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RESEARCH ARTICLE

Rectangular group congruences on a semigroup

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Abstract We study rectangular group congruences on an arbitrary semigroup. Some of our results are an extension of the results obtained by Masat (Proc. Am. Math. Soc. 50:107–114, 1975). We show that each rectangular group congruence on a semigroup S is the intersection of a group congruence and a matrix congruence and vice versa, and this expression is unique, when S is E-inversive. Finally, we prove that every rectangular group congruence on an E-inversive semigroup is uniquely determined by its kernel and trace.

Keywords Rectangular group congruence · Group congruence · Matrix congruence

1 Introduction and preliminaries

A groupoid S is called a *left* [*right*] *zero semigroup* if it satisfies the identity $xy = x$ [$xy = y$]. Further, by a *rectangular band* we shall mean the direct product of a left zero and a right zero semigroup. Moreover, a semigroup S is said to be a *rectangular group* if it is isomorphic to the direct product $G \times M$ of a group G and a rectangular band M.

Let $\mathcal{C}$ be a class of semigroups. We say that a congruence ρ on a semigroup S is a $\mathcal{C}$-*congruence* if $S/\rho \in \mathcal{C}$. For example, if $\mathcal{C}$ is the class of groups, then ρ is called a *group* congruence on S if S/ρ is a group. In way of an exception, a congruence ρ on a semigroup S is said to be a *matrix* congruence if S/ρ is a rectangular band. Note that every left [right] zero semigroup is a rectangular band, so every left [right] zero congruence on S is a matrix congruence. Also, clearly the least matrix congruence

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on any semigroup S exists. Denote it by ψ. Furthermore, every group congruence and every matrix congruence is a rectangular group congruence. Hence we say that a rectangular group congruence is *proper* if it is neither a group nor a matrix congruence. We first give necessary and sufficient conditions on a semigroup S in order that it will have a proper rectangular group congruence. Furthermore, we show that every rectangular group congruence on S is the intersection of a group congruence and a matrix congruence. In addition, if S is E-inversive, then this expression is unique. Moreover, we prove that each rectangular group congruence on an E-inversive semigroup is uniquely determined by its kernel and trace. Before we start our study, we recall some definitions.

Let S be a semigroup and $a \in S$. The set $W(a) = \{x \in S : x = xax\}$ is called the set of all *weak inverses* of a and so the elements of $W(a)$ will be called *weak inverse elements* of a. A semigroup S is said to be *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$, where E_S (or briefly E) is the set of idempotents of S (more generally, if $A \subseteq S$, then E_A denotes the set of idempotents of A). It is easy to see that a semigroup S is E-inversive if and only if $W(a)$ is nonempty for all $a \in S$. Hence if S is E-inversive, then for every $a \in S$ there is $x \in S$ such that $ax, xa \in E_S$ [7, 8]. Further, by $Reg(S)$ we shall mean the set of *regular elements* of S (an element a of S is called *regular* if $a \in aSa$) and by $V(a) = \{x \in S : a = axa, x = xax\}$ the set of all *inverse elements* of a. It is well known that an element a of S is regular if and only if $V(a) \neq \emptyset$, so a semigroup S is regular if and only if $V(a) \neq \emptyset$ for every $a \in S$. Finally, a regular semigroup S is said to be *orthodox* if E_S forms a subsemigroup of S.

The following result seems to belong to folklore.

Result 1.1 *The following conditions concerning a semigroup S are equivalent*:

(i) *S is a rectangular band*;
(ii) *S is nowhere commutative*, *i.e.*, $\forall a, b \in S\ [ab = ba \Longrightarrow a = b]$;
(iii) $\forall a, b \in S\ [aba = a]$;
(iv) $\forall a, b, c \in S\ [a^2 = a, abc = ac]$.

Recall from [9] that a nonempty subset A of a semigroup S is called *left* [*right*] *dense* if the condition $ab \in A$ implies that $a \in A$ [$b \in A$] for all $a, b \in S$. Further, A is said to be *quasi dense* if the following two conditions hold:

(i) $\forall a \in S\ [a \in A \iff a^2 \in A]$;
(ii) $\forall a, b \in S\ [ab \in A \iff aSb \subseteq A]$.

Finally, we say that A is a *quasi ideal* of S if $AS \cap SA \subseteq A$. For the connections between left [right] zero, matrix congruences on a semigroup S and left dense right [right dense left] ideals, quasi dense subsemigroups of S (respectively), the reader is referred to [9]. We note only some results of [9]. Firstly, denote by X the set of all left dense right ideals of a semigroup S and all right dense left ideals of a semigroup S with the empty set included and S excluded. Let 2^X be a family of all subsets of X and $\mathcal{MC}(S)$ be the set of all matrix congruences on S. Define the map $\phi : 2^X \to \mathcal{MC}(S)$ by $\mathcal{X}\phi = \rho_{\mathcal{X}}$ ($\mathcal{X} \in 2^X$), where

$$\rho_{\mathcal{X}} = \big\{(a, b) \in S \times S : \forall A \in \mathcal{X}\ [a, b \in A \text{ or } a, b \notin A]\big\}.$$

Result 1.2 (Theorem 5 [9]) *The map* ϕ *is antitone* (*i.e.*, $\mathcal{X} \subseteq \mathcal{Y} \Longrightarrow \rho_{\mathcal{Y}} \subseteq \rho_{\mathcal{X}}$) *and maps* 2^X *onto* $\mathcal{MC}(S)$.

Result 1.3 (Corollary to Theorem 5 [9]) *The relation* ρ_X *is the least matrix congruence on a semigroup* S. *Moreover, we may replace* (*in the present result*) *the set* X *by the set* Y *of all quasi dense subsemigroups of* S.

Result 1.4 (A part of Proposition 4, Theorem 9 [9]) *The following conditions concerning a congruence* ρ *on a semigroup* S *are equivalent*:

(i) ρ *is a matrix congruence on* S;
(ii) *every* ρ-*class of* S *is a quasi dense subsemigroup of* S;
(iii) *every* ρ-*class of* S *is a quasi ideal of* S.

Conversely, a subsemigroup A *of* S *is quasi dense, when* A *is a matrix of some* ψ-*classes of* S. *Thus* A *is quasi dense if and only if* A *is a* ρ-*class of some matrix congruence* ρ *on* S.

Result 1.5 (Theorem 14 [9]) *Let* S *be a matrix of semigroups* $S_{\imath\lambda}$, *where* $\imath \in I$, $\lambda \in \Lambda$, *such that every* $S_{\imath\lambda}$ *has an identity element* $e_{\imath\lambda}$ *and the set* M (*say*) *of elements* $e_{\imath\lambda}$ ($\imath \in I$, $\lambda \in \Lambda$) *forms a subsemigroup of* S. *Then* M *is a rectangular band. Further,* $S_{\imath\lambda} \cong S_{\jmath\mu}$ *for all* $\imath, \jmath \in I, \lambda, \mu \in \Lambda$ *and if we suppose that* $1 \in I, \Lambda$, *then* S *is isomorphic to the direct product* $M \times S_{11}$ *of a rectangular band* M *and a semigroup* S_{11}. *Moreover, the semigroups* $S_{\imath\lambda}$ *are precisely the* ψ-*classes of* S *and* $S_{\imath\lambda} = e_{\imath\lambda} S_{\imath\lambda} e_{\imath\lambda} = e_{\imath\lambda} S e_{\imath\lambda}$ *for all* $\imath \in I, \lambda \in \Lambda$.

Notice that if S is a rectangular group (that is, $S \cong M \times G$, where M is a rectangular band and G is a group), then we shall write rather $S = M \times G$ than $S \cong M \times G$. The following theorem is known but for example: Masat considered in [5, 6] a regular semigroup S such that E_S forms a rectangular band, and he did not know that S is a rectangular group, and so we include a simple proof for the completeness. Green's relations on a semigroup S are denoted by $\mathcal{L}^S$, $\mathcal{R}^S$, $\mathcal{H}^S$, $\mathcal{D}^S$ and $\mathcal{J}^S$. For undefined terms the reader is referred to the books [3, 4].

Theorem 1.6 *The following conditions concerning a semigroup* S *are equivalent*:

(i) S *is a rectangular group*;
(ii) S *is completely simple and orthodox*;
(iii) S *is completely regular and satisfies the identity*: $x^{-1}yy^{-1}x = x^{-1}x$;
(iv) S *is regular and* E_S *forms a rectangular band.*

Consequently, if S *is a rectangular group, then* $S \cong E_S \times \mathcal{H}_e = E_S \times eSe$ *for some* (*all*) $e \in E_S$.

Proof (ii) $\Longrightarrow$ (i). If S is completely simple, then S is a matrix of groups $\mathcal{H}_e$ ($e \in E_S$), since Lemma III.2.4 [4] implies that $\mathcal{H}$ is a matrix congruence on S, so $\mathcal{H} = \psi$. Clearly, every $\mathcal{H}_e$ has an identity element e. Also, $(efe, e) \in \mathcal{H}$ for all $e, f \in E_S$, that is, $e = efe$ for all $e, f \in E_S$ (since $efe \in E_S$). Hence E_S is a rectangular band. Thus $S \cong E_S \times \mathcal{H}_e$ for some (all) $e \in E_S$ (by Result 1.5).

(i) $\Longrightarrow$ (iii). Let $S = M \times G$, where M is a rectangular band and G is a group (with an identity 1). Define the mapping $^{-1} : S \to S$ by $(a, g)^{-1} = (a, g^{-1})$ for all $(a, g) \in S$. One can easily verify that $(S, \cdot, ^{-1})$ is completely regular, that is, $(x^{-1})^{-1} = x, xx^{-1} = x^{-1}x$ and $xx^{-1}x = x$ for every $x \in S$. Further, suppose that $x = (a, g), y = (b, h) \in S$. Then

$$x^{-1}yy^{-1}x = (ab^2a, 1) = (a^2, 1) = (a, g^{-1})(a, g) = x^{-1}x.$$

(iii) $\Longrightarrow$ (iv). Let $e \in E_S$. Since $ee^{-1} = e^{-1}e$, $(e^{-1})^{-1} = e$, then

$$e = ee^{-1}e = e^{-1}e = (e^{-1}e)^2 = (e^{-1}e)(ee^{-1}) = e^{-1}ee^{-1} = e^{-1}.$$

Hence $e^{-1}ff^{-1}e = e^{-1}e$, i.e., $efe = e$ for all $e, f \in E_S$. Thus E_S is a rectangular band. Consequently, the condition (iv) holds.

(iv) $\Longrightarrow$ (ii). Clearly, each idempotent of S is primitive and $\mathcal{D}^{E_S} = E_S \times E_S$. Since S is regular, then every element of S is $\mathcal{D}$-related with some of its idempotent. It follows that $\mathcal{D}^S = \mathcal{J}^S = S \times S$. Thus S is completely simple and orthodox. □

By the *trace* $\operatorname{tr} \rho$ of a relation ρ on a semigroup S we shall mean the restriction of ρ to the set E_S.

The following result will be useful in the proof of Theorems 2.2(iii), 2.5, 2.6.

Result 1.7 (Corollary 2.7 [1]) *If ρ is a matrix congruence on an E-inversive semigroup S, then every ρ-class of S is E-inversive. Also, every matrix congruence on an E-inversive semigroup is uniquely determined by its trace.*

Further, some preliminaries about group congruences on a semigroup S are needed. A subset A of S is called (respectively) *full*; *reflexive* and *dense* if $E_S \subseteq A$; $\forall a, b \in S$ $[ab \in A \Longrightarrow ba \in A]$ and $\forall s \in S\ \exists x, y \in S$ $[sx, ys \in A]$. Also, define the *closure operator* ω on S by $A\omega = \{s \in S : \exists a \in A\ [as \in A]\}$ $(A \subseteq S)$. We shall say that $A \subseteq S$ is *closed* (in S) if $A\omega = A$. Finally, a subsemigroup N of a semigroup S is said to be *normal* if it is full, dense, reflexive and closed (if N is normal, then we shall write $N \lhd S$). Moreover, if a subsemigroup of S is full, dense and reflexive, then it is called *seminormal*.

By the *kernel* $\ker \rho$ of a congruence ρ on a semigroup S we shall mean the set $\{x \in S : (x, x^2) \in \rho\}$.

The following two results follow directly from the definition of the group.

Lemma 1.8 *Let ρ be a group congruence on a semigroup S. Then* $\ker \rho \lhd S$.

Lemma 1.9 *Let ρ_1, ρ_2 be group congruences on a semigroup S. Then $\rho_1 \subset \rho_2$ if and only if* $\ker \rho_1 \subset \ker \rho_2$.

Let B be a nonempty subset of a semigroup S. Consider four relations on S:

$$\rho_{1,B} = \{(a, b) \in S \times S : \exists x \in S\ [ax, bx \in B]\};$$
$$\rho_{2,B} = \{(a, b) \in S \times S : \exists x, y \in B\ [ax = yb]\};$$
$$\rho_{3,B} = \{(a, b) \in S \times S : \exists x \in S\ [xa, xb \in B]\};$$
$$\rho_{4,B} = \{(a, b) \in S \times S : \exists x, y \in B\ [xa = by]\}.$$

Lemma 1.10 [2] *Let a subsemigroup B of a semigroup S be dense and reflexive. Then* $\rho_{1,B} = \rho_{2,B} = \rho_{3,B} = \rho_{4,B}$.

If B is a seminormal subsemigroup of S, then we denote the above four relations by ρ_B.

The following theorem says that there exists an inclusion-preserving bijection between the set of all normal subsemigroups of a semigroup S and the set of all group congruences on S.

Theorem 1.11 [2] *Let B be a seminormal subsemigroup of a semigroup S. Then the relation ρ_B is a group congruence on S. Moreover, $B \subseteq B\omega = \ker \rho_B$. If B is normal, then $B = \ker \rho_B$.*

Conversely, if ρ is a group congruence on S, then there exists a normal subsemigroup N of S such that $\rho = \rho_N$ (*in fact*, $N = \ker \rho$). *Thus there exists an inclusion-preserving bijection between the set of all normal subsemigroups of S and the set of all group congruences on S.*

Finally, the following remark will be useful.

Remark 1 Denote by σ the least group congruence on a semigroup (if it exists). One can easily seen that if S is an E-inversive semigroup (and so E_S is dense), then there exists the least normal subsemigroup of S. In the light of the above theorem, every E-inversive semigroup possesses a least group congruence.

2 Rectangular group congruences

The following theorem gives necessary and sufficient conditions on a semigroup S in order that it has a proper rectangular group congruence. (Notice that a normal subsemigroup N of S is called *proper* if $N \neq S$.)

Theorem 2.1 *Let S be a semigroup. The following conditions are equivalent*:

(i) *there exists a proper rectangular group congruence on S*;
(ii) *S is a disjoint union of two or more quasi dense subsemigroups of S and contains a proper normal subsemigroup of S*;
(iii) *there exists a non-universal group and a non-universal matrix congruence on S.*

Proof (i) $\Longrightarrow$ (ii). Let ρ be a proper rectangular group congruence on S, say S/ρ is equal $M \times G$, where M is a rectangular band, G is a group (with identity 1). Note that $M \cong E_{M\times G} = \{(m, 1) : m \in M\}$. Further, for all $m \in M$, define Q_m to be the preimage of $\{m\} \times G$ by the canonical epimorphism $\rho^{\natural}$ from S onto S/ρ. It follows easily from Result 1.1(iv) that $\{m\} \times G$ is a quasi dense subsemigroup of $M \times G$. Thus the preimage of $\{m\} \times G$ by $\rho^{\natural}$ is also such a subsemigroup of S (by $M \cong E_{M\times G}$). Since ρ is not a group congruence, then $|M| > 1$, and so S has a proper matrix congruence (Result 1.3). Hence S is a disjoint union of two or more quasi dense subsemigroups of S, see Result 1.4 (notice that $S = \bigcup\{Q_m : m \in M\}$, where the union is disjoint, and this decomposition of S induced, by the first part of Result 1.4, a matrix congruence on S). Let $N = \ker\rho$. Then $a \in N$ if and only if $a\rho \in E_{M\times G}$, that is, if and only if $a\rho \in M \times \{1\}$. Clearly, N is a full subsemigroup of S. Also, $M \times \{1\}$ is reflexive in $M \times G$, so N is reflexive in S. Furthermore, N is dense, since S/ρ is E-inversive. Finally, N is closed in S, since $M \times \{1\}$ is closed in $M \times G$. Consequently, N is a normal subsemigroup of S and since S/ρ is not a rectangular band, then N is proper.

(ii) $\Longrightarrow$ (iii). This follows from Result 1.4 and Theorem 1.11.

(iii) $\Longrightarrow$ (i). Let ρ_1 be a non-universal matrix congruence on S and ρ_2 be a non-universal group congruence on S. We show that $\rho = \rho_1 \cap \rho_2$ is a proper rectangular group congruence on a semigroup S. Indeed, let $a \in S$. Since S/ρ_2 is regular, then $(axa, a) \in \rho_2$ for some $x \in S$, so $(axa, a) \in \rho$. Therefore S/ρ is regular. Clearly, $x\rho_2$, where $x \in \ker\rho_2$ is an identity of the group S/ρ_2 and so $(xyx, x) \in \rho_2$ for all $x, y \in \ker\rho_2$. Hence $(xyx, x) \in \rho$ for all $x, y \in \ker\rho_2$. On the other hand, if $x\rho \in E_{S/\rho}$, then $x \in \ker\rho_2$. It follows that $E_{S/\rho}$ forms a rectangular band, therefore, S/ρ is a rectangular group (Theorem 1.6(iv)). Finally, suppose by way of contradiction that ρ is a matrix congruence on S, that is, $(aba, a) \in \rho$ for all $a, b \in S$. Then $(aba, a) \in \rho_2$ for all $a, b \in S$. Hence S/ρ_2 must be a trivial group. Thus $\rho_2 = S \times S$, a contradiction from the assumption that ρ_2 is a non-universal congruence on S. Similarly, if ρ is a group congruence, then ρ_1 is a group congruence (since $\rho \subseteq \rho_1$), so S/ρ_1 must be trivial. Hence ρ_1 is the universal relation, but this is impossible. Consequently, ρ is a proper rectangular group congruence on S. □

We have just seen that the intersection of a group congruence on a semigroup S and a matrix congruence on S is a rectangular group congruence on S. Conversely, the part (i) of the following theorem (together with Theorem 2.1) implies that any rectangular group congruence on S can be expressed in this way.

Theorem 2.2 *Let ρ be a rectangular group congruence on a semigroup S (and so $S/\rho = M \times G$, where M is a rectangular band, G is a group). Also, let Q_m $(m \in M)$ be the preimage of $\{m\} \times G$ by the canonical epimorphism $\rho^\natural$ from S onto S/ρ, and put $N = \{s \in S : s\rho \in E_{M\times G}\}$. Moreover, denote by υ the matrix congruence on S, induced by the partition $\{Q_m : m \in M\}$ of S (see the proof of* "(i) $\Longrightarrow$ (ii)" *in Theorem* 2.1). *Then*:

(i) $\rho = \upsilon \cap \rho_N$;
(ii) $S/\rho \cong S/\upsilon \times S/\rho_N$.

If in addition S is E-inversive, then:

(iii) $\forall m \in M\ [N \cap Q_m \lhd Q_m]$;
(iv) $\forall m \in M\ [S/\rho_N \cong Q_m/\rho_{(N\cap Q_m)}]$.

Proof Firstly, notice that N is the preimage of $M \times \{1_G\}$ by $\rho^\natural$. Secondly, every Q_m is a quasi dense subsemigroup of S (see Result 1.4). Also, if S is E-inversive, then each Q_m is an E-inversive subsemigroup of S (Result 1.7).

(i). Let $(a, b) \in \rho$ and $a\rho = (m, g)$, where $(m, g) \in M \times G$. Take $x = (m, g^{-1})$, where g^{-1} is a group inverse of g in G. Then clearly $xa, xb \in N$ and so $(a, b) \in \rho_N$ (see Remark 1). Also, $a\rho = (m, g) = b\rho \in \{m\} \times G$. Hence $a, b \in Q_m$, so $(a, b) \in \upsilon$.

Thus $(a, b) \in \upsilon \cap \rho_N$. Consequently, $\rho \subset \upsilon \cap \rho_N$. Conversely, let $(a, b) \in \upsilon \cap \rho_N$. Then $a\rho, b\rho \in \{m\} \times G$ ($a\rho = (m, g_1)$, $b\rho = (m, g_2)$), $xa, xb \in N$ for some $m \in M$ and $x \in Q_n$, where $n \in M$ (say $x\rho = (n, g)$), and so $(xa)\rho = (nm, gg_1)$. On the other hand, $(xa)\rho \in M \times \{1_G\}$. Hence $g_1 = g^{-1}$. We may equally well show that $g_2 = g^{-1}$. Thus $(a, b) \in \rho$. Consequently, $\rho = \upsilon \cap \rho_N$, as exactly required.

(ii). Indeed, $\rho = \upsilon \cap \rho_N$ by (i). Define the mapping $\phi : S/\rho \to S/\upsilon \times S/\rho_N$ by $(a\rho)\phi = (a\upsilon, a\rho_N)$ $(a \in S)$. Clearly, ϕ is a monomorphism. We show that ϕ is surjective. Let $(a\upsilon, b\rho_N) \in S/\upsilon \times S/\rho_N$. Then $a \in Q_m$, where $m \in M$. Take any element $n \in N \cap Q_m$. Then

$$\big((nbn)\rho\big)\phi = \big((nbn)\upsilon, (nbn)\rho_N\big) = (n\upsilon, b\rho_N) = (a\upsilon, b\rho_N).$$

(iii). Let $m \in M$. Put $N_m = N \cap Q_m$. Evidently, N_m is a full, reflexive and closed subsemigroup of Q_m (even if S is an arbitrary semigroup). By Result 1.7, N_m is dense in Q_m. Thus $N_m \lhd Q_m$.

(iv). Let $m \in M$. Define the map $\phi : Q_m/\rho_{N_m} \to S/\rho_N$ by $(a\rho_{N_m})\phi = a\rho_N$ $(a \in Q_m)$. Clearly, ϕ is a well-defined homomorphism. Furthermore, if $a \in S$ and $n \in N_m \subseteq N$, then $nan \in Q_m$ and $((nan)\rho_{N_m})\phi = (nan)\rho_N = a\rho_N$. Thus ϕ is surjective. Finally, we show that ϕ is injective. Let $a, b \in Q_m$, $(a\rho_{N_m})\phi = (b\rho_{N_m})\phi$. Then $(a, b) \in \rho_N$, so ax, $bx \in N$ for some $x \in S$. Hence for every $n \in N_m \subseteq N$, anx, $bnx \in N$. Thus $n(anx)n \in N \cap (Q_m N Q_m) \subseteq N \cap Q_m = N_m$ and similarly: $n(bnx)n \in N_m$. Since $na, nb, nxn \in Q_m$, then $(na, nb) \in \rho_{N_m}$, and so $(a, b) \in \rho_{N_m}$, because $n \in N_m = \ker \rho_{N_m}$. □

Corollary 2.3 *If the least group congruence exists on a semigroup S, then the relation $\psi \cap \sigma$ is the least rectangular group congruence on S. In particular, in any E-inversive semigroup, $\psi \cap \sigma$ is the least rectangular group congruence.*

Remark 2 If S is not E-inversive, then the least rectangular group congruence on a semigroup S may not exist. Indeed, consider the additive semigroup of non-negative integers $\mathbb{N}$. It is well known that every group congruence on $\mathbb{N}$ is of the following form: $\rho_n = \{(k, l) \in \mathbb{N} \times \mathbb{N} : n|(k - l)\}$ $(n > 0)$. Further, since $\mathbb{N}$ has identity, then the least matrix congruence on $\mathbb{N}$ is the universal relation, so any rectangular group congruence on $\mathbb{N}$ is a group congruence (Theorem 2.2(i)). Consequently, $\mathbb{N}$ has no least rectangular group congruence.

Let $\mathcal{C}$ be a class of semigroups which is closed under homomorphic images. Note that if the least $\mathcal{C}$-congruence $\rho_{\mathcal{C}}$ on a semigroup S exists, then the interval $[\rho_{\mathcal{C}}, S \times S]$ consists of all $\mathcal{C}$-congruences on S and it is a complete sublattice of the complete lattice $\mathcal{C}(S)$ of congruences on S. Evidently, the class of all groups [rectangular bands] is closed under homomorphic images. Using Theorem 1.6(iv) one can prove without difficulty that the class of all rectangular groups has this property. Denote by θ the least rectangular group congruence on an E-inversive semigroup. In particular, the intervals $[\psi, S \times S]$, $[\sigma, S \times S]$, $[\theta, S \times S]$ consist of all matrix, group, rectangular group congruences on an E-inversive semigroup S, respectively, and they are complete sublattices of $\mathcal{C}(S)$. Denote them by $\mathcal{MC}(S)$, $\mathcal{GC}(S)$, $\mathcal{RGC}(S)$, respectively. Clearly, the direct product $\mathcal{MC}(S) \times \mathcal{GC}(S)$ is a complete sublattice of $\mathcal{C}(S) \times \mathcal{C}(S)$ (see [3, p. 37]).

For terminology and elementary facts about lattices the reader is referred to the book [10, Sect. I.2]. The following simple result will be useful (see Lemma I.2.8 and Exercise I.2.15(iii) in [10]).

Result 2.4 *If ϕ is an order isomorphism of a lattice L onto a lattice M, then ϕ is a lattice isomorphism. Moreover, every lattice isomorphism of complete lattices is a complete lattice isomorphism.*

We show that each rectangular group congruence on an E-inversive semigroup can be expressed as the unique intersection of a group and a matrix congruence.

Theorem 2.5 *Every rectangular group congruence on an E-inversive semigroup S is of the form $\upsilon \cap \rho_N$, where υ is a matrix congruence on S, $N \lhd S$, and this expression is unique.*

Moreover, there exists an inclusion-preserving bijection ϕ between the complete lattice $\mathcal{MC}(S) \times \mathcal{GC}(S)$ and the complete lattice $\mathcal{RGC}(S)$. In fact, ϕ is defined by:

$$(\upsilon, \rho_N)\phi = \upsilon \cap \rho_N$$

for every $(\upsilon, \rho_N) \in \mathcal{MC}(S) \times \mathcal{GC}(S)$. Furthermore, ϕ^{-1} is an inclusion-preserving bijection (by the proof of Theorem 2.2(i)), *so ϕ is an order isomorphism of the complete lattice $\mathcal{MC}(S) \times \mathcal{GC}(S)$ onto the complete lattice $\mathcal{RGC}(S)$. Consequently, ϕ is a complete lattice isomorphism between the lattices $\mathcal{MC}(S) \times \mathcal{GC}(S)$ and $\mathcal{RGC}(S)$, respectively.*

Proof Let ρ be a rectangular group congruence on S. Then ρ is the intersection of some matrix and some group congruence on S (Theorem 2.2(i)). Next, suppose that $\rho = \upsilon_1 \cap \rho_{N_1} = \upsilon_2 \cap \rho_{N_2}$, where υ_i is a matrix congruence on S, $N_i \lhd S$ $(i = 1, 2)$. Let $(a, b) \in \upsilon_1$. Since $\upsilon_1 \cap \upsilon_2$ is a matrix congruence on S, then there are idempotents e, f of S such that $(a, e) \in \upsilon_1 \cap \upsilon_2$, $(e, f) \in \rho_{N_1}$, $(f, b) \in \upsilon_1 \cap \upsilon_2$ (Result 1.7), so $(e, f) \in \upsilon_1 \cap \rho_{N_1} = \upsilon_2 \cap \rho_{N_2} \subseteq \upsilon_2$. Hence $(a, b) \in \upsilon_2$, i.e., $\upsilon_1 \subset \upsilon_2$. We may equally well show the opposite inclusion. Put $\upsilon_1 = \upsilon_2 = \upsilon$, so that $\rho = \upsilon \cap \rho_{N_1} = \upsilon \cap \rho_{N_2}$. If $(a, b) \in \rho_{N_1}$, then $(aab, abb) \in \upsilon \cap \rho_{N_1} \subset \rho_{N_2}$, so $(a, b) \in \rho_{N_2}$ (by cancellation). Hence $\rho_{N_1} \subset \rho_{N_2}$. By symmetry, $\rho_{N_2} \subset \rho_{N_1}$. Thus $\rho_{N_1} = \rho_{N_2}$, as exactly required.

The second part of the theorem follows directly from the above considerations and Result 2.4. □

Finally, from Result 1.7 we obtain the following theorem.

Theorem 2.6 *Every rectangular group congruence on an E-inversive semigroup S is uniquely determined by its kernel and trace.*

Proof Let $\rho_1, \rho_2 \in \mathcal{GC}(S)$, $\upsilon_1, \upsilon_2 \in \mathcal{MC}(S)$ be such that $\ker(\upsilon_1 \cap \rho_1) \subset \ker(\upsilon_2 \cap \rho_2)$ and $\operatorname{tr}(\upsilon_1 \cap \rho_1) \subset \operatorname{tr}(\upsilon_2 \cap \rho_2)$. Then

$$\ker(\upsilon_1 \cap \rho_1) = \ker \upsilon_1 \cap \ker \rho_1 = S \cap \ker \rho_1 = \ker \rho_1 \subset \ker \rho_2.$$

In the light of Lemma 1.9, $\rho_1 \subset \rho_2$. Similarly, we obtain that $\operatorname{tr} \upsilon_1 \subset \operatorname{tr} \upsilon_2$. Hence $\upsilon_1 \subset \upsilon_2$ (this follows from the proof of Result 1.7, see [1]). Thus $\upsilon_1 \cap \rho_1 \subset \upsilon_2 \cap \rho_2$. This implies the thesis of the theorem. □

References

1. Gigoń, R.S.: Some results on E-inversive semigroups. Quasigroups Relat. Syst. **20**, 53–60 (2012)
2. Gigoń, R.S.: Congruences and group congruences on a semigroup. Semigroup Forum. doi:10.1007/s00233-012-9425-z
3. Howie, J.M.: Fundamentals of Semigroup Theory. Oxford University Press, New York (1995)
4. Howie, J.M.: An Introduction to Semigroup Theory. Academic Press, London (1984)
5. Masat, F.E.: Right group congruences on a semigroup. Proc. Am. Math. Soc. **50**, 107–114 (1975)
6. Masat, F.E.: Right group and group congruences on a regular semigroups. Duke Math. J. **40**, 393–402 (1973)
7. Mitsch, H., Petrich, M.: Basic properties on E-inversive semigroups. Commun. Algebra **28**, 5169–5182 (2000)
8. Mitsch, H., Petrich, M.: Restricting idempotents in E-inversive semigroups. Acta Sci. Math. (Szeged) **67**, 555–570 (2001)
9. Petrich, M.: The maximal matrix decomposition of a semigroup. Port. Math. **25**, 15–33 (1966)
10. Petrich, M.: Inverse Semigroups. Wiley, New York (1984)