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SHORT NOTE

Growth Estimates for $\exp(A^{-1}t)$ on a Hilbert Space

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Abstract

Let A be the infinitesimal generator of an exponentially stable, strongly continuous semigroup on the Hilbert space H. Since A^{-1} is a bounded operator, it is the infinitesimal generator of a strongly continuous semigroup. In this paper we show that the growth of this semigroup is bounded by a constant time $\log(t)$.

1. Introduction

Over the last five years there is a growing interest in the behavior of the semigroup generated by A^{-1}, where A is the infinitesimal generator of a bounded semigroup. Note that throughout the paper we assume that A^{-1} exists as a closed, densely defined operator. This interest was raised by three questions. The first one is coming from systems theory, see Curtain [2].

Within infinite-dimensional systems theory one studies the following set of equations:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \qquad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{1}$$

where A: $D(A) \subset H \to H$ is the infinitesimal generator of a C_0-semigroup on the Hilbert space H, B is a bounded linear operator from the Hilbert space U to the dual of the domain of A^*, i.e., $B \in \mathcal{L}(U, D(A^*)')$, $C \in \mathcal{L}(D(A), Y)$, where Y is a third Hilbert space, and $D \in \mathcal{L}(U, Y)$. In [2] the following related system was introduced:

$$\begin{aligned} \dot{x}_1(t) &= A^{-1}x_1(t) + A^{-1}Bu_1(t) \\ y_1(t) &= -CA^{-1}x_1(t) + (D - CA^{-1}B)u_1(t). \end{aligned} \tag{2}$$

This system has the nice property that $A^{-1}B \in \mathcal{L}(U, H), CA^{-1} \in \mathcal{L}(H, Y)$. Hence this system is a bounded linear system as studied in the text book by Curtain and Zwart [3]. Furthermore, the systems (1) and (2) share many properties. For instance, (1) is input-state stable if and only if (2) is. Here input-state stability means that for all inputs $u \in L^2((0, \infty); U)$ the solution of (1) exists and is (uniformly) bounded on $[0, \infty)$.

The only stability property which is not known is the state-state stability, i.e., if the semigroup generated by A is (strongly) stable if and only if the semigroup generated by A^{-1} (strongly) stable.

The second motivation for our problem comes from numerical analysis. Consider the (abstract) differential equation

$$\dot{x}(t) = Ax(t), \qquad x(0) = x_0. \tag{3}$$

A standard method for solving this differential equation is by using the Crank-Nicolson method. In this method the differential equation (3) is replaced by the difference equation

$$x_d(n+1) = (I + \Delta A/2)(I - \Delta A/2)^{-1} x_d(n), \qquad x_d(0) = x(0), \tag{4}$$

where Δ is the time step.

If H is finite-dimensional, and thus A is a matrix, then it is easy to show that the solutions of (3) are bounded if and only if the solutions of (4) are bounded. In Azizov, Barsukov, and Dijksma [1], Gomilko [4], and in Guo and Zwart [6], it is shown that the solutions of (4) are bounded if both A and A^{-1} generate a uniformly bounded semigroup. The question whether the uniform boundedness of the semigroup generated by A is sufficient is still open. The answer to this question will be positive, if the uniform boundedness of the semigroup generated by A implies the uniform boundedness of the semigroup generated by A^{-1}.

The problem whether the inverse of the generator of a bounded semigroup is again a generator of a bounded semigroup was posed as an open problem by deLaubenfels in [7]. This serves as our third motivation.

In the above, we assumed that A generates a uniformly bounded semigroup on a Hilbert space. In Zwart [8] it was shown that if A generates a uniformly bounded semigroup on a Banach space, then the growth of $\exp(A^{-1}t)$ is bounded by a constant times $\sqrt[4]{t}$. It is even shown that this estimate is sharp, i.e., there exists a Banach space and a generator of a nilpotent semigroup, such that $\|\exp(A^{-1}t)\| = m\sqrt[4]{t}$ for some m. In this paper we show that for Hilbert spaces the situation is less dramatic. In Section 2 we prove that the growth is always bounded by a constant time $\log(t)$. However, before we can prove this, we need the following result on Lyapunov equations. For the proof we refer to Curtain and Zwart [3, section 4.1 and 5.1].

Lemma 1.1. *Let A, with domain $D(A)$, generate a strongly continuous semigroup $(\exp(At))_{t\geq 0}$ on a Hilbert space H, and let C be a bounded operator from H to a Hilbert space Y. Then the following are equivalent:*

1. *There exists an $m \geq 0$ such that for all $x \in D(A)$ there holds*

$$\int_0^\infty \|C\exp(At)x\|^2\,dt \leq m\|x\|^2.$$

2. *There exists a self-adjoint, non-negative $Q \in \mathcal{L}(H)$ such that*

$$\langle Ax, Qz\rangle_H + \langle Qx, Ay\rangle_H = -\langle Cx, Cz\rangle_Y, \qquad x, z \in D(A). \tag{5}$$

3. *There exists a self-adjoint, non-negative $Q \in \mathcal{L}(H)$ such that $QD(A) \subset D(A^*)$*

$$A^*Q + QA = -C^*C \qquad \text{on } D(A). \tag{6}$$

Furthermore, the following additional results hold:

1. *If item 2. or 3. holds, then $\langle x, Qy\rangle \geq \int_0^\infty \langle C\exp(At)x, C\exp(At)y\rangle\, dt$ for all $x, y \in H$, where equality holds if exp(At) is strongly stable, i.e. when $\lim_{t\to 0}\exp(At)x = 0$ for all $x \in H$.*

2. *If item 1. holds, then one can choose Q as*

$$\langle x, Qy\rangle = \int_0^\infty \langle C\exp(At)x, C\exp(At)y\rangle\, dt, \qquad x, y \in H.$$

2. New results on bounded semigroups

We begin with a lemma which is based on Lemma 1.1.

Lemma 2.1. *Let A generate a bounded C_0-semigroup $\exp(At)$ on a Hilbert space H, and let M equal $\sup_{t\geq 0}\|\exp(At)\|$. Then for all $\varepsilon > 0$, $\gamma > 0$, and all $x \in H$, we have that*

$$\int_0^\infty \|(\gamma A - \varepsilon I)^{-1}\exp((\gamma A - \varepsilon I)^{-1}t)x\|^2\, dt \leq M^2\|x\|^2\frac{1}{2\varepsilon}.$$

The same estimate holds for the adjoint.

Proof. Since $\|\exp(At)\|$ is bounded by M, we have that for $\gamma, \varepsilon > 0$

$$\int_0^\infty \|\exp((\gamma A - \varepsilon I)t)x_0\|^2\, dt \leq \frac{M^2}{2\varepsilon}\|x_0\|^2.$$

Hence by Lemma 1.1, there exists a non-negative bounded operator $Q_{\gamma,\varepsilon}$ satisfying $Q_{\gamma,\varepsilon}D(A) \subset D(A^*)$,

$$(\gamma A - \varepsilon I)^*Q_{\gamma,\varepsilon} + Q_{\gamma,\varepsilon}(\gamma A - \varepsilon I) = -I \qquad \text{on } D(A), \tag{7}$$

and $\|Q_{\gamma,\varepsilon}\| \leq \frac{M^2}{2\varepsilon}$.

Multiplying this equation from the right by $(\gamma A - \varepsilon I)^{-1}$ and from the left by $\left((\gamma A - \varepsilon I)^{-1}\right)^* = (\gamma A^* - \varepsilon I)^{-1}$, we obtain

$$(\gamma A^* - \varepsilon I)^{-1}Q_{\gamma,\varepsilon} + Q_{\gamma,\varepsilon}(\gamma A - \varepsilon I)^{-1} = -(\gamma A^* - \varepsilon I)^{-1}(\gamma A - \varepsilon I)^{-1}. \tag{8}$$

By Lemma 1.1 this implies that

$$\int_0^\infty \|(\gamma A-\varepsilon I)^{-1}\exp((\gamma A-\varepsilon I)^{-1}t)x\|^2\,dt \le \langle x, Q_{\gamma,\varepsilon}x\rangle \le \frac{M^2}{2\varepsilon}\|x\|^2$$

which proves the result. ∎

Using this lemma, we obtain a growth estimate for the semigroup generated by $(\gamma A-I)^{-1}$.

Theorem 2.2. *Let A be the infinitesimal generator of the bounded semigroup $(\exp(At))_{t\ge 0}$ on the Hilbert space H, and let $M=\sup_{t\ge 0}\|\exp(At)\|$. Then for all $\gamma>0$,*

$$\|\exp((\gamma A-I)^{-1}t)\| \le \begin{cases} 1+\frac{M}{\sqrt{2}}\sqrt{e}+\frac{M^2(e-1)}{2\sqrt{e}}\log(t) & t\ge e\\ 1+\frac{M}{\sqrt{2}}\sqrt{t} & t\in[0,e]. \end{cases} \tag{9}$$

Proof. The proof consists out of several steps. The estimate on $[0,e]$ is proved in Step 1. In the second step we compare the semigroups $\exp((\gamma A-\varepsilon_1 I)^{-1}t_1)$ and $\exp((\gamma A-\varepsilon_2 I)^{-1}t_1)$. This is used in the third step to compare $\exp((\gamma A- I)^{-1}t_1)$ and $\exp((\gamma A-e^{-N}I)^{-1}t_1)$, for $N\in\mathbb{N}$. In the last step, we combine step 3 with step 1, and derive the estimates in (9) on $[e,\infty)$.

Step 1. For $t\in\mathbb{R}$ we have that

$$\exp((\gamma A-I)^{-1}t)x = x+\int_0^t(\gamma A-I)^{-1}\exp((\gamma A-I)^{-1}s)x\,ds.$$

Using Cauchy-Schwarz, and Lemma 2.1 we find

$$\|\exp((\gamma A-I)^{-1}t)x\| \le \|x\|+\sqrt{t}\frac{M}{\sqrt{2}}\|x\|.$$

Thus we have proved the estimate on $[0,e]$.

Step 2. Let $t_1>0$ be fixed, then by the variation of constant formula, we find

$$\begin{aligned}&\exp((\gamma A-\varepsilon_1 I)^{-1}t_1)x-\exp((\gamma A-\varepsilon_2 I)^{-1}t_1)x\\ &\quad=\int_0^{t_1}\exp((\gamma A-\varepsilon_1 I)^{-1}(t_1-s))\left[(\gamma A-\varepsilon_1 I)^{-1}-(\gamma A-\varepsilon_2 I)^{-1}\right]\\ &\qquad\quad\times\exp((\gamma A-\varepsilon_2 I)^{-1}s)x\,ds.\end{aligned} \tag{10}$$

Since

$$(\gamma A-\varepsilon_1 I)^{-1}-(\gamma A-\varepsilon_2 I)^{-1}=(\gamma A-\varepsilon_1 I)^{-1}\left[\varepsilon_1-\varepsilon_2\right](\gamma A-\varepsilon_2 I)^{-1},$$

we can use Lemma 2.1, to find that

$$\begin{aligned}
&\langle y, \exp((\gamma A - \varepsilon_1 I)^{-1} t_1) x - \exp((\gamma A - \varepsilon_2 I)^{-1} t_1) x \rangle \\
&\quad = \int_0^{t_1} \langle y, \exp((\gamma A - \varepsilon_1 I)^{-1}(t_1 - s))(\gamma A - \varepsilon_1 I)^{-1} \cdot \\
&\qquad [\varepsilon_1 - \varepsilon_2] \exp((\gamma A - \varepsilon_2 I)^{-1} s)(\gamma A - \varepsilon_2 I)^{-1} x \rangle \, ds \\
&\quad = [\varepsilon_1 - \varepsilon_2] \int_0^{t_1} \langle \exp((\gamma A^* - \varepsilon_1 I)^{-1}(t_1 - s))(\gamma A^* - \varepsilon_1 I)^{-1} y, \\
&\qquad\qquad \exp((\gamma A - \varepsilon_2 I)^{-1} s)(\gamma A - \varepsilon_2 I)^{-1} x \rangle \, ds \\
&\quad \leq |\varepsilon_2 - \varepsilon_1| \frac{M^2}{2\sqrt{\varepsilon_2 \cdot \varepsilon_1}} \|x\| \|y\|,
\end{aligned}$$

where we have used the Cauchy-Schwarz inequality. Since the norm of an operator S equals $\sup_{x,y \neq 0} \frac{|\langle y, Sx \rangle|}{\|x\|\|y\|}$, we find that

$$\| \exp((\gamma A - \varepsilon_1 I)^{-1} t_1) - \exp((\gamma A - \varepsilon_2 I)^{-1} t_1) \| \leq |\varepsilon_2 - \varepsilon_1| \frac{M^2}{2\sqrt{\varepsilon_2 \cdot \varepsilon_1}}. \qquad (11)$$

Step 3. Let $N \in \mathbb{N}$ be given and choose $\varepsilon_n = e^{-n}, n \in \{0, 1, 2, \ldots, N\}$. Then

$$\begin{aligned}
&\| \exp((\gamma A - e^{-N} I)^{-1} t_1) - \exp((\gamma A - I)^{-1} t_1) \| \\
&\quad = \| \sum_{n=1}^{N} \exp((\gamma A - e^{-n} I)^{-1} t_1) - \exp((\gamma A - e^{-n+1} I)^{-1} t_1) \| \\
&\quad \leq \sum_{n=1}^{N} \frac{e-1}{2\sqrt{e}} M^2, \qquad (12)
\end{aligned}$$

where we have used (11). Hence we have that

$$\| \exp((\gamma A - e^{-N} I)^{-1} t_1) - \exp((\gamma A - I)^{-1} t_1) \| \leq \frac{M^2 (e-1)}{2\sqrt{e}} N. \qquad (13)$$

Step 4. Let $t \in [e, \infty)$ and choose $N \in \mathbb{N}$ such that $e^N \leq t < e^{N+1}$. Furthermore, define t_1 as $t * e^{-N}$. By the definition of N, we see that $1 \leq t_1 < e$.

Since $\exp((\gamma A - I)^{-1} t) = \exp((e^{-N} \gamma A - e^{-N} I)^{-1} t_1)$ we have that, see (13),

$$\begin{aligned}
\| \exp((\gamma A - I)^{-1} t) - \exp((e^{-N} \gamma A - I)^{-1} t_1) \| &\leq \frac{M^2 (e-1)}{2\sqrt{e}} N \\
&\leq \frac{M^2 (e-1)}{2\sqrt{e}} \log(t). \qquad (14)
\end{aligned}$$

Now since $t_1 \in [1, e)$ we may use step 1. to majorize $\exp((e^{-N}\gamma A - I)^{-1}t_1)$. Doing so, we find

$$\left\|\exp((e^{-N}\gamma A - I)^{-1}t_1)\right\| \le 1 + \sqrt{t_1}\frac{M}{\sqrt{2}} \le \left[1 + M\frac{\sqrt{e}}{\sqrt{2}}\right].$$

Combining this with (14), proves (9). ∎

From this theorem we derive two corollaries. Since the inverse generator of an exponentially stable semigroup is a bounded operator, it is clear that this inverse is the infinitesimal generator of a strongly continuous semigroup. In the first corollary we show that this semigroup can grow at most like the logarithm of t.

If A generates a bounded semigroup, then it is not a priori clear whether A^{-1} generates a strongly continuous semigroup, even when A^{-1} exists as a closed and densely defined operator. A natural approach to this problem, would be to consider $A - \varepsilon I$, and letting ε approach zero. In the second corollary, we show that we only have the estimate $\|\exp((A - \varepsilon I)^{-1})\| = O(|\log(\varepsilon)|)$ for $\varepsilon \downarrow 0$. Hence we cannot conclude that A^{-1} is the generator of a strongly continuous semigroup.

Corollary 2.3. *Let A generate a strongly continuous semigroup $(\exp(At))_{t\ge 0}$ on a Hilbert space H. Assume further that this semigroup is exponentially stable and satisfies $\|\exp(At)\| \le Me^{-\omega t}$ with $M \ge 1$ and $\omega > 0$. Then*

$$\|\exp(A^{-1}t)\| \le \begin{cases} 1 + \frac{M}{\sqrt{2}}\sqrt{e} + \frac{M^2(e-1)}{2\sqrt{e}}\log(t/\omega) & t \ge e\omega \\ 1 + \frac{M}{\sqrt{2}}\sqrt{\frac{t}{\omega}} & t \in [0, e\omega]. \end{cases} \tag{15}$$

Proof. We define A_0 as $A_0 = A + \omega I$. By the assumptions it is clear that A_0 generates a semigroup on H which is uniformly bounded by M. Simple manipulation gives

$$\exp(A^{-1}t) = \exp((A_0 - \omega I)^{-1}t) = \exp((\omega^{-1}A_0 - I)^{-1}t\omega^{-1}). \tag{16}$$

Using Theorem 2.2 gives the desired result. ∎

Hence if A generates an exponentially stable semigroup, then $\exp(A^{-1}t)$ can grow at most like $\log(t)$. We remark that similar result holds for the powers of $A_d = (A + I)(A - I)^{-1}$, see Gomilko [4].

Corollary 2.4. *Let A be the infinitesimal generator of a strongly continuous semigroup on the Hilbert space H. Assume that the semigroup is uniformly* *bounded by M. For $\varepsilon \in (0, e^{-1})$ we have that*

$$\| \exp((A - \varepsilon I)^{-1}) \| \leq 1 + \frac{M}{\sqrt{2}} \sqrt{e} + \frac{M^2(e-1)}{2\sqrt{e}} |\log(\varepsilon)| \tag{17}$$

Proof. We have that

$$\exp((A - \varepsilon I)^{-1}) = \exp((\varepsilon^{-1} A - I)^{-1} \varepsilon^{-1})$$

Applying Theorem 2.2 proves the assertion. ∎

From the above result, we conclude that the problem whether A^{-1} generates a strongly continuous semigroup is still open. In [5] Gomilko shows that by putting some additional conditions on the resolvent of A the operator A^{-1} generates a uniformly bounded, strongly continuous semigroup.

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