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OBITUARY

David Rees 1918–2013

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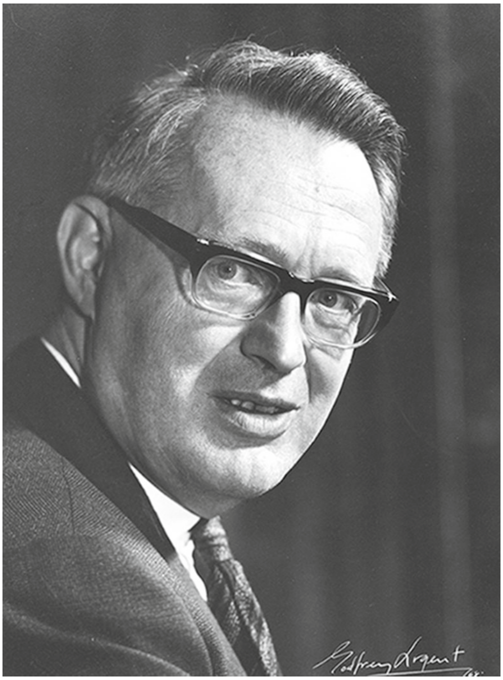

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1 Introduction

David Rees achieved recognition in three very different endeavours. For the readers of this journal, he was, of course, one of the pioneers of semigroup theory, commemorated in Rees matrix semigroups, the Rees theorem, and Rees quotients; in the wider mathematical world, he was best known as one of the leading figures in the post-war development of commutative algebra; and, finally, in the obituaries in the British press, it was his work as a code-breaker at Bletchley Park during the Second World War which attracted most attention. While in this short tribute we cannot hope to do justice to all these aspects of his life and work, we have tried to give some indication of the depth and influence of his ideas.

2 Brief biography

David Rees was born on the 29th of May 1918 in the Welsh market town of Abergavenny[1] the fourth of five children to David, a miller and corn dealer, and Florence Gertrude Rees (née Powell). Education was greatly valued in Wales at that time[2] and, as a result, the secondary school David attended, the King Henry VIII grammar school in Abergavenny, was one of a number of grammar schools in the Principality which provided a first-rate education to able children of all backgrounds and access to the top universities in Britain. David showed particular mathematical promise from an early age. For example, returning to school after several months away due to illness, he was able to catch up with his mathematics, under the guidance of mathematics master L. F. Porter[3], after only a couple of weeks, and in the sixth form, he won the school prize for mathematics. In October 1936, he went up to Cambridge to study mathematics at Sidney Sussex College where he was an outstanding student, winning a number of prizes and distinctions before graduating with his BA in June 1939.

Although David began his graduate work at Cambridge, it was never completed. With the outbreak of war on the 3rd September 1939, personal priorities were changed irrevocably. In December 1940, David was recruited by his undergraduate director of studies at Sidney, Gordon Welchman, to join the code-breakers at Bletchley Park. In fact, the invitation he received from Welchman and fellow don Dennis Babbage was couched in the vaguest possible terms. Initially, he was simply told they had a job for him to do but they would not say where. This raised obvious difficulties. In the end, he was simply told to report to the railway station at Bletchley. Welchman, along with John Jeffreys, ran Hut 6 responsible for breaking the German Army and Airforce Enigma ciphers. Another Sidney recruit at this time was mathematician John Herivel, who would work alongside David. It was Herivel who formulated an approach to breaking the Luftwaffe Enigma code (nicknamed *Red*). Known as the *Herivel tip*, it exploited procedural errors in the way operators used their Enigma machines. For some months it yielded nothing, but towards the end of May 1940, David made Herivel's idea

[1] Welsh *Y Fenni*.

[2] A consequence of the Welsh dissenting tradition.

[3] Father of the mathematician Timothy Porter.

work, leading to Red being broken at a crucial moment in the war. David subsequently worked in Max Newman's group at Bletchley, the *Newmanry*, where the first electronic computer, Colossus, was used in decoding the German teletype ciphers.

After the war, Newman became one of the pioneers in the development of British computing when he set up the Royal Society Computing Machine Laboratory at Manchester University. He was joined by David, who was made an assistant lecturer in mathematics. During this time, he attended the famous *Moore School Lectures* at the University of Pennsylvania, which might be described as the first conference on computers. In 1948, David was appointed to a University Lectureship in Mathematics at Cambridge, and in 1950, he became a Fellow of Downing College. It was whilst at Cambridge that David met Joan Cushen, also a mathematician, and they married in 1952. They would have four children, Mary, Rebecca, Sarah and Debbie, and three grandchildren, Rachel, Peter and Christopher. David left Cambridge in 1958 to take up a Chair in Mathematics at the University of Exeter, where he remained until his retirement in 1983. He was made an FRS in 1968 and an Honorary Fellow of Downing College in 1970. In 1993, he was awarded the prestigious Pólya Prize by the London Mathematical Society. David Rees died on the 16th August 2013 and Joan shortly afterwards on the 28th August.

3 Contributions to semigroup theory

David wrote only five papers on semigroup theory [12–16], and one of them [13] is just a brief addendum to [12], but along with Sushkevich and Clifford, he can be regarded as one of the subject's founding fathers.

David's first paper [12] proves a theorem, the Rees-Sushkevich theorem, which, appropriately enough, is also the first that any beginning student of semigroup theory encounters. This theorem, and the Rees matrix construction which underlies it, has become one of the subject's most influential results[4]. The paper is clearly motivated by the structure theory of algebras as described in Albert's book [1] and initially translates well-known ideas from ring theory to semigroup theory. However, in the construction he gives of what we would now call completely 0-simple semigroups, the differences between semigroups and rings become apparent and perhaps it was here that semigroup theory as an autonomous area of algebra became a possibility. The paper itself is satisfyingly complete, in that the structure of completely 0-simple semigroups is fully resolved. In addition, the earlier work of Sushkevich from 1928 is shown to be just the finite case of David's more general result. But the paper also contains something else in that there are hints of what would become known as Green's relations. In fact, Green himself was another Newmanry alumnus and David would became his third supervisor. Green's famous 1951 paper [7] represents the specialization of his own ideas in universal algebra to semigroups.

David's work on completely 0-simple semigroups, along with Clifford's 1941 work on completely regular semigroups, led to a flowering of semigroup research. Rees

[4] Therefore fully deserving the fanfare of trumpets wickedly suggested by Rhodes in a famous, if not infamous, review. See [41].

matrix semigroups in general became a basic tool of the subject, culminating in two important developments, both of which were motivated by Allen's important paper of 1971 [2]: first, the synthesis theorem, uniting the Rees theorem, with its roots in classical algebra, with the Krohn-Rhodes theorem, with its roots in automata theory [3]; second, Don McAlister's sequence of papers that developed a Morita theory for regular semigroups in all but name, thus placing the Rees construction in its proper context. See, for example, [10]. David's next two semigroup papers explore the mathematical possibilities of semigroups as structures in their own right, and both deal with semigroups having cancellation properties.

Cancellative semigroups arise naturally as subsemigroups of groups. Although the problem of characterizing just which cancellative semigroups can be embedded in groups is a hard one, solved by Malcev in 1939, it is still interesting to find usable necessary conditions for embeddability and to provide direct proofs. One such necessary condition can be read off from a ring-theoretic result by Ore in 1931: every cancellative semigroup in which any two principal left ideals intersect has a group of left fractions and so, in particular, can be embedded in a group. In his 1947 paper [14], David gives a new proof of Ore's theorem which is strikingly semigroup-theoretic. He considers the set of all partial bijections of the semigroup generated by right translations by elements of the semigroup. This forms what we would now call an inverse monoid. The role of the reversibility condition is to ensure that this monoid does not contain a zero. He then defines a group congruence on this inverse monoid and proves that the original cancellative monoid embeds into it. It was this proof of Ore's theorem that found its way into Clifford and Preston [5]. This paper contains the germs of three important ideas. First, it contains the idea of an inverse semigroup and even that of the minimum group congruence on an inverse semigroup. In his thesis, Gordon Preston, yet another Newmanry alumnus, attempted to axiomatize such semigroups of partial bijections, motivated by David's paper. Although he was not initially successful, he persevered and by the end of 1953, Preston had written the first British papers on inverse semigroups and coined the name they would be known by in English[5]. Preston would later emigrate to Australia and set about establishing Australian semigroup theory. Second, the inverse semigroups that arise in this way from cancellative monoids are E-unitary. In fact, it is this property which implies that the original cancellative monoids can be embedded in the group. This class of semigroups would play an important role in the development of inverse semigroup theory, beginning with McAlister's seminal papers on E-unitary inverse semigroups and E-unitary covers in the 1970s. Third, it introduces the theme of constructing groups from inverse semigroups[6] which may yet turn out to be one of the most important ways in which inverse semigroups interact with the wider mathematical world.

In 1948, David wrote his last solo paper on semigroups [15]. This deals with the structure of left cancellative monoids but treats them as interesting structures in their own right rather than as potential candidates for embedding into groups. The paper is

[5] Such semigroups were also independently introduced by Wagner in the Soviet Union and Ehresmann in France.

[6] See the article by John Meakin entitled *Groups and semigroups: connections and contrasts*. This is available to download from Meakin's homepage.

based on a simple observation about the structure of the poset of principal right ideals of a left cancellative monoid: any two principal order ideals of this poset are isomorphic. David calls any poset satisfying this condition *uniform*. Today, we would use a much more suggestive term to describe such posets: we would say that they are *self-similar*. Thus this paper represents a very early, perhaps the first, use of this idea in algebra. David does not reveal his motivations but it is possible that he first became aware of this property by observing it in free monoids. Such monoids are, of course, nothing other than finite symbol sequences under the operation of concatenation. Having introduced the idea of a uniform poset he proves his main theorem: every uniform poset with a maximum element arises in this way. The proof is very easy. With each uniform poset, he associates the monoid consisting of all order isomorphisms from the poset to its principal order ideals. This turns out to be a left cancellative monoid with the desired property. Next, he shows that this monoid enjoys a certain universal property. To achieve this, he develops a theory of a special class of homomorphisms which are determined by what he calls *right normal divisors* and generalize normal subgroups. As an application of his theory, he determines the structure of those left cancellative monoids in which the principal left ideals form an infinite, countable descending chain, that is, the same principal left ideal structure as the natural numbers under addition. This paper influenced the development of inverse semigroup theory via the theory of bisimple inverse monoids and it contains the germ of the idea that every inverse semigroup has a fundamental image constructed from the Munn semigroup of its semilattice: this directly extends David's result on left cancellative monoids which have no left normal divisors. But the most striking, and sadly little known, developments of this paper are to be found in J.-F. Perrot's work in the early 1970s. In Chapter 6 of his thesis [11], Perrot makes what looks like a small extension of Rees's main theorem. He studies those left cancellative monoids whose poset of principal left ideals is order-isomorphic to that of the free monoid generated by a set X. He calls these monoids X*-monoids*. In these terms, David's paper deals with the case where X contains only one element. Perrot determines that their structure is what we would now call a Zappa-Szép product of a group and the free monoid generated by X. It was only much later, and by chance, that it became clear that these monoids are nothing other than self-similar group actions re-encoded as certain types of left cancellative monoids, with the faithful actions corresponding to the case where the monoids have no right normal divisors [9]. The theory of self-similar group actions emerged officially in the 1980s from a completely different direction.

David's last paper in semigroup theory was written with Sandy Green in 1952 [16] and is motivated by the Burnside problem in group theory. Recall that the free Burnside groups $B_{n,r-1}$ are the groups freely generated by n elements subject only to the requirement that $w^{r-1} = 1$ for all $w \in B_{n,r-1}$. In their paper, they define a class of semigroup analogues of the Burnside groups. These are the semigroups S_{nr} freely generated by n elements subject only to the requirement that $w^r = w$ for all elements $w \in S_{nr}$. Their main result is striking: they prove that S_{nr} is finite if and only if $B_{n,r-1}$ is finite. The case where $r = 2$ is especially interesting, because S_{n2} is nothing other than the free band on n generators. They explicitly calculate its cardinality and so, in particular, show that the free band on 4 generators has 332,380 elements. This paper is a beautiful piece of combinatorial semigroup theory and, once again, develops the theme of studying the relationship between groups and semigroups. Some of their proofs found their way into John Howie's famous book [8] in the section dealing with the structure of free bands.

It was now that David left semigroup theory for commutative ring theory where he went on to establish an international reputation. However his influence on the field he helped to create lives on in his papers and through his personal influence. We have already mentioned Green and Preston but there was one more person who benefitted from David's advice, and that was Douglas Munn. Douglas went up to Cambridge in 1951 and attended lectures given by David. Douglas's PhD supervisor does not seem to have taken any interest in semigroups, but Douglas recalled useful discussions with David when still in the early stages of the project that led to his PhD thesis. In fact, he later reminisced that five minutes discussing mathematics with Rees was worth its weight in gold.

"All beginnings are difficult" as the saying goes, and no more so than in a new branch of mathematics. David Rees very quickly saw new mathematical possibilities in semigroups, and laid the foundations for a number of important avenues of future research. Perhaps this could only have been accomplished by someone who was young enough to be without mathematical prejudices, and at the same time endowed with considerable mathematical ability. Bletchley Park itself may also have played some small role. Not only did it bring together a whole generation of British mathematicians, it did so in a way that encouraged the unfettered development of new ideas.

4 Contributions to commutative algebra

When David moved back to Cambridge in 1948, he joined a working seminar led by Douglas Northcott on André Weil's now classic text *The Foundations of Algebraic Geometry*, published in 1946 [44]. Weil's book formed the perfect backdrop for David's brilliant work to come on local and graded commutative algebra, work which was always informed by a deep appreciation of the subject's strong links with algebraic geometry. It is also worth noting that another formative influence around this time was the invasion of homological algebra into what might be termed classical algebra, as summarized by Henri Cartan and Samuel Eilenberg's famous 1956 book [4].

We begin by sketching briefly, and informally, the links between commutative algebra and algebraic geometry. A *local ring* $(R, \mathfrak{m})$ is by definition a commutative Noetherian ring R with identity element, having a unique maximal ideal $\mathfrak{m}$. A proper ideal $\mathfrak{q}$ of R is called $\mathfrak{m}$-*primary* if $\mathfrak{m}^t$ lies inside $\mathfrak{q}$ for some positive integer t. Such rings arise as the rings of rational functions defined at a given point on an algebraic variety, this being a geometrical object specified by the vanishing of a set of polynomial equations, with coefficients in a fixed field, in a finite number of variables. Indeed, a general local ring $(R, \mathfrak{m})$ is thought of geometrically as the coordinate ring of functions defined at a given point P on an algebraic variety V, with $\mathfrak{m}$ being considered as an algebraic version of the geometrical object P (and vice versa), and an $\mathfrak{m}$-primary ideal $\mathfrak{q}$ being thought of as a 'thickened' version of P, the amount of thickening involved being specified by the so-called multiplicity $e(\mathfrak{q})$ of $\mathfrak{q}$. In 'good' situations, for example, when R is a so-called Cohen-Macaulay ring and $\mathfrak{q}$ is generated by a finite sequence of successive non-zero-divisors, $e(\mathfrak{q})$ is given by the length $l(R/\mathfrak{q})$ of $R/\mathfrak{q}$, but in general $e(\mathfrak{q})$ has an expression involving the asymptotic length $l(R/\mathfrak{q}^n)$, $n \gg 0$, which is a much more intractable object. There is another interpretation of $e(\mathfrak{q})$ as the intersection multiplicity at P of the variety V with another variety having ideal of definition $\mathfrak{q}$ on the coordinate ring R of V.

David's first paper [17] in commutative algebra, written jointly with Douglas Northcott, addressed precisely this complicated general problem of specifying $e(\mathfrak{q})$ for an arbitrary $\mathfrak{m}$-primary ideal $\mathfrak{q}$, and clarified completely how to reduce $\mathfrak{q}$ to a simpler ideal $\mathfrak{r}$ lying inside $\mathfrak{q}$ ($\mathfrak{r}$ being a so-called reduction of $\mathfrak{q}$), having the same asymptotic behaviour as $\mathfrak{q}$ and hence with $e(\mathfrak{r}) = e(\mathfrak{q})$; $\mathfrak{q}$ is then integrally dependent on $\mathfrak{r}$, 'integral dependence' being a natural generalization to the case of ideals of the familiar notion of algebraic dependence of field extensions. To this day, this paper is one of the most widely quoted and widely used papers in the subject, in part because if R is Cohen-Macaulay, the defining property of a wide and tractable class of rings, then necessarily $\mathfrak{r}$, if chosen minimally, is generated by a finite sequence of successive non-zero-divisors, so $e(\mathfrak{q}) \equiv e(\mathfrak{r})$ is given by a simple length, namely the length $l(R/\mathfrak{r})$ of $R/\mathfrak{r}$. In 1961, in the equally famous paper [31], David established a converse in the case where R satisfies a natural condition, and one holding in the geometrical context: namely, that two $\mathfrak{m}$-primary ideals $\mathfrak{q}$ and $\mathfrak{q}'$, with $\mathfrak{q} \supseteq \mathfrak{q}'$, have the same multiplicity if and only if $\mathfrak{q}'$ is a reduction of $\mathfrak{q}$.

The reason for going into some detail on these matters is that David returned again and again to the themes of reductions and multiplicities, eventually formulating a symmetrized notion of reduction of a module M with respect to a finite set of ideals $\mathcal{I}$, namely that of a joint reduction of M with respect to $\mathcal{I}$, with applications to the so-called mixed multiplicities of Jean-Jacques Risler and Bernard Teissier developed in the context of algebraic geometry (cf. [36]). Full analogues of the foundational results mentioned above were established, and the theory extended further, in joint papers with Rodney Sharp [33] and with David Kirby [38–40]. Indeed, David's final papers, written jointly with David Kirby, developed a broad theory of 'Buchsbaum-Rim' multiplicities and mixed multiplicities, and developed earlier work of Kirby's on multiplicity theory in the very general setting of Euler-Poincaré characteristics of complexes.

In algebraic geometry, a thickened point, or, more generally, a singularity, can be teased apart into component simpler singularities via a process known as blowing up. And it was here, early in his career, in a sequence of papers devoted to valuations on local rings [18,19,21–23], that David made his second fundamental contribution to the subject: namely, the introduction of what are now known as the restricted and unrestricted Rees rings. Geometrically, these appear as the coordinate rings of blowups, and David used these in part to track how multiplicities behave during the process of blowing up, with valuations entering into the formulae describing this behaviour.

However, the above survey does not do full justice to the power of David's work on these matters. The fact is that these notions of reductions and Rees rings have powerful purely algebraic uses. In good situations, passage to a minimal reduction can transform a rather abstract problem into a straightforward, concrete matter where, for example, an inductive argument can be deployed. Similarly, use of the extended Rees ring in certain cases can allow a general ideal to be replaced by a principal ideal, which is a very useful simplification. Rees rings also allow the behaviour of powers of an ideal or indeed of a finite collection of ideals and the integral closures of these powers to be examined all together, as it were, and also to permit simple passage to associated graded rings and so-called special fibre rings (which, in the geometrical setting, comprise the homogeneous coordinate rings of tangent cones). See [27,35]. But the most famous such use for Rees rings first appeared in David's 1956 paper *Two classical theorems of ideal theory* [20] (a note described as 'brilliant' by Irving Kaplansky in his well-known monograph *Commutative Rings*). Here the central result, now known as the Artin-Rees Lemma, is given a simple proof: simple, that is, once one has the notion of a Rees ring to hand. This Lemma is then used to give elegant, crisp and now standard proofs of two foundational results of Wolfgang Krull: namely, his Intersection Theorem and his Principal Ideal Theorem, the latter being 'probably the most important single theorem in the theory of Noetherian rings', to quote Kaplansky again.

David's versatility is further emphasized by a small but very influential group of papers on homological algebra that appeared in the period 1956–1961, the final one written jointly with Peter Hilton, another Bletchley Park colleague. See [24,25,32] and also [26,28]. Irving Kaplansky in his monograph gives an excellent introduction to this work and its background in a non-homological joint paper with Douglas Northcott. The homological approach allows for a very swift proof of the invariance of certain important parameters attached to an ideal in a commutative Noetherian ring with identity element (e.g, the so-called grade, type, ...). This circle of ideas, including others arising in joint work with Douglas Northcott during this period, laid to a crucial extent the basis for Hyman Bass' seminal 1963 paper *On the ubiquity of Gorenstein rings*.

While David's work in commutative algebra is too rich for every important aspect to be mentioned, one particular paper [29] deserves especial mention. In it, there is a simply astonishing call on an entirely unexpected algebro-geometric *deus ex machina* that unlocks the desired result. This paper, which appeared in 1958, answered in the negative a conjecture by Oscar Zariski concerning a generalization of David Hilbert's 14th problem. Here the surprise lies in the use of the extended Rees ring of the ideal defining a point on a projective complex elliptic curve C, over the base ring given by the homogeneous coordinate ring of C, in order to construct the desired counterexample via an analysis requiring delicate geometrical and algebraic reasoning. This is beautiful mathematics of the highest quality.

"*By mathematics we shall come to heaven.*" *From Bach for the Cello by Rowan Williams*

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on Sandy Green, Douglas Munn, Max Newman, David Rees and Gordon Preston. It also contains an article by Preston on the early history of semigroups. Information on John Herivel and Gordon Welchman may be found on wikipedia. Obituaries to David appeared in the main national British newspapers *The Times*, *The Guardian* and *The Telegraph*. *Semigroup Forum* did not allow us to include a complete bibliography of David's work, only allowing us to include the papers we explicitly referred to. All his papers can be found via *MathSciNet* with the exception of [30,34]. We also felt we had to mention here David's one book [37]. David Rees on his election as a Fellow of the Royal Society in 1968. Photograph: Godfrey Argent Studio. The photograph at the beginning of this article shows David Rees.

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