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RESEARCH ARTICLE

A general theorem on generation of moments-preserving cosine families by Laplace operators in $C[0,1]$

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Abstract We use Kelvin's method of images (Bobrowski in J. Evol. Equ. 10(3):663–675, 2010; Semigroup Forum 81(3):435–445, 2010) to show that given two non-negative integers $i \neq j$ there exists a unique cosine family generated by a restriction of the Laplace operator in $C[0,1]$, that preserves the moments of order i and j about 0, if and only if precisely one of these integers is zero.

Keywords Method of images · Cosine families · Strongly-continuous semigroup · Differential operators with integral conditions

1 Introduction

Following the seminal work of J.R. Cannon [6], a semigroup-theoretical study of diffusion and wave equations associated with one-dimensional Laplace operators equipped with integral conditions has recently been commenced in [9], where an abstract framework for studying such problems in Hilbert spaces has been proposed. Paper [5] presents a different approach, applicable apparently in a broader context: it shows that the recently developed Lord Kelvin's method of images [3, 4] provides natural tools for constructing moments-preserving cosine families. In particular, the main theorem of [5] states that there is a unique cosine family generated by a Laplace operator in $C[0,1]$ that preserves the moments of order zero and 1 (about 0).

In this context, a natural question arises of whether and to what extent the main result of [5], may be generalized to the case of two arbitrary moments of order, say i

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and j, where $i < j$ are two non-negative integers. Our main theorem (Theorem 2.1) provides the following answer to this question: *a necessary and sufficient condition for existence of a cosine family generated by a Laplace operator in* $C[0, 1]$ *that preserves the moments of order* i *and* j *(about 0) is that* $i = 0$. In particular, there are no Laplace-operator-generated cosine families that preserve two moments of order larger than 0. Moreover, the cosine family that preserves the moments of order 0 and $j \geq 1$ is uniquely determined, and can be constructed (almost) explicitly by means of the abstract Kelvin formula (3.1). Extensions to non-integer moments are also discussed.

2 Preservation of moments about 0

Let $C[0, 1]$ be the Banach space of continuous functions on the unit interval, and let $C^2[0, 1]$ be its subspace of twice continuously differentiable functions. (In what follows we think of real-valued functions, but this is merely to fix attention; the same analysis can be performed in the space of complex functions, as well.) By $\mathfrak{L}_c$ we denote the class of restrictions of the Laplace operator $Lf = f''$, $D(L) = C^2[0, 1]$ to various domains that generate (strongly continuous) cosine families in $C[0, 1]$. In other words, a member of $\mathfrak{L}_c$ is a closed linear operator A that generates a strongly continuous cosine family in $C[0, 1]$ and on its domain coincides with L. The cosine family generated by A will be denoted $C_A = (C_A(t))_{t\in\mathbb{R}}$. We recall (see, e.g., [1, Proof of Theorem 3.14.17] or [8, Theorem 8.7]) that $A \in \mathfrak{L}_c$ generates also a strongly continuous semigroup $S_A = (S_A(t))_{t\geq 0}$; the latter semigroup is given by the abstract Weierstrass formula

$$S_A(t)f = \frac{1}{\sqrt{\pi t}} \int_0^\infty \mathrm{e}^{-\tau^2/4t} C_A(\tau) f \,\mathrm{d}\tau \quad (t > 0,\ f \in C[0, 1]), \tag{2.1}$$

and $S_A(0) = Id_{C[0,1]}$ (identity operator in $C[0, 1]$). In particular, $\mathfrak{L}_c$ is a subclass of the class $\mathfrak{L}_s$ of restrictions of the Laplace operator that generate strongly continuous semigroups in $C[0, 1]$.

Let $\mathbb{N}$ be the set of non-negative integers, and let F_i denote the moment of order i about 0, i.e., let it be the linear functional on $C[0, 1]$ defined by

$$F_i f := \int_0^1 \mathrm{k}_i(x) f(x) \,\mathrm{d}x, \quad i \in \mathbb{N}, \tag{2.2}$$

where

$$\mathrm{k}_i(x) := x^i.$$

Given $A \in \mathfrak{L}_c$ we say that the related cosine family C_A preserves the ith moment iff for all $t \in \mathbb{R}$ and $f \in C[0, 1]$, we have $F_i C_A(t) f = F_i f$. Analogously, for $A \in \mathfrak{L}_s$ we say that the related semigroup preserves the ith moment iff for all $t \geq 0$ and $f \in C[0, 1]$, we have $F_i S_A(t) f = F_i f$. Observe that, by the Weierstrass formula, if C_A preserves F_i then so does S_A (see Proposition 2.2, later on).

Theorem 2.1 *Let $i, j \in \mathbb{N}$ with $i < j$ be given.*

(a) *If $i = 0$, then there is exactly one $A \in \mathfrak{L}_c$ such that the related cosine family preserves F_i and F_j.*
(b) *If $i > 0$, then there is no $A \in \mathfrak{L}_s$ such that the related semigroup preserves F_i and F_j.*

This theorem will be proved in two steps. First, following [5] we will relate preservation of moments with boundary conditions to show that a generator $A \in \mathfrak{L}_s$ of a semigroup that preserves two moments of order $i, j \geq 1$ could not be densely defined (thus establishing (b)). Then, in the next section, we will construct the moments preserving cosine family of point (a).

Before continuing, we note that in case (a), a generation theorem for moments-preserving semigroups has been obtained in [9, Theorem 3.4].

Proposition 2.2 *Let A be a member of $\mathfrak{L}_c$ and let $i \in \mathbb{N}$. The following statements are equivalent.*

(a) *The cosine family C_A preserves F_i.*
(b) *The semigroup S_A preserves F_i.*
(c) *For $f \in D(A)$, we have $F_i(f'') = 0$.*
(d) *For $f \in D(A)$, we have*:

$$\begin{array}{ll} f'(0) = f'(1) & \text{if } i = 0, \\ f'(1) = f(1) - f(0) & \text{if } i = 1, \\ if(1) = i(i-1)F_{i-2}f + f'(1) & \text{if } i \geq 2. \end{array} \tag{2.3}$$

Proof If we assume that C_A preserves F_i, then since F_i is bounded,

$$F_i S_A(t) f = \frac{1}{\sqrt{\pi t}} \int_0^\infty \mathrm{e}^{-\tau^2/4t} F_i C_A(\tau) f \, \mathrm{d}\tau = F_i f,$$

by (2.1) and because $\int_0^\infty \mathrm{e}^{-\tau^2/4t} \, \mathrm{d}\tau = \sqrt{\pi t}$. This proves that (a) implies (b).

In order to prove that (b) implies (c) let $f \in D(A)$ and consider

$$u_i : [0, \infty) \ni t \mapsto F_i(S_A(t) f) \in \mathbb{R}.$$

The scalar-valued function u_i is differentiable with $u_i'(t) = F_i(S_A(t)Af) = F_i(S_A(t)f'')$. If S_A preserves F_i the function u_i is constant, and hence $u_i'(t) = 0$ for $t \in [0, \infty)$. Thus in particular $u_i'(0) = F_i(f'') = 0$.

The equivalence of (c) and (d) is evident since

$$F_i(f') = f(1) - iF_{i-1} f \tag{2.4}$$

holds for all $i \geq 1$ and all $f \in C^1[0, 1]$.

Finally, assume condition (c) holds. We show that the cosine family C_A preserves F_i. First observe that if $f \in D(A)$ then $F_i(\frac{\mathrm{d}^2}{\mathrm{d}t^2} C_A(t) f) = F_i(AC_A(t) f) = 0$, because $C_A(t)(D(A)) \subset D(A)$. Then, similarly as in the proof of (b) $\Rightarrow$ (c),

$$v_i : [0, \infty) \ni t \mapsto F_i(C_A(t) f) \in \mathbb{R}$$

is a scalar-valued twice differentiable function with $v_i'(t) = F_i(\int_0^t C_A(s)\,\mathrm{d}s f'')$ and $v_i''(t) = F_i(C_A(t)f'') = 0$. Therefore, since $v_i'(0) = 0$, v_i is constant, i.e., $F_i(C_A(t)f) = F_i(C_A(0)f) = F_i f$ for $f \in D(A)$. This completes the proof because $D(A)$ is dense in $C[0,1]$. □

Corollary 2.3 *For $i, j \geq 1$, $i \neq j$, the set*

$$D_{i,j} = \{f \in C^2[0,1] \mid F_i(f'') = F_j(f'') = 0\}$$

is not dense in $C[0,1]$.

Proof Since (c) and (d) in Proposition 2.2 are equivalent,

$$D_{i,j} \subset \operatorname{Ker} H_{i,j},$$

where $H_{i,j}$ is a bounded linear functional on $C[0,1]$ given by

$$H_{i,j} f = (j-1)f(1) + f(0) - j(j-1)F_{j-2} f$$

for $i = 1$, and

$$H_{i,j} f = (i-j)f(1) - i(i-1)F_{i-2} f + j(j-1)F_{j-2} f$$

for $i \geq 2$. We note that $H_{i,j}$ is non-zero, because

$$H_{1,j}\mathrm{k}_2 = j - 1 - j(j-1)\frac{1}{j+1} = \frac{j-1}{j+1} \neq 0,$$

and for $i \geq 2$,

$$H_{i,j}\mathrm{k}_2 = i - j - \frac{i(i-1)}{i+1} + \frac{j(j-1)}{j+1} = \frac{2(i-j)}{(i+1)(j+1)} \neq 0.$$

Hence, the corollary follows since $\operatorname{Ker} H_{i,j}$ is closed and not equal to $C[0,1]$. □

This corollary clearly implies (b) in Theorem 2.1. For, if the semigroup generated by $A \in \mathfrak{L}_s$ preserves moments F_i and F_j then, by Proposition 2.2, $D(A) \subset D_{i,j}$ which contradicts the fact that $D(A)$ is dense in $C[0,1]$.

3 Proof of the case $i = 0$, $j \geq 1$

Let $C(\mathbb{R})$ be the Fréchet space of continuous functions on $\mathbb{R}$ with topology of almost uniform convergence, and let $(C(t))_{t\in\mathbb{R}}$ be the basic cosine family in $C(\mathbb{R})$ given by the D'Alembert formula,

$$C(t)f(x) := \frac{1}{2}(f(x+t) + f(x-t)), \quad t, x \in \mathbb{R}.$$

Also, let F_j be the linear functional on $C(\mathbb{R})$ defined by

$$F_j f := \int_0^1 \mathrm{k}_j(x) f(x)\,\mathrm{d}x, \quad j \in \mathbb{N}.$$

The fact that two distinct objects, the functional in $C(\mathbb{R})$ defined here, and the functional on $C[0,1]$ defined in (2.2), are denoted by the same letter, should not lead to misunderstanding. Clearly, F_i is continuous both on $C[0,1]$ and $C(\mathbb{R})$ for all $i \in \mathbb{N}$.

In the theory of semigroups of linear operators and the related theory of cosine families, Lord Kelvin's method of images can be thought of as a way of constructing families of operators generated by an operator with a boundary condition by means of families generated by the same operator in a larger space, where no boundary conditions are imposed (cf. [3, 4]). In our particular context, the method boils down to constructing a cosine family $C_{\mathrm{mp}} = (C_{\mathrm{mp}}(t))_{t\in\mathbb{R}}$ in $C[0,1]$ via the formula

$$C_{\mathrm{mp}}(t) f(x) = C(t)\tilde{f}(x), \quad x \in [0,1],\ t \in \mathbb{R},\ f \in C[0,1], \tag{3.1}$$

where 'mp' stands for 'moments-preserving' and, more importantly, $\widetilde{f} \in C(\mathbb{R})$ is a certain extension of f, chosen in such a way that C_{mp} preserves both F_0 and F_j. To be more specific: Given $f \in C[0,1]$, and $j \geq 1$, we are looking for an $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ such that

(A) $\widetilde{f} \in C(\mathbb{R})$ and $\widetilde{f}(x) = f(x)$ for all $x \in [0,1]$,
(B) $F_0 C(t)\widetilde{f} = F_0 f$ for all $t \in \mathbb{R}$, and
(C) $F_j C(t)\widetilde{f} = F_j f$ for all $t \in \mathbb{R}$.

Existence of such an extension is secured by Proposition 3.2, later on.

Before proceeding, we need to introduce some notations. For a function f defined on $[0,1]$ let f^e be its symmetric reflection about $\frac{1}{2}$, that is $f^e(x) = f(1-x)$ for $x \in [0,1]$. Moreover, given two functions $f, \phi \in C[0,1]$, let $f * \phi \in C[0,1]$ be their convolution:

$$(f * \phi)(x) = \int_0^x f(x-y)\phi(y)\,\mathrm{d}y, \quad x \in [0,1].$$

Observe that

$$\|f * \phi\|_{C[0,1]} \leq \|f\|_{C[0,1]} \|\phi\|_{C[0,1]}.$$

Finally, if $f \in C^1[0,1]$, then $f * \phi \in C^1[0,1]$ with

$$(f * \phi)' = f(0)\phi + f' * \phi.$$

For the proof of Proposition 3.2, we need the following lemma.

Lemma 3.1 *For $g, \phi \in C[0,1]$, there exists a unique $f \in C[0,1]$ such that*

$$f - \phi * f = g.$$

Moreover, if $\phi \in C^1[0,1]$ and $g \in C^2[0,1]$, then $f \in C^2[0,1]$.

Proof Let us define the Bielecki-type norm [2, 7]

$$\|f\|_\lambda = \sup_{x\in[0,1]} |\mathrm{e}^{-\lambda x} f(x)|$$

in $C[0,1]$ with $\lambda > 0$. We note that this norm is equivalent to the usual supremum norm in this space. Consider the operator $T : C[0,1] \to C[0,1]$ given by

$$Tf = g + \phi * f.$$

If $f_1, f_2 \in C[0,1]$, then

$$\begin{aligned}\|Tf_1 - Tf_2\|_\lambda &= \|(f_1 - f_2) * \phi\|_\lambda \\ &\le \sup_{x\in[0,1]} \int_0^x \mathrm{e}^{-\lambda(x-y)} \|f_1 - f_2\|_\lambda |\phi(x-y)|\,\mathrm{d}y \\ &\le \int_0^1 \mathrm{e}^{-\lambda x}\,\mathrm{d}x\, \|\phi\|_{C[0,1]} \|f_1 - f_2\|_\lambda.\end{aligned}$$

Hence, λ can be chosen so large that $\|Tf_1 - Tf_2\|_\lambda \le C_\lambda \|f_1 - f_2\|_\lambda$ for some $C_\lambda \in (0,1)$, and by the Banach fixed point theorem, there exists a unique $f \in C[0,1]$ such that $Tf = f$.

In order to prove the second part of the lemma recall that the unique $f \in C[0,1]$ constructed above satisfies $f = \lim_{n\to\infty} f_n$ in $C[0,1]$, where

$$f_n = T^n g = g + \sum_{k=1}^n \phi_k * g,$$

and $\phi_1 := \phi$, $\phi_{k+1} := \phi_k * \phi$, $k \ge 1$. Observe that for $\phi \in C^1[0,1]$ we have $\phi_k \in C^1[0,1]$,$k \ge 1$, by induction. Hence, if we assume that g is twice continuously differentiable, then so is f_n, $n \ge 1$. Moreover, since $\phi_{k+1}' = \phi(0)\phi_k + \phi' * \phi_k$, $k \ge 1$, we have

$$\begin{aligned}Lf_n = f_n'' &= g'' + g'(0)\sum_{k=1}^n \phi_k + g(0)\sum_{k=1}^n \phi_k' + g'' * \sum_{k=1}^n \phi_k \\ &= g'' + g(0)\phi' + g'(0)\phi_n + g'' * \phi_n \\ &\quad + \big(g'(0) + g(0)\phi(0)\big)\sum_{k=1}^{n-1} \phi_k \\ &\quad + \big(g(0)\phi' + g''\big) * \sum_{k=1}^{n-1} \phi_k.\end{aligned}$$

Therefore, $(f_n'')_{n\ge1}$ converges in $C[0,1]$, for

$$\|\phi_k\|_{C[0,1]} \le \frac{\|\phi\|_{C[0,1]}^k}{(k-1)!}, \quad k \ge 1.$$

Since L is closed, $f = \lim_{n\to\infty} f_n$ is twice continuously differentiable, which completes the proof. □

Proposition 3.2 *For $f \in C[0,1]$, an extension $\widetilde{f}$ that fulfills conditions (A)–(C), listed above, exists and is uniquely determined.*

Proof It suffices to find for all $n \in \mathbb{N}$, functions $g_n, h_n \in C[0,1]$ related to $\widetilde{f}$ as follows:

$$g_n(x) = \widetilde{f}(x+n), \qquad h_n(x) = \widetilde{f}(1-x-n), \quad x \in [0,1]. \tag{3.2}$$

Since we want $\widetilde{f}$ to be well-defined and continuous, these functions must satisfy compatibility conditions:

$$h_{n+1}(0) = h_n(1), \qquad g_{n+1}(0) = g_n(1), \quad n \in \mathbb{N}. \tag{3.3}$$

The proof of [5, Proposition 2.2] shows that condition (B) is satisfied if and only if

$$g_{n+1} + h_{n+1} = g_n + h_n, \quad n \in \mathbb{N}. \tag{3.4}$$

On the other hand, condition (C) holds if and only if

$$\int_t^{1+t} \mathrm{k}_j(y-t)\widetilde{f}(y)\,\mathrm{d}y + \int_{-t}^{1-t} \mathrm{k}_j(y+t)\widetilde{f}(y)\,\mathrm{d}y = 2\int_0^1 \mathrm{k}_j(x)f(x)\,\mathrm{d}x,$$

for $t \ge 0$. Differentiating with respect to t and then writing $t = n + x$, $x \in [0,1]$, we see that this is equivalent to

$$\begin{aligned} g_{n+1}(x) - j\int_{n+x}^{1+n+x} \mathrm{k}_{j-1}(y-n-x)\widetilde{f}(y)\,\mathrm{d}y \\ - h_n(x) + j\int_{-n-x}^{1-n-x} \mathrm{k}_{j-1}(y+n+x)\widetilde{f}(y)\,\mathrm{d}y = 0, \end{aligned}$$

which is satisfied if and only if

$$\begin{aligned} g_{n+1}(x) - j\left[\int_x^1 \mathrm{k}_{j-1}(y-x)g_n(y)\,\mathrm{d}y + \int_0^x k^e_{j-1}(x-y)g_{n+1}(y)\,\mathrm{d}y\right] \\ - h_n(x) + j\left[\int_0^x \mathrm{k}_{j-1}(x-y)h_{n+1}(y)\,\mathrm{d}y + \int_x^1 \mathrm{k}^e_{j-1}(y-x)h_n(y)\,\mathrm{d}y\right] = 0, \end{aligned}$$

for $x \in [0,1]$ and $n \in \mathbb{N}$. Finally, since $h_{n+1} = g_n + h_n - g_{n+1}$ by (3.4), condition (C) holds if and only if

$$\begin{aligned} &g_{n+1} - j(k_{j-1} + \mathrm{k}^e_{j-1}) * g_{n+1} \\ &\quad = j(g^e_n * \mathrm{k}_{j-1})^e - j\mathrm{k}_{j-1} * (g_n + h_n) - j(h^e_n * \mathrm{k}^e_{j-1})^e + h_n, \quad n \in \mathbb{N}. \end{aligned} \tag{3.5}$$

By Lemma 3.1, g_{n+1} is uniquely determined by the pair (g_n, h_n), and by (3.4) so is h_{n+1}.

It remains to prove that for g_n and h_n defined recursively by $h_0 = f^e, g_0 = f$, (3.4) and (3.5), conditions (3.3) are satisfied. To this end, let

$$d_n := jF_{j-1}(h_n^e - g_n), \quad n \in \mathbb{N}.$$

Then, by (3.5),

$$g_{n+1}(0) = h_n(0) - d_n \quad \text{and} \quad g_{n+1}(1) = h_n(1) - d_{n+1}. \tag{3.6}$$

By (3.4) this yields

$$h_{n+1}(0) = g_n(0) + d_n \quad \text{and} \quad h_{n+1}(1) = g_n(1) + d_{n+1}. \tag{3.7}$$

By induction argument, it follows that for all $n \in \mathbb{N}$,

$$h_n(1) - g_n(0) = h_n(0) - g_n(1) = d_n. \tag{3.8}$$

For, the formula is evidently true for $n = 0$ since $d_0 = jF_{j-1}0 = 0$, $h_0(0) = f(1) = g_0(1)$, and $h_0(1) = f(0) = g_0(0)$, and relations (3.6) and (3.7) allow proving the induction step.

Equalities (3.8) in turn show

$$g_{n+1}(0) = h_n(0) - d_n = g_n(1) + d_n - d_n = g_n(1),$$

and

$$h_{n+1}(0) = g_n(0) + d_n = h_n(1) - d_n + d_n = h_n(1),$$

establishing (3.3) and completing the proof. □

Definition 3.3 Let $f \in C[0, 1]$. The function $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ defined in accordance with the rules (3.2), (3.4) and (3.5) is called the *integral extension* of f.

We note that the extension operator

$$E : C[0, 1] \ni f \mapsto Ef := \widetilde{f} \in C(\mathbb{R})$$

is continuous. Let

$$D_j = \{f \in C^2[0, 1] : F_0(f'') = F_j(f'') = 0\}.$$

Lemma 3.4 *Let $f \in D_j$. Then its integral extension Ef is twice continuously differentiable on $(-1, 2)$.*

Proof The case $j = 1$ is proved in [5, Lemma 2.4], hence we restrict ourselves to $j \geq 2$. By (3.4), (3.5), and Lemma 3.1, restrictions of Ef to the intervals $[-1, 0]$, $[0, 1]$, and $[1, 2]$ are twice continuously differentiable. We need to show that

$$g_1'(0) = f'(1), \qquad g_1''(0) = f''(1), \qquad h_1'(0) = -f'(0), \quad \text{and} \quad h_1''(0) = f''(0).$$

Differentiating (3.5) with $n = 0$ we obtain

$$g_1'(0) = 2jf(1) - 2j(j-1)F_{j-2}f - f'(1).$$

Since $f \in D_j$, by the equivalence of conditions (c) and (d) in Proposition 2.2, we see that

$$f'(1) = jf(1) - j(j-1)F_{j-2}f,$$

which proves $g_1'(0) = f'(1)$, the first of the desired equalities. The third equality now follows by $F_0(f'') = f'(1) - f'(0) = 0$ and (3.4) with $n = 0$:

$$h_1'(0) = f'(0) - f'(1) - g_1'(0) = -f'(1).$$

Turning to the second equality we consider the cases $j = 2$ and $j > 2$ separately. If $j = 2$, then (3.5) gives

$$\begin{aligned} g_1''(0) &= jf'(0) + j(j-1)f(0) - j(j-1)(f(0) + f(1)) \\ &\quad - jf'(1) + j(j-1)f(1) + f''(1) \\ &= f''(1). \end{aligned}$$

Similarly, if $j > 2$, then

$$\begin{aligned} &g_1''(0) - jf'(1) + j(j-1)f(1) \\ &\quad = j(j-1)(j-2)F_{j-3}f - jf'(1) + j(j-1)f(1) \\ &\quad\quad - j(j-1)(j-2)F_{j-3}f + f''(1), \end{aligned}$$

hence $g_1''(0) = f''(1)$. Finally, by (3.4) with $n = 0$, it follows that

$$h_1''(0) = f''(0) + f''(1) - g_1''(0) = f''(0),$$

which completes the proof. □

Theorem 3.5 *The abstract Kelvin formula* (3.1) *defines a strongly continuous cosine family* $(C_{\mathrm{mp}}(t))_{t\in\mathbb{R}}$ *on* $C[0,1]$. *This family preserves both functionals* F_0 *and* F_j. *Moreover, the generator* A *of* $(C_{\mathrm{mp}}(t))_{t\in\mathbb{R}}$ *is a member of* $\mathfrak{L}_c$ *and its domain is* D_j.

Proof Let $R : C(\mathbb{R}) \to C[0,1]$ map a member of $C(\mathbb{R})$ to its restriction to $[0,1]$. Then (3.1) takes the form

$$C_{\mathrm{mp}}(t) = RC(t)E, \quad t \in \mathbb{R}. \tag{3.9}$$

By (3.4) and (3.5), a pair (g_{n+1}, h_{n+1}) is obtained from (g_n, h_n) by means of a bounded linear operator mapping $C[0,1] \times C[0,1]$ into itself. Since for any t, $RC(t)Ef$ depends merely on the finite number of such pairs, it follows that $C_{\mathrm{mp}}(t)$ is a bounded linear operator in $C[0,1]$. That the operators $C_{\mathrm{mp}}(t)$ preserve functionals F_0 and F_j is clear by Proposition 3.2.

Fix $f \in C[0,1]$ and $s \in \mathbb{R}$. Clearly, $C(s)Ef$ extends $RC(s)Ef$ and, by the cosine equation for C and the definition of Ef, we have

$$F_i C(t)C(s)Ef = F_i f = F_i RC(s)Ef, \quad i = 0, j, t \in \mathbb{R}.$$

By uniqueness of integral extensions, this shows that $C(s)Ef$ is the integral extension of $RC(s)Ef$:

$$ERC(s)Ef = C(s)Ef, \quad s \in \mathbb{R}.$$

Using this and the cosine equation for C, we check that

$$2C_{\mathrm{mp}}(t)C_{\mathrm{mp}}(s)f = C_{\mathrm{mp}}(t+s)f + C_{\mathrm{mp}}(t-s)f, \quad t, s \in \mathbb{R},$$

i.e., that C_{mp} is a cosine family. This family is strongly continuous, i.e., we have $\lim_{t\to 0} RC(t)Ef = f$ for all $f \in C[0,1]$, since Ef, as restricted to any compact interval, is a uniformly continuous function, and on $[0,1]$ it coincides with f.

Turning to the characterization of the generator: Lemma 3.4 and the Taylor formula imply that for $f \in D_j$,

$$\lim_{t\to 0} \frac{2}{t^2}(C(t)\widetilde{f}(x) - \widetilde{f}(x)) = \widetilde{f}''(x), \quad x \in (-1, 2);$$

the limit is uniform in $x \in [0,1]$ since $\widetilde{f}''$ is uniformly continuous in any compact subinterval of $(-1,2)$. By (3.9) this proves that f belongs to $D(A)$ and we have $Af = f''$.

Finally, we observe that Proposition 2.2 implies $D(A) \subset D_j$, which shows that $D(A) = D_j$, and completes the proof. □

When combined with Proposition 2.2, Theorem 3.5 proves not only existence of the cosine family that preserves moments of order 0 and $j \geq 1$, but also its uniqueness (in the class of cosine families generated by members of $\mathfrak{L}_c$). For, by Proposition 2.2, the domain of the generator of a cosine family preserving these moments is contained in D_j. Since no member of $\mathfrak{L}_c$ is a proper extension of another member, this generator must coincide with the generator described in Theorem 3.5. In particular, we have completed the proof of Theorem 2.1.

We conclude this section with a remark on symmetries in the moments-preserving cosine families. We say that a function $f \in C[0,1]$ is *symmetric* about $\frac{1}{2}$ if $f = f^e$ and similarly, we say that f is *asymmetric* about $\frac{1}{2}$ if $f = -f^e$. By $C_{\mathrm{even}}[0,1]$ and $C_{\mathrm{odd}}[0,1]$ we denote the spaces of symmetric and asymmetric functions, respectively.

In [5, Proposition 3.2] it is proved that in the case $i = 0$, $j = 1$, the moments-preserving cosine family C_{mp} leaves the spaces $C_{\mathrm{even}}[0,1]$ and $C_{\mathrm{odd}}[0,1]$ invariant. This allows decomposition of C_{mp} into ‘smaller’ pieces which are easier to handle. (For example, one of the pieces is the cosine family related to the Neumann boundary conditions.) As we shall see now, such a decomposition is not possible in general. More specifically, for $j \geq 2$ the space $C_{\mathrm{odd}}[0,1]$ is invariant for C_{mp} (the reason for that is formula (3.4) showing that integral extensions of asymmetric functions are asymmetric) but the space $C_{\mathrm{even}}[0,1]$ does not possess this property.

Indeed, suppose that, contrary to our claim, $C_{\mathrm{mp}}C_{\mathrm{even}}[0,1] \subset C_{\mathrm{even}}[0,1]$. If A is the generator of C_{mp}, then the generator of C_{mp} restricted to $C_{\mathrm{even}}[0,1]$ is A_p, the part of A in $C_{\mathrm{even}}[0,1]$. Hence, for $f \in D(A_p)$ the first and the third conditions in (2.3) hold and, f being even, $f'(1) = f'(0) = 0$. Therefore $D(A_p) \subset \operatorname{Ker} H_j$, where H_j is the bounded linear functional on $C_{\mathrm{even}}[0,1]$ given by

$$H_j f = j(j-1)F_{j-2}f - jf(1).$$

Since $D(A_p)$ is dense in $C_{\mathrm{even}}[0,1]$ and $\operatorname{Ker} H_j$ is closed, $\operatorname{Ker} H_j = C_{\mathrm{even}}[0,1]$. This contradicts the fact that for $f \in C_{\mathrm{even}}[0,1]$ given by $f(x) = (x-\frac{1}{2})^2$, $H_j f = -\frac{j-1}{j+1} \neq 0$, $j \geq 2$, completing the proof of the claim.

In this context it is natural to ask whether there is a subspace of $C[0,1]$ that is complementary to $C_{\mathrm{odd}}[0,1]$ and invariant for C_{mp}. However, at present an answer to this question eludes us. In fact, as we have noted above, the requirement of preservation of the first moment forces the cosine family to leave the space $C_{\mathrm{odd}}[0,1]$ invariant, yet in general it is unclear how to relate a moment to be preserved with an invariant subspace for the cosine family.

4 Extensions

The article focuses on non-negative integer moments. However, the main theorem (Theorem 2.1) may be extended to the case of non-negative real moments, as follows.

Because k_i is integrable over $[0,1]$ for real $i > -1$, formula (2.2) defines a bounded linear functional on $C[0,1]$ for all such i. Hence, relation (2.4) remains true for $i > 0$ and Proposition 2.2 may be extended to real $i \geq 0$ by writing (2.3) in the form:

$$\begin{array}{ll} f'(0) = f'(1) & \text{if } i = 0, \\ f'(1) = iF_{i-1}f' & \text{if } i \in (0,1), \\ f'(1) = f(1) - f(0) & \text{if } i = 1, \\ if(1) = i(i-1)F_{i-2}f + f'(1) & \text{if } i > 1. \end{array}$$

Next, for real $i, j \geq 1$, Corollary 2.3 remains valid, but for i or j in $(-1,1)$, it does not (see below). Finally, for real positive j, unless $j = 1$ or $j = 2$, the argument used in Lemma 3.4 requires existence of $F_{j-3}f$, and does not work for $j \in (0,1) \cup (1,2)$. To recapitulate, we have the following theorem.

Theorem 4.1 *Let $i, j \geq 0$ be two real numbers with $i < j$.*

(a) *If $i = 0$, and $j = 1$ or $j \geq 2$, then there is exactly one $A \in \mathfrak{L}_c$ such that the related cosine family preserves F_i and F_j.*
(b) *If $i \geq 1$, then there is no $A \in \mathfrak{L}_s$ such that the related semigroup preserves F_i and F_j.*

If $C[0,1]$ is the space of complex functions, similar extensions are possible for moments of complex order i with $\Re i \geq 0$.

Except for the claim that $D_{i,j}$ of Corollary 2.3 is dense for i or j in $(-1, 1)$, all the statements presented above are proved precisely as in the case where i and j are integers. Hence, we restrict ourselves to showing the former.

First, let $-1 < i < j < 1$. Given $C > 0$ let $\alpha, \beta \in [-C, C]$. Observe that there is a constant $K = K(i, j) > 0$ and a function $f_{\alpha,\beta,C} \in C[0, 1]$, such that $F_i f_{\alpha,\beta,C} = \alpha$, $F_j f_{\alpha,\beta,C} = \beta$ and $\|f_{\alpha,\beta,C}\|_{C[0,1]} \le CK$. Indeed, let $f_{\alpha,\beta,C}(x) = (x-1)(ax+b)$, where real numbers a and b are chosen so that

$$\begin{cases} F_i f_{\alpha,\beta,C} = \alpha, \\ F_j f_{\alpha,\beta,C} = \beta, \end{cases} \quad \text{i.e.,} \quad \begin{cases} (i+1)a + (i+3)b = -\alpha P(i), \\ (j+1)a + (j+3)b = -\beta P(j), \end{cases}$$

where $P(x) = (x+1)(x+2)(x+3)$. Precisely, we have

$$a = \frac{(j+3)P(i)\alpha - (i+3)P(j)\beta}{2(j-i)}, \qquad b = \frac{(i+1)P(j)\beta - (j+1)P(i)\alpha}{2(j-i)}.$$

Then, for $g_{\alpha,\beta,C}$ defined by $g_{\alpha,\beta,C}(x) = \int_1^x \int_1^y f_{\alpha,\beta,C}(z)\,\mathrm{d}z\,\mathrm{d}y$, $x \in [0, 1]$, we obtain $g_{\alpha,\beta,C}(1) = g'_{\alpha,\beta,C}(1) = g''_{\alpha,\beta,C}(1) = 0$, $F_i g''_{\alpha,\beta,C} = \alpha$, $F_j g''_{\alpha,\beta,C} = \beta$, and $\|g_{\alpha,\beta,C}\|_{C[0,1]} < CK$.

Given $f \in C^2[0, 1]$, let $c > 0$ be chosen so that $c + j - 1 < 0$. For $k > 1$ we define

$$f_k(x) = f(x) - \frac{1}{k^c} h_k(kx), \quad x \in [0, 1] \tag{4.1}$$

where

$$h_k(x) = \begin{cases} g_{\alpha,\beta,C}(x), & x \in [0, 1], \\ 0, & x \ge 1, \end{cases}$$

for $C := \max(|F_i f''|, |F_j f''|)$, $\alpha := k^{c+i-1} F_i f''$, and $\beta := k^{c+j-1} F_j f''$. Then

$$F_i f_k'' = F_i f'' - \frac{1}{k^c} \int_0^1 k^2 h_k''(kx) x^i\,\mathrm{d}x = F_i f'' - \frac{1}{k^{c+i-1}} \int_0^1 h_k''(y) y^i\,\mathrm{d}y = 0,$$

and similarly $F_j f_k'' = 0$, that is $f_k \in D_{i,j}$. Furthermore $\|f - f_k\|_{C[0,1]} = \frac{1}{k^c}\|h_k\|_{C[0,1]} < \frac{1}{k^c} CK$, where C depends merely on f, and K depends on i and j. Since $k > 1$ is arbitrary, this shows that $D_{i,j}$ is dense in $C^2[0, 1]$, and the claim follows in this case.

For the proof of the other case, where $i \in (-1, 1)$ and $j \ge 1$, we claim first that for $f \in C[0, 1]$ and $k > 1$, there is $f_j \in C^2[0, 1]$ such that $F_j f_j'' = 0$ and $\|f - f_j\|_{C[0,1]} < \frac{1}{k}$. Since $C^2[0, 1]$ is dense in $C[0, 1]$, it suffices to show this for $f \in C^2[0, 1]$. Given $k > 1$ let $g_a \in C^2[0, 1]$, $a \ge 1$, be defined by $g_a(x) = \frac{1}{k} x^a$. Then $\|g_a\|_{C[0,1]} < \frac{1}{k}$ and $F_j g_a'' = \frac{1}{k}\frac{a(a-1)}{a-1+j}$. Hence, $a \mapsto F_j g_a'' \in \mathbb{R}^+$ is a continuous function satisfying $F_j g_1'' = 0$ and $\lim_{a\to\infty} F_j g_a'' = +\infty$. Thus, by the intermediate value theorem there exists $a_0 \ge 1$ such that $F_j g_{a_0}'' = |F_j f''|$. Finally, let $f_j = f - \operatorname{sgn}(F_j f'')\, g_{a_0}$. Then $f_j \in C^2[0, 1]$, $\|f - f_j\|_{C[0,1]} \le \frac{1}{k}$ and $F_j f_j'' = 0$, as desired.

To complete the proof, we fix $k > 1$. Defining f_k via (4.1) with $c > 0$, $c+i-1 < 0$, $C := |F_i f_j''|$, $\alpha = k^{c+i-1} F_i f_j''$, and $\beta = 0$, and with f replaced by f_j, we obtain $f_k \in C^2[0, 1]$ and $F_i f_k'' = 0$ as before, and

$$F_j f_k'' = F_j f_j'' - k^{1-i-c} \int_0^k h_k''(y) y^j \, \mathrm{d}y = F_j f_j'' = 0,$$

since $\beta = 0$. Thus $f_k \in D_{i,j}$, and $\|f - f_k\|_{C[0,1]} < \frac{1}{k} C K$, where K is defined as before. This completes the proof of the claim.

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