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RESEARCH ARTICLE

Semigroup approach to diffusion and transport problems on networks

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Abstract Models describing transport and diffusion processes occurring along the edges of a graph and interlinked by its vertices have been recently receiving a considerable attention. In this paper we generalize such models and consider a network of transport or diffusion operators defined on one dimensional domains and connected through boundary conditions linking the end-points of these domains in an arbitrary way (not necessarily in the way the edges of a graph are connected). We prove the existence of C_0-semigroups solving such problems and provide conditions fully characterizing when they are positive.

Keywords Networks · Diffusion · Transport · Strongly continuous semigroups · Positive semigroups

1 Introduction

Recently there has been an interest in dynamical problems on graphs, where some evolution operators, such as transport or diffusion, act on the edges of a graph and

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interact through its nodes. One can mention here quantum graphs, see e.g. [1–4], diffusion on graphs in probabilistic context, [2,5,6], transport problems, both linear and nonlinear, [7–12], migrations, [13], and several other applications discussed in e.g. [4,14]. In particular, the recent monograph [14] is a rich source of network models and methods. However, most of these works focus on particular problems. For instance, in the quantum graph theory the main interest is to determine whether the operators defined on the edges of a graph are self-adjoint and the work is confined to the Hilbert space setting. Most papers on the linear transport theory on graphs focus on long term dynamics of the flow. Papers such as [2,5,6], motivated by probabilistic applications, look at Feller or Markov processes on graphs.

The present paper, which provides the theoretical foundation for [15], is similar in spirit to [5,6] in the sense that we prove the existence of strongly continuous semigroups in the space of continuous functions, as well as in the space of integrable functions, that solve the diffusion problem on a network. However, we extend the existing results of [5,6] by considering processes that are more general than the diffusion along edges of a metric graph with Robin boundary conditions at its vertices. More precisely, we allow for communication between domains that are not necessarily physically connected. In fact, the models we analyse can be also interpreted as diffusion on a hypergraph, [16], but we shall not pursue this line of research here. For completeness, we also present similar results for transport problems, generalizing [7] in a similar way.

To explain the idea of our extension, in the next section we consider two examples, see [5,17].

1.1 Motivation

First, let us introduce basic notation which will help to formulate the problems and results. We will work in a finite dimensional space, say, $\mathbb{R}^m$. The boldface characters will typically denote vectors in $\mathbb{R}^m$, e.g. $\mathbf{u} = (u_1, \ldots, u_m)$. We denote $\mathcal{M} = \{1, \ldots, m\}$. Further, for any Banach space X, we will use the notation $\mathbf{X} = \underbrace{X \times \ldots \times X}_{m\,times}$, e.g. for $X = L_1(I), I = [0, 1]$ we denote $\mathbf{L}_1(I) = \underbrace{L_1(I) \times \ldots \times L_1(I)}_{m\,times}$.

Let $(\mathbf{A}, D(\mathbf{A}))$ be an operator in $\mathbf{X}$. If $\mathbf{A}$ is a generator, we will denote by $\{e^{t\mathbf{A}}\}_{t\geq 0}$ the semigroup generated by $\mathbf{A}$.

1.1.1 Diffusion

We consider a finite metric graph without loops and isolated edges, $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with, say, n vertices and m edges. On each edge there is a substance with density u_j, $j \in \mathcal{M}$, which diffuses along this edge according to

$$\partial_t u_j = \sigma_j \partial_{xx} u_j, \quad x \in]0, 1[, \quad j \in \mathcal{M}, \tag{1}$$

where $\sigma_j > 0$ are constant diffusion coefficients, and can also enter the adjacent edges. To simplify considerations, each edge is identified with the unit interval I. In the model of [5], the particles can permeate between the edges across the vertices that join them according to a version of the Fick law. To write down its analytical form, first we note that, since diffusion does not have a preferred direction, we can assign the tail, or the left endpoint, (that is, 0) and the head, or the right endpoint (that is, 1) to the endpoints of the edge in an arbitrary way. Let l_i and r_i be the rates at which the substance leaves e_i through, respectively, the left and the right endpoints and l_{ik} and r_{ik} be the rates of at which it subsequently enters the edge e_k. Then the Fick law at, respectively, the head and the tail of e_i, gives

$$\begin{aligned} -\partial_x u_i(1) &= r_i u_i(1) - \sum_{j\neq i} r_{ij} u_j(v), \\ \partial_x u_i(0) &= l_i u_i(0) - \sum_{j\neq i} l_{ij} u_j(v), \end{aligned} \tag{2}$$

where we have written $u_j(v)$ as v may be either the tail or the head of the incident edge e_j. In particular, if there are no edges incident to the tail of e_i, or there are no edges incident to the head of e_i, then the Fick's laws take the form

$$\partial_x u_i(0) = l_i u_i(0), \qquad -\partial_x u_i(1) = r_i u_i(1), \tag{3}$$

respectively, where either coefficient on the right hand side can be 0.

It is clear that if $r_{ij} \neq 0$ or $l_{ij} \neq 0$, the edges e_i and e_j are incident and thus we can define an $m \times m$ matrix $\mathbb{A} = \{a_{ij}\}_{1\le i\le m, 1\le j\le m}$ by setting $a_{ij} = 1$ if either $r_{ij} \neq 0$ or $l_{ij} \neq 0$ and zero otherwise. Then $\mathbb{A}$ is the adjacency matrix of the line graph $L(\mathcal{G})$ of $\mathcal{G}$, see e.g. [18]. However, it turns out that such a matrix is not easy to use. On the other hand, introducing, for any $i, j \in \mathcal{M}$,

$$\begin{aligned} k_{ij}^{00} &= -l_{ij} \quad \text{if } v = 0, \qquad k_{ij}^{01} = -l_{ij} \quad \text{if } v = 1, \qquad k_{ii}^{00} = l_i, \\ k_{ij}^{10} &= r_{ij} \quad \text{if } v = 0, \qquad k_{ij}^{11} = r_{ij} \quad \text{if } v = 1, \qquad k_{ii}^{11} = -r_i, \end{aligned} \tag{4}$$

where v is, respectively, the tail or the head of the edge under consideration, the problem can be written as

$$\begin{aligned} \partial_t \mathbf{u}(x,t) &= \mathbb{D}\partial_{xx}\mathbf{u}(x,t), \qquad (x,t) \in]0,1[\times\mathbb{R}_+, \\ \partial_x \mathbf{u}(0,t) &= \mathbb{K}^{00}\mathbf{u}(0,t) + \mathbb{K}^{01}\mathbf{u}(1,t), \qquad t > 0, \\ \partial_x \mathbf{u}(1,t) &= \mathbb{K}^{10}\mathbf{u}(0,t) + \mathbb{K}^{11}\mathbf{u}(1,t), \qquad t > 0, \\ \mathbf{u}(x,0) &= \mathring{\mathbf{u}}(x), \qquad x \in]0,1[, \end{aligned} \tag{5}$$

where $\mathbf{u} = (u_1, \ldots, u_m)$, $\mathbb{D} = \operatorname{diag}\{\sigma_i\}_{1\le i\le m}$ and $\mathring{\mathbf{u}}$ is the initial distribution. This form of the problem allows for its analysis.

It is also clear that there is no mathematical reason why the matrices K^ω, $\omega \in \Omega = \{00, 01, 10, 11\}$, in (5) should be restricted to the matrices given by (4) which, indeed, form a strictly smaller class, see [17]. In this paper we study the well-posedness of (5) for arbitrary matrices K^{ω} in spaces $\mathbf{C}(I)$ and $\mathbf{L}_1(I)$, extending and simplifying the results of [5,6]. We also find necessary and sufficient conditions for the semigroup solving (5) to be positive.

1.1.2 Transport problems

We consider a digraph $G = (V(G), E(G)) = (\{v_1, \ldots, v_n\}, \{e_1, \ldots, e_m\})$ with n vertices and m edges. We suppose that none of the vertices is isolated. As before, each edge is normalized so as to be identified with I with the head at 1 and the tail at 0. Following [7,9–12], we consider a substance of density $u_j(x, t)$ on the edge e_j, moving with speed c_j along this edge. The conservation of mass at each vertex is expressed by the Kirchhoff law,

$$\sum_{j=1}^{m} \phi_{ij}^{-} c_j u_j(0, t) = \sum_{j=1}^{m} \phi_{ij}^{+} c_j u_j(1, t), \quad t > 0, i \in 1, \ldots, n, \tag{6}$$

where $\Phi^{-} = \{\phi_{ij}^{-}\}_{1\le i\le n, 1\le j\le m}$ and $\Phi^{+} = \{\phi_{ij}^{+}\}_{1\le i\le n, 1\le j\le m}$ are, respectively, the outgoing and incoming incidence matrices; that is, matrices with the entry ϕ_{ij}^{-} (resp. ϕ_{ij}^{+}) equals 1 if there is edge e_j outgoing from (res. incoming to) the vertex v_i, and zero otherwise. Note that due to definitions of the matrices Φ^{-} and Φ^{+}, the summation on the right hand side is over all incoming edges of the vertex v_i and on the left hand side over all outgoing edges of v_i.

In [7] we considered a slightly more general model

$$\begin{aligned} &\partial_t u_j(x, t) + c_j \partial_x u_j(x, t) = 0, \quad x \in (0, 1), \quad t \ge 0,\\ &u_j(x, 0) = \mathring{u}_j(x),\\ &\phi_{ij}^{-} \xi_j c_j u_j(0, t) = w_{ij} \sum_{k=1}^{m} \phi_{ik}^{+} (\gamma_k c_k u_k(1, t)), \end{aligned} \tag{7}$$

where $\gamma_j > 0$ and $\xi_j > 0$ are the absorption/amplification coefficients at, respectively, the head and the tail of e_j. Here the matrix $\{w_{ij}\}_{1\le i\le n, 1\le j\le m}$ describes the distribution of the incoming flow at the vertex v_i into the edges outgoing from it; it is a column stochastic matrix, [11]. We denote $\mathbb{C} = \mathrm{diag}\{c_j\}_{1\le j\le m}$, $\Xi = \mathrm{diag}\{\xi_j\}_{1\le j\le m}$ and $\Gamma = \mathrm{diag}\{\gamma_j\}_{1\le j\le m}$.

It follows, [7], that if G has a sink, than there is no C_0-semigroup solving (7). Hence we discard this case and then it can be proved, e.g. [9, Proposition 3.1], that (7) can be written as an abstract Cauchy problem

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}, \qquad \mathbf{u}(0) = \mathring{\mathbf{u}}, \tag{8}$$

in $\mathbf{X} = \mathbf{L}_1(I)$, where $\mathbf{A}$ is the realization of $\mathsf{A} = \mathrm{diag}\{-c_j \partial_x\}_{1\le j\le m}$ on the domain

$$D(\mathbf{A}) = \left\{\mathbf{u} \in \mathbf{W}_1^1(I);\ \mathbf{u}(0) = \Xi^{-1}\mathbb{C}^{-1}\mathbb{B}\Gamma\mathbb{C}\mathbf{u}(1)\right\}, \tag{9}$$

where $\mathbb{B}$ is the (transposed) adjacency matrix of the line graph of G.

As with the diffusion problems, there is no mathematical reason to restrict our analysis to the matrices of the form $\Xi^{-1}\mathbb{C}^{-1}\mathbb{B}\Gamma\mathbb{C}$ in the boundary conditions which, in fact, see [17], form a strict subset of the set of all matrices. Thus, we consider the following generalization of (8), (9),

$$\mathbf{u}_t = \mathbf{A}\mathbf{u}, \qquad \mathbf{u}(0) = \mathbb{K}\mathbf{u}(1), \qquad \mathbf{u}(0) = \mathring{\mathbf{u}}, \tag{10}$$

where $\mathbb{K}$ is an arbitrary matrix.

Example 1.1 The main difference between (10) with arbitrary $\mathbb{K}$ and the model with $\mathbb{K}$ given in (9) is that in the former, the exchange of the substance can occur instantaneously between any edges, while in the latter the edges must be physically connected by vertices for the exchange to take place. So, for instance, if there is a connection $e_1 \to e_2$ and $e_2 \to e_3$, then there is no connection $e_1 \to e_3$. So, while in general (10) cannot model a flow in a physical network, it can describe e.g. a mutation process. Indeed, let a population of cells be divided into m subpopulations, with $\mathbf{v}(x) = (v_1(x), \ldots, v_m(x))$, where $v_j(x)$, $j \in \mathcal{M}$, $x \in [0, 1]$, is the density of cells of age x whose genotype belongs to a class j (for instance, having j copies of a gene of a particular type). We assume that cells of class j mature and divide upon reaching maturity at $x = 1$, with offspring, due to mutations, appearing in class i with probability k_{ij}, $i \in \mathcal{M}$. In such a case $(k_{ij})_{1\le i,j\le m}$ is a column stochastic matrix. We note that a particular case of this model is the discrete Rotenberg-Rubinov-Lebowitz model, [19], where the cells are divided into classes according to their maturation velocity.

A similar interpretation can be given to (5) where the variable x, instead of the age, denotes the size of the organism, e.g. [20].

Moreover, problems of the form (10) arise e.g. in queuing theory, [21].

2 Well-posedness of the diffusion problem

We shall consider solvability of (5) in $\mathbf{X} = \mathbf{C}(I)$ and $\mathbf{X} = \mathbf{L}_1(I)$. For technical reasons, we also shall need the solvability of (5) in $\mathbf{W}_1^1(I)$. The norms in these spaces will be denoted, respectively, by $\|\cdot\|_\infty$, $\|\cdot\|_0$, $\|\cdot\|_1$. However, if it does not lead to any misunderstanding, we will use $\mathbf{X}$ to denote any of these spaces and $\|\cdot\|$ or $\|\cdot\|_{\mathbf{X}}$ to denote the norm in $\mathbf{X}$. Similarly, by $|||\cdot|||_{\mathbf{X}}$ we denote the operator norm in the space of bounded linear operators from $\mathbf{X}$ to $\mathbf{X}$.

Our results are based on [22] and thus, introducing relevant spaces and operators, we try to keep notation consistent with *op.cit.* First, consider

$$\mathbf{X} \ni \mathbf{u} \to L\mathbf{u} = (\gamma_0\partial_x\mathbf{u}, \gamma_1\partial_x\mathbf{u}) \in \mathbf{Y} = \mathbb{R}^m \times \mathbb{R}^m, \tag{11}$$

where γ_i, $i = 0, 1$, is the trace operator at $x = i$ (taking the value at $x = i$ if $\mathbb{X} = \mathbf{C}(I)$). The domains of L are $D(L) = \mathbf{C}^1(I)$ if $\mathbf{X} = \mathbf{C}(I)$ and $D(L) = \mathbf{W}_1^2(I)$ in two other cases. Then we define the operator

$$\mathbf{X} \ni \mathbf{u} \to \Phi\mathbf{u} = \mathbb{K}(\gamma_0\mathbf{u}, \gamma_1\mathbf{u}) = \begin{pmatrix} \mathbb{K}^{00} & \mathbb{K}^{01} \\ \mathbb{K}^{10} & \mathbb{K}^{11} \end{pmatrix} \begin{pmatrix} \gamma_0\mathbf{u} \\ \gamma_1\mathbf{u} \end{pmatrix} \in \mathbb{R}^m \times \mathbb{R}^m.$$

Further, let us denote

$$\Phi^*\mathbf{u} = \mathbb{K}^*(\gamma_0\mathbf{u}, \gamma_1\mathbf{u}) = \begin{pmatrix} \mathbb{K}^{00\,T} & -\mathbb{K}^{10\,T} \\ -\mathbb{K}^{01\,T} & \mathbb{K}^{11\,T} \end{pmatrix} \begin{pmatrix} \gamma_0\mathbf{u} \\ \gamma_1\mathbf{u} \end{pmatrix}, \tag{12}$$

where $\mathbb{K}^T$ denotes the transpose of $\mathbb{K}$. Clearly $(\Phi^*)^* = \Phi$.

Let A denote the differential expression $\mathsf{A}\mathbf{u} := \mathbb{D}\partial_{xx}\mathbf{u}$. Then we define the operators $\mathbf{A}^\alpha_\Phi$, $\alpha = \infty, 0, 1$ by the restriction of A to the domains

$$\begin{aligned} D(\mathbf{A}^\infty_\Phi) &= \left\{\mathbf{u} \in \mathbf{C}^2(I);\ L\mathbf{u} = \Phi\mathbf{u}\right\}, \\ D(\mathbf{A}^0_\Phi) &= \left\{\mathbf{u} \in \mathbf{W}^2_1(I);\ L\mathbf{u} = \Phi\mathbf{u}\right\}, \\ D(\mathbf{A}^1_\Phi) &= \left\{\mathbf{u} \in \mathbf{W}^3_1(I);\ L\mathbf{u} = \Phi\mathbf{u}\right\}, \end{aligned} \tag{13}$$

respectively. As before, we drop the indices from the notation if it will not lead to any misunderstanding.

2.1 Basic estimates in the scalar case

Consider the general resolvent equation for (5)

$$\lambda\mathbf{u} - \mathbb{D}\partial_{xx}\mathbf{u} = \mathbf{f}, \quad x \in]0, 1[. \tag{14}$$

Since without the boundary conditions the system is uncoupled, its solution $\mathbf{u}$ is given by

$$\mathbf{u}(x) = \mathbb{E}(-\boldsymbol{\mu}x)\mathbf{C}_1 + \mathbb{E}(\boldsymbol{\mu}x)\mathbf{C}_2 + \mathbf{U}_{\boldsymbol{\mu}}(x) \tag{15}$$

where $\mathbb{E}(\pm\boldsymbol{\mu}x) = \mathrm{diag}\{e^{\pm\mu_i x}\}_{1\le i\le m}$, $0 \neq \lambda/\sigma_i = \mu_i^2 = |\lambda/\sigma_i|e^{i\theta}$ with $\Re\mu_i > 0$, $\mathbf{U}_{\boldsymbol{\mu}}(x) = (U_{\mu_1}(x), \dots, U_{\mu_m}(x))$ with

$$U_{\mu_i}(x) = \frac{1}{2\mu_i\sigma_i}\int\limits_0^1 e^{-\mu_i|x-s|} f_i(s)ds, \tag{16}$$

and the vector constants $\mathbf{C}_1$ and $\mathbf{C}_2$ are determined by the boundary conditions. Further, denote

$$\Sigma_{\alpha,\theta_0} = \left\{\lambda = |\lambda|e^{i\theta} \in \mathbb{C};\ |\lambda| \ge \alpha, |\theta| \le \theta_0 < \pi\right\}$$

and $\Sigma_{0,\theta_0} = \Sigma_{\theta_0}$.

The starting point are well-known estimates for the scalar Dirichlet and Neumann problems, e.g. [23]. We briefly recall them here in a slightly more precise form, similarly to [24]. In the scalar case we can use $\sigma = 1$, as will become clear when the estimate is derived. It follows that for the Dirichlet problem we have

$$C_1^D = \frac{U_\mu(1) - U_\mu(0)e^{\mu}}{e^{\mu} - e^{-\mu}}, \qquad C_2^D = \frac{U_\mu(0)e^{-\mu} - U_\mu(1)}{e^{\mu} - e^{-\mu}}, \tag{17}$$

and for the Neumann problem

$$C_1^N = \frac{U_\mu(0)e^{\mu} + U_\mu(1)}{e^{\mu} - e^{-\mu}}, \qquad C_2^N = \frac{U_\mu(1) + U_\mu(0)e^{-\mu}}{e^{\mu} - e^{-\mu}}.$$

Further, [23],

$$\|U_\mu\|_\infty \le \frac{\|f\|_\infty}{|\mu|\Re\mu} \le \frac{\|f\|_\infty}{|\lambda|\cos\theta/2}, \qquad \|U_\mu\|_0 \le \frac{\|f\|_0}{|\mu|\Re\mu} \le \frac{\|f\|_0}{|\lambda|\cos\theta/2}, \tag{18}$$

and, for $i = 0, 1$,

$$|U_\mu(i)| \le \frac{\|f\|_\infty(1 - e^{-\Re\mu})}{2|\mu|\Re\mu}, \quad |U_\mu(i)| \le \frac{\|f\|_0}{2|\mu|}. \tag{19}$$

The following result plays an essential role in deriving precise estimates,

$$\left|\frac{1}{e^{\mu} - e^{-\mu}}\right| = e^{-\Re\mu}\left|\sum_{n=0}^{\infty} e^{-2n\mu}\right| \le e^{-\Re\mu}\sum_{n=0}^{\infty} e^{-2n\Re\mu} = \frac{e^{-\Re\mu}}{1 - e^{-2\Re\mu}}. \tag{20}$$

Then, for either Dirichlet or Neumann problem in $C(I)$, for $\lambda \in \Sigma_{\theta_0}$ for any $\theta_0 < \pi$, we have for $\omega = D, N$

$$\begin{aligned}
\|u^\omega\|_\infty &\le |C_1^\omega| + |C_2^\omega|e^{\Re\mu} + \frac{\|f\|_\infty}{|\lambda|\cos\theta_0/2} \\
&\le \frac{\|f\|_\infty}{|\lambda|\cos\theta_0/2}\left(\frac{1}{1 - e^{-2\Re\mu}}\left(\frac{1 - e^{-2\Re\mu}}{2} + \frac{1 - e^{-2\Re\mu}}{2}\right) + 1\right) \\
&\le \frac{2\|f\|_0}{|\lambda|\cos\theta_0/2}.
\end{aligned} \tag{21}$$

Similarly, in $L_1(I)$, we have

$$\begin{aligned}
\|u^\omega\|_0 &\le |C_1^\omega|\frac{1 - e^{-\Re\mu}}{\Re\mu} + |C_2^\omega|\frac{e^{\Re\mu} - 1}{\Re\mu} + \frac{\|f\|_0}{|\lambda|\cos\theta_0/2} \\
&\le \frac{\|f\|_0}{|\lambda|\cos\theta_0/2}\left(\frac{1}{1 - e^{-2\Re\mu}}\left(\frac{1 - e^{-2\Re\mu}}{2} + \frac{1 - e^{-2\Re\mu}}{2}\right) + 1\right) \\
&\le \frac{2\|f\|_0}{|\lambda|\cos\theta_0/2}.
\end{aligned} \tag{22}$$

Consider now A_0^1, see (13); that is, the operator corresponding to the Neumann boundary conditions in one dimension. We have

Proposition 2.1 A_0^1 *generates an analytic semigroup in* $W_1^1(I)$ *with the resolvent* $R(\lambda, A_0^1)$ *satisfying the estimate*

$$\|R(\lambda, A_0^1)f\|_1 \leq \frac{2\|f\|_1}{|\lambda|\cos\theta_0/2}, \qquad \lambda \in \Sigma_{\theta_0}. \tag{23}$$

Proof Consider the resolvent equation

$$\lambda u - \partial_{xx}u = f, \quad \partial_x u(0) = \partial_x u(1) = 0.$$

If $f \in W_1^1(I)$, then $u \in W_1^3(I)$ and we can differentiate the differential equation getting for $v := \partial_x u$,

$$\lambda v - \partial_{xx}v = \partial_x f, \quad v(0) = v(1) = 0; \tag{24}$$

that is, v satisfies the resolvent equation for the Dirichlet problem. Hence

$$\|u\|_1 = \|u\|_0 + \|\partial_x u\|_0 \leq \frac{2\|f\|_0}{|\lambda|\cos\theta_0/2} + \frac{2\|\partial_x f\|_0}{|\lambda|\cos\theta_0/2} = \frac{2\|f\|_1}{|\lambda|\cos\theta_0/2}.$$

□

Since $\lambda u - \sigma\partial_{xx}u = f$ is equivalent to $\sigma^{-1}\lambda u - \partial_{xx}u = \sigma^{-1}f$, we see that the estimates above are independent of σ.

2.2 Solvability of (5)

The ideas in this section are based on [5,6] but the analysis is simplified by using the analyticity of the semigroup and the application of [22, Theorem 2.4].

Since in the vector case and for the Neumann boundary conditions the system of the resolvent equations decouples, we obtain

$$\|R(\lambda, \mathbf{A}_0)f\|_{\mathbf{X}} \leq \frac{2\|f\|_{\mathbf{X}}}{|\lambda|\cos\theta_0/2}, \qquad \lambda \in \Sigma_{\theta_0} \tag{25}$$

for any $\theta_0 < \pi$ and $\mathbf{X} = \mathbf{C}(I), \mathbf{L}_1(I), \mathbf{W}_1^1(I)$. Hence, in particular, $\mathbf{A}_0$ generates an analytic semigroup in $\mathbf{X}$.

We begin with a straightforward consequence of Ref. [22].

Theorem 2.2 *Let* $\mathbf{X}$ *be either* $\mathbf{C}(I)$, *or* $\mathbf{W}_1^1(I)$. *Then the operator* $\mathbf{A}_\Phi$ *generates an analytic semigroup in* $\mathbf{X}$ *with the resolvent* $R(\lambda, \mathbf{A}_\Phi)$ *satisfying the estimate*

$$|||R(\lambda, \mathbf{A}_\Phi)|||_{\mathbf{X}} \leq 2|||R(\lambda, \mathbf{A}_0)|||_{\mathbf{X}}, \qquad \lambda \in \Sigma_{\alpha,\theta_0} \tag{26}$$

for some $\alpha \geq 0$.

Proof The result follows directly from [22, Theorem 2.4], since $\mathbf{A}_0$ generates an analytic semigroup in $\mathbf{X}$, L is an unbounded operator on $\mathbf{X}$ which is, however, bounded as an operator from $D(\mathbf{A})$ to $\mathbf{Y}$, and it is a surjection (the right inverse in each case is given by $[L_r^{-1}(\mathbf{y}_0, \mathbf{y}_1)](x) = \frac{1}{2}(\mathbf{y}_1 - \mathbf{y}_0)x^2 - \mathbf{y}_0 x$. Furthermore, Φ is a bounded operator on $\mathbf{X}$ (if $\mathbf{X} = \mathbf{W}_1^1(I)$ this follows since $\mathbf{W}_1^1(I)$-functions are absolutely continuous on I). Hence, the assumptions [22, (1.13)] are satisfied and the theorem follows from [22, Theorem 2.4]. □

In $\mathbf{X} = \mathbf{L}_1(I)$ the situation is more complicated as Φ is not bounded on $\mathbf{X}$. First we show that $R(\lambda, \mathbf{A}_\Phi^1)$ extends to a resolvent on $\mathbf{L}_1(I)$.

Lemma 2.3 *We have*

$$\overline{\mathbf{A}_\Phi^1}^{\mathbf{L}_1(I)} = \mathbf{A}_\Phi^0$$

and the resolvent set of $\mathbf{A}_\Phi^0$ *satisfies* $\rho(\mathbf{A}_\Phi^0) \cap \mathbb{R}_+ \neq \emptyset$.

Proof Let $D(\mathbf{A}_\Phi^1) \ni \mathbf{u}_n \to \mathbf{u} \in \mathbf{L}_1(I)$ and $\mathbf{A}_\Phi^0 \mathbf{u}_n = \mathsf{A}\mathbf{u}_n \to \mathbf{v} \in \mathbf{L}_1(I)$. This shows that $\mathbf{u}_n \to \mathbf{u}$ in $\mathbf{W}_1^2(I)$ and thus $\mathbf{v} = \mathsf{A}\mathbf{u}$. Since taking the trace of a $\mathbf{W}_1^2(I)$ function and of its derivative is continuous in $\mathbf{W}_1^2(I)$, the boundary values of $\mathbf{u}_n$ and $\partial_x \mathbf{u}_n$ are preserved in the limit and thus $\mathbf{u} \in D(\mathbf{A}_\Phi^0)$. Hence

$$\overline{\mathbf{A}_\Phi^1}^{\mathbf{L}_1(I)} \subset \mathbf{A}_\Phi^0.$$

On the other hand, let $\mathbf{u} \in D(\mathbf{A}_\Phi^0)$. Then $\mathbf{v} = \mathbf{u} - \mathbf{f} \in \overset{0}{\mathbf{W}}{}_1^2(I)$, where

$$\begin{aligned}\mathbf{f}(x) &= (\mathbf{u}(0) + \mathbf{u}'(0) - 2\mathbf{u}(1) + \mathbf{u}'(1))x^3 \\ &\quad -(2\mathbf{u}(0) + 2\mathbf{u}'(0) - 3\mathbf{u}(1) + \mathbf{u}'(1))x^2 + \mathbf{u}'(0)x + \mathbf{u}(0) \in D(\mathbf{A}_\Phi^0).\end{aligned}$$

Thus there is a sequence $(\mathbf{h}_n)_{n\in\mathbb{N}} \subset \mathbf{C}_0^\infty(I)$ converging to $\mathbf{v}$ in $\mathbf{W}_1^2(I)$ and hence $\mathbf{u}$ is the limit in $\mathbf{W}_1^2(I)$ of functions $\mathbf{u}_n = \mathbf{h}_n + \mathbf{f} \in D(\mathbf{A}_\Phi^1)$. Since the convergence in $\mathbf{W}_1^2(I)$ implies the convergence of both $\mathbf{u}_n$ and $\mathbf{A}\mathbf{u}_n$ in $\mathbf{L}_1(I)$ we see that also

$$\overline{\mathbf{A}_\Phi^1}^{\mathbf{L}_1(I)} \supset \mathbf{A}_\Phi^0.$$

Finally, we see that, as in the scalar case (36), the solvability of the resolvent problem for (5) is equivalent to solvability of the linear system

$$\begin{aligned} -\mathbb{M}\mathbf{C}_1 + \mathbb{M}\mathbf{C}_2 &= \mathbb{K}^{00}(\mathbf{C}_1 + \mathbf{C}_2) + \mathbb{K}^{01}(e^{-\mu}\mathbf{C}_1 + e^{\mu}\mathbf{C}_2) \\ &\quad -\mathbb{M}\mathbf{U}_{\boldsymbol{\mu}}(0) + \mathbb{K}^{00}\mathbf{U}_{\boldsymbol{\mu}}(0) + \mathbb{K}^{01}\mathbf{U}_{\boldsymbol{\mu}}(1), \\ -\mathbb{M}\mathbb{E}(-\boldsymbol{\mu})\mathbf{C}_1 + \mathbb{M}\mathbb{E}(\boldsymbol{\mu})\mathbf{C}_2 &= \mathbb{K}^{10}(\mathbf{C}_1 + \mathbf{C}_2) + \mathbb{K}^{11}(\mathbb{E}(-\boldsymbol{\mu})\mathbf{C}_1 + \mathbb{E}(\boldsymbol{\mu})\mathbf{C}_2) \\ &\quad +\mathbb{M}\mathbf{U}_{\boldsymbol{\mu}}(1) + \mathbb{K}^{01}\mathbf{U}_{\boldsymbol{\mu}}(0) + \mathbb{K}^{11}\mathbf{U}_{\boldsymbol{\mu}}(1), \end{aligned} \tag{27}$$

for the vectors $\mathbf{C}_1, \mathbf{C}_2$, where $\mathbb{M} = \mathrm{diag}\{\boldsymbol{\mu}\}$. Once the constants are found, the solution is given by the vector version of (15) and thus belongs to $\mathbf{W}_1^2(I) \subset \mathbf{L}_1(I)$. The system above is solvable provided its determinant is different from zero. However, the determinant is clearly an entire function in μ and thus can have only isolated zeros in $\mathbb{C}$. Hence, there must be positive values of $\lambda \in \rho(\mathbf{A}^0_\Phi)$. □

Theorem 2.4 *The operator* $\mathbf{A}^0_\Phi$ *generates an analytic semigroup in* $\mathbf{L}_1(I)$.

Proof We use the formula from Lemma 1.4 of Ref. [22] stating that

$$R(\lambda, \mathbf{A}^1_\Phi) = (I - L_\lambda \Phi)^{-1} R(\lambda, \mathbf{A}^1_N) \tag{28}$$

in $\mathbf{X} = \mathbf{W}^1_1(I)$, as $(I - L_\lambda \Phi)^{-1}$ is an isomorphism of $\mathbf{X}$ for large $|\lambda|$; here $L_\lambda = (L|_{Ker(\lambda - \mathbf{A})})^{-1}$, $\lambda \in \rho(\mathbf{A}^1_N)$, see the proof of [22, Theorem 2.4].

However, $R(\lambda, \mathbf{A}^1_N)$ extends by density to $R(\lambda, \mathbf{A}^0_N)$ (the resolvent on $\mathbf{L}_1(I)$) which is a bounded linear operator from $\mathbf{L}_1(I)$ to $D(\mathbf{A}^1_N) \subset \mathbf{W}^2_1(I)$. Since the latter space is continuously embedded in $\mathbf{W}^1_1(I)$, $R(\lambda, \mathbf{A}^1_\Phi)$ extends to a bounded linear operator, say $R_\Phi(\lambda)$, on $\mathbf{L}_1(I)$. Since $R(\lambda, \mathbf{A}^1_\Phi)$ is a resolvent on $\mathbf{W}^1_1(I)$, we obtain, by density, that $R_\Phi(\lambda)$ is a pseudoresolvent on $\mathbf{L}_1(I)$. Furthermore, $\mathbf{C}^\infty_0(]0, 1[) \subset D(\mathbf{A}^1_\Phi)$, thus it is also a subset of the range of $R_\Phi(\lambda)$ and therefore the range is dense in $\mathbf{L}_1(I)$. Also, since the range of $R(\lambda, \mathbf{A}^0_N)$ is in $\mathbf{W}^2_1(I) \subset \mathbf{W}^1_1(I)$, there is no need to extend $(I - L_\lambda \Phi)^{-1}$. Hence $R_\Phi(\lambda)$ is a one-to-one operator as a composition of two injective operators. Thus, by Proposition III.4.6 in Ref. [23], $R_\Phi(\lambda)$ is the resolvent of a densely defined operator, say $\hat{\mathbf{A}}_\Phi$. Clearly, $\hat{\mathbf{A}}_\Phi$ is a closed extension of $\mathbf{A}^1_\Phi$ and thus $\hat{\mathbf{A}}_\Phi \supset \mathbf{A}^0_\Phi = \overline{\mathbf{A}^1_\Phi}^{\mathbf{L}_1(I)}$. From Lemma 2.3, there is $\lambda \in \rho(\hat{\mathbf{A}}_\Phi) \cap \rho(\mathbf{A}^0_\Phi)$, thus $R(\lambda, \hat{\mathbf{A}}_\Phi) \supset R(\lambda, \mathbf{A}^0_\Phi)$ and hence $\hat{\mathbf{A}}_\Phi = \mathbf{A}^0_\Phi$ and, consequently, $R(\lambda, \mathbf{A}^0_\Phi)$ is defined in some sector. To prove that is a sectorial operator, we use the idea of [6] but in a somewhat simpler way. We consider the operator $\mathbf{A}^\infty_{\Phi*}$. Let $R^\#_\lambda$ denote the adjoint to $R(\lambda, \mathbf{A}^\infty_{\Phi*})$ for $\lambda \in \rho(\mathbf{A}^\infty_{\Phi*})$, which acts in the space of signed (vector) Borel measures on I, [25]. $R(\lambda, \mathbf{A}^\infty_{\Phi*})$ is given by (15) with $\mathbf{C}_1$, $\mathbf{C}_2$ given by (27) with the matrices $\mathbb{K}^\omega$ replaced by the corresponding matrices in (12). Hence, apart from $\mathbf{U}_{\boldsymbol{\mu}}(x)$, $R(\lambda, \mathbf{A}^\infty_{\Phi*})$ is a composition of an algebraic operator coming from inverting the matrix in (27) with a vector of functionals acting on $\mathbf{f}$. Thus, if $\mathbf{f} \in \mathbf{L}_1(I)$ is the density of an absolutely continuous measure, a standard calculation shows that

$$R^\#_\lambda \mathbf{f} = R(\bar{\lambda}, \mathbf{A}^0_\Phi)\mathbf{f}.$$

From the definition of the norm of a signed Borel measure, if the latter is absolutely continuous, its norm is equal to the L_1 norm of its density. Since taking the adjoint preserves the norm of the operator, we obtain

$$|||R(\lambda, \mathbf{A}^0_\Phi)|||_{\mathbf{L}_1(I)} = |||R(\lambda, \mathbf{A}^\infty_{\Phi*})|||_{\mathbf{C}(I)}. \tag{29}$$

Hence, by Theorem 2.2, $\mathbf{A}^0_\Phi$ is sectorial and, being densely defined, it generates an analytic semigroup on $\mathbf{L}_1(I)$. □

2.3 Positivity of the semigroup

Let us recall that for an element $\mathbf{u}$ of a Banach lattice, we write $\mathbf{u} > 0$ if $0 \neq \mathbf{u} \geq 0$. We have the following result

Theorem 2.5 *The semigroup* $\{e^{t\mathbf{A}_\Phi^\infty}\}_{t\geq 0}$ *is resolvent positive if and only if*

$$\begin{aligned} -\mathbb{K}^{00}, \quad & \mathbb{K}^{11} \text{ are nonnegative off-diagonal} \\ -\mathbb{K}^{01}, \quad & \mathbb{K}^{10} \text{ are nonnegative.} \end{aligned} \tag{30}$$

Proof To prove the result we use Theorem B-II.1.6 in [26], which states that if an operator A on $C(K)$ (where K is compact) generates a semigroup, then the semigroup is positive (or, equivalently, A is resolvent positive) if and only if A satisfies the positive minimum principle: for every $0 \leq f \leq D(A)$ and $x \in K$, if $f(x) = 0$, then $(Af)(x) \geq 0$. However, to rephrase the problem at hand in the language of the positive maximum principle, we have to work in the space of real valued continuous functions. To achieve this, we identify $\mathbf{C}(I)$ with the space $C(\mathbf{I})$, where $\mathbf{I} = [0_1, 1_1] \cup \cdots \cup [0_m, 1_m]$; that is, instead of considering a vector function on I, we consider a scalar function on a disconnected compact space composed of m disjoint closed intervals. In particular, each edge e_j, $j \in \mathcal{M}$, is identified with the closed interval $[0_j, 1_j]$. Then $\mathbf{A}_\Phi^\infty$ will be changed to A_Φ^∞ which is the restriction of

$$(Au)(x) = \sum_{j\in\mathcal{M}} \chi_{[0_j,1_j]}(x)\sigma_j \partial_{xx} u(x)$$

to the space of twice differentiable functions on $Int\mathbf{I} =]0_1, 1_1[\cup \cdots \cup]0_m, 1_m[$, differentiable on $\mathbf{I}$ and satisfying

$$\begin{aligned} \partial_x u(0_j) &= \sum_{k=1}^m k_{jk}^{00} u(0_k) + \sum_{k=1}^m k_{jk}^{01} u(1_k), \\ \partial_x u(1_j) &= \sum_{k=1}^m k_{jk}^{10} u(0_k) + \sum_{k=1}^m k_{jk}^{11} u(1_k). \end{aligned}$$

Thus, if $0 \leq u \in D(A_\Phi^\infty)$ takes the value 0 at some $x \in \mathbf{I}$, then either $x \in Int\mathbf{I}$, and then classically $\partial_{xx} u(x) \geq 0$, or $x = 0_j$ or $x = 1_j$ for some $j \in \mathcal{M}$. If $x = 0_j$, then $\partial_x u(0_j) \geq 0$. If $\partial_x u(0_j) = 0$, then $\partial_{xx} u(0_j) \geq 0$ follows as in Example B-II.1.24 of Ref. [26] (or simply by noting that the even extension to $[-1, 0]$ gives a C^2 function on $[-1, 1]$ with minimum at $x = 0$). If $\partial_x u(0_j) < 0$ then, since $u(0_j) = 0$, such a function cannot be nonnegative on $[0_j, 1_j]$. Analogous considerations hold if $u(1_j) = 0$. Therefore the positive minimum principle is satisfied and (30) yields the positivity of $\{e^{tA_\Phi^\infty}\}_{t\geq 0}$ and thus of $\{e^{t\mathbf{A}_\Phi^\infty}\}_{t\geq 0}$.

To prove the converse we introduce the following notation. For any $\boldsymbol{\alpha} := (\boldsymbol{\alpha}^0, \boldsymbol{\alpha}^1) \geq 0$ with $\alpha_i^j = 0$ for some $i \in \mathcal{M}$, $j = 0, 1$, we have

$$\Xi := \begin{pmatrix} -\mathbb{K}^{00} & -\mathbb{K}^{01} \\ \mathbb{K}^{10} & \mathbb{K}^{11} \end{pmatrix}. \tag{31}$$

Accordingly, we denote

$$(\Xi\boldsymbol{\alpha})^r_s = (-1)^{r+1} \sum_{l=0,1} \left(\sum_{j\in\mathcal{M}} k^{rl}_{sj}\alpha^l_j \right), \quad r = 0, 1, s \in \mathcal{M}.$$

Let us assume that (30) is not satisfied. Then there is a non-diagonal element of Ξ which is strictly negative. Suppose $(-1)^{r+1}k^{rs}_{ij} < 0$ for some $i \neq j$ and $r, s = 0, 1$, and consider a vector $\boldsymbol{\alpha}$ with

$$\alpha^s_j = 1, \alpha^r_i = 0, \alpha^t_l = \delta > 0, \quad t = 0, 1, \quad l \neq j \text{ if } t = s \text{ and } l \neq i \text{ if } t = s. \tag{32}$$

Then for $r = s$ we obtain

$$(\Xi\boldsymbol{\alpha})^r_i = (-1)^{r+1} \left(k^{rr}_{ij} + \delta \left(\sum_{l\in\mathcal{M}, l\neq j,i} k^{rr}_{il} + \sum_{l\in\mathcal{M}} k^{rt}_{il} \right) \right) < 0, \tag{33}$$

where $t = 0$ if $r = 1$ and $t = 1$ if $r = 0$, while for $r \neq s$

$$(\Xi\boldsymbol{\alpha})^r_i = (-1)^{r+1} \left(k^{rt}_{ij} + \delta \left(\sum_{l\in\mathcal{M}, l\neq i} k^{rr}_{il} + \sum_{l\in\mathcal{M}, l\neq j} k^{rt}_{il} \right) \right) < 0 \tag{34}$$

for sufficiently small $\delta > 0$. We shall prove that there exists a function $0 \leq u \in C^\infty(\mathbf{I})$ satisfying $u(r_i) = \alpha^r_i$ and $(-1)^{r+1}\partial_x u(r_i) = (\Xi\boldsymbol{\alpha})^r_i$ which additionally satisfies $\partial_{xx}u(r_i) < 0$.

For a given constants $\alpha^r_i, \beta^r_i = (\Xi\boldsymbol{\alpha})^r_i, r = 0, 1$, we consider auxiliary functions $f^r_i(x) = \beta^r_i x(x-1) + \alpha^r_i$. We have $f^r_i(r) = \alpha^r_i$, $\partial_x f_i(r) = (-1)^{r+1}\beta^r_i$ and $\partial_{xx} f^r_i = 2\beta^r_i$. We observe that as long as $\alpha^r_i > 0$, there is a one-sided interval ($]0, \omega_i[$ if $r = 0$ and $]\omega_i, 1[$ if $r = 1$), where $f^r_i \geq 0$ irrespective of the sign of β^r_i. On the other hand, if $\alpha^r_i = 0$, for local nonnegativity we need $\beta^r_i \leq 0$. Now let ϕ be a nonnegative C^∞ function which is 1 on $[-a, a]$ and 0 outside $[-2a, 2a]$ where $0 < 2a < \min_{i\in\mathcal{M}} \omega_i$ and define

$$u(x) = \phi(x) f^0_j(x) + \phi(1-x) f^1_j(x), \quad x \in [0_j, 1_j], j \in \mathcal{M}.$$

is a $C^\infty(\mathbf{I})$ function that satisfies:

1. $u \geq 0$;
2. $u(0_j) = \alpha^0_j$ and $\partial_x u(0_j) = \partial_x f^0_j(0_j) = -\beta^0_j = (\Xi\boldsymbol{\alpha})^0_j$;
3. $u(1_j) = \alpha^1_j$ and $\partial_x u(1_j) = \partial_x f^1_j(1_j) = \beta^1_j = (\Xi\boldsymbol{\alpha})^1_j$,

so that $0 \leq u \in D(A_\Phi^\infty)$. Recalling that we assumed $(-1)^{r+1}k_{ij}^{rs} < 0$ for some $i \neq j$ and $r, s = 0, 1$, we consider coefficients $\{\alpha_j^t\}_{t=0,1, j\in\mathcal{M}}$ satisfying (32). Then we have $u(r_i) = 0$. On the other hand, $\partial_{xx}u(r_i) = 2\beta_i^s = 2(\Xi\boldsymbol{\alpha})_i^r < 0$ by (33) or (34). Thus, there is a nonnegative element $u \in D(A_\Phi^\infty)$ for which $\partial_{xx}u < 0$ at a point where the global minimum of zero is attained. □

We can use this result to prove an analogous result in $\mathbf{L}_1(I)$.

Corollary 2.6 *The operator A_Φ^0 generates a positive semigroup if and only if the assumptions of Theorem 2.5 are satisfied.*

Proof In one direction the result immediately follows by density of $\mathbf{C}(I)$ in $\mathbf{L}^1(I)$. Conversely, if $\{e^{tA_\Phi^0}\}_{t\geq 0} \geq 0$ then, in particular, for any $0 \leq \mathring{\mathbf{u}} \in \mathbf{C}(I)$ we have $e^{tA_\Phi^0}\mathring{\mathbf{u}} = e^{tA_\Phi^\infty}\mathring{\mathbf{u}} \geq 0$. □

3 Solvability of (8)

We return to the problem (8) where, we emphasize, $\mathbb{K}$ is an arbitrary matrix. In this section we restrict our attention to $\mathbf{X} = \mathbf{L}_1(I)$ as in $\mathbf{C}(I)$ the operator $\mathbf{A}$ is not densely defined. Let us recall that $\mathbf{A}$ is the realization of $\mathsf{A} = \mathrm{diag}\{-c_j\partial_x\}_{1\leq j\leq m}$ on the domain $D(\mathbf{A}) = \{\mathbf{u} \in \mathbf{W}_1^1(I);\ \mathbf{u}(0) = \mathbb{K}\mathbf{u}(1)\}$.

The following theorem for (7) has been proved in [7] (see also [27]) but the proof for any nonnegative matrix $\mathbb{K}$ is practically the same. Here we extend this proof to an arbitrary $\mathbb{K}$. However, for the proof in the general case we need to provide basic steps of the proof for nonnegative matrices.

Theorem 3.1 *The operator $(\mathbf{A}, D(\mathbf{A}))$ generates a C_0-semigroup on $\mathbf{L}_1(I)$. The semigroup is positive if and only if $\mathbb{K} \geq 0$.*

Proof Clearly, $\mathbf{C}_0^\infty(]0, 1[) \subset D(\mathbf{A})$ and hence $D(\mathbf{A})$ is dense in $\mathbf{X}$. To find the resolvent of $\mathbf{A}$, the first step is to solve

$$\lambda u_j + c_j\partial_x u_j = f_j, \quad j = 1, \ldots, m, \quad x \in]0, 1[, \tag{35}$$

with $(u_1, \ldots, u_m) = \mathbf{u} \in D(\mathbf{A})$. Integrating, we find

$$\mathbf{u}(x) = \mathbb{E}_\lambda(x)\mathbf{v} + \mathbb{C}^{-1}\int_0^x \mathbb{E}_\lambda(x-s)\mathbf{f}(s)ds, \tag{36}$$

where $\mathbf{v} = (v_1, \ldots, v_m)$ is an arbitrary vector and $\mathbb{E}_\lambda(s) = \mathrm{diag}\left\{e^{-\frac{\lambda}{c_j}s}\right\}_{1\leq j\leq m}$. To determine $\mathbf{v}$ so that $\mathbf{u} \in D(\mathbf{A})$, we use the boundary conditions. At $x = 1$ and at $x = 0$ we obtain, respectively

$$\mathbf{u}(1) = \mathbb{E}_\lambda(1)\mathbf{v} + \mathbb{C}^{-1}\int_0^1 \mathbb{E}_\lambda(1-s)\mathbf{f}(s)ds, \qquad \mathbf{u}(0) = \mathbf{v}.$$

Then, using the boundary condition $\mathbf{u}(0) = \mathbb{K}\mathbf{u}(1)$ we obtain

$$(\mathbb{I} - \mathbb{K}\mathbb{E}_\lambda(1))\mathbf{v} = \mathbb{K}\mathbb{C}^{-1}\int_0^1 \mathbb{E}_\lambda(1-s)\mathbf{f}(s)ds. \tag{37}$$

Since the norm of $\mathbb{E}_\lambda(1)$ can be made as small as one wishes by taking large λ, we see that $\mathbf{v}$ is uniquely defined by the Neumann series provided λ is sufficiently large and hence the resolvent of $\mathbf{A}$ exists.

Let us first consider $\mathbb{K} \geq 0$. Then the Neumann series expansion ensures that $\mathbf{A}$ is a resolvent positive operator and hence we can consider only $\mathbf{f} \geq 0$. Adding together the rows in (37) we obtain

$$\sum_{j=1}^m v_j = \sum_{j=1}^m \kappa_j e^{-\frac{\lambda}{c_j}} v_j + \sum_{j=1}^m \frac{\kappa_j}{c_j}\int_0^1 e^{\frac{\lambda}{c_j}(s-1)} f_j(s)ds, \tag{38}$$

where $\kappa_j = \sum_{i=1}^m k_{ij}$. Renorming $\mathbf{X}$ with the norm $\|\mathbf{u}\|_c = \sum_{j=1}^m c_j^{-1}\|u_j\|_{L_1(I)}$ and using (36) and (38), we obtain

$$\|\mathbf{u}\|_c = \frac{1}{\lambda}\sum_{j=1}^m v_j e^{-\frac{\lambda}{c_j}}(\kappa_j - 1) + \frac{1}{\lambda}\sum_{j=1}^m \frac{\kappa_j - 1}{c_j}\int_0^1 e^{\frac{\lambda}{c_j}(s-1)} f_j(s)ds + \frac{1}{\lambda}\|\mathbf{f}\|_c.$$

We consider three cases.

(a) $\kappa_j \leq 1$ for $j \in \mathcal{M}$. Then $\|\mathbb{E}_\lambda(-1)\mathbb{K}\| < 1$ and thus $\mathbf{v}$, and hence $R(\lambda, \mathbf{A})$, are defined and positive for any $\lambda > 0$. Further, dropping the first two terms in (39) we get

$$\|\mathbf{u}\|_c \leq \frac{1}{\lambda}\sum_{j=1}^m \frac{1}{c_j}\int_0^1 f_j(s)ds = \frac{1}{\lambda}\|\mathbf{f}\|_c, \quad \lambda > 0.$$

Hence $(\mathbf{A}, D(\mathbf{A}))$ generates a positive semigroup of contractions in $(\mathbf{X}, \|\cdot\|_c)$.

(b) $\kappa_j \geq 1$ for $j \in \mathcal{M}$. Then (39) implies that for some $\lambda > 0$ and $c = 1/\lambda$ we have

$$\|R(\lambda, \mathbf{A})\mathbf{f}\|_c \geq c\|\mathbf{f}\|_c$$

and, by density of $D(\mathbf{A})$, the application of the Arendt-Batty-Robinson theorem [28,29], gives the existence of a positive semigroup generated by $\mathbf{A}$ in $(\mathbf{X}, \|\cdot\|_c)$. Since $\|\cdot\|_c$ is equivalent to $\|\cdot\|_{\mathbf{X}}$, $\mathbf{A}$ generates a positive semigroup in $\mathbf{X}$.

(c) $\kappa_j < 1$ for $j \in I_1$ and $\kappa_j \geq 1$ for $j \in I_2$, where $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = \{1, \ldots, m\}$. Let $\mathbb{L} = (l_{ij})_{1\leq i,j\leq m}$, where $l_{ij} = k_{ij}$ for $j \in I_2$ and $l_{ij} = 1$

for $j \in I_1$. Denoting by $\mathbf{A}_{\mathbb{L}}$ the operator given by the differential expression A restricted to $D(\mathbf{A}_{\mathbb{L}}) = \{\mathbf{u} \in \mathbf{W}_1^1(I);\ \mathbf{u}(0) = \mathbb{L}\mathbf{u}(1)\}$ we see, by (37), that

$$0 \leq R(\lambda, \mathbf{A}) \leq R(\lambda, \mathbf{A}_{\mathbb{L}}) \tag{39}$$

for any $\lambda \in \rho(\mathbf{A}_{\mathbb{L}})$. By item (b), $\mathbf{A}_{\mathbb{L}}$ generates a positive C_0-semigroup and thus satisfies the Hille–Yosida estimates. Since clearly (39) yields $R^k(\lambda, \mathbf{A}) \leq R^k(\lambda, \mathbf{A}_{\mathbb{L}})$ for any $k \in \mathbb{N}$, for some $\omega > 0$ and $M \geq 1$ we have

$$\|R^k(\lambda, \mathbf{A})\| \leq \|R^k(\lambda, \mathbf{A}_{\mathbb{L}})\| \leq M(\lambda - \omega)^{-k}, \qquad \lambda > \omega, k \in \mathbb{N},$$

and hence we obtain the generation of a semigroup by $\mathbf{A}$.

Assume now that $\mathbb{K}$ is arbitrary. The analysis up to (37) remains valid. Then (37) can be expanded as

$$\mathbf{v} = \sum_{n=0}^{\infty} (\mathbb{K}\mathbb{E}_\lambda(1))^n \mathbb{K}\mathbb{C}^{-1} \int_0^1 \mathbb{E}_\lambda(1-s)\mathbf{f}(s)ds \tag{40}$$

and hence

$$\mathbf{u}(x) = \mathbb{E}_\lambda(x) \sum_{n=0}^{\infty} (\mathbb{K}\mathbb{E}_\lambda(1))^n \mathbb{K}\mathbb{C}^{-1} \int_0^1 \mathbb{E}_\lambda(1-s)\mathbf{f}(s)ds + \mathbb{C}^{-1} \int_0^x \mathbb{E}_\lambda(x-s)\mathbf{f}(s)ds. \tag{41}$$

Denoting now $|\mathbf{u}| = (|u_1|, \ldots, |u_n|)$ and $|\mathbb{K}| = (|k_{ij}|)_{1 \leq i,j \leq m}$, and using the fact that only $\mathbb{K}$ may have non positive entries, we find

$$\begin{aligned} |\mathbf{u}(x)| &\leq \mathbb{E}_\lambda(x) \sum_{n=0}^{\infty} (|\mathbb{K}|\mathbb{E}_\lambda(1)|)^n |\mathbb{K}|\mathbb{C}^{-1} \int_0^1 \mathbb{E}_\lambda(1-s)|\mathbf{f}|(s)ds \\ &\quad + \mathbb{C}^{-1} \int_0^x \mathbb{E}_\lambda(x-s)|\mathbf{f}|(s)ds = R(\lambda, \mathbf{A}_{|\mathbb{K}|})|\mathbf{f}|, \end{aligned}$$

where $\mathbf{A}_{|\mathbb{K}|}$ denotes A restricted to $D(\mathbf{A}_{|\mathbb{K}|}) = \{\mathbf{u} \in \mathbf{W}_1^1(I);\ \mathbf{u}(0) = |\mathbb{K}|\mathbf{u}(1)\}$. So, we can write

$$|R(\lambda, \mathbf{A}_{\mathbb{K}})\mathbf{f}| \leq R(\lambda, \mathbf{A}_{|\mathbb{K}|})|\mathbf{f}| \tag{42}$$

and, iterating,

$$|R(\lambda, \mathbf{A}_{\mathbb{K}})^k \mathbf{f}| \leq R(\lambda, \mathbf{A}_{|\mathbb{K}|})^k |\mathbf{f}|.$$

Using the fact that taking the modulus does not change the norm, we find

$$\|R(\lambda, \mathbf{A}_{\mathbb{K}})^k \mathbf{f}\| \leq \|R(\lambda, \mathbf{A}_{|\mathbb{K}|})^k |\mathbf{f}|\| \leq \frac{M}{\lambda - \omega}\|\mathbf{f}\|.$$

with M and ω following from the Hille–Yosida estimates for $A_{|\mathbb{K}|}$.

The fact that $\mathbb{K} \geq 0$ yields the positivity of the semigroup follows from the first part of the proof. To prove the converse, let $k_{ij} < 0$ for some i, j and consider the initial condition $\mathbf{f}(x) = (f_1(x), \dots, f_m(x))$ with $f_k = 0$ for $k \neq j$ and $f_j \in C^1([0, 1])$ with $f_j(0) = f_j(1) = 0$, so that $\mathbf{f} \in D(A)$, and $f_j(x) > 0$ for $0 < x < 1$. Then, at least for $t < \min_{1 \leq j \leq m}\{1/c_j\}$, u_i satisfies

$$\partial_t u_i = c_i \partial_x u_i, \quad u_i(x, 0) = 0, u_i(0, t) = k_{ij} f_j(1 - c_j t);$$

that is,

$$u_i(x, t) = k_{ij} f_j \left(1 + \frac{c_j x}{c_i} - c_j t\right), \qquad t \geq \frac{x}{c_i},$$

and we see that the solution is negative for such t. This ends the proof. □

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References

1. Kant, U., Klauss, T., Voigt, J., Weber, M.: Dirichlet forms for singular one-dimensional operators and on graphs. J. Evol. Equ. **9**, 637–659 (2009)
2. Kostrykin, V., Potthoff, J., Schrader, R.: Contraction semigroups on metric graphs. In: Proceedings of Symposia in Pure Mathematics, vol. 77, pp. 423–458. AMS (2008)
3. Kuchment, P.: Quantum graphs: I. Some basic structures. Waves Random Media **14**, S107–S128 (2004)
4. Kuchment, P.: Analysis on Graphs and Its Applications. In: Proceedings of Symposia in Pure Mathematics, pp. 291–314. AMS (2008)
5. Bobrowski, A.: From diffusion on graphs to markov chains via asymptotic state lumping. Ann. Henri Poincaré **13**, 1501–1510 (2012)
6. Gregosiewicz, A.: Asymptotic behaviour of diffusion on graphs. In: Banek, T., Kozłowski, E. (eds.) Probability in Action, pp. 83–96. Lublin University of Technology Press, Lublin (2014)
7. Banasiak, J., Namayanja, P.: Asymptotic behaviour of flows on reducible networks. Netw. Heterog. Media **9**(2), 197–216 (2014)
8. Bressan, A., Čanić, S., Garavello, M., Herty, M., Piccoli, B.: Flows on networks: recent results and perspectives. EMS Surv. Math. Sci. **1**, 47–111 (2014)
9. Dorn, B.: Semigroups for flows in infinite networks. Semigroup Forum **76**, 341–356 (2008)
10. Dorn, B., Kramar Fijavž, M., Nagel, R., Radl, A.: The semigroup approach to transport processes in networks. Physica D **239**, 1416–1421 (2010)
11. Kramar, M., Sikolya, E.: Spectral properties and asymptotic periodicity of flows in networks. Math. Z. **249**, 139–162 (2005)

12. Matrai, T., Sikolya, E.: Asymptotic behaviour of flows in networks. Forum Math. **19**, 429–461 (2007)
13. Knopoff, D.: On the modelling of migration phenomena on small networks. Math. Models Methods Appl. Sci. **23**(3), 541–563 (2013)
14. Mugnolo, D.: Semigroup Methods for Evolution Equations on Networks. Springer, Cham (2014)
15. Banasiak, J., Falkiewicz, A., Namayanja, P.: Asymptotic state lumping in network transport and diffusion problems. Math. Models Methods Appl. Sci. (2016, accepted). arXiv:1503.00683
16. Berge, C.: Hypergraphs: Combinatorics of Finite Sets. North-Holland, Amsterdam (1989)
17. Banasiak, J., Falkiewicz, A.: Some transport and diffusion processes on networks and their graph realizability. Appl. Math. Lett. **45**, 25–30 (2015)
18. Hemminger, R.L., Beineke, L.W.: Line graphs and line digraphs. In: Beineke, L.W., Wilson, R.J. (eds.) Selected Topics in Graph Theory I, pp. 271–305. Academic Press, London (1978)
19. Rotenberg, M.: Transport theory for growing cell population. J. Theor. Biol. **103**, 181–199 (1983)
20. Hadeler, K.P.: Structured populations with diffusion in state space. Math. Biosci. Eng. **7**(1), 37–49 (2010)
21. D'Apice, C., El Habil, B., Rhandi, A.: Positivity and stability for a system of transport equations with unbounded boundary perturbations. Electron. J. Differ. Equ. **137**, 1–13 (2009)
22. Greiner, G.: Perturbing the boundary conditions of a generator. Houst. J. Math. **13**(2), 213–228 (1987)
23. Engel, K.-J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (1999)
24. Bobrowski, A.: Generalized telegraph equation and the Sova–Kurtz version of the Trotter–Kato theorem. Ann. Polon. Math. **LXIV.1**, 37–45 (1996)
25. Bobrowski, A.: Functional Analysis for Probability and Stochastic Processes. Cambridge University Press, Cambridge (2005)
26. Nagel, R. (ed.): One Parameter Semigroups of Positive Operators. Lecture Notes in Mathematics, vol. 1184. Springer, Berlin (1986)
27. Banasiak, J.: Kinetic models in natural sciences. In: Banasiak, J., Mokhtar-Kharroubi, M. (eds.) Evolutionary Equations with Applications in Natural Sciences. Lecture Notes in Mathematics, vol. 2126, pp. 133–198. Springer, Heidelberg (2015)
28. Arendt, W.: Resolvent positive operators. Proc. Lond. Math. Soc. **54**, 321–349 (1987)
29. Banasiak, J., Arlotti, L.: Positive Perturbations of Semigroups with Applications. Springer, London (2006)
30. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)