq M(t). \tag{4.1}$$

Proof The fact that for any $t \geq 0$ we have $\mathcal{T}(t) \in \mathcal{B}(X)$ and that (4.1) holds follows immediately from Proposition 3.5. For any $f \in X$, $s, t \geq 0$, $j \in \mathbb{N}$ we have

$$(\mathcal{T}(s+t)f)_j = \mathcal{T}_j(s+t)f_j = \mathcal{T}_j(s)\mathcal{T}_j(t)f_j = \mathcal{T}_j(s)\left(\mathcal{T}(t)f\right)_j = \left(\mathcal{T}(s)\mathcal{T}(t)f\right)_j,$$

i.e. $\mathcal{T}(s+t) = \mathcal{T}(s)\mathcal{T}(t)$, and $\left(\mathcal{T}(0)f\right)_j = \mathcal{T}_j(0)f_j = f_j$, i.e. $\mathcal{T}(0) = I$. □

4.2 The C_0-semigroup case and the generators

A priori, the uniform in j estimate by $M(t)$—the asumption of Remark 4.2—possesses an arbitrary dependence on t. But in the assertion of this remark we get the same estimate for the operator function $\mathcal{T}$ being the direct sum of the operator functions $\mathcal{T}_j$. And it is a classical result, that in the case when the operator function $\mathcal{T}$ is a C_0-semigroup, its estimate $M(t)$ must have very special form: $Ce^{\omega t}$ with constants $C \geq 1$ and $\omega \in \mathbb{R}$. So, restricting ourselves only to the C_0-semigroup case, we are doomed to this type of estimates, both in the assertion as well as assumption.

Theorem 4.3 *Assume that* $(X, \|\cdot\|)$ *is an M-BDS of a sequence* $\{(X_j, \|\cdot\|_j)\}_{j\in\mathbb{N}}$ *of Banach spaces, and for any* $j \in \mathbb{N}$ *let* $\mathcal{T}_j$ *be a* C_0*-semigroup on* $X_j \neq \{0\}$. *Suppose that* $A_j \in \mathcal{L}(X_j)$ *is the generator of* $\mathcal{T}_j$ *for any* $j \in \mathbb{N}$. *If*

$$\exists_{\substack{C\geq 1\\ \omega\in\mathbb{R}}} \forall_{\substack{j\in\mathbb{N}\\ t\geq 0}} \quad \|\mathcal{T}_j(t)\|_{\mathrm{Op}} \leq Ce^{\omega t}, \tag{4.2}$$

then $\mathcal{T} := \operatorname{diag}_{j\in\mathbb{N}} \mathcal{T}_j$ *is a* C_0*-semigroup on* X *satisfying*

$$\forall_{t\geq 0} \quad \|\mathcal{T}(t)\|_{\mathrm{Op}} \leq Ce^{\omega t} \tag{4.3}$$

and $\operatorname{diag}_{j\in\mathbb{N}} A_j$ *is the generator of* $\mathcal{T}$.

Proof $\mathcal{T}$ is a semigroup on X satisfying (4.3) by Remark 4.2. We are going to prove that

$$\lim_{t\to 0+} \mathcal{T}(t)f = f \tag{4.4}$$

for any $f \in X$. But thanks to (4.3) it suffices to prove (4.4) for $f \in X_{\mathrm{fin}}$ only (see [5, Chapter I, Proposition 1.3]), because X satisfies **(den)**. Fix $f \in X_{\mathrm{fin}}$ and suppose that $0 < t_n \xrightarrow[n\to+\infty]{} 0$. We have $f \in X^{\leq r}$ for some $r \in \mathbb{N}$, thus using $\mathcal{T} = \operatorname{diag}_{j\in\mathbb{N}} \mathcal{T}_j$ for any n we get $\mathcal{T}(t_n)f \in X^{\leq r}$ and

$$(\mathcal{T}(t_n)f)_j = \mathcal{T}_j(t_n)f_j \xrightarrow[n\to+\infty]{X_j} f_j, \qquad j = 1, \ldots, r,$$

because $\mathcal{T}_j$ is a C_0-semigroup on X_j. Hence by Proposition 2.11 part 10. we get (4.4), and $\mathcal{T}$ is a C_0-semigroup on X.

Consider the generator $B \in \mathcal{L}(X)$ of $\mathcal{T}$, and denote $A := \operatorname{diag}_{j\in\mathbb{N}} A_j$. We are going to prove that $A = B$. Let $f \in D(B)$ and consider an arbitrary real sequence $\{t_n\}$ such that $0 < t_n \xrightarrow[n\to+\infty]{} 0$. By the definition of the generator we have

$$\frac{1}{t_n}\left(\mathcal{T}(t_n)f - f\right) \xrightarrow[n\to+\infty]{X} Bf.$$

Hence, by **(coor)**, for any $j \in \mathbb{N}$

$$\frac{1}{t_n}\left(\mathcal{T}_j(t_n)f_j - f_j\right) = \frac{1}{t_n}\left((\mathcal{T}(t_n)f) - f\right)_j \xrightarrow[n\to+\infty]{X_j} (Bf)_j,$$

which means that $f_j \in D(A_j)$ and $(Bf)_j = A_j f_j$. Therefore we get $\left\{A_j f_j\right\}_{j\in\mathbb{N}} = Bf \in X$ and thus $f \in D(A)$ with $Af = Bf$, i.e. we obtain $B \subset A$.

We have already proved that the semigroup $\mathcal{T}$ generated by B satisfies the growth estimate (4.3), which is also satisfied by all the semigroups $\mathcal{T}_j$ generated by A_j, $j = 1, \ldots$. Thus by General Generation Theorem for C_0-semigroups [5, Chapter II,Theorem 3.8] we have

$$\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\} \subset \rho(B) \cap \bigcap_{j\in\mathbb{N}} \rho(A_j),$$

with ω as in (4.3). Let $\lambda_0 := \omega + 1$. We have $\lambda_0 \in \rho(B) \cap \bigcap_{j\in\mathbb{N}} \rho(A_j)$, and using again General Generation Theorem we get

$$\forall_{j\in\mathbb{N}} \, \|(A_j - \lambda_0)^{-1}\|_{\mathrm{Op}} \leq C, \tag{4.5}$$

with C as in (4.3). So, by Proposition 3.6 **(ii)** we obtain $\lambda_0 \in \rho(B) \cap \rho(A)$, which gives $A = B$ (see, e.g. [2, Lemma p. 184]). □

A natural question to ask is whether the sufficient conditions for diagonal operator guaranteeing that it generates a C_0-semigroup, formulated in Theorem 4.3, are also necessary conditions. As we shall see, the answer is positive.

Denote by $\mathcal{G}(Y)$ the family of all the operators from $\mathcal{L}(Y)$ which generate a C_0-semigroup on Y and for $C \geq 1$, $\omega \in \mathbb{R}$ let $\mathcal{G}(Y, C, \omega)$ denote the family of all the operators from $\mathcal{G}(Y)$ for which the generated C_0-semigroup $\mathcal{T}$ satisfies the growth estimate

$$\forall_{t\geq 0} \quad \|\mathcal{T}(t)\|_{\mathrm{Op}} \leq Ce^{\omega t}.$$

Recall that for any Banach space Y and $A \in \mathcal{G}(Y)$ we have $A \in \mathcal{G}(Y, C, \omega)$ for some $C \geq 1$, $\omega \in \mathbb{R}$ (see e.g.[5, Chapter I, Proposition 1.4]).

We can summarize the relations between generation properties for $A := \operatorname{diag}_{j\in\mathbb{N}} A_j$ and for all the individual A_j in the following theorem, being the main result of the paper.

Theorem 4.4 *Assume that* $(X, \|\cdot\|)$ *is an M-BDS of a sequence* $\big\{(X_j, \|\cdot\|_j)\big\}_{j\in\mathbb{N}}$ *of Banach spaces and* $A := \operatorname{diag}_{j\in\mathbb{N}} A_j$, *where* $X_j \neq \{0\}$ *and* $A_j \in \mathcal{L}(X_j)$ *for any* $j \in \mathbb{N}$.

If $C \geq 1$, $\omega \in \mathbb{R}$, *then* $A \in \mathcal{G}(X, C, \omega)$ *iff* $A_j \in \mathcal{G}(X_j, C, \omega)$ *for any* $j \in \mathbb{N}$.

In particular: $A \in \mathcal{G}(X)$ *iff*

$$\exists_{\substack{C\geq 1\\ \omega\in\mathbb{R}}} \forall_{j\in\mathbb{N}} \, A_j \in \mathcal{G}(X_j, C, \omega). \tag{4.6}$$

Proof The part "$\Longleftarrow$" is contained in Theorem 4.3. Suppose that $A \in \mathcal{G}(X, C, \omega)$ and let $\mathcal{T}$ be the C_0-semigroup on X generated by A. Fix $r \in \mathbb{N}$. By Proposition 2.11 part 11. $\widetilde{X_r}$ is a closed subspace of X. By Chernoff-Post-Widder Product Formula [5, Chapter IV, Corollary 2.5] for any $t > 0$ and $f \in \widetilde{X_r}$ we have

$$R_n(t) f \xrightarrow[n\to+\infty]{X} \mathcal{T}(t) f, \tag{4.7}$$

where $R_n(t) \in \mathcal{B}(X)$ is defined for sufficiently large n, say for $n \geq N(t)$, by the following formula:

$$R_n(t) := (-n/t)^n \left(\left(A - \frac{n}{t}\right)^{-1}\right)^n, \quad n \geq N(t).$$

In particular the choice of $N(t)$ is such that $\frac{n}{t} \in \rho(A)$ for all $n \geq N(t)$, so by (3.6) in Proposition 3.6 $\left(A - \frac{n}{t}\right)^{-1}$ is a diagonal operator, and thus also $R_n(t)$ is a diagonal operator for any $n \geq N(t)$. As a consequence we get $R_n(t)f \in \widetilde{X_r}$ for $n \geq N(t)$, and hence, we get $\mathcal{T}(t)f \in \widetilde{X_r}$ by closedness of $\widetilde{X_r}$ and by (4.7). So $\widetilde{X_r}$ is $\mathcal{T}$-invariant, and we can consider the restriction $\widetilde{\mathcal{T}}$ of $\mathcal{T}$ to $\widetilde{X_r}$, which is a C_0-semigroup on $\widetilde{X_r}$ defined by

$$\widetilde{\mathcal{T}}(t) \in \mathcal{B}(\widetilde{X_r}), \qquad \widetilde{\mathcal{T}}(t) := \mathcal{T}(t)|_{\widetilde{X_r}} \quad \text{for} \quad t \geq 0 \tag{4.8}$$

and generated by $\widetilde{A} \in \mathcal{L}(\widetilde{X_r})$ given by

$$\widetilde{A} \in \mathcal{L}(\widetilde{X_r}), \qquad D(\widetilde{A}) := D(A) \cap \widetilde{X_r}, \quad \widetilde{A} := A|_{D(\widetilde{A})} \tag{4.9}$$

(see [5, I.1.11 and Corollary in II.2.3]). By (4.8) we have

$$\|\widetilde{\mathcal{T}}(t)\|_{\mathrm{Op}} \leq \|\mathcal{T}(t)\|_{\mathrm{Op}} \leq Ce^{\omega t}, \tag{4.10}$$

hence $\widetilde{A} \in \mathcal{G}(\widetilde{X_r}, C, \omega)$. Moreover, by (4.9) and (3.1),

$$D(\widetilde{A}) = \{f \in \widetilde{X_r} : f_r \in D(A_r)\} = \{\mathcal{I}_r u : u \in D(A_r)\} = \mathcal{I}_r(D(A_r))$$

and by (3.2)

$$\widetilde{A}(\mathcal{I}_r u) = A(\mathcal{I}_r u) = \mathcal{I}_r(A_r u). \tag{4.11}$$

Finally by **(iso)** $\mathcal{I}_r : X_r \longrightarrow \widetilde{X_r}$ is an isomorphism, and we can consider the C_0-semigroup $\hat{\mathcal{T}}$ on X_r transfering of $\widetilde{\mathcal{T}}$ from $\widetilde{X_r}$ onto X_r by $\mathcal{I}_r^{-1}$, i.e.

$$\hat{\mathcal{T}}(t) \in \mathcal{B}(X_r), \qquad \hat{\mathcal{T}}(t) := \mathcal{I}_r^{-1}\widetilde{\mathcal{T}}(t)\mathcal{I}_r \quad \text{for} \quad t \geq 0. \tag{4.12}$$

It is generated by $\hat{A} := \mathcal{I}_r^{-1}\widetilde{A}\,\mathcal{I}_r$ (see, e.g., [5, II.2.1]), so by)4.11) $\hat{A} = A_r$, and this gives $A_r \in \mathcal{G}(X_r)$. But by **(sim)**, using Lemma 6.1 and (4.10) we also have

$$\|\hat{\mathcal{T}}(t)\|_{\mathrm{Op}} \leq \|\mathcal{I}_r^{-1}\|_{\mathrm{Op}}\|\widetilde{\mathcal{T}}(t)\|_{\mathrm{Op}}\|\mathcal{I}_r\|_{\mathrm{Op}} \leq Ce^{\omega t}, \quad t \geq 0,$$

hence $A_r \in \mathcal{G}(X_r, C, \omega)$. □

Example 4.5 (Scalar diagonal generators) Consider the "scalar product case", i.e., an arbitrary $(X, \|\cdot\|)$ being an M-BDS of countable copies of $\mathbb{C}$: $X_j = \mathbb{C}$ for any $j \in \mathbb{N}$. Let us try to find, some explicit conditions for a diagonal operator A on X, which are equivalent to the condition $A \in \mathcal{G}(X)$. The equivalent condition (4.6) from Theorem 4.4, with $A = \operatorname*{diag}_{j\in\mathbb{N}} A_j$ means in particular that each A_j is densely defined, i.e. $D(A_j) = \mathbb{C}$ in this case. Hence each A_j is the multiplication by a number $a_j \in \mathbb{C}$: $A_j x = a_j x$, $x \in \mathbb{C}$. Thus, by the definition of $\operatorname*{diag}_{j\in\mathbb{N}} A_j$, we have to consider only the operators of the form $A = M_a$ where M_a denotes operator of multiplication by a sequence $a = \{a_j\}_{j\in\mathbb{N}}$:

$$D(M_a) := \{f \in X : af \in X\}, \qquad M_a f := af \quad \text{for} \quad f \in D(M_a)$$

and af denotes the usual product of sequences: $\{a_j f_j\}_{j\in\mathbb{N}}$. Let us express now the conditions of (4.6) in terms of the scalar sequence a. Obviously the semigroup $\mathcal{T}_j$ generated by A_j in C is given by $\mathcal{T}_j(t)x = \exp(a_j t)x$ for any $x \in \mathbb{C}$, $t \geq 0$. But

$$\|\mathcal{T}_j(t)\|_{\mathrm{Op}} = |\exp(a_j t)| = e^{t\mathrm{Re}\, a_j},$$

and hence (4.6) holds iff

$$\{\mathrm{Re}\, a_j\}_{j\in\mathbb{N}} \qquad \text{is bounded from above.} \tag{4.13}$$

Finally a diagonal operator A is in $\mathcal{G}(X)$ iff $A = M_a$ for some $a = \{a_j\}_{j\in\mathbb{N}}$ satisfying (4.13). This result generalizes the well-known result on diagonal operators generating C_0-semigroups in all the standard sequence spaces $\ell^p(\mathbb{N})$ with $1 \leq p < +\infty$ and $c_0(\mathbb{N})$.

5 Application to a stochastic particle system

5.1 The general model

Let us consider a system of (large) number N of individuals of a population (see [1,8] and references therein). Every k–th individual, $k = 1, \ldots, N$, is characterized by a parameter

$$\mathbf{u}_k \in \mathbf{U},$$

describing its biological (or physical) inner state, and $(\mathbf{U}, \mu)$ is a space with a σ–finite measure μ. In many particular applications $\mathbf{U}$ is a product of a discrete set and a Lebesgue–measurable subset of $\mathbb{R}^d$, $d \geq 1$, and the measure μ is a product of counting measure and the Lebesgue measure.

We assume that the evolution of the system is defined by the Markov jump processes corresponding to N interacting individuals (see [1,8] and references therein) and that the evolution of probability densities is given here by an evolution equation (the so–called modified Liouville equation) defined by a (linear) generator. The standard procedure related to taking the limit $N \to \infty$ leads to the following infinite system of equations (usually referred to as a hierarchy of equations)

$$(\partial_t f_j)(t) = \Theta_{j+1}(f_{j+1}(t)), \qquad t \geq 0, \quad j = 1, 2, \ldots, \tag{5.1}$$

where for each $t \geq 0$ the sequence $\{f_j(t)\}_{j\geq 1}$ satisfies $f_j(t) \in L^1(\mathbf{U}^j)$ for any j, with the product measure μ^j on $\mathbf{U}^j$, and $\Theta_{j+1} : L^1(\mathbf{U}^{j+1}) \longrightarrow L^1(\mathbf{U}^j)$ is given by

$$(\Theta_{j+1}h)(\mathbf{u}_1, \ldots, \mathbf{u}_j)$$
$$= \sum_{s=1}^{j} \int_{\mathbf{U}^2} A(\mathbf{u}_s; \mathbf{v}, \mathbf{w}) a(\mathbf{v}, \mathbf{w})\, h(\mathbf{u}_1, \ldots, \mathbf{u}_{s-1}, \mathbf{v}, \mathbf{u}_{s+1}, \ldots, \mathbf{u}_j, \mathbf{w})\, \mathrm{d}\mu(\mathbf{v})\, \mathrm{d}\mu(\mathbf{w})$$

$$-\sum_{s=1}^{j} \int_{\mathbf{U}} a(\mathbf{u}_s, \mathbf{v})\, h(\mathbf{u}_1, \ldots, \mathbf{u}_j, \mathbf{v})\, \mathrm{d}\mu(\mathbf{v})$$

for $h \in L^1(\mathbf{U}^{j+1})$. The problem concerns the existence of a solution of (5.1), i.e. a function: $[0; +\infty) \ni t \longmapsto \{f_j(t)\}_{j\geq 1} := f(t)$ satisfying (5.1) with a certain "rational" sense of the derivative ∂_t, for a given initial data. The approach of [8] (see also [1] and references therein) gives for some initial conditions a kind of weak solutions—i.e. solutions which all j-th terms are differentiable (as vector functions of t with values in Banach space $L^1(\mathbf{U}^j)$), separately for $j \in \mathbb{N}$—see also the notion *separately-weak solution* in the next subsection. We ask now for some stronger result—at least with some extra assumptions and some particular initial conditions.

The above model was determined in [8] by a pair of functions A and a, where

- $a(\mathbf{u}, \mathbf{v})$ has the meaning of *the rate of interaction* of individual with state $\mathbf{u} \in \mathbf{U}$ and individual with state $\mathbf{v} \in \mathbf{U}$; a is a measurable bounded function

$$a : \mathbf{U}^2 \to [0; +\infty)\,; \tag{5.2}$$

- $A(\mathbf{u}; \mathbf{v}, \mathbf{w})$ possesses the meaning of *transition probability* into state $\mathbf{u} \in \mathbf{U}$, of individual with state $\mathbf{v} \in \mathbf{U}$ due to the interaction with individual with state $\mathbf{w} \in \mathbf{U}$; A is a measurable function

$$A : \mathbf{U}^3 \to [0; +\infty)\,, \tag{5.3}$$

such that for any $\mathbf{v}, \mathbf{w} \in \mathbf{U}$

$$\int_{\mathbf{U}} A(\mathbf{u}; \mathbf{v}, \mathbf{w})\, \mathrm{d}\mu(\mathbf{u}) = 1\,. \tag{5.4}$$

Referring to the mentioned above standard procedure (related to "the transition $N \to \infty$") we are interested only in such solutions $t \longmapsto f(t)$ which are so-called admissible hierarchies[8] for any $t \geq 0$.

Definition 5.1 *Admissible hierarchy* is a sequence $\{f_j\}_{j\geq 1}$ such that for any $j \geq 1$

(i) f_j is a probability density on $\mathbf{U}^j$ with respect to the measure μ^j, i.e., $f_j \geq 0$ and $\int_{\mathbf{U}^{\mathbf{j}}} f_j\, \mathrm{d}\mu^j = 1$;

(ii) $f_j(\mathbf{u}_1, \ldots, \mathbf{u}_j) = f_j(\mathbf{u}_{r_1}, \ldots, \mathbf{u}_{r_j})$, for $(\mathbf{u}_1, \ldots, \mathbf{u}_j) \in \mathbf{U}^j$ and for any permutation $r = \{r_1, \ldots, r_j\}$ of the set $\{1, \ldots, j\}$;

(iii) $f_j(\mathbf{u}_1, \ldots, \mathbf{u}_j) = \int_{\mathbf{U}} f_{j+1}(\mathbf{u}_1, \ldots, \mathbf{u}_{j+1})\, \mathrm{d}\mu(\mathbf{u}_{j+1})$, for $(\mathbf{u}_1, \ldots, \mathbf{u}_j) \in \mathbf{U}^j$.

[8] Note the double meaning of the notion "hierarchy" here: (1)—a special kind of system of equations and (2)—a special kind of sequence of functions (classes of functions) with growing number of variables.

5.2 A simplified model

In the present paper, to obtain the above mentioned stronger kind of solutions, we impose a simplifying assumption:

Assumption 5.2 The function a does not depend on the second variable, i.e.,

$$a(\mathbf{u}, \mathbf{v}) = \tilde{a}(\mathbf{u}), \qquad \text{for all } (\mathbf{u}, \mathbf{v}) \in \mathbf{U}^2,$$

for some measurable bounded nonnegative function $\tilde{a}$ on $\mathbf{U}$, and the function A does not depend on the third variable, i.e.,

$$A(\mathbf{u}; \mathbf{v}, \mathbf{w}) = \tilde{A}(\mathbf{u}; \mathbf{v}), \qquad \text{for all } (\mathbf{u}, \mathbf{v}, \mathbf{w}) \in \mathbf{U}^3,$$

for some measurable nonnegative function $\tilde{A}$ such that

$$\int_{\mathbf{U}} \tilde{A}(\mathbf{u}; \mathbf{v})\, \mathrm{d}\mu(\mathbf{u}) = 1, \qquad \text{for all } \mathbf{v} \in \mathbf{U}. \tag{5.5}$$

Assumption 5.2 means that during the interaction between two individuals the new state of the first individual is chosen with probability that is independent of the current state of the second individual.

One can easily see that making this assumption and restricting ourselves only to admissible hierarchies, (i.e. assuming that $f(t)$ is an admissible hierarchy for any $t \geq 0$) we can rewrite (5.1) into the form

$$(\partial_t f_j)(t) = \tilde{\Theta}_j(f_j(t)), \qquad t \geq 0, \qquad j = 1, 2, \ldots, \tag{5.6}$$

where $\tilde{\Theta}_j : L^1(\mathbf{U}^j) \longrightarrow L^1(\mathbf{U}^j)$ is given by

$$(\tilde{\Theta}_j h)(\mathbf{u}_1, \ldots, \mathbf{u}_j) = \sum_{s=1}^{j} \int_{\mathbf{U}} \tilde{A}(\mathbf{u}_s; \mathbf{v}) \tilde{a}(\mathbf{v}) h(\mathbf{u}_1, \ldots, \mathbf{u}_{s-1}, \mathbf{v}, \mathbf{u}_{s+1}, \ldots, \mathbf{u}_j) \mathrm{d}\mu(\mathbf{v})$$
$$- \left(\sum_{s=1}^{j} \tilde{a}(\mathbf{u}_s) \right) h(\mathbf{u}_1, \ldots, \mathbf{u}_j) \tag{5.7}$$

for each $h \in L^1(\mathbf{U}^j)$. So we obtain a case of "j - separation" of all the equations of the system.

We can first study a more general problem related to system (5.6):

Can this system be solved, without restricting only to solutions $t \longmapsto f(t)$ being admissible hierarchies for any $t \geq 0$?

Having general solutions $t \longmapsto f(t)$—for **all** such initial conditions $f(0)$ which satisfy $f_j(0) \in L^1(\mathbf{U}^j)$ for any j, we can study also solution satisfying a particular initial condition which is an admissible hierarchy. And we can ask then, whether $f(t)$ is an admissible hierarchy for all $t \geq 0$. We shall now show easily, that this problem can be immediately (and positively) solved by analyzing the equations of system (5.6) separately for each j, in the standard C_0-semigroup sense in the Banach spaces $L^1(\mathbf{U}^j)$.

By Tonelli and Fubini theorems it is easy to see that under Assumption 5.2 each operator $\tilde{\Theta}_j$ is a bounded operator in $L_1(\mathbf{U}^j)$. In fact, by (5.7), we have

$$\|\tilde{\Theta}_j h\|_{L^1(\mathbf{U}^j)} \leq 2j\|\tilde{a}\|_{L^\infty(\mathbf{U})}\|h\|_{L^1(\mathbf{U}^j)}, \tag{5.8}$$

for each $h \in L^1(\mathbf{U}^j)$.

Now, for each separate j, let us define $\mathcal{T}_j$ to be the C_0-semigroup in $L_1(\mathbf{U}^j)$ generated by $\tilde{\Theta}_j$. These semigroups are operator norm continuous, by the boudedness of the generators, and

$$\mathcal{T}_j(t) = e^{t\tilde{\Theta}_j} \quad \text{for any } t \geq 0. \tag{5.9}$$

Let us also fix the initial data—a sequence $\{F_j\}_{j\geq 1}$ such that

$$F_j \in L^1(\mathbf{U}^j) \quad \text{for any } j \geq 1 \tag{5.10}$$

and for each $t \geq 0$ define the sequence $f(t) = \{f_j(t)\}_{j\geq 1}$ as follows:

$$f_j(t) := \mathcal{T}_j(t)F_j, \quad j \geq 1.$$

Let $(X, \|\cdot\|)$ be a fixed normed space with $X \subset \bigoplus_{j\in\mathbb{N}} L^1(\mathbf{U}^j)$. We shall use the following terminology to make the formulations of our statements precise: a function $[0;+\infty) \ni t \longmapsto f(t) = \{f_j(t)\}_{j\geq 1}$ with $f_j(t) \in L^1(\mathbf{U}^j)$ for any $j \geq 1$, $t \geq 0$ is

- a *separately-weak solution* of (5.6) iff $[0;+\infty) \ni t \longmapsto f_j(t) \in L^1(\mathbf{U}^j)$ are differentiable as $L^1(\mathbf{U}^j)$ vector-valued functions for all $j \geq 1$ and (5.6) holds with ∂_t meaning the derivative of an $L^1(\mathbf{U}^j)$ vector-valued function for each j.
- a X-*strong solution* of (5.6) iff $f(t) \in X$ for any $t \geq 0$ and $[0;+\infty) \ni t \longmapsto f(t)$ is differentiable as X vector-valued function and

$$(\partial_{X,t} f)(t) = \left\{\tilde{\Theta}_j(f_j(t))\right\}_{j\geq 1}, \qquad t \geq 0, \tag{5.11}$$

holds with $\partial_{X,t}$ meaning the derivative of an X vector-valued function.

Note, that to be an X-strong solution, **is not** only the differentiability kind problem for $t \longmapsto f(t)$. It is also the requirement that $\left\{\tilde{\Theta}_j(f_j(t))\right\}_{j\geq 1} \in X$ for any $t \geq 0$. Obviously each X-strong solution is automatically a separately-weak solution, if X is such a space that all the coordinate maps $X \ni g \longmapsto g_j \in L^1(\mathbf{U}^j)$ are continuous.

Corollary 5.3 *Suppose that Assumption 5.2 and (5.10) hold. The function* $[0;+\infty) \ni t \longmapsto f(t)$ *defined above is a separately-weak solution of (5.6), and it is the unique separately-weak solution for the initial data* $F = \{F_j\}_{j\geq 1}$. *If, moreover, F is an admissible hierarchy, then $f(t)$ is an admissible hierarchy for any $t \geq 0$.*

Before the proof we need the following two lemmas.

Lemma 5.4 *For each $j \geq 1$ the semigroup $\mathcal{T}_j$ is positive*[9] *and for any $h \in L^1(\mathbf{U}^j)$*

$$\int_{\mathbf{U}^j} \mathcal{T}_j(t)h \, \mathrm{d}\mu^j = \int_{\mathbf{U}^j} h \, \mathrm{d}\mu^j. \tag{5.12}$$

Proof By (5.7) the operator $\tilde{\Theta}_j + \gamma_j I$ is a positive (and bounded) operator in $L^1(\mathbf{U}^j)$, with I—the identity operator and $\gamma_j := j\|\tilde{a}\|_{L^\infty}$. By the series formula for the exponent $e^{t(\tilde{\Theta}_j+\gamma_j I)}$ is also positive for any $t \geq 0$. And hence the operator

$$\mathcal{T}_j(t) = e^{t(\tilde{\Theta}_j+\gamma_j I)} e^{-t\gamma_j}$$

is positive for any $t \geq 0$.

For each $h \in L^1(\mathbf{U}^j)$, by (5.7) and (5.5) we have

$$\int\limits_{\mathbf{U}^j} \tilde{\Theta}_j h \, \mathrm{d}\mu^j = 0 \tag{5.13}$$

and thus, again by the series formula for the exponent, we get (5.12). □

We have:

Lemma 5.5 *Suppose that X and Y are Banach spaces, A is a generator of the C_0-semigroup $\mathcal{T}_A$ in X, B is a generator of the C_0-semigroup $\mathcal{T}_B$ in Y and $S : X \longrightarrow Y$ is a bounded operator satisfying*

$$\forall_{x \in D(A)} \; Sx \in D(B) \& SAx = BSx.$$

Then for any $t \geq 0$ $S\mathcal{T}_A(t) = \mathcal{T}_B(t)S$.

The above is obvious e.g. by the constantness of the function $[0; +\infty) \ni t \longmapsto S\mathcal{T}_A(t)x - \mathcal{T}_B(t)Sx$ for $x \in D(A)$.

Proof of Corollary 5.3 The first part of the assertion, is just obvious from the basic C_0-semigroup theory separately on each "level" j; to get the uniqueness we use the abstract Cauchy problem theory for the C^1 solutions (see [10, Th. 1.3 Chapter 4])—it suffices to observe that for each separately-weak solution $[0; +\infty) \ni t \longmapsto f(t)$ for any $j \geq 1$ the function $[0; +\infty) \ni t \longmapsto f_j(t)$ is automatically a C^1 function (as $L^1(\mathbf{U}^j)$ vector-valued function), because of the continuity of the generator $\tilde{\Theta}_j$. Now assume that $\{F_j\}_{j\geq 1}$ is an admissible hierarchy. Using Lemma 5.4 we see that $f(t) = \{f_j(t)\}_{j\geq 1}$ satisfies part (i) of the definition of admissible hierarchy for any $t \geq 0$.

[9] i.e., for any $t \geq 0$ and for any $0 \leq h \in L^1(\mathbf{U}^j)$ we have $\mathcal{T}_j(t)h \geq 0$.

Observe that, if $g \in L^1(\mathbf{U}^{j+1})$ and $h \in L^1(\mathbf{U}^j)$ is given by

$$h(\mathbf{u}_1, \ldots, \mathbf{u}_j) = \int\limits_{\mathbf{U}} g(\mathbf{u}_1, \ldots, \mathbf{u}_{j+1})\, \mathrm{d}\mu(\mathbf{u}_{j+1})$$

for $(\mathbf{u}_1, \ldots, \mathbf{u}_j) \in \mathbf{U}^j$, then by (5.7) and (5.5)

$$\int\limits_{\mathbf{U}} (\tilde{\Theta}_{j+1} g)(\mathbf{u}_1, \ldots, \mathbf{u}_{j+1})\, \mathrm{d}\mu(\mathbf{u}_{j+1}) = \tilde{\Theta}_j h(\mathbf{u}_1, \ldots, \mathbf{u}_j)\,, \tag{5.14}$$

for $\mathbf{u}_1, \ldots, \mathbf{u}_j \in \mathbf{U}$. Hence, defining $S_j : L^1(\mathbf{U}^{j+1}) \longrightarrow L^1(\mathbf{U}^j)$ by the formula

$$(S_j g)(\mathbf{u}_1, \ldots, \mathbf{u}_j) := \int\limits_{\mathbf{U}} g(\mathbf{u}_1, \ldots, \mathbf{u}_{j+1})\, \mathrm{d}\mu(\mathbf{u}_{j+1})$$

we see that S_j is bounded and $S_j \tilde{\Theta}_{j+1} = \tilde{\Theta}_j S_j$ by (5.14). Therefore using Lemma 5.5 and the fact that $S_j F_{j+1} = F_j$ we see that $f(t) = \left\{f_j(t)\right\}_{j\geq 1}$ satisfies part (iii) of the definition of admissible hierarchy for any $t \geq 0$. To check part (ii) of the definition we use a similar reasoning with any „variable permutation operator" $P_r : L^1(\mathbf{U}^j) \longrightarrow L^1(\mathbf{U}^j)$, which is an isometry of $L^1(\mathbf{U}^j)$ and satisies $P_r \tilde{\Theta}_j = \tilde{\Theta}_j P_r$ for any permutation r of $\{1, \ldots, j\}$ (i.e. we use Lemma 5.5 with $S = P_r$ and $X = Y = L^1(\mathbf{U}^j)$). □

As we could see, using the standard C_0-semigroup approach separately in each space $L^1(\mathbf{U}^j)$ we got separately-weak solutions. The main goal of this section is to show, that thanks to the abstract results of the previous sections, the system 5.6 can be also solved in a different way, which can guarantee a "stronger kind"of solutions than separately weak solutions obtained in Corolary 5.3. In particular we shall get a stronger "j - joint" sense of the derivative ∂_t, provided we restrict ourselves to an apropriate class of initial conditions. To formulate the above more precisely and to prove it we shall apply the theory of Sect. 4.2, namely Theorems 4.3 and 4.4. But observe that we do not have any j-joint upper bound for $\|\tilde{\Theta}_j h\|_{L^1(\mathbf{U}^j)}$ (conversly—our estimate in (5.8) grows linearly in j...). Hence, obtaining a j-joint upper bound for the growth of the semigroups $\mathcal{T}_j$ needs a more delicate argumentation, based on the positivity of operators.

Lemma 5.6 $\|\mathcal{T}_j(t)\|_{\mathrm{Op}} \leq 2\sqrt{2}$ *for any* $j \geq 1$, $t \geq 0$.

Proof By Lemma 5.4, if $h \in L^1(\mathbf{U}^j)$ is positive, then $\mathcal{T}_j(t)h$ is also positive and thus

$$\|\mathcal{T}_j(t)h\|_{L^1(\mathbf{U}^j)} = \int_{\mathbf{U}^j} \mathcal{T}_j(t)h\, \mathrm{d}\mu^j = \int_{\mathbf{U}^j} h\, \mathrm{d}\mu^j = \|h\|_{L^1(\mathbf{U}^j)}.$$

Decomposing a real–valued function $L^1(\mathbf{U}^j) \ni h = h_+ - h_-$, where $h_+ = \frac{|h|+h}{2}$ and $h_- = \frac{|h|-h}{2}$ we have

$$\mathcal{T}_j(t)h = \mathcal{T}_j(t)h_+ - \mathcal{T}_j(t)h_-\,,$$

and

$$|\mathcal{T}_j(t)h| \le \mathcal{T}_j(t)h_+ + \mathcal{T}_j(t)h_- = 2\mathcal{T}_j(t)|h|\,.$$

Thus

$$\|\mathcal{T}_j(t)h\|_{L^1(\mathbf{U}^j)} \le 2\|h\|_{L^1(\mathbf{U}^j)}\,,$$

for each real–valued $h \in L^1(\mathbf{U}^j)$. Finally, for a complex–valued function $h \in L^1(\mathbf{U}^j)$, decomposing it onto the real and imaginary part, we analogously obtain

$$\|\mathcal{T}_j(t)h\|_{L^1(\mathbf{U}^j)} \le 2\sqrt{2}\|h\|_{L^1(\mathbf{U}^j)}\,. \tag{5.15}$$

□

Now we may adopt the theory developed in the previous sections. Let $\left(X_j\,,\ \|\,.\,\|_j\right)$, $j \in \mathbb{N}$, be defined as

$$X_j := L^1(\mathbf{U}^j)\,, \qquad \|\,.\|_j := \|\,.\,\|_{L^1(\mathbf{U}^j)}\,.$$

Let $(X, \|\cdot\|)$ be any of the corresponding M–BDS of the sequence $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ (cf. Definition 2.9). Let us define in X:

$$\tilde{\Theta} := \operatorname*{diag}_{j\in\mathbb{N}} \tilde{\Theta}_j\,, \qquad \mathcal{T} := \operatorname*{diag}_{j\in\mathbb{N}} \mathcal{T}_j\,.$$

By Theorems 4.3 and 4.4 $\mathcal{T}$ is a C_0-semigroup in X and $\tilde{\Theta}$ is the generator of $\mathcal{T}$.

Recall that the function $[0; +\infty) \ni t \longmapsto f(t)$ was defined by the initial data—a sequence $\left\{F_j\right\}_{j\ge1}$. Suppose now, that

$$\left\{F_j\right\}_{j\ge1}\,, \left\{\tilde{\Theta}_j F_j\right\}_{j\ge1} \in X, \tag{5.16}$$

and define $[0; +\infty) \ni t \longmapsto \hat{f}(t)$ by the formula $\hat{f}(t) := \mathcal{T}(t)\left\{F_j\right\}_{j\ge1}$ for $t \ge 0$. This function can be treated as the announced earlier new solution of the system (5.6) "stronger than f". In particular, for any $t \ge 0$ we have $\hat{f}(t) \in X$.

Theorem 5.7 *Suppose that X is an M–BDS of the sequence $\left\{(X_j, \|\cdot\|_j)\right\}_{j\in\mathbb{N}}$ and that (5.16) holds. Then the function $t \longmapsto f(t)$ is an X-strong solution of (5.6) and*

$f(t) = \hat{f}(t)$ for any $t \geq 0$. In particular $f(t) \in X$ fot any $t \geq 0$ and the function $t \longmapsto f(t)$ is differentiable on $[0; +\infty)$ as an X-vector function. Moreover for any t

$$\|f(t)\| \leq 2\sqrt{2}\,\|\{F_j\}_{j\geq 1}\|.$$

Proof Observe first that $\hat{f}(t) = f(t)$ for any t, because by the definition of the direct sum of operator functions (see Definition 4.1) we have $\hat{f}_j(t) = \mathcal{T}_j(t)F_j = f_j(t)$. And (5.16) means exactly that $\{F_j\}_{j\geq 1} \in D(\tilde{\Theta})$. Hence the assertion follows directly from Theorem 4.3 and from the generator domain-strong differentiability property of C_0-semigroups (see e.g. [10, Th. 1.2.4]). □

Referring to Example 2.18 we may note that the appropriate M–BDS X can be constructed for instance on the basis of any

$$\ell^p_w(\mathbb{N}, \mathcal{X}), \qquad 1 \leq p < \infty, \tag{5.17}$$

or

$$c_{0,w}(\mathbb{N}, \mathcal{X}). \tag{5.18}$$

Note also, that a reasonable choice of the weight sequence $w = \{w_j\}_{j\in\mathbb{N}}$ should guarantee that each admissible hierarchy is in X. We shall obtain this condition choosing such w that:

$$\sum_{j\in\mathbb{N}} w_j < +\infty,$$

in case of (5.17), or

$$\lim_{j\to\infty} w_j = 0,$$

in case of (5.18). And if we want more—namely—that the results of Theorem 5.7 could be applied to any initial condition being an admissible hierarchy, due to (5.8) it suffices to assume

$$\sum_{j\in\mathbb{N}} j w_j < +\infty,$$

in case of (5.17), or

$$\lim_{j\to\infty} j w_j = 0$$

in case of (5.18).

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Appendix

A lemma on norm-monotonic and on similaritity linear maps

The result below is used mainly in the proofs of Proposition 2.11 and Theorem 4.4.

Lemma 6.1 *Suppose that $T : Y_1 \longrightarrow Y_2$ is a linear map. Then the following three conditions are mutually equivalent:*

(a) *T is norm-monotonic;*
(b) *for any $y, y' \in Y_1$ $\big(\|y\|_1 = \|y'\|_1 \Longrightarrow \|Ty\|_2 = \|Ty'\|_2\big)$;*
(c) *T is a similarity.*

If moreover $\operatorname{Ran} T = Y_2 \neq \{0\}$*, then each of the above conditions is equivalent to:*

(d) *T is invertible, T and T^{-1} are bounded and*

$$\|T\|_{\mathrm{Op}} \cdot \|T^{-1}\|_{\mathrm{Op}} = 1. \tag{6.1}$$

The obvious proof is omitted.

Remark 6.2 The condition (6.1) can be equivalently replaced by

$$\|T\|_{\mathrm{Op}} \cdot \|T^{-1}\|_{\mathrm{Op}} \leq 1, \tag{6.2}$$

because always $\|T\|_{\mathrm{Op}} \cdot \|T^{-1}\|_{\mathrm{Op}} \geq \|TT^{-1}\|_{\mathrm{Op}} = \|I\|_{\mathrm{Op}} = 1$.

The proof of Proposition 2.11

Proof 1. If $r \geq s \geq 1$ and $f \in \bigoplus_{j\in\mathbb{N}} X_j$, then $\|(\mathcal{P}_r f)_j\|_j = \|(\mathcal{P}_s f)_j\|_j$ for $s \geq j \geq 1$ and $\|(\mathcal{P}_r f)_j\|_j \geq 0 = \|(\mathcal{P}_s f)_j\|_j$ for $j > s$. This gives $\|\mathcal{P}_r f\| \geq \|\mathcal{P}_s f\|$ by **(mono)**.
2. For increasing sequences "lim" and "sup" coincide.
3. It suffices to use the triangle inequality:

$$|\|f\| - \|\mathcal{P}_r f\|| \leq \|f - \mathcal{P}_r f\|$$

and the boundedness of each convergent sequence.
4. **(coor)** and **(inj)** give the continuity of $\mathcal{P}_r|_X$, because $\pi_j = \mathcal{I}_j \circ p_j$ for $j \in \mathbb{N}$, and thus

$$\mathcal{P}_r|_X = \sum_{j=1}^{r} \pi_j|_X = \sum_{j=1}^{r} \mathcal{I}_j \circ (p_j|_X).$$

5. It suffices to use Banach – Steinhaus Theorem to the family $\{\mathcal{P}_n|_X\}_{n\geq 1}$ of linear operators, being bounded thank to (**proj** $--$).
6. By **(appr)** and part 4. we have $\|f\| = \lim_{r\to+\infty}\|\mathcal{P}_r f\| \leq \sup_{r\in\mathbb{N}}\|\mathcal{P}_r f\|$, so it suffices to use (**proj**$-$).
7. The family of linear operators $\{I-\mathcal{P}_n|_X\}_{n\geq 1}$ on the space X is equibounded by (**proj**$-$). Moreover, if $f \in X_{\text{fin}}$, then $(I-\mathcal{P}_n)f = 0$ for n large enough, so we get (2.11) by **(den)**.
8. It is obvious by part 3.
9. It is obvious by parts 1., 2. and 3.
10. By $f, f^{(n)} \in X^{\leq r}$ we have $f^{(n)} = \sum_{j=1}^{r}\mathcal{I}_j f_j^{(n)}$ for any $n \in \mathbb{N}$ and $f = \sum_{j=1}^{r}\mathcal{I}_j f_j$. So by $f_j^{(n)} \xrightarrow[n\to+\infty]{X_j} f_j$ for any $j = 1,\dots,r$ and by the continuity of all the maps $\mathcal{I}_j$ we get $f^{(n)} \xrightarrow[n\to+\infty]{X} f$.
11. If $f \in X$, $f^{(n)} \in X^{\leq r}$ and $f^{(n)} \xrightarrow[n\to+\infty]{X} f$, then by **(coor)** for any $s > r$ we have $0 = f_s^{(n)} \xrightarrow[n\to+\infty]{X_s} f_s$, which means that $f_s = 0$. So $f \in X^{\leq r}$. The proof for $\widetilde{X_r}$ is analogic.
12. The map $\mathcal{I}_r : X_r \longrightarrow \widetilde{X_r}$ is a linear bijection, which is continuous by **(inj)**. By **(coor)** and by part 11. $\widetilde{X_r}$ is a closed subspace of X, so by **(ban)** $\widetilde{X_r}$ is also a Banach space. But X_r is a Banach space, hence it suffices to use the inverse mapping theorem.
13. By **(mono)** $\mathcal{I}_r : X_r \longrightarrow \widetilde{X_r}$ is norm-monotonic, thus by Lemma 6.1 it is a similarity and, also by this lemma, it is an isomorphism (in the case $X_r \neq \{0\}$, but when $X_r = \{0\}$ this fact is obvious).
14. We have

$$p_r|_X = \mathcal{I}_r^{-1}\circ(\mathcal{P}_r - \mathcal{P}_{r-1})|_X,$$

with $\mathcal{P}_0 = 0$ and with $\mathcal{I}_r$ treated as a map into $\widetilde{X_r}$. Thus we get the continuity of $p_r|_X$ from the continuity of $\mathcal{I}_r^{-1}$ which follows from part 13., and from the continuity of all the $\mathcal{P}_j|_X$, which follows from (**proj** $--$).
15. Let $r \geq s \geq 1$ and $f \in \bigoplus_{j\in\mathbb{N}} X_j$. By **(proj)** we have $\|\mathcal{P}_s f\| = \|\mathcal{P}_s\mathcal{P}_r f\| \leq \|\mathcal{P}_r f\|$.
16. It follows from the fact that each C-sequence in a norm space is bounded.
17. If $f \in \bigoplus_{j\in\mathbb{N}} X_j$ and $\{\mathcal{P}_r f\}_{r\geq 1}$ is a C-sequence, then by **(ban)** $\mathcal{P}_r f \xrightarrow[r\to+\infty]{X} g$ for some $g \in X$. By **(coor)** $(\mathcal{P}_r f)_j \xrightarrow[r\to+\infty]{X_j} g_j$ for any $j \in \mathbb{N}$, but for any j we have $(\mathcal{P}_r f)_j = f_j$ for $r \geq j$, which gives $f_j = g_j$. Hence $f = g$.
18. Let $0 \leq m \leq n$. Observe that for any $h \in \bigoplus_{j\in\mathbb{N}} X_j$ and $j \in \mathbb{N}$

$$\|((\mathcal{P}_n - \mathcal{P}_m)h)_j\|_j = \begin{cases} \|h_j\|_j & \text{for } j \in (m;n] \\ 0 & \text{for } j \notin (m;n] \end{cases}, \tag{6.3}$$

where we denote $\mathcal{P}_0 = 0$. Hence if $f, g \in \bigoplus_{j\in\mathbb{N}} X_j$ and $f \preceq g$, then for any $j \in \mathbb{N}$

$$\|((\mathcal{P}_n - \mathcal{P}_m)f)_j\|_j \leq \|((\mathcal{P}_n - \mathcal{P}_m)g)_j\|_j,$$

i. e. $(\mathcal{P}_n - \mathcal{P}_m)f \preceq (\mathcal{P}_n - \mathcal{P}_m)g$, which by **(mono)** gives

$$\|(\mathcal{P}_n - \mathcal{P}_m)f\| \leq \|(\mathcal{P}_n - \mathcal{P}_m)g\|. \tag{6.4}$$

When $g \in X$, then by **(appr)** $\mathcal{P}_r g \xrightarrow[r\to+\infty]{X} g$, hence $\{\mathcal{P}_r g\}_{r\geq 1}$ is a C-sequence, and by (6.4) also $\{\mathcal{P}_r f\}_{r\geq 1}$ is a C-sequence. By (**bel**−) we have $f \in X$ and by **(appr)** and (6.4) (with $m = 0$) we get

$$\|f\| = \lim_{r\to+\infty} \|\mathcal{P}_r f\| \leq \lim_{r\to+\infty} \|\mathcal{P}_r g\| = \|g\|.$$

□

The proof of Theorem 2.16

Proof Observe first that ${}^F\bigoplus_{j\in\mathbb{N}} X_j$ is a normed base subspace of $\bigoplus_{j\in\mathbb{N}} X_j$ thanks to $\ell_{\text{fin}}(\mathbb{N}) \subset F$. By Proposition 2.12 and by Proposition 2.15 we must prove that **(mono)** and **(appr)** hold. Let $f, g \in X_{\text{fin}}$ with $f \preceq g$. Then by (2.19) we have

$$\|f\|_{F,\mathcal{X}} = \|\left\{\|f_j\|_j\right\}_{j\in\mathbb{N}}\|_F \leq \|\left\{\|g_j\|_j\right\}_{j\in\mathbb{N}}\|_F = \|g\|_{F,\mathcal{X}}$$

which proves (**mono**). To get **(appr)** consider any $f \in {}^F\bigoplus_{j\in\mathbb{N}} X_j$ and observe first that

$$\|f - \mathcal{P}_n f\|_{F,\mathcal{X}} = \|(I - \mathcal{P}_n)f\|_{F,\mathcal{X}} = \|R_n\left\{\|f_j\|_j\right\}_{j\in\mathbb{N}}\|_F, \quad n \in \mathbb{N} \tag{6.5}$$

where $R_n : F \longrightarrow F$ is the linear operator given by

$$(R_n h)_j = \begin{cases} 0 & \text{for } j \leq n \\ h_j & \text{for } j > n \end{cases}, \quad h \in F,\ j \in \mathbb{N}.$$

Note, that the fact that the above formula properly defines the element $R_n h$ from F follows from (2.19), which also gives $\|R_n\|_{\text{Op}} \leq 1$ for any n. In particular $\{R_n\}_{n\geq 1}$ is an equibounded family of operators from $\mathcal{B}(F)$. Moreover if $h \in \ell_{\text{fin}}(\mathbb{N})$ then $R_n h$ is the zero vector from F for n sufficiently large, so $R_n h \xrightarrow[n\to+\infty]{F} 0$. Thus by the equiboundance and by the density of $\ell_{\text{fin}}(\mathbb{N})$ in F we get $R_n h \xrightarrow[n\to+\infty]{F} 0$ for any $h \in F$, including $h = \left\{\|f_j\|_j\right\}_{j\in\mathbb{N}}$, which gives **(appr)** by (6.5). □

Generalized inversion of diagonal operators and the proof of Corollary 3.7

To prove Corollary 3.7 it will be convenient to prove first the following lemma.

Lemma 6.3 *Suppose that* $(X, \|\cdot\|) \sqsubset \bigoplus_{j\in\mathbb{N}} X_j$. *If* $A_j \in \mathcal{L}(X_j)$ *for any* $j \in \mathbb{N}$ *and* $A = \operatorname{diag}_{j\in\mathbb{N}} A_j$, *then*

(i) $\operatorname{Ker} A = \{0\} \iff \forall_{j\in\mathbb{N}} \operatorname{Ker} A_j = \{0\}$;
(ii) *If* $\operatorname{Ker} A = \{0\}$, *then* $A^{-1\bullet} = \operatorname*{diag}_{j\in\mathbb{N}} \left((A_j)^{-1\bullet}\right)$.

Proof "$\Leftarrow$" for **(i)** is obvious from the definition of $\operatorname*{diag}_{j\in\mathbb{N}} A_j$. To get "$\Rightarrow$" assume $\operatorname{Ker} A = \{0\}$ and suppose that $u \in \operatorname{Ker} A_{j_0}$. Then we have $\left\{A_j(\mathcal{I}_{j_0}u)_j\right\}_{j\in\mathbb{N}} = \{0\}_{j\in\mathbb{N}} \in X$, hence $\mathcal{I}_{j_0}u \in D(A)$ and $A(\mathcal{I}_{j_0}u) = 0$. Thus $\mathcal{I}_{j_0}u = 0$, which gives also $u = 0$.

To prove **(ii)**, let's again assume $\operatorname{Ker} A = \{0\}$. By **(i)** both operators $A^{-1\bullet}$ and $\operatorname*{diag}_{j\in\mathbb{N}} \left((A_j)^{-1\bullet}\right)$ are well-defined, thus to prove their equality it suffices to check that $A^{-1\bullet} f = g \iff \operatorname*{diag}_{j\in\mathbb{N}} \left((A_j)^{-1\bullet}\right) f = g$ for any $f, g \in X$. But taking $f, g \in X$ and using twice both (3.3) and (2.1) we get

$$A^{-1\bullet} f = g \iff f = Ag \iff \forall_{j\in\mathbb{N}}\, f_j = A_j g_j \iff \forall_{j\in\mathbb{N}}\, (A_j)^{-1\bullet} f_j = g_j$$
$$\iff \operatorname*{diag}_{j\in\mathbb{N}} \left((A_j)^{-1\bullet}\right) f = g.$$

□

Proof of Corollary 3.7 Let $A := \operatorname*{diag}_{j\in\mathbb{N}} A_j$. By Proposition 3.2 $A \in \mathcal{C}(X)$. Thus $\sigma(A)$ is a closed subset of $\mathbb{C}$ and by Proposition 3.6 **(iii)** we have $\sigma(A) \supset \overline{\bigcup_{j\in\mathbb{N}} \sigma(A_j)}$. Let $\lambda \in \sigma(A)$; we shall prove that $\lambda \in \overline{\bigcup_{j\in\mathbb{N}} \sigma(A_j)}$. It suffices to study such λ that $\lambda \notin \bigcup_{j\in\mathbb{N}} \sigma(A_j)$, and in such a case we have $\lambda \in \rho_\infty\left(\left\{A_j\right\}_{j\in\mathbb{N}}\right)$ by Proposition 3.6 **(iii)**. By (3.7) we also have

$$\sup_{j\in\mathbb{N}} \|(A_j - \lambda)^{-1}\|_{\mathrm{sp}} = +\infty.$$

Let $\epsilon > 0$. By the definition of the spectral norm there exists $j \in \mathbb{N}$ and $\nu \in \sigma((A_j - \lambda)^{-1})$ satisfying $|\nu| > \frac{1}{\epsilon}$. But by "spectral mapping theorem" for resolvents of closed operators in Banach space (see e.g. [5, Chapter V, Theorem 1.13])

$$\sigma((A_j - \lambda)^{-1}) = \begin{cases} \{(z-\lambda)^{-1} : z \in \sigma(A_j)\} & \text{for } D(A_j) = X_j \\ \{(z-\lambda)^{-1} : z \in \sigma(A_j)\} \cup \{0\} & \text{for } D(A_j) \neq X_j \end{cases},$$

thus there exists $j \in \mathbb{N}$ and $z \in \sigma(A_j)$ such that $|(z-\lambda)^{-1}| > \frac{1}{\epsilon}$. The above means that there exists $z \in \bigcup_{j\in\mathbb{N}} \sigma(A_j)$ with $|z - \lambda| < \epsilon$. Hence $\lambda \in \overline{\bigcup_{j\in\mathbb{N}} \sigma(A_j)}$. □

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