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RESEARCH ARTICLE

Congruences and group congruences on a semigroup

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Abstract We show that there is an inclusion-preserving bijection between the set of all normal subsemigroups of a semigroup S and the set of all group congruences on S. We describe also group congruences on E-inversive (E-)semigroups. In particular, we generalize the result of Meakin (J. Aust. Math. Soc. 13:259–266, 1972) concerning the description of the least group congruence on an orthodox semigroup, the result of Howie (Proc. Edinb. Math. Soc. 14:71–79, 1964) concerning the description of $\rho \vee \sigma$ in an inverse semigroup S, where ρ is a congruence and σ is the least group congruence on S, some results of Jones (Semigroup Forum 30:1–16, 1984) and some results contained in the book of Petrich (Inverse Semigroups, 1984). Also, one of the main aims of this paper is to study of group congruences on E-unitary semigroups. In particular, we prove that in any E-inversive semigroup, $\mathcal{H} \cap \sigma \subseteq \kappa$, where κ is the least E-unitary congruence. This result is equivalent to the statement that in an arbitrary E-unitary E-inversive semigroup S, $\mathcal{H} \cap \sigma = 1_S$.

Keywords Group congruence · E-inversive semigroup · E-semigroup · Idempotent-surjective semigroup · Eventually regular semigroup · Idempotent pure congruence · Idempotent-separating congruence · E-unitary congruence

1 Introduction and preliminaries

Let S be an inverse semigroup with semilattice of idempotents E. Define an inverse subsemigroup N of S to be *normal* if it is *full* (i.e., $E \subseteq N$), *closed* (i.e., $N\omega = N$, where $\omega : 2^S \to 2^S$ is a *closure operator* given by $A\omega = \{s \in S : \exists a \in A\ [as \in A]\}$

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for all $A \subseteq S$), and *self-conjugate* (i.e., $s^{-1}Ns \subseteq N$ for every $s \in S$). It follows from [11] (see p. 181) that there exists an inclusion-preserving bijection between the set of normal subsemigroups of S and the set of group congruences on S. In fact, the relation $\rho_N = \{(a,b) \in S \times S : ab^{-1} \in N\}$ is a group congruence on S and $\ker \rho_N = N$. These results were generalized in [9] and [16]. It is easy to see that $(a,b) \in \rho_N$ if and only if $ax, bx \in N$ for some $x \in S$.

The main purpose of the next section is a description of group congruences on a semigroup S in the terms of some special subsemigroups of S. Our description is simpler than that of Dubreil (see 10.2 [1]) and a little more general than the description of Gomes [6] (*nota bene* our proof is simpler). We apply this description to determine group congruences (in particular, the least group congruence) on some special classes of semigroups; namely: E-inversive (E-)semigroups (in particular, idempotent-surjective (E-)semigroups), eventually regular semigroups.

We divide this paper into seven sections. In Sect. 2 we describe group congruences on an arbitrary semigroup S in the terms of *normal* subsemigroups of S (see below for the definition). In Sect. 3 we investigate group congruences on E-inversive semigroups. In particular, we show that the least group congruence on an E-inversive semigroup exists (in general, this is false: see Example 1.2). In Sect. 4 and 5 we study group congruences on E-inversive E-semigroups and E-unitary E-inversive semigroups, respectively. Further, in Sect. 6 we use the results of Sect. 2 for an easy description of all group congruences on eventually regular semigroups (in terms of full, closed and self-conjugate subsemigroups) and we give some remarks on group congruences on inverse semigroups. Finally, in Sect. 7, some remarks on the hypercore of a semigroup are given (see [8]).

Let S be a semigroup. Denote by $Reg(S)$ the set of *regular elements* of S, that is, $Reg(S) = \{a \in S : a \in aSa\}$ and by $V(a)$ the set of *inverses* of $a \in S$, i.e., the set $\{x \in S : a = axa, x = xax\}$. Note that if $a \in S$ is regular, say $a = axa$ for some $x \in S$, then $xax \in V(a)$. Also, S is called *regular* if $V(a) \neq \emptyset$ for every $a \in S$. Further, S is said to be *eventually regular* if every element a of S has a regular power. In such a case, by $r(a)$ we shall mean the *regular index* of a, i.e., the least positive integer n for which $a^n \in Reg(S)$.

Let S be a semigroup, $a \in S$. The set $W(a) = \{x \in S : x = xax\}$ is called the set of all *weak inverses* of a and so the elements of $W(a)$ will be called *weak inverse elements* of a. A semigroup S is said to be *E-inversive* if for every $a \in S$ there exists $x \in S$ such that $ax \in E_S$, where E_S (or briefly E) is the set of idempotents of S (more generally, if $A \subseteq S$, then E_A denotes the set of idempotents of A). It is easy to see that a semigroup S is E-inversive if and only if $W(a)$ is nonempty for all $a \in S$. Hence if S is E-inversive, then for every $a \in S$ there is $x \in S$ such that $ax, xa \in E_S$ (see [19, 20]). Clearly, eventually regular semigroups are E-inversive. We remark that the class of eventually regular semigroups is very wide and contains the class of regular, group-bound (in particular, periodic, finite) semigroups. In [7] Hall observed that the set $Reg(S)$ of a semigroup S with $E_S \neq \emptyset$ forms a regular subsemigroup of S if and only if the product of any two idempotents of S is regular. In that case, S is said to be an *R-semigroup*. Also, we say that S is an *E-semigroup* if E_S is a subsemigroup of S. Evidently, every E-semigroup is an R-semigroup. Finally, [eventually] regular E-semigroups are called [*eventually*] *orthodox*.

A generalization of the concept of eventually regular will also prove convenient. Define a semigroup S to be *idempotent-surjective* if whenever ρ is a congruence on S and $a\rho$ is an idempotent of S/ρ, then $a\rho$ contains some idempotent of S. It is well known that eventually regular semigroups are idempotent-surjective [2]. Further, we have the following known result [10] (we include a simple proof for completeness).

Result 1.1 *Every idempotent-surjective semigroup S is E-inversive.*

Proof Let $a \in S$. From the definition of a Rees congruence on S follows that the ideal SaS has at least one idempotent, that is, $xay = e \in E_S$, where $x, y \in S$. Hence $exaye = e$. Thus $yex = (yex)a(yex)$, so $yex \in W(a)$, as required. □

A subset A of S is said to be (respectively) *full*; *reflexive* and *dense* if $E_S \subseteq A$; $\forall a, b \in S\ [ab \in A \Longrightarrow ba \in A]$ and $\forall s \in S\ \exists x, y \in S\ [sx, ys \in A]$. Also, define the *closure operator* ω on S by $A\omega = \{s \in S : \exists a \in A\ [as \in A]\}$ $(A \subseteq S)$. We shall say that $A \subseteq S$ is *closed* (in S) if $A\omega = A$. Finally, a subsemigroup N of a semigroup S is *normal* if it is full, dense, reflexive and closed (if N is normal, then we shall write $N \lhd S$). Moreover, if a subsemigroup of S is full, dense and reflexive, then it is called *seminormal* [6].

By the *kernel* $\ker\rho$ of a congruence ρ on a semigroup S we shall mean the set $\{x \in S : (x, x^2) \in \rho\}$. Finally, denote by $\mathcal{C}(S)$ the complete lattice of all congruences on a semigroup S.

Example 1.2 Consider the semigroup of positive integers $(\mathbb{N}, +)$ (with respect to addition). It is well known that every group congruence on $\mathbb{N}$ is of the following form: $\rho_n = \{(k, l) \in \mathbb{N} \times \mathbb{N} : n|(k - l)\}$ $(n > 0)$. Note that $E_{\mathbb{N}} = \emptyset$, so a semigroup without idempotents possesses group congruences but $\mathbb{N}$ has not least group congruence. Also, $\ker\rho_n = n\rho_n = \{n, 2n, 3n, \ldots\}$.

2 Group congruences—general case

Let S be a semigroup, $\rho \in \mathcal{C}(S)$. We say that ρ is a *group* congruence if S/ρ is a group. Denote by $\mathcal{GC}(S)$ the set of group congruences on S. Clearly, if $\rho \in \mathcal{GC}(S)$, then $\ker\rho$ is the identity of the group S/ρ. Finally, by $\mathcal{N}(S)$ we shall mean the set of all normal subsemigroups of S.

The following two lemmas are almost evident and we omit their easy proofs.

Lemma 2.1 *Let ρ be a group congruence on a semigroup S. Then* $\ker\rho \lhd S$.

Lemma 2.2 *Let ρ_1, ρ_2 be group congruences on a semigroup S. Then $\rho_1 \subset \rho_2$ if and only if* $\ker\rho_1 \subset \ker\rho_2$.

Let B be a nonempty subset of a semigroup S. Consider four relations on S:

$$\rho_{1,B} = \{(a, b) \in S \times S : \exists x \in S\ [ax, bx \in B]\};$$

$$\rho_{2,B} = \{(a,b) \in S \times S : \exists x, y \in B\ [ax = yb]\};$$

$$\rho_{3,B} = \{(a,b) \in S \times S : \exists x \in S\ [xa, xb \in B]\};$$

$$\rho_{4,B} = \{(a,b) \in S \times S : \exists x, y \in B\ [xa = by]\}.$$

Lemma 2.3 *Let a subsemigroup B of a semigroup S be dense and reflexive. Then* $\rho_{1,B} = \rho_{2,B} = \rho_{3,B} = \rho_{4,B}$.

Proof Let $(a,b) \in \rho_{2,B}$. Then $ax = yb$ for some $x, y \in B$. Also, $as \in B$ for some $s \in S$, since B is dense and so $sa \in B$, since B is reflexive. Hence $asy \in B$ and so $(sy)a \in B$. It follows that $(sy)b = s(yb) = s(ax) = (sa)x \in B$. Thus $(sy)a$, $(sy)b \in B$. We have just shown that $\rho_{2,B} \subset \rho_{3,B}$.

Conversely, if $xa, xb \in B$ for some $x \in S$, then $ax, bx \in B$ (since B is reflexive), so $a(xb) = (ax)b$, where $ax, xb \in B$. Hence $(a,b) \in \rho_{2,B}$. Thus $\rho_{2,B} = \rho_{3,B}$.

Dually, $\rho_{1,B} = \rho_{4,B}$. Since B is reflexive, then $\rho_{1,B} = \rho_{3,B}$. □

If B is a dense, reflexive subsemigroup of S, then we denote the above four relations by ρ_B. We have the following theorem.

Theorem 2.4 *Let B be a dense and reflexive subsemigroup of a semigroup S. Then the relation ρ_B is a group congruence on S. Moreover, $B \subseteq B\omega = \ker \rho_B$. If B is normal, then $B = \ker \rho_B$.*

Conversely, if ρ is a group congruence on S, then there exists a normal subsemigroup N of S such that $\rho = \rho_N$ (*in fact, $N = \ker \rho$*). *Thus there exists an inclusion-preserving bijection between the set of all normal subsemigroups of S and the set of all group congruences on S.*

Proof Let $a \in S$. Since B is dense, then there exists $x \in S$ such that $xa \in B$. Hence ρ_B is reflexive. Obviously, ρ_B is symmetric. Also, since B is a semigroup, then ρ_B is transitive. Consequently, ρ_B is an equivalence relation on S. Moreover, ρ_B is a left congruence on S. Indeed, let $(a,b) \in \rho_B$, $c \in S$. Then $ax, bx \in B$ and $zc \in B$ for some $x, z \in S$, so $zcax, zcbx \in B$. It follows that $(ca)(xz), (cb)(xz) \in B$, since B is reflexive. Therefore $(ca, cb) \in \rho_B$. By symmetry, ρ_B is a right congruence on S. Finally, S/ρ_B is a group. Indeed, let $a \in S, b \in B$ and $ax, xa \in B$ for some $x \in S$. Then $bax \in B$. Hence $xa, x(ba) \in B$, so $(ba, a) \in \rho_B$. Since B is dense, then S/ρ_B is a group, as required.

Since $b(bb) = (bb)b$ for every $b \in B$, then $B \subset \ker \rho_B$. Also, $B\omega = \ker \rho_B$. Indeed, let $s \in \ker \rho_B$. Then $(s,b) \in \rho_B$ for some $b \in B$. Hence $b_1 s = b b_2$ for some $b_1, b_2 \in B$. Thus $s \in B\omega$, so $\ker \rho_B \subset B\omega$. Conversely, let $s \in B\omega$. Then $bs \in B$ for some $b \in B$. Since $bb \in B$, then $(s,b) \in \rho_B$, so $s \in \ker \rho_B$. Thus $B\omega \subset \ker \rho_B$, as exactly required. Finally, if B is normal, then $B = B\omega$. Hence $B = \ker \rho_B$.

Conversely, let ρ be a group congruence on S. By Lemma 2.1, $\ker \rho \lhd S$. Put $\ker \rho = N$. Then by Lemma 2.2, $\rho = \rho_N$, since $N = \ker \rho_N = \ker \rho$. It is now easy to see that the map $\phi : \mathcal{N}(S) \to \mathcal{GC}(S)$, where $N\phi = \rho_N$ for every $N \in \mathcal{N}(S)$, is an inclusion-preserving bijection between the set of all normal subsemigroups of S and the set of all group congruences on S (with the inverse $\phi^{-1} : \mathcal{GC}(S) \to \mathcal{N}(S)$, where

$\rho\phi^{-1} = \ker\rho$ for all $\rho \in \mathcal{GC}(S)$). Note that ϕ^{-1} is an inclusion-preserving mapping, too. □

Since the first part of Theorem 2.4 is true for an arbitrary dense and reflexive subsemigroup of S, then we get the following corollary.

Corollary 2.5 *Let B be a dense and reflexive subsemigroup of S. Then $B\omega \lhd S$.*

Example 2.6 Let $S = \{a, b, c, e, f\}$ be the semigroup with the multiplication table given below:

$\cdot$	e	f	a	b	c
e	e	e	e	b	b
f	e	f	a	b	b
a	e	a	f	b	b
b	b	b	b	e	e
c	b	c	c	e	e

It is easy to see that E is a dense and reflexive subsemigroup of S but E is not closed, since $ea \in E$ and $a \notin E$. Also, $N = \{a, e, f\}$ is normal. Indeed, the group congruence ρ_E has two ρ_E-classes: N and $\{b, c\}$, since $ae, ee, bb, bc \in E$ and $(e, b) \notin \rho_E$. Note also that $E \subset \ker\rho_N = N$, $E \neq N$ and $\rho_E = \rho_N$. It follows that there is no a one-to-one correspondence between the set of all seminormal subsemigroups of S and the set of all group congruences on S.

Remark 1 Obviously, every subgroup of a group is full and unitary but not every subgroup of a group is reflexive (for example: each two element subgroup of the group of all permutations of the six-element set X is not reflexive). It is well known that a subgroup H of a group G is normal if and only if the relation ρ_H is a congruence on G. We have a corresponding result:

Let A be a closed subsemigroup of a semigroup S. Then A is normal if and only if $\rho_A \in \mathcal{GC}(S)$.

Indeed, let $\rho_A \in \mathcal{GC}(S)$. From $A = A\omega$ and the second paragraph of the proof of Theorem 2.4 we obtain that $A = \ker\rho_A$. Thus $A \lhd S$ (Lemma 2.1). The converse of the result follows from Theorem 2.4.

The set of all group congruences on a semigroup S (in general) does not form a lattice. Indeed, let $(\mathbb{R}, +)$ be the semigroup of real positive numbers with respect to addition. Put $M = \mathbb{N}$ and $N = \{x, 2x, 3x, \ldots\}$, where $x \in \mathbb{R} \setminus \mathbb{Q}$. Then $M, N \lhd S$ but $M \cap N = \emptyset$.

We generalize now the results of Howie [12], LaTorre [16] and Hanumantha Rao and Lakshmi [9].

Theorem 2.7 *Let B be a seminormal subsemigroup of a semigroup S, $\rho \in \mathcal{C}(S)$. Then*:

(i) $\rho \vee \rho_B = \rho_B\rho\rho_B$;

(ii) $\rho \vee \rho_B \in \mathcal{GC}(S)$;
(iii) $(x, y) \in \rho \vee \rho_B$ *if and only if* $(ax, yb) \in \rho$ *for some* $a, b \in B$.

Proof (i). Since $\rho, \rho_B \subset \rho \vee \rho_B$, $\rho_B\rho\rho_B \subset \rho \vee \rho_B$. Also, $\rho_B\rho\rho_B$ is a reflexive, symmetric and compatible relation on S. We show that $\rho_B\rho\rho_B$ is transitive. Then $\rho \vee \rho_B = \rho_B\rho\rho_B$. Indeed, let $(r, s), (s, t) \in \rho_B\rho\rho_B$. Then (a) $(w, s), (s, x) \in \rho_B$; (b) $(y, w), (x, z) \in \rho$; (c) $(r, y), (z, t) \in \rho_B$ for some $w, x, y, z \in S$. From (a) we obtain $(w, x) \in \rho_B$, so $aw = xb$ for some $a, b \in B$. From (b) follows that $(aw, ay), (xb, zb) \in \rho$. Hence $(ay, zb) \in \rho$, since $aw = xb$. Finally, by (c), (r, ay), $(zb, t) \in \rho_B$, since $B \subset \ker \rho_B$, so $(r, ay) \in \rho_B, (ay, zb) \in \rho, (zb, t) \in \rho_B$. Thus $(r, t) \in \rho_B\rho\rho_B$, as required.

(ii). This is evident.

(iii). Let $(x, y) \in \rho \vee \rho_B$. Then $(x, r) \in \rho_B, (r, s) \in \rho$ and $(s, y) \in \rho_B$ for some $r, s \in S$. Hence $ax = rb, cs = yd$ for some elements a, b, c, d of B. Therefore $(ca)x = c(ax) = c(rb) = (crb)\rho(csb) = (cs)b = (yd)b = y(db)$, where $ca, db \in B$. Conversely, let $(ax, yb) \in \rho$ for some $a, b \in B$. Since $(x, ax), (yb, y) \in \rho_B$, then $(x, y) \in \rho_B\rho\rho_B = \rho \vee \rho_B$ (by (i)). □

Let A be a nonempty subset of a semigroup S, $\rho \in \mathcal{C}(S)$. Put

$$A\rho = \{s \in S : \exists a \in A\ [(s, a) \in \rho]\}.$$

Corollary 2.8 *Let B be a seminormal subsemigroup of a semigroup S, $\rho \in \mathcal{C}(S)$. Then* $\ker(\rho \vee \rho_B) = (B\rho)\omega$. *In particular,* $(B\rho)\omega \lhd S$.

Proof Let $x \in \ker(\rho \vee \rho_B)$. Then there exists $b \in B$ such that $(x, b) \in \rho \vee \rho_B$, since $B \subset \ker(\rho \vee \rho_B)$. Hence $(ax, bc) \in \rho$ for some $a, c \in B$ (by Theorem 1.6(iii)). Thus $ax \in B\rho$. It follows that $x \in (B\rho)\omega$. Conversely, if $x \in (B\rho)\omega$, then $ax \in B\rho$ for some $a \in B\rho$, so $(ax, b), (a, c) \in \rho$ for some $b, c \in B$. It follows that $(cx, b) \in \rho$. Hence $((cc)x, cb) \in \rho$. Thus $(x, c) \in \rho \vee \rho_B$. Consequently, $x \in \ker(\rho \vee \rho_B)$. □

Also, by Theorem 1.6(i) and Proposition 2.3(ii) in [15] we obtain the following (see Corollary 3.2 [15]) corollary.

Corollary 2.9 *Every group congruence on a semigroup S is dually right modular element of $\mathcal{C}(S)$.*

Corollary 2.10 *Let B be a seminormal subsemigroup of a semigroup S, $\rho \in \mathcal{C}(S)$. Then $\rho \vee \rho_B = S \times S$ if and only if $(B\rho)\omega = S$.*

Let B be a seminormal subsemigroup of a semigroup S, $\rho_1, \rho_2 \in \mathcal{C}(S)$. Suppose that $(x, y) \in (\rho_1 \vee \rho_B) \cap (\rho_2 \vee \rho_B)$. Then $(ax)\rho_2(yb)$, where $a, b \in B$. Moreover, $ax(\rho_1 \vee \rho_B)x, x(\rho_1 \vee \rho_B)y, y(\rho_1 \vee \rho_B)yb$, so $ax(\rho_1 \vee \rho_B)yb$. Thus $(cax, ybd) \in \rho_1$, where $c, d \in B$. It follows that $(caxd, cybd) \in \rho_1$. Moreover, $(caxd, cybd) \in \rho_2$. Hence $(xd, cy) \in (\rho_1 \cap \rho_2) \vee \rho_B$. Thus $(x, y) \in (\rho_1 \cap \rho_2) \vee \rho_B$, since $(\rho_1 \cap \rho_2) \vee \rho_B$ is a group congruence on S and $c, d \in B \subset \ker((\rho_1 \cap \rho_2) \vee \rho_B)$. We have just shown that $(\rho_1 \vee \rho_B) \cap (\rho_2 \vee \rho_B) \subset (\rho_1 \cap \rho_2) \vee \rho_B$. The converse inclusion is evident. Thus we may conclude that $(\rho_1 \vee \rho_B) \cap (\rho_2 \vee \rho_B) = (\rho_1 \cap \rho_2) \vee \rho_B$.

We have the following theorem (see Theorem III.5.6 [21] and Theorem 4 [23]).

Theorem 2.11 *Let B be a seminormal subsemigroup of a semigroup S. Then the mapping $\phi : \mathcal{C}(S) \to \mathcal{GC}(S)$, where*

$$\rho\phi = \rho \vee \rho_B$$

for every $\rho \in \mathcal{C}(S)$, is a (lattice) homomorphism of $\mathcal{C}(S)$ onto the (modular) lattice $[\rho_B, S \times S]$ of all group congruences on S containing ρ_B.

Proof We have just proved that $(\rho_1 \cap \rho_2)\phi = \rho_1\phi \cap \rho_2\phi$ for all $\rho_1, \rho_2 \in \mathcal{C}(S)$. Clearly, $(\rho_1 \vee \rho_2)\phi = \rho_1\phi \vee \rho_2\phi$ for all $\rho_1, \rho_2 \in \mathcal{C}(S)$ and evidently ϕ is onto $[\rho_B, S \times S]$. □

We have the following corollary (see Theorem 4.5 [15]).

Corollary 2.12 *Let B be a seminormal subsemigroup of a semigroup S. Then ρ_B distributes over meet.*

Let S be a semigroup, $N \lhd S$. Put

$$\mathcal{P}(S; N) = \left\{A \subseteq S : A^2 \subseteq A, N \subseteq A, A\omega = A\right\}.$$

Also, denote S/ρ_N by S/N. In particular, $\mathcal{P}(S/N; \{N\})$ is the set of all subgroups of the group S/N. Remark that if $A \in \mathcal{P}(S; N)$, then A is full and dense.

The proofs of the following two propositions are standard and so we omit the proofs.

Proposition 2.13 *Let S be a semigroup, $N \lhd S$. Then there exists an inclusion-preserving bijection ϕ between the set $\mathcal{P}(S; N)$ and the set $\mathcal{P}(S/N; \{N\})$. Moreover, $M \in \mathcal{P}(S; N)$ and $M \lhd S$ if and only if $M\phi \lhd S/N$.*

Proposition 2.14 *Let ϕ be an epimorphism of a semigroup S onto a group $(G, \cdot, 1)$. Then*:

(i) $\mathrm{Ker}(\phi) = \phi\phi^{-1}$ *is a group congruence on S*;
(ii) $N = \{1\}\phi^{-1} \lhd S$;
(iii) $\mathrm{Ker}(\phi) = \rho_N$.

Conversely, if $N \lhd S$, then N is the kernel of the canonical homomorphism of S onto S/N.

Example 2.15 We now describe all normal subsemigroups of the bicyclic semigroup $S = \mathbb{N}_0 \times \mathbb{N}_0$, where $(k, l)(m, n) = (k - l + \max\{l, m\}, n - m + \max\{l, m\})$. It is known that every (non-identical) homomorphic image of the bicyclic semigroup is a cyclic group. Also, it is almost evident that $E_S = \{(0, 0), (1, 1), (2, 2), \ldots\} \lhd S$

and $(k,l)\rho_E(m,n)$ if and only if $k+n=l+m$, so $S/\rho_E \cong (\mathbb{Z},+)$. It follows that $(i\mathbb{Z})\phi^{-1} = \{(m,n) \in S : (m)_i = (n)_i\}$ for every $i \in \mathbb{N}$. The conclusion is that every cyclic group is a homomorphic image of the bicyclic semigroup.

We have also the following well known proposition (from group theory).

Proposition 2.16 *Let S be a semigroup*; $M, N \lhd S$ *and* $M \subseteq N$. *Then*:

(i) $M \lhd N$;
(ii) $N/M \lhd S/M$;
(iii) $(S/M)/(N/M) \cong S/N$.

Every full and closed subsemigroup A of an E-inversive semigroup S is itself E-inversive. Indeed, let $a \in A$. Then $ax \in E_S = E_A$ for some $x \in S$, so $x \in A\omega = A$. Consequently, there is $x \in A$ such that $ax \in E_A$.

Finally, by way of contrast, we prove in the present section the following proposition which is valid for the class of all E-inversive semigroups.

Proposition 2.17 *Let S be an E-inversive semigroup*, $N \lhd S$. *Suppose also that a subsemigroup M of S is full and closed. Then*:

(i) $M \cap N \lhd M$;
(ii) $N \lhd (MN)\omega$;
(iii) $M/(M \cap N) \cong (MN)\omega/N$.

Proof (i). It is clear that $E_S \subset M \cap N$, so $M \cap N$ is a full subsemigroup of M. Let $a, b \in M$ be such that $ab \in M \cap N$. Then $ba \in M$ and $ba \in N$ (since N is reflexive in S). Hence $ba \in M \cap N$. Hence $M \cap N$ is reflexive in M. Further, if $x \in (M \cap N)\omega$, then $yx \in M \cap N$ for some $y \in M \cap N$, so $x \in M \cap N$ (because N and M are closed). Since $M \cap N$ is full and closed, then it is E-inversive, so it is dense in M. Thus $M \cap N \lhd M$.

(ii). We show that $(MN)\omega$ is a subsemigroup of S. Let $a, b \in (MN)\omega$. Then $m_1n_1a = m_2n_2$ for some $m_1, m_2 \in M, n_1, n_2 \in N$. Since S is E-inversive, then $W(m_1) \neq \emptyset$. Hence $mm_1, m_1m \in E_S \subset M$ for some $m \in S$. Thus $m \in M$ (since M is closed), $(mm_1)n_1a = (mm_2)n_2$. Therefore $(n_1a, mm_2) \in \rho_N$, since $mm_1 \in E_S \subset N$, so $(a, m_3) \in \rho_N$ ($m_3 \in M$). Similarly, $(b, m_4) \in \rho_N$ for some $m_4 \in M$. It follows that $(ab, m_5) \in \rho_N$, where $m_5 \in M$. Hence $n_3ab = m_5n_4$ for some $n_3, n_4 \in N$. Thus $(m_5n_3)ab = (m_5m_5)n_4$. Consequently, $ab \in (MN)\omega$. Furthermore, $N \subset (MN)\omega$. Indeed, let $n \in N$. Then $n_1n = en_2$ for some $e \in E_S, n_1, n_2 \in N$. Hence we have $(en_1)n = en_2 \in MN$, so $n \in (MN)\omega$. Consequently, $N \lhd (MN)\omega$ (since $N \lhd S$).

The proof of the condition (iii) is standard. □

3 Group congruences on an E-inversive semigroup

Note that if a semigroup S is E-inversive, then every full subsemigroup of S is dense (since E_S is dense), so a subsemigroup A of S is normal if and only if A is full, reflexive and closed. It follows that S has a least normal subsemigroup U. Thus the least group congruence on an arbitrary E-inversive semigroup exists. Denote it by σ or σ_S. Then $\sigma = \rho_U$ and $\ker \sigma = U$ (Theorem 2.4).

Firstly, we have the following proposition.

Proposition 3.1 *Let S be an E-inversive semigroup. Then $\mathcal{GC}(S) = [\sigma, S \times S]$. Thus $\mathcal{GC}(S)$ is a complete sublattice of $\mathcal{C}(S)$.*

Also, $\rho_M \vee \rho_N = \rho_M \rho_N = \rho_N \rho_M$ for all $M, N \lhd S$. Hence the lattice

$$\big(\mathcal{GC}(S), \subseteq, \cap, \circ\big)$$

is modular.

Proof The first part of the above proposition is clear. We show its second part. Let $a(\rho_M \rho_N)b$. Then $(a,c) \in \rho_M$, $(c,b) \in \rho_N$, where $c \in S$. Take any $x \in W(c)$. Then $xc, cx \in E_S$, $(cxa)\rho_N(bxa)$, $(bxa)\rho_M(bxc)$, so $(a, bxa) \in \rho_N$, $(bxa, b) \in \rho_M$. Hence $(a,b) \in \rho_N \rho_M$. Therefore $\rho_M \rho_N \subset \rho_N \rho_M$. We may equally well show the opposite inclusion. Consequently, $\rho_M \vee \rho_N = \rho_M \rho_N = \rho_N \rho_M$. In the light of Proposition I.8.5 [11], the lattice $(\mathcal{GC}(S), \subseteq, \cap, \circ)$ is modular. □

Let M, N be normal subsemigroups of a semigroup S. From Proposition 3.1 and Corollary 2.8 we obtain that $\ker(\rho_M \rho_N) = \ker(\rho_N \rho_M) = (M\rho_N)\omega = (N\rho_M)\omega$. In fact, if S is E-inversive, then $\ker(\rho_M \rho_N) = \ker(\rho_N \rho_M) = M\rho_N = N\rho_M$. Indeed, let $x \in \ker(\rho_M \rho_N)$. Then $(x,e) \in \rho_M \rho_N$ for some $e \in E_S$. Hence $(x,n) \in \rho_M$, $(n,e) \in \rho_N$, where $n \in S$ (in fact, $n \in \ker \rho_N = N$). Thus $x \in N\rho_M$. Conversely, if $x \in N\rho_M$, then $(x,n) \in \rho_M$ for some $n \in N$. Hence $(x,n) \in \rho_M$, $(n,e) \in \rho_N$, where $e \in E_S$. Thus $(x,e) \in \rho_M \rho_N$, that is, $x \in \ker(\rho_M \rho_N)$, so $\ker(\rho_M \rho_N) = N\rho_M$. Similarly, $\ker(\rho_N \rho_M) = M\rho_N$. This implies the required equalities. Also, $\ker(\rho_M \rho_N) = (MN)\omega$. Indeed, let $x \in M\rho_N$. Then $n_1 x = m n_2$ for some $n_1, n_2 \in N$, $m \in M$. Hence $(mn_1)x \in MN$. Thus $x \in (MN)\omega$. We have proved that $\ker(\rho_M \rho_N) \subset (MN)\omega$. Conversely, let $x \in (MN)\omega$. Then $m_1 n_1 x = m_2 n_2$ for some $m_1, m_2 \in M$, $n_1, n_2 \in N$. Since S is E-inversive, then $mm_1 = e \in E_S \subset M$ for some $m \in S$. It follows that $m \in M$ (since M is closed), so $en_1 x = mm_2 n_2$. Hence $(x, mm_2) \in \rho_N$. Thus $x \in M\rho_N = \ker(\rho_M \rho_N)$, so $(MN)\omega \subset \ker(\rho_M \rho_N)$, as exactly required.

In fact, we have just shown that in an arbitrary E-inversive semigroup S, $\rho_{(MN)\omega} = \rho_M \rho_N = \rho_N \rho_M = \rho_{(NM)\omega}$ for all $M, N \lhd S$. Moreover, notice that $\ker(\rho_M \cap \rho_N) = \ker \rho_M \cap \ker \rho_N = M \cap N$ $(M, N \lhd S)$, so $\rho_M \cap \rho_N = \rho_{M \cap N}$ for $M, N \lhd S$. Consequently, the lattice $(\mathcal{N}(S), \subseteq, \cap, \vee)$, where $M \vee N = (MN)\omega$ for all $M, N \lhd S$, is isomorphic to the lattice $(\mathcal{GC}(S), \subseteq, \cap, \circ)$ (by the inclusion-preserving bijection ϕ, see the proof of Theorem 2.4). Note also that the lattice $(\mathcal{N}(S), \subseteq, \cap, \vee)$ is complete (since it has the greatest element S and the intersection of any nonempty family of normal subsemigroups of S is a normal subsemigroup of S).

For terminology and elementary facts about lattices the reader is referred to the book [21] (Sect. I.2). The following result will be useful (see Exercise I.2.15(iii) in [21]).

Lemma 3.2 *Every lattice isomorphism of complete lattices is a complete lattice isomorphism.*

From the above consideration we obtain the following theorem.

Theorem 3.3 *Let S be an E-inversive semigroup. Then there exists a (lattice) isomorphism ϕ between the lattice $(\mathcal{N}(S), \subseteq, \cap, \vee)$, where $M \vee N = (MN)\omega$ for all $M, N \lhd S$, and the lattice $(\mathcal{GC}(S), \subseteq, \cap, \circ)$. In fact, ϕ is defined by $N\phi = \rho_N$ for every $N \in \mathcal{N}(S)$. Moreover, ϕ is a complete lattice isomorphism.*

Finally, we have the following proposition.

Proposition 3.4 *Let S be an E-inversive semigroup, $N \lhd S$. Then $(a, b) \in \rho_N$ if and only if $ab^* \in N$ for some (all) $b^* \in W(b)$.*

Proof ($\Longrightarrow$). Let $na = bm$, where $n, m \in N$, and $b^* \in W(b)$. Then $nab^* = bmb^*$. Since $b^*bm \in N$ and N is reflexive, then $nab^* \in N$. Hence $ab^* \in N\omega = N$.

($\Longleftarrow$). Let $ab^* = n \in N$ for some $b^* \in W(b)$. Then $a(b^*b) = nb$, so $(a, b) \in \rho_N$ (by Lemma 2.3). □

4 Group congruences on an E-semigroup

First, we "generalize" some results from orthodox semigroups to E-semigroups (see Theorem VI.1.1 [11]).

Proposition 4.1 *Let S be a semigroup. The following conditions are equivalent:*

(i) *S is an E-semigroup;*
(ii) $\forall a, b \in S\ [W(b)W(a) \subseteq W(ab)]$.

Moreover, the condition (i) *implies the following condition:*

(iii) $\forall e \in E_S\ [W(e) \subseteq E_S]$.

If in addition S is an R-semigroup, then the conditions (i)–(iii) *are equivalent.*

Proof The proof is closely similar to the proof of Theorem VI.1.1 [11]. □

Corollary 4.2 *Let S be an E-semigroup. Then:*

(i) $\forall e \in E_S\ [W(e), V(e) \subseteq E_S]$;
(ii) $\forall a \in S, a^* \in W(a), e \in E_S\ [aea^*, a^*ea \in E_S]$;
(iii) $\forall a \in S, a^* \in W(a), e, f \in E_S\ [ea^*, a^*e, ea^*f \in W(a)]$.

Proof (i). This follows from Proposition 4.1.

(ii). This follows from the proof of Proposition VI.1.4 [11].

(iii). Let $a \in S, a^* \in W(a), e, f \in E_S$. Since $e \in W(e)$ and $f \in W(f)$, then $ea^* \in W(e)W(a) \subseteq W(ae)$. Hence $ea^* = ea^*aeea^* = (ea^*)a(ea^*)$. Therefore $ea^* \in W(a)$. Similarly, $a^*e \in W(a)$. Finally, $ea^*f \in W(e)W(a)W(f) \subseteq W(fae)$ and so $ea^*f = ea^*ffaeea^*f = (ea^*f)a(ea^*f)$. Hence $ea^*f \in W(a)$. □

Proposition 4.3 *Let S be an E-inversive E-semigroup. Then*

$$\rho_{1,E} = \rho_{2,E} = \rho_{3,E} = \rho_{4,E}.$$

Proof Let $(a,b) \in \rho_{2,E}$ and $a^* \in W(a)$. Then $ae = fb$ for some $e, f \in E$. Moreover, $a^*f \in W(a)$ (Corollary 4.2(iii)), so $(a^*f)a, a(a^*f) \in E$. Further, $a^*fb = a^*ae \in E$. We have just shown that $xa, ax, xb \in E$ for some $x \in S$. Thus $\rho_{2,E} \subset \rho_{4,E}$.

On the other hand, if $xa, xb \in E$ for some $x \in S$, say $xa = e, xb = f$, then $(efx)a(efx) = ef(xa)efx = efx$, so $efx \in W(a)$. Also, $fxbfx = f(xb)fx = fx$, i.e., $fx \in W(b)$. Hence $efx \in W(b)$ (Corollary 4.2(iii)). Thus $W(a) \cap W(b) \neq \emptyset$. It follows that $ay, by, ya, yb \in E$ for some $y \in S$. Dually, if $ax, bx \in E$ for some $x \in S$, then $ay, by, ya, yb \in E$ for some $y \in S$. Thus $\rho_{4,E} = \rho_{1,E}$. In fact, we get $\rho_{4,E} = \rho_{1,E} = \{(a,b) \in S \times S : W(a) \cap W(b) \neq \emptyset\}$. Finally, if $x \in W(a) \cap W(b)$, then $a(xb) = (ax)b$ and $xb, ax \in E$. Thus $\rho_{2,E} = \rho_{4,E} = \rho_{1,E}$. We may equally well show that $\rho_{3,E} = \rho_{4,E} = \rho_{1,E}$. Consequently, $\rho_{1,E} = \rho_{2,E} = \rho_{3,E} = \rho_{4,E}$. □

Lemma 4.4 *Let S be an E-inversive E-semigroup. Then*:

(i) $\forall a \in S\ \exists e, f \in E_S\ [ea, af \in Reg(S)]$;
(ii) $\forall a \in S\ \exists r \in Reg(S)\ [W(a) \cap W(r) \neq \emptyset]$.

Proof Let $a \in S, x \in W(a)$. Then $(ax)a, a(xa) \in Reg(S)$, where $ax, xa \in E_S$, so (i) holds. Also, $r = axa \in Reg(S)$ and $xrx = x$. Thus $x \in W(a) \cap W(r)$. □

Denote the above four relations from Proposition 4.3 by ρ_E. Recall that from the proof of Proposition 4.3 follows that $\rho_E = \{(a,b) \in S \times S : W(a) \cap W(b) \neq \emptyset\}$.

Theorem 4.5 *In any E-inversive E-semigroup, $\sigma = \rho_E$. Moreover,* $\ker \sigma = E_S\omega$. *Thus $E_S\omega \lhd S$.*

Proof It is clear that ρ_E is an equivalence relation on S. Let $(a,b) \in \rho_E, c \in S$. Then $x \in W(a) \cap W(b)$. Take any $y \in W(c)$. In the light of Proposition 4.1,

$$xy \in W(a)W(c) \cap W(b)W(c) \subseteq W(ca) \cap W(cb).$$

Hence $(ca, cb) \in \rho_E$. Thus ρ_E is a left congruence on S. We may equally well show that ρ_E is a right congruence on S. Also, if $e, f \in E_S$, then $ee, ef \in E_S$. Consequently, $(e,f) \in \rho_E$ for all $e, f \in E_S$. Lemma 4.4(ii) says that every ρ_E-class of S contains a regular element. This implies that S/ρ_E is a group.

Furthermore,

$$x \in \ker \rho_E \ \Leftrightarrow\ \exists e \in E_S\ [(x,e) \in \rho_E] \ \Leftrightarrow\ \exists e, f, g \in E_S\ [fx = eg] \ \Leftrightarrow\ x \in E_S\omega,$$

so $\ker \sigma = E_S\omega$. Thus $E_S\omega \lhd S$ (Theorem 2.4). Finally, $\rho_E \subseteq \rho_N$ for ever $N \lhd S$. Indeed, $E_S \subseteq N$. Hence $E_S\omega \subseteq N\omega = N$. Thus $\rho_E = \rho_{E_S\omega} \subseteq \rho_N$ (Theorem 2.4). Consequently, $\sigma = \rho_E$. □

Corollary 4.6 *The least group congruence σ on an E-inversive E-semigroup is given by*

$$\sigma=\{(a,b)\in S\times S:\exists\, e\in E_S\ [eae=ebe]\}.$$

Remark 2 Note that the condition "$\exists\, e\in E_S\ [eae=ebe]$" from the above corollary is equivalent to the apparently weaker condition "$\exists\, s\in S\ [sas=sbs]$".

From Result 1.1 and Theorem 4.5 we obtain the following theorem.

Theorem 4.7 *In any idempotent-surjective E-semigroup, $\sigma=\rho_E$.*

Let S be a semigroup. A congruence ρ on S is called *idempotent pure* if $e\rho\subseteq E_S$ for every $e\in E_S$. Note that if S is idempotent-surjective, then ρ is idempotent pure if and only if $\ker\rho=E_S$. Let $\mathcal{E}$ be an equivalence relation on S induced by the partition: $\{E_S, S\setminus E_S\}$. Then $\mathcal{E}^\flat$ (defined in [13], see p. 27) is the greatest idempotent pure congruence on S. Put $\tau=\mathcal{E}^\flat$. Then (see [13], p. 28)

$$\tau=\{(a,b)\in S\times S:\forall x,y\in S^1\ [xay\in E_S\iff xby\in E_S]\}.$$

Finally, if S is E-inversive, then $\tau\subseteq\sigma$. Indeed, let $(a,b)\in\tau$ and $b^*\in W(b)$. Then $bb^*\in E_S$, $(ab^*,bb^*)\in\tau$. Hence $ab^*\in E_S\subseteq\ker\sigma$. In the light of Proposition 3.4, $(a,b)\in\sigma$, as exactly required. In the following corollary we give an alternative proof of this fact.

Corollary 4.8 *If ρ is a congruence on an idempotent-surjective E-semigroup S, then* $\ker(\rho\vee\sigma)=(\ker\rho)\omega$. *In particular, $\tau\subseteq\sigma$.*

Proof By Corollary 2.8, $\ker(\rho\vee\sigma)=(E_S\rho)\omega=(\ker\rho)\omega$. In particular,

$$\ker(\tau\vee\sigma)=E_S\omega\subseteq\ker\sigma.$$

Hence $\tau\vee\sigma=\sigma$. Thus $\tau\subseteq\sigma$. □

Let ρ be a congruence on a semigroup S. By the *trace* $\operatorname{tr}\rho$ of ρ we shall mean the restriction of ρ to E_S. Also, we say that ρ is *idempotent-separating* if $\operatorname{tr}\rho=1_{E_S}$. Edwards in [3] shows that if S is an eventually regular semigroup, then the relation $\theta=\{(\rho_1,\rho_2)\in\mathcal{C}(S)\times\mathcal{C}(S):\operatorname{tr}\rho_1=\operatorname{tr}\rho_2\}$ is a complete congruence on $\mathcal{C}(S)$ and proves that every θ-class $\rho\theta$ is a complete sublattice of $\mathcal{C}(S)$ with the maximum element

$$\mu(\rho)=\{(a,b)\in S\times S:(a\rho,b\rho)\in\mu_{S/\rho}\}$$

and the minimum element $1(\rho)$. Edwards generalizes some of these results for the class of all idempotent-surjective semigroups [4]. In fact, if S is an arbitrary idempotent-surjective semigroup, then every θ-class $\rho\theta$ is the interval $[1(\rho),\mu(\rho)]$, where μ is the maximum idempotent-separating congruence on S (see [4] for more details).

It is easily seen that the class of idempotent-surjective semigroups is closed under homomorphic images [10]. Using the obvious terminology we show next that every homomorphism of idempotent-surjective E-semigroups can be factored into a homomorphism preserving the maximal group homomorphic images and an idempotent-separating homomorphism. Firstly, we have need the following lemma.

Lemma 4.9 *Let ρ be a congruence on an idempotent-surjective E-semigroup S, $a, b \in S$. Then $(a\rho, b\rho) \in \sigma$ in S/ρ implies $(a, b) \in \sigma$ if and only if $\rho \subseteq \sigma$.*

Proof The proof is closely similar to the proof of Lemma III.5.9 [21]. □

Let S be an idempotent-surjective E-semigroup, $\rho \in \mathcal{C}(S)$. Clearly, $(a, b) \in \sigma$ implies $(a\rho, b\rho) \in \sigma$. In the light of Lemma 4.9, if $\rho \subseteq \sigma$, then $(a, b) \in \sigma$ if and only if $(a\rho, b\rho) \in \sigma$. Hence $S/\sigma \cong (S/\rho)/\sigma$, that is, S and S/ρ have isomorphic maximal group homomorphic images. In that case, we may say that ρ *preserves the maximal group homomorphic images*. Since for any congruence ρ on S we have $1(\rho) \subseteq \rho$, then we obtain the following factorization:

$$S \to S/1(\rho) \to S/\rho \cong \big(S/1(\rho)\big)/\big(\rho/1(\rho)\big).$$

The following proposition generalizes Proposition III.5.10 [21].

Proposition 4.10 *Every homomorphism of idempotent-surjective E-semigroups can be factored into a homomorphism preserving the maximal group homomorphic images and an idempotent-separating homomorphism.*

Proof Let ρ be any congruence on an idempotent-surjective E-semigroup S. Since $\rho \subseteq S \times S$, then $1(\rho) \subseteq 1(S \times S)$. Clearly, $\sigma \in [1(S \times S), S \times S]$ and so $1(\rho) \subseteq \sigma$. It follows that the canonical epimorphism of S onto $S/1(\rho)$ preserves the maximal group homomorphic images. Finally, an epimorphism $\phi : S/1(\rho) \to S/\rho$ (defined by the obvious way) is idempotent-separating, since $\operatorname{tr} \rho = \operatorname{tr}(1(\rho))$. The thesis of the proposition is a consequence of the above factorization. □

5 Group congruences on an E-unitary semigroup

A nonempty subset A of a semigroup S is called *left* [*right*] *unitary* if $as \in A$ [$sa \in A$] implies $s \in A$ for every $a \in A, s \in S$. Also, we say that A is *unitary* if it is both left and right unitary. Finally, a semigroup S with $E_S \neq \emptyset$ is said to be *E-unitary* if E_S is unitary.

Proposition 5.1 *Let S be a semigroup with $E_S \neq \emptyset$. The following conditions are equivalent*:

(i) *S is E-unitary*;
(ii) *E_S is left unitary*;
(iii) *E_S is right unitary.*

Also, if S is an E-unitary E-inversive semigroup, then S is an E-semigroup.

Proof (i) $\Longrightarrow$ (ii). This is trivial.

(ii) $\Longrightarrow$ (iii). Let $s \in S, e \in E_S$. If $se = f \in E_S$, then $fsef = f$ and so we get $(efs)(efs) = efs$, that is, $efs \in E_S$. Hence $fs \in E_S$. Thus $s \in E_S$.

(iii) $\Longrightarrow$ (i). We may equally well show like above that E_S is left unitary. Thus the condition (i) holds.

Finally, let S be an E-unitary E-inversive semigroup. If $e, f \in E_S, x \in W(ef)$, then $xef \in E_S$. Hence $xef, x \in E_S$. Thus $ef \in E_S$. □

Corollary 5.2 *Let S be an E-inversive semigroup. Then the following conditions are equivalent*:

(i) *S is E-unitary*;
(ii) $\ker\sigma = E_S$;
(iii) $\tau = \sigma$.

In particular, if S is an E-unitary E-inversive semigroup, then $E_S \lhd S$.

Proof (i) $\Longrightarrow$ (ii). In the light of Proposition 5.1 and Theorem 4.5, $\ker\sigma = E_S\omega$. Also, S is left unitary, that is, E_S is closed. Thus $\ker\sigma = E_S$.

(ii) $\Longrightarrow$ (iii). We have mentioned above that $\tau \subseteq \sigma$. On the other hand, the main assumption implies that σ is idempotent pure. Hence $\sigma \subseteq \tau$. Thus $\tau = \sigma$.

(iii) $\Longrightarrow$ (i). Let $a \in S, e, f \in E_S$. If $ea = f$, then $a \in \ker\sigma = \ker\tau = E_S$, that is, E_S is left unitary. In the light of Proposition 5.1, S is E-unitary. □

Remark 3 Notice that if a semigroup is not E-inversive, then Corollary 5.2 is false. Indeed, let ${F_X}^1$ be the free monoid on the set X. Then ${F_X}^1$ is E-unitary but τ is induced by the partition $\{F_X, \{1\}\}$. Thus τ is not a group congruence.

From Proposition 3.4 and Corollary 5.2 we obtain the following proposition.

Proposition 5.3 *Let S be an E-unitary E-inversive semigroup. Then $(a, b) \in \sigma$ if and only if $ab^* \in E_S$ for some (all) $b^* \in W(b)$.*

Corollary 5.4 *Let A be an E-inversive subsemigroup of an E-unitary E-inversive semigroup S. Then $\sigma_A = \sigma_S \cap (A \times A)$.*

Proof Clearly, $\sigma_A \subset \sigma_S \cap (A \times A)$. The converse follows from Proposition 5.3. □

In [14] Howie and Lallement showed that $\sigma \cap \mathcal{H} = 1_S$, when S is an E-unitary regular semigroup. We prove a corresponding result.

Theorem 5.5 *Let S be an E-unitary E-inversive semigroup. Then $\sigma \cap \mathcal{H} = 1_S$. Moreover, if in addition E_S forms a semilattice, then $\sigma \cap \mathcal{L} = \sigma \cap \mathcal{R} = 1_S$.*

Proof Let S be an E-unitary E-inversive semigroup. Suppose also that E_S forms a semilattice. Then E_S is normal (Corollary 5.2), so if $(a, b) \in \sigma \cap \mathcal{L}$, then $ax = e$,

$bx = f \in E_S$ for some $x \in S$ (see Proposition 5.3) and $sa = b, tb = a$ for some $s, t \in S$. Hence $se = sax = bx = f \in E_S, tf = tbx = ax = e \in E_S$. Thus $s, t \in E_S$ (since E_S is unitary), so since idempotents commute and $ta = tb$,

$$a = tb = t(sa) = (ts)a = (st)a = s(ta) = s(tb) = sa = b.$$

We may equally well show that $\sigma \cap \mathcal{R} = 1_S$.

If S is E-unitary, then E_S is normal, too. Let $(a, b) \in \sigma \cap \mathcal{H}$. By the above proof and its dual we conclude that $a = eb = bf$ and $b = ga = ah$ for some $e, f, g, h \in E_S$. In the light of Proposition 2 in [18], $a = b$. □

Remark 4 The assumption that S is an E-inversive semigroup is important. Indeed, let $S = (\mathbb{R}_0, +)$ be the semigroup of nonnegative real numbers with respect to addition. Then S is an E-unitary commutative semigroup. Put $M = \mathbb{N}_0$ and $N = \{0, x, 2x, 3x, \ldots\}$ (where $x \in \mathbb{R} \setminus \mathbb{Q}$). Then $M, N \lhd S$ but $M \cap N = \{0\}$ is not normal, so S has no least group congruence.

The converse of Theorem 4.15 is not valid (in general). Indeed, let $S = \langle x \rangle$, where $x = (2\,3\,4\,5\,6\,7\,5)$ is a mapping of $\mathcal{T}(\{1, 2, \ldots, 7\})$. Then $S = M(4, 3)$ is the monogenic semigroup with index 4 and period 3, say $S = \{x, x^2, \ldots, x^6\}$. Also, the cyclic subgroup K_x of S with the unit e is equal $\{x^4, x^5, x^6 = e\}$. Since $x^3 e = x^7 x^2 = x^4 x^2 = e$, then S is not E-unitary. On the other hand, σ is induced by the partition: $\{\{x, x^4\}, \{x^2, x^5\}, \{x^3, e\}\}$ and $\mathcal{H}$ by the partition: $\{K_x, \{x\}, \{x^2\}, \{x^3\}\}$. Thus $\sigma \cap \mathcal{H} = 1_S$.

From Theorem 5.5 and Corollary 5.2 we have the following corollary.

Corollary 5.6 *Let S be an E-unitary E-inversive semigroup. Then*

$$\sigma \cap \mathcal{H} = \tau \cap \mathcal{H} = 1_S.$$

Moreover, if in addition E_S forms a semilattice, then

$$\sigma \cap \mathcal{L} = \tau \cap \mathcal{L} = \sigma \cap \mathcal{R} = \tau \cap \mathcal{R} = 1_S.$$

Recall that a congruence ρ on a semigroup S is *E-unitary* if S/ρ is E-unitary. In [5] the author described the least E-unitary congruence κ on an idempotent-surjective semigroup. Also, for every congruence ρ on an idempotent-surjective semigroup S there exists the least E-unitary congruence κ_ρ on S containing ρ [5].

Let S be an idempotent-surjective semigroup, $N \lhd S$. Define the relation $\hat{\rho_N}$ on $\mathcal{C}(S)$ by the following rule: $(\rho_1, \rho_2) \in \hat{\rho_N} \Leftrightarrow \rho_1 \vee \rho_N = \rho_2 \vee \rho_N$ $(\rho_1, \rho_2 \in \mathcal{C}(S))$. Then $\hat{\rho_N}$ is a congruence on $\mathcal{C}(S)$, since $\phi\phi^{-1} = \hat{\rho_N}$ (see Theorem 2.11).

Also, we prove the following proposition.

Proposition 5.7 *Let S be an idempotent-surjective semigroup, $N \lhd S$, $\rho \in \mathcal{C}(S)$. Then the elements $\rho, \kappa_\rho, \rho \vee \rho_N$ are $\hat{\rho_N}$-equivalent and $\rho \subseteq \kappa_\rho \subseteq \rho \vee \rho_N$. Moreover, the element $\rho \vee \rho_N$ is the largest in the $\hat{\rho_N}$-class $\rho\hat{\rho_N}$.*

Proof Since κ_ρ is the least E-unitary congruence containing ρ and clearly $\rho \vee \rho_N$ is E-unitary, then $\rho \subseteq \kappa_\rho \subseteq \rho \vee \rho_N$. Hence $\rho \vee \rho_N \subseteq \kappa_\rho \vee \rho_N \subseteq \rho \vee \rho_N$. Therefore $\rho \vee \rho_N = \kappa_\rho \vee \rho_N$. Thus $(\rho, \kappa_\rho) \in \hat{\rho_N}$. Evidently, $(\rho, \rho \vee \rho_N) \in \hat{\rho_N}$. This implies the first part of the proposition. The second part is clear. □

Remark 5 Recall from [22] that in the class of inverse semigroups not every $\hat{\sigma}$-class has a least element.

Finally, it is easy to see that the least E-unitary congruence κ on an arbitrary E-inversive semigroup exists, too. We show that $\mathcal{H} \cap \sigma \subseteq \kappa$ in any E-inversive semigroup. Firstly, we have need the following useful proposition.

Proposition 5.8 *Let B be the least seminormal subsemigroup of an E-inversive semigroup S. If ϕ is an epimorphism of S onto an E-unitary semigroup T, then $B\phi \subseteq E_T$.*

Proof Put $A = (E_T)\phi^{-1}$. Clearly, A is a full subsemigroup of S, so A is dense. Further, if $xy \in A$, then $E_T \ni (xy)\phi = x\phi \cdot y\phi = y\phi \cdot x\phi = (yx)\phi$ (since E_T is reflexive), so $yx \in A$. Hence $B \subseteq A$. Thus $B\phi \subseteq A\phi \subseteq ((E_T)\phi^{-1})\phi \subseteq E_T$. □

We may now prove the following equivalent theorem to Theorem 5.5.

Theorem 5.9 *In any E-inversive semigroup S, $\mathcal{H} \cap \sigma \subseteq \kappa$. If in addition E_S forms a semilattice, then $\mathcal{L} \cap \sigma \subseteq \kappa$ and $\mathcal{R} \cap \sigma \subseteq \kappa$.*

Proof Indeed, $\sigma = \rho_B$, where B is the least seminormal subsemigroup of S. Let $(a, b) \in \mathcal{H} \cap \sigma$. Then clearly $(a\kappa, b\kappa) \in \mathcal{H}^{S/\kappa}$. Also, $ax = yb$ for some $a, b \in B$. In the light of Proposition 5.8, $(a\kappa)(x\kappa) = (y\kappa)(b\kappa)$, where $a\kappa, b\kappa \in E_{S/\kappa}$. Hence $(a\kappa, b\kappa) \in \mathcal{H}^{S/\kappa} \cap \sigma_{S/\kappa} = 1_{S/\kappa}$ (Theorem 5.5). Thus $\mathcal{H} \cap \sigma \subseteq \kappa$, as required. □

6 Group congruences on an eventually regular semigroup

Group congruences on eventually regular semigroups were described in [9] by Hanumantha Rao and Lakshmi. In the paper [9] the following definition was introduced: a subset A of S is called *self-conjugate* if $x^{r(x)-1}(x^{r(x)})^* Ax \subseteq A$ and $xAx^{r(x)-1}(x^{r(x)})^* \subseteq A$ for all $x \in S$, $(x^{r(x)})^* \in V(x^{r(x)})$. We say that A is *self-conjugate* if the former condition holds.

Lemma 6.1 *Let N be a subsemigroup of an eventually regular semigroup S. Then N is normal if and only if N is full, self-conjugate and closed.*

Proof Let N be normal, $x \in S$, $(x^{r(x)})^* \in V(x^{r(x)})$. Then N is full and closed. Also, $x^{r(x)}(x^{r(x)})^* N \subseteq EN \subseteq N$, so $x^{r(x)-1}(x^{r(x)})^* Nx \subseteq N$, since N is reflexive.

Let N be full, self-conjugate and closed, $xy \in N$, $(x^{r(x)})^* \in V(x^{r(x)})$. Then $x^{r(x)-1}(x^{r(x)})^*(xy)x \in x^{r(x)-1}(x^{r(x)})^* Nx \subseteq N$, i.e., $(x^{r(x)-1}(x^{r(x)})^* x)(yx) \in N$, where $x^{r(x)-1}(x^{r(x)})^* x \in E_S \subseteq N$. Hence $yx \in N\omega = N$, so N is reflexive. Thus $N \lhd S$. □

Lemma 6.2 *Let S be an eventually regular semigroup, $N \lhd S$. Then*

$$\rho_N = \{(a,b) \in S \times S : \exists \left(b^{r(b)}\right)^* \in V\left(b^{r(b)}\right) \left[ab^{r(b)-1}\left(b^{r(b)}\right)^* \in N\right]\}.$$

Proof Let $(a,b) \in \rho_N$ and $(b^{r(b)})^* \in V(b^{r(b)})$. Then $na = bm$ for some $n, m \in N$. Hence $nab^{r(b)-1}(b^{r(b)})^* = bmb^{r(b)-1}(b^{r(b)})^*$. Also, since $b^{r(b)-1}(b^{r(b)})^*b \in E_S$, then $mb^{r(b)-1}(b^{r(b)})^*b \in NE_S \subseteq N$, so $nab^{r(b)-1}(b^{r(b)})^* = bmb^{r(b)-1}(b^{r(b)})^* \in N$, since N is reflexive. Consequently, $ab^{r(b)-1}(b^{r(b)})^* \in N\omega = N$.

Conversely, let $a, b \in S$, $(b^{r(b)})^* \in V(b^{r(b)})$ and $ab^{r(b)-1}(b^{r(b)})^* = n \in N$. Then $a(b^{r(b)-1}(b^{r(b)})^*b) = nb$, where $b^{r(b)-1}(b^{r(b)})^*b \in E_S \subseteq N$. Hence $(a,b) \in \rho_N$. □

We have the following corollary (see Theorem 1 [9]).

Corollary 6.3 *Let S be an eventually regular semigroup, $N \lhd S$. Then*

$$\rho_N = \{(a,b) \in S \times S : \exists \left(b^{r(b)}\right)^* \in V\left(b^{r(b)}\right) \left[ab^{r(b)-1}\left(b^{r(b)}\right)^* \in N\right]\}$$

is a group congruence on S.

Finally, we give some remarks concerning group congruences on inverse semigroups. Firstly, consider the following result (see Exercise 7(ii) [11], p. 181).

Statement 6.4 *An inverse subsemigroup N of an inverse semigroup S is normal if and only if $(Nx)\omega = (xN)\omega$ for every $x \in S$.*

This result is false. Indeed, let S be a Clifford semigroup. Put $N = \mathcal{Z}(S)$, where $\mathcal{Z}(S) = \{s \in S : \forall a \in S\ [sa = as]\}$. Clearly, N is a full subsemigroup of S. Also, N is self-conjugate. If the result is valid, then N is normal (since $Nx = xN$ for every $x \in S$). Hence $\rho_N = S \times S = \rho_S$, when $S = S^0$. It follows that every Clifford semigroup is commutative, a contradiction. Consequently, we conclude that the above result is false. Moreover, the assumptions of the result and the conditions: "N is full" and "N is self-conjugate" do not imply that $(Nx)\omega = (xN)\omega$ for every $x \in S$.

It is clear that every subgroup of a group is full and closed. We prove now a correct version of the above statement.

Proposition 6.5 *A full and closed inverse subsemigroup N of an inverse semigroup S is normal if and only if $(Nx)\omega = (xN)\omega$ for every $x \in S$.*

Proof It is easy to see that if N is normal, then $(Nx)\omega = (xN)\omega$ for every $x \in S$.

Conversely, let $(Nx)\omega = (xN)\omega$ for every $x \in S$. It is easy to check that two relations $\rho_1 = \{(a,b) \in S \times S : ab^{-1} \in N\}$ and $\rho_2 = \{(a,b) \in S \times S : a^{-1}b \in N\}$ are equivalences on S and that $x\rho_1 = (Nx)\omega$, $x\rho_2 = (xN)\omega$ for every $x \in S$. Also, ρ_1 is right compatible and ρ_2 is left compatible. Indeed, we show first that the equality $(A(B\omega))\omega = (AB)\omega$ holds for all $A, B \subseteq S$. Recall from [11] that

$$H\omega = \{s \in S : \exists h \in H\ [h \leq s]\} \quad (H \subseteq S),$$

where $\leq$ is the so-called *natural partial order* on (an inverse semigroup) S (i.e., $a \leq b \iff \exists e \in E_S\ [a = eb]$). Notice that $\leq$ is compatible. Let $x \in (A(B\omega))\omega$. Then $ay \leq x$ for some $a \in A, y \in B\omega$ (that is, $b \leq y$ for some $b \in B$). Hence $ab \leq ay \leq x$. Thus $x \in (AB)\omega$. We have just proved that $(A(B\omega))\omega \subset (AB)\omega$. The opposite inclusion is clear. Let now $(a, b) \in \rho_2, c \in S$. Then $(aN)\omega = (bN)\omega$ and so $(c(aN)\omega)\omega = (c(bN)\omega)\omega$. Therefore $(caN)\omega = (cbN)\omega$. Thus ρ_2 is a left congruence on S. We may equally well show that ρ_1 is a right congruence on S. Since $(Nx)\omega = (xN)\omega$ and $x\rho_1 = (Nx)\omega$, $x\rho_2 = (xN)\omega$ for every $x \in S$, then $\rho_1 = \rho_2$ is a congruence on S. Put for simplicity $\rho = \rho_1 = \rho_2$. Finally, if $e \in E_S$, then $E_S \subseteq N = N\omega = (eN)\omega$. Hence ρ is a group congruence on S and $\ker \rho = N$. Thus $N \lhd S$, as required. □

Corollary 6.6 *A Clifford semigroup S is commutative if and only if $\mathcal{Z}(S)$ is closed in S (i.e., if and only if for every $s \in S$ there exists $z \in \mathcal{Z}(S)$ such that $z \leq s$).*

Lemma 6.7 *Let S be a finite inverse semigroup with semilattice of idempotents E. Then $E\omega = S$ if and only if S has zero.*

Proof It is clear that if S has zero, then $E\omega = S$. Conversely, let $E\omega = S$. Since E is finite, then E has the least idempotent with respect to the natural partial order, say 0. Let $s \in S = E\omega$. Then $e = fs$ and $e = sg$ for some $e, f, g \in E$ (see Proposition V.2.2 in [11]). Hence $0 = 0s = s0$. Thus $S = S^0$, as required. □

By an analogy to groups we may introduce the concept of a *σ-simple inverse semigroup* in the class of finite inverse semigroups without 0. From Lemma 6.7 follows that every finite inverse semigroup S without zero has at least one non-universal group congruence, so S has exactly one non-universal group congruence if and only if $S/E\omega$ is a simple group. Hence we may say that a finite inverse semigroup S without zero is *σ-simple* if $S/E\omega$ is a simple group. This definition is equivalent to the following definition: S is σ-simple if S has exactly two normal subsemigroups, namely: $E\omega$ and S.

Example 6.8 Let $(E, \leq)$ be a chain with the least element 0. Put $S = E \cup \{a\}$, where $a \notin E$ and $aaa = a$. Assume also that $aa = 0$. Hence $a = aaa = 0a = a0$. It is easy to see that if a binary operation on S is associative, then $ea = ae = a$ for every $e \in E_S$. For example, $ea = e(0a) = (e0)a = 0a = a$. Conversely, it is straightforward to verify that such defined binary operation is associative. Thus S is a semigroup. Since $a = a^{-1}$, then S is an inverse semigroup. Finally, $E = E\omega$, so $S/E = \{E, \{a\}\}$.

7 The hypercore of a semigroup

In [8] Hall and Munn studied the hypercore of a semigroup. In this section we give some remarks on the hypercore of E-inversive E-semigroups and inverse semigroups.

Let S be a semigroup with $E_S \neq \emptyset$. Denote by $\wp_S$ the set of all subsemigroups A of S such that A has no cancellative congruences except the universal congruence.

Note that $\{e\} \in \wp_S$ for every $e \in E_S$. Define the *hypercore* hyp(S) of S, as follows: $\mathrm{hyp}(S) = \langle \bigcup \{A : A \in \wp_S\} \rangle$ [8]. Furthermore, by the *core* core(S) of an E-inversive semigroup S we shall mean $\ker \sigma$.

In [8] the authors showed the following two results.

Result 7.1 *Let S be an E-inversive semigroup. Then*:

(i) $\mathrm{hyp}(S) \in \wp_S$;
(ii) $\mathrm{hyp}(S)$ *is full and unitary*;
(iii) $\forall \rho \in \mathcal{GC}(S)\ [\mathrm{hyp}(S) \subseteq \ker \rho]$.

Result 7.2 *In any E-inversive semigroup S*, hyp(S) *is the greatest E-inversive subsemigroup of S with no non-universal group congruence*.

Let U be the least full unitary subsemigroup of an E-inversive semigroup S. Clearly, $U \subseteq \mathrm{hyp}(S) \subseteq \mathrm{core}(S)$.

Finally, we have the following proposition.

Proposition 7.3 *Let S be an E-inversive E-semigroup such that $1_S \notin \mathcal{GC}(S)$. Then $U = \mathrm{hyp}(S) = \mathrm{core}(S) = E_S\omega$. In particular, $E_S\omega$ has no non-universal group congruence*.

If in addition S is an inverse semigroup and $E_S\omega$ is finite, then $E_S\omega$ is an inverse semigroup with zero. In particular, every finite inverse semigroup S (which is not a group) contains exactly one normal inverse subsemigroup with zero.

Proof Let S be an E-inversive E-semigroup. Then $\mathrm{core}(S) = E_S\omega$ (Theorem 4.5). Since $E_S \subseteq U$ and U is closed, then $E_S\omega \subseteq U$, so $U = \mathrm{hyp}(S) = \mathrm{core}(S) = E_S\omega$. In the light of Result 7.2, $E_S\omega$ has no non-universal group congruence.

If S is an inverse semigroup, then obviously $U = \mathrm{hyp}(S) = \mathrm{core}(S) = E_S\omega$ has no non-universal group congruence. Finally, if $E_S\omega$ is finite, then $E_S\omega$ has zero (Lemma 6.7). The rest of the proposition is now immediate. □

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