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Strongly tilting truncated path algebras

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Abstract. For any truncated path algebra Λ, we give a structural description of the modules in the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, consisting of the finitely generated (resp. arbitrary) Λ-modules of finite projective dimension. We deduce that these categories are contravariantly finite in Λ-mod and Λ-Mod, respectively, and determine the corresponding minimal $\mathcal{P}^{<\infty}$-approximation of an arbitrary Λ-module from a projective presentation. In particular, we explicitly construct—based on the Gabriel quiver Q and the Loewy length of Λ—the basic strong tilting module ${}_\Lambda T$ (in the sense of Auslander and Reiten) which is coupled with $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in the contravariantly finite case. A main topic is the study of the homological properties of the corresponding tilted algebra $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$, such as its finitistic dimensions and the structure of its modules of finite projective dimension. In particular, we characterize, in terms of a straightforward condition on Q, the situation where the tilting module $T_{\widetilde{\Lambda}}$ is strong over $\widetilde{\Lambda}$ as well. In this Λ-$\widetilde{\Lambda}$-symmetric situation, we obtain sharp results on the submodule lattices of the objects in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$, among them a certain heredity property; it entails that any module in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ is an extension of a projective module by a module all of whose simple composition factors belong to $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

1. Introduction and terminology

We let $\Lambda = KQ/I$ be a truncated path algebra of Loewy length $L+1$ for some positive integer L, meaning that KQ is the path algebra of a quiver Q with coefficients in a field K and $I \subseteq KQ$ the ideal generated by all paths of length $L+1$. Provided that K is algebraically closed, the class of truncated path algebras includes all basic hereditary algebras, as well as all basic algebras with vanishing radical square. Since we place no restrictions beyond finiteness on the quiver Q, algebraic closedness of the base field moreover entails that every finite dimensional K-algebra is Morita equivalent to a factor algebra of a truncated path algebra. Our results do not require any hypothesis on K, however. By Λ-mod (resp. Λ-Mod), we denote the category of all finitely generated (resp. all) left Λ-modules.

In Sect. 3, we structurally characterize the objects in the subcategories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, consisting of the modules of finite projective

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dimension in the categories Λ-mod and Λ-Mod, respectively. Our description rests on the following two facts: Every Λ-module M contains a unique largest submodule $U(M)$ all of whose composition factors have finite projective dimension. Moreover, there are finitely many local Λ-modules $\mathcal{A}_i$ giving rise to the following test for finiteness of the projective dimension: Namely, M belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ if and only if $M/U(M)$ is a direct sum of copies of the $\mathcal{A}_i$; see Theorem 3.1 and Corollary 3.3 for more precision. As one byproduct of this result, we see that the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ is closed under top-stable submodules (a module $N \subseteq M$ is a *top-stable submodule of* M in case $JN = JM \cap N$, where J is the Jacobson radical of Λ).

As another consequence of the mentioned "homological subdivision" of Λ-modules, we find that the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ are contravariantly finite in Λ-mod and Λ-Mod, respectively (Theorems 4.1, 4.2). The $\mathcal{A}_i$ mentioned above turn out to be the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the simple modules of infinite projective dimension; their structure is immediate from the quiver and Loewy length of Λ. This adds another instance to the short list of known classes of finite dimensional algebras A whose categories $\mathcal{P}^{<\infty}(A\text{-mod})$ are consistently contravariantly finite: So far, this has been established whenever A is stably equivalent to a hereditary algebra (see [4, p. 130]), or else when A is left serial (see [7]); the former class, in turn, contains the radical-square zero algebras. Over a truncated path algebra Λ, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$-approximation of any Λ-module M is readily accessible from a minimal projective presentation of M; the connection is described in Theorem 4.2. We conclude Sect. 4 by showing that contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ can alternatively be derived from a theorem of Smalø in [19].

The homological picture of Λ, which started emerging in [11], where the homological dimensions of Λ were pinned down in terms of the quiver of the algebra and of its Loewy length, is based on a bicoloring of the vertices of the quiver Q, precyclic versus non-precyclic. We call a vertex *e precyclic* if there is an oriented path which starts in e and ends on an oriented cycle. It is easily seen that a vertex e_i of Q is precyclic if and only if the corresponding simple left module $S_i = \Lambda e_i/Je_i$ has infinite projective dimension (for a more general result on projective dimensions of local Λ-modules in terms of their tops, see [11, Theorem 2.6]). The bicoloring continues to be pivotal in our description of the unique basic tilting module T of Λ which is Ext-injective in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. In [4, Sect. 6], such a tilting module was proved to exist, over a finite dimensional algebra A say, precisely when $\mathcal{P}^{<\infty}(A\text{-mod})$ is contravariantly finite; in case of existence, it was dubbed the *strong* (*basic*) *tilting module* in A-mod—see the beginning of Sect. 5 for background. In our scenario, that is, over a truncated path algebra Λ, the structure of the strong tilting module ${}_\Lambda T$ can be pinned down (see Theorem 5.3). In particular, our characterization permits us to construct T from the basic data, Q and L; for concrete illustrations we refer to Examples 5.6. As a consequence, the quiver and relations of the tilted algebra $\widetilde{\Lambda} := \mathrm{End}_\Lambda(T)^{\mathrm{op}}$ can in turn be determined from these data, albeit with some computational effort; instead of giving a cumbersome formal algorithm, we include two examples at the end (Sect. 9).

Typically, the tilted algebra $\widetilde{\Lambda}$ has higher Loewy length than Λ, and the basic oriented cycles of its quiver $\widetilde{Q}$ may increase in number and length when compared with those of Q. In Sect. 9, we present an example of a truncated path algebra Λ with Loewy length 3 and a quiver having only one basic oriented cycle, while $\widetilde{\Lambda}$ has Loewy length 7 and four distinct basic oriented cycles. Moreover: Whereas path length in Q clearly induces a grading of Λ, the analogue for $\widetilde{Q}$ and $\widetilde{\Lambda}$ fails, in general. However, an in-depth study of $\widetilde{\Lambda}$ does reveal a natural grading that stems from a valuation of $\widetilde{Q}$ in general, a point which will be solidified in a sequel to this article.

In Sect. 6, we describe filtrations of the objects in the categories ${}_{\Lambda}T^{\perp} \subseteq \Lambda$-mod and ${}^{\perp}({}_{\widetilde{\Lambda}}DT) \subseteq \widetilde{\Lambda}$-mod, finding parallels with the theory of quasi-hereditary algebras. As is the case for the latter algebras, any truncated path algebra Λ is standardly stratified (in the weak sense of Cline et al. [8]), relative to a suitable pre-order on the set of simples. While such stratifications are much coarser than those introduced by Dlab [9] under the same name, Frisk has recently shown that many results known for Dlab's standardly stratified algebras can be extended to the more general situation [12]. We illustrate this theory in the case of a truncated path algebra (see Theorem 6.1, Remark 6.2) and refer to Remarks 3.4 and 8.6 for more information on the connection.

Next, we proceed to a structural exploration of the objects in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. The most symmetric and transparent situation occurs when Q has no precyclic source. In Sect. 7, we show this condition to be equivalent to the requirement that the tilting bimodule ${}_{\Lambda}T_{\widetilde{\Lambda}}$ be strong on both sides (Corollary 7.2). Thus, the algebras $\widetilde{\Lambda}$ obtained by strongly tilting truncated path algebras with quivers devoid of precyclic sources constitute yet another class of algebras A enjoying contravariant finiteness of $\mathcal{P}^{<\infty}(\text{mod-}A)$. On one hand, these algebras $\widetilde{\Lambda}$ are considerably more complex in structure than the aforementioned examples. On the other hand, further applications of tilting theory yield substantial information on the objects of $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. The sharpest structural results can be found in Theorems 8.2 and 8.5. We remark that the results of Sects. 7–9 do not depend on Sect. 6, but are linked directly to Sects. 3–5.

1.1. Terminology

Let K be an arbitrary field, Q a finite quiver and L an integer ≥ 1. *Throughout, $\Lambda = KQ/I$ will stand for a truncated path algebra* with radical J and $J^{L+1} = 0$; in other words, $I \subseteq KQ$ will denote the ideal generated by all paths of length $L+1$ in Q. (Whenever we address algebras that are not necessarily of this type, we will use a different notation.) The set of vertices of Q will be identified with the paths of length zero in KQ, and further with a full sequence $e_1, \dots, e_n$ of primitive idempotents of Λ. Our convention for multiplying paths $p, q \in KQ$ is as follows: pq stands for "p after q". In keeping with this convention, we call a path p' an *initial* (or *terminal*) *subpath* of a path p if $p = p''p'$ (or $p = p'p''$). Moreover, a *path in* Λ is any residue class $p+I$, where p is a path in $KQ \setminus I$. It is clearly unambiguous to carry over the notions of length, starting point and end point from paths in KQ to paths in Λ, since the ideal I is homogeneous with respect to the path-length grading of KQ. Representatives of the simples in Λ-mod are $S_i = \Lambda e_i/Je_i$, $1 \leq i \leq n$.

A vertex e_i of Q is called *precyclic* in case there exists a path in Q which starts in e_i and terminates in a vertex lying on an oriented cycle. Dually, e_i is *postcyclic* if e_i is the endpoint of a path in Q that starts on an oriented cycle. Correspondingly, we also refer to the simple module S_i as *precyclic* or *postcyclic*.

An auxiliary concept we use is that of a *sequence of top elements* of $M \in \Lambda$-Mod: We call an element $m \in M \setminus JM$ a *top element* of M if it is normed by one of the primitive idempotents, i.e., $e_i m = m$ for some $i \in \{1, \dots, n\}$. A family $(m_r)_{r \in R}$ of top elements will be called a *sequence of top elements of M* provided that the residue classes $m_r + JM$ form a basis for M/JM over K. For instance $e_1, \dots, e_n$ is a sequence of top elements of the left regular module Λ.

By a *tilting module* ${}_A T$ over any finite dimensional algebra A we mean a finitely generated module of finite projective dimension with $\mathrm{Ext}^i_A(T, T) = 0$ for all $i \geq 1$ such that the regular left A-module A has a finite (exact) coresolution

$$0 \to {}_A A \to M_0 \to \cdots \to M_t \to 0$$

with $M_i \in \mathrm{add}(T)$ for all $i \geq 0$.

For any subcategory $\mathcal{C}$ of A-mod, we define its left and right perpendicular subcategories to be

$${}^{\perp}\mathcal{C} = \left\{ M \in A\text{-}\,\mathrm{mod} \mid \mathrm{Ext}^i_A(M, C) = 0 \text{ for all } C \in \mathcal{C} \text{ and } i \geq 1 \right\}$$

and

$$\mathcal{C}^{\perp} = \left\{ M \in A\text{-}\,\mathrm{mod} \mid \mathrm{Ext}^i_A(C, M) = 0 \text{ for all } C \in \mathcal{C} \text{ and } i \geq 1 \right\},$$

respectively. As usual, given a module C, we write $C^{\perp}$ for $\{C\}^{\perp}$.

Finally, we call the invariants

$$\mathrm{l.\,findim}\, A = \sup\{\mathrm{p\,dim}\, M \mid M \in \mathcal{P}^{<\infty}(\text{A-mod})\}$$

and

$$\mathrm{l.\,Findim}\, A = \sup\{\mathrm{p\,dim}\, M \mid M \in \mathcal{P}^{<\infty}(\text{A-Mod})\}$$

the left *little* and *big finitistic dimensions* of A, respectively.

2. Prerequisites

Let A be any finite dimensional algebra and $\mathcal{C}$ a subcategory of the category A-mod of finitely generated left A-modules which is closed under direct summands. Recall that, according to Auslander and Smalø [5], the subcategory $\mathcal{C}$ is said to be contravariantly finite in A-mod in case every object M in A-mod has a (right) $\mathcal{C}$-approximation: Such an approximation is a morphism $\psi : B \to M$ with B in $\mathcal{C}$, such that every homomorphism $C \to M$ with C in $\mathcal{C}$ factors through ψ. By a slight abuse of language, one also refers to the object B as a $\mathcal{C}$-approximation of M in this case. Whenever M has a (right) $\mathcal{P}^{<\infty}$(A-mod)-approximation, there is a "best", that is, *minimal* one, say $\phi : B(M) \to M$; it is characterized by the property that, for any endomorphism g of $B(M)$, the equality $\phi \circ g = \phi$ forces g to be an automorphism of $B(M)$. It is easily checked that $B(M)$ is isomorphic to a direct summand of any $\mathcal{C}$-approximation of M; in particular, $B(M)$ has minimal K-dimension among the approximations of M.

Subsequently, this terminology was extended to arbitrary summand-closed subcategories $\mathcal{C}$ of the big module category A-Mod. For $\mathcal{C}$ to be *contravariantly finite* in A-Mod, we require that every left A-module should have a (right) $\mathcal{C}$-approximation; such an approximation is defined as above, on waiving all conditions involving K-dimensions. The definition of minimal approximations, in turn, follows the above pattern.

We will briefly refer to a "$\mathcal{C}$-approximation" when we mean a "right $\mathcal{C}$-approximation". Since we do not consider any left approximations in this paper, this will not lead to ambiguities.

Here, we are primarily interested in the special cases $\mathcal{C} = \mathcal{P}^{<\infty}$(A-mod) and $\mathcal{C} = \mathcal{P}^{<\infty}$(A-Mod). By [4], the category $\mathcal{P}^{<\infty}$(A-mod) is contravariantly finite in A-mod if and only if every simple left A-module has a $\mathcal{P}^{<\infty}$(A-mod)-approximation. In the positive case, suppose that $\mathcal{A}_1, \dots, \mathcal{A}_n$ are the minimal approximations of the simple left A-modules. The objects in $\mathcal{P}^{<\infty}$(A-mod) are then precisely the direct summands of modules that have (finite) filtrations with consecutive factors in $\{\mathcal{A}_1, \dots, \mathcal{A}_n\}$. As a consequence, l. findim A equals the maximum of the projective dimensions of the $\mathcal{A}_i$. Due to [16], the objects in $\mathcal{P}^{<\infty}$(A-Mod) are direct limits of finitely generated modules of finite projective dimension in case $\mathcal{P}^{<\infty}$(A-mod) is contravariantly finite, which yields l. Findim A = l. findim A.

We now specialize to the situation where $A = \Lambda$ is a truncated path algebra as above. In this context, we recall a useful homological tool, the skeleton of a Λ-module M. (It is defined in general—see [6]—but the general definition can be simplified in the truncated case.) A skeleton is a special basis, reflecting the KQ-structure of M. As we will see, it provides a convenient coordinate system for the exploration of the structure of M—of homological properties in the present context. In the case of a truncated path algebra, any skeleton of M completely determines the syzygies of M up to isomorphism, for instance; see Theorem 2.3 below.

Definition 2.1. *(Skeleton of a Λ-module M).* Fix a projective cover P of M, say $P = \bigoplus_{r \in R} \Lambda z_r$, where each z_r is one of the idempotents in $\{e_1, \dots, e_n\}$ tagged with a place number r. We will refer to the family $(z_r)_{r \in R}$ as *the distinguished sequence of top elements of P*. A *path of length l* in P is any nonzero element $pz_r \in P$, where p is a path of length l in Λ (see the first paragraph under terminology above; moreover, note that $pz_r \neq 0$ forces p to start in the vertex $e(r)$ norming the top element z_r of P, that is, satisfying $e(r)z_r = z_r$). Given any set σ of paths in P, we denote by σ_l the subset consisting of the paths of length l in σ.

(a) A set σ of paths of length at most L in P is a *skeleton* of M (in P), in case there exists an epimorphism $f : P \to M$ such that, for each $l \leq L$, the family of residue classes $f(pz_r) + J^{l+1}M$, where pz_r traces the paths in σ_l, is a K-basis for $J^l M / J^{l+1} M$. Moreover, we require that σ be closed under initial subpaths, that is, if $p_2 p_1 z_r \in \sigma$, then $p_1 z_r$ in σ.

(b) A path qz_r in $P \setminus \sigma$ is called *σ-critical* if it is of the form $\alpha p z_r$, where α is an arrow and pz_r a path in σ (possibly of length zero).

In particular, the definition entails that any skeleton σ of M in P contains the distinguished sequence of top elements of P (as the subset σ_0). We will typically identify M with a quotient P/C, where $C \subseteq JP$, and focus on subsets $\sigma \subseteq P$ which are skeleta of M with respect to the canonical epimorphism $P \to P/C$. For any such skeleton σ, the set of residue classes $\{pz_r + C \mid p \in \sigma\}$ is clearly a basis for M, the subsets $\{pz_r + C \mid p \in \sigma_l\}$ inducing bases for the radical layers $J^l M/J^{l+1}M$ for $l \geq 0$. Note that the isomorphism class of P/C as a Λ-module is completely determined by the expansion coefficients of the elements $qz_r + C$ relative to the basis $\{pz_r + C \mid p \in \sigma\}$, where qz_r runs through the σ-critical paths in P.

It is easily checked that every Λ-module M has at least one skeleton (in any given projective cover P with distinguished sequence of top elements). On the other hand, when P is finite dimensional, the collection of all skeleta of modules P/C with $C \subseteq JP$ is clearly finite (provided that the distinguished sequence of top elements of P is fixed). Note that $M = \Lambda$, endowed with the distinguished top elements $e_1, \dots, e_n$, has precisely one skeleton, namely the set of all paths in Λ.

Concerning existence, the following strengthened observation will be useful in Sect. 3. The final statement of the upcoming lemma is only relevant when the module M_1 fails to be finitely generated.

Lemma 2.2. *Let M_1, M_2 be Λ-modules, not necessarily finitely generated, and let σ'' be a skeleton of M_2 (in some projective cover P_2 of M_2). If M_2 is an epimorphic image of M_1, then σ'' can be supplemented to a skeleton $\sigma = \sigma' \sqcup \sigma''$ of M_1 (in a suitable projective cover of the form $P_1 \oplus P_2$ of M_1).*

Moreover, given any epimorphism $\pi : M_1 \to M_2$, the skeleton $\sigma = \sigma' \sqcup \sigma''$ of M_1 may be chosen (dependent on π) in such a fashion that σ' is empty precisely when π is an isomorphism.

Proof. Without loss of generality, $M_2 = M_1/U$, and $\pi : M_1 \to M_2$ is the canonical epimorphism. Moreover, it is harmless to start with a projective cover $f : P \to M_1$ such that $P = P_1 \oplus P_2$, for a suitable projective module P_1, with the property that $f_2 := \pi \circ f|_{P_2} : P_2 \to M_2$ is a projective cover of M_2 satisfying condition (a) of the definition, relative to the skeleton σ'' of M_2. In other words, we assume that the elements $f_2(q) + J^{l+1}M_2$, with $q \in \sigma''_2$ form a basis for $J^l M_2/J^{l+1}M_2$. Say $(z_r)_{r \in R_1}$ and $(z_r)_{r \in R_2}$ are the distinguished sequences of top elements of P_1 and P_2, where R_1 and R_2 are disjoint index sets; then the union of all the z_r is the distinguished sequence of P. Note that $\sigma''_0 = \{z_r \mid r \in R_2\}$, set $\sigma'_0 = \{z_r \mid r \in R_1\}$, and define $\sigma_0 = \sigma'_0 \cup \sigma''_0$. The set

$$\left\{f(q) + J^2M_1 \mid q \in \sigma''_1\right\} \cup \left\{f(q) + J^2M_1 \mid q \text{ is a path of length 1 in } P_1\right\}$$

generates JM_1/J^2M_1, and the first of the two subsets is linearly independent by hypothesis. Therefore, we may choose a set σ'_1 of paths q of length 1 in P_1 such that the images $f(q) + J^2M_1$ with $q \in \sigma'_1 \cup \sigma''_1$ constitute a basis for JM_1/J^2M_1. Set $\sigma_1 = \sigma'_1 \cup \sigma''_1$. Next we find that the union of $\{f(q) + J^3M_1 \mid q \in \sigma''_2\}$ with the set of those residue classes $f(q) + J^3M_1$, which correspond to the paths q of length 2 in P_1 that contain some path in σ'_1 as a right subpath, generates J^2M_1/J^3M_1; again, the first of the listed sets is linearly independent by hypothesis. This permits us to choose a subset σ'_2 of the set

$$\{q \mid q \text{ is a path of length } 2 \text{ in } P_1,\ q = \alpha p \text{ for an arrow } \alpha \text{ and } p \in \sigma'_1\},$$

so as to obtain a basis $\{f(q) + J^3 M_1 \mid q \in \sigma_2' \cup \sigma_2''\}$ for $J^2 M_1 / J^3 M_1$. We define $\sigma_2 = \sigma_2' \cup \sigma_2''$, and continue inductively. Our construction then guarantees that the resulting set $\sigma = \bigcup_{0 \le l \le L} \sigma_l$ is a skeleton of M_1 with the required properties. □

As announced, over a truncated path algebra, any skeleton of a module determines its syzygies. More precisely, we obtain:

Theorem 2.3. A known fact. [6, Lemma 5.10] *If M is a nonzero left Λ-module, not necessarily finitely generated, and σ any skeleton of M (in a suitable projective cover of M), then*

$$\Omega^1(M) \cong \bigoplus_{q\ \sigma\text{-critical}} \Lambda q.$$

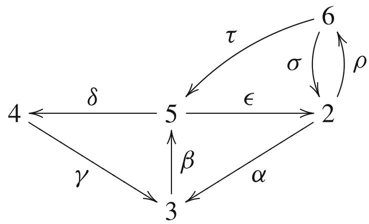
In particular, $\Omega^1(M)$ is isomorphic to a direct sum of cyclic left ideals generated by nonzero paths of positive length in Λ. □

The following observation was already used in [11].

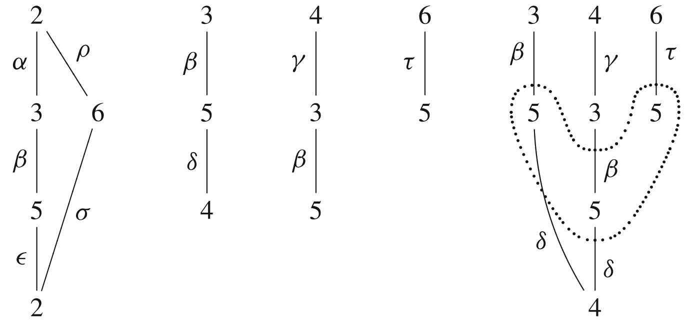
Observation 2.4. *Given a path q of positive length in Λ, the cyclic left ideal Λq has finite projective dimension if and only if the endpoint of q is not precyclic. As a consequence, a simple module $\Lambda e / J e$ has finite projective dimension precisely when it is not precyclic.* □

A major asset of truncated path algebras lies in the ease with which computations implicit in the theory can be carried out graphically via the *layered and labeled graphs* of modules as described in [14,15]. The following illustration of module graphs (over a nontruncated finite dimensional algebra $\widetilde{\Lambda}$) is to provide an informal reminder of all that is relevant for the present article. The algebra we use for this purpose will resurface in Example 9.1.

Example 2.5. Let $A = K\widetilde{Q}/\widetilde{I}$, where $\widetilde{Q}$ is the quiver



and $\widetilde{I} \subseteq K\widetilde{Q}$ is the ideal generated by the following relations: $\alpha\epsilon\beta\alpha$, $\tau\rho$, $\rho\epsilon\beta$, $\delta\beta\alpha$, $\rho\sigma\rho$, $\beta\alpha\epsilon\beta$, $\delta\beta\gamma\delta$, $\delta\tau$, $\epsilon\beta\alpha - \sigma\rho$, and $\alpha\epsilon - \gamma\delta$. The following are examples of layered and labeled graphs of certain left A-modules:



Ae_2 M_1 M_2 M_3 M

The leftmost graph represents the indecomposable projective left A-module Ae_2. Its shape shows that $\epsilon\beta\alpha = k\sigma\rho$ for some $k \in K^*$; in the present situation, the scalar is $k = 1$, due to the relations of A. The labeling of the edges is redundant in this example, since the quiver $\widetilde{Q}$ does not contain multiple edges between any pair of vertices. (For graphs of the remaining indecomposable projective left A-modules, we refer to Example 9.1.)

The three tree graphs in the center depict uniserial modules M_i, $i = 1, 2, 3$. For instance, $M_1 \cong Ae_3/A\delta\beta$. If x_i is a top element of M_i (see *Terminology* in Sect. 1), the rightmost graph represents the isomorphism class of the module

$$M = (M_1 \oplus M_2 \oplus M_3) / A\,(k_1\beta x_1 + k_2\beta\gamma x_2 + k_3\tau x_3)$$

with $k_1, k_2, k_3 \in K^*$ (the choice of these scalars clearly does not impinge on the isomorphism type of M); the dotted line enclosing the vertices representing βx_1, $\beta\gamma x_2$, τx_3 signifies that these elements are K-linearly dependent, while any two of them are K-linearly independent. The relation $\delta\beta x_1 \in K^*\delta\beta\gamma x_2$ in M, shown in the graph, is a consequence of the relation $\delta\tau = 0$.

Clearly, a graph of a module X need not determine X up to isomorphism, unless it is a tree. Conversely, the isomorphism class of X will typically not lead to a unique graph representing it, but will do so only once a sequence of top elements of X has been specified. An alternate graph of the module M above, for example, is a disjoint union of two nontrivial subgraphs, thus displaying decomposability of M at first glance.

We remark, moreover, that any skeleton σ of a finitely generated module M can be pinned down in a visually suggestive format by a finite forest of tree graphs, one tree for each element in the chosen sequence of top elements z_r in the distinguished projective cover P of M; we give an illustration below. The skeleton σ can be retrieved from its graph as the set of all edge paths of length ≥ 0 that start in a vertex representing an element $z_r \in \sigma_0$. Any such graph of a skeleton displays a composition series of M, recording, from the top down, the simple composition factors in the radical layering $\mathbb{S}(M) = \left(J^l M / J^{l+1} M\right)_{l \geq 0}$. Below, we give examples of skeleta for two of the modules displayed in Example 2.5. The left-hand graph depicts the (unique) skeleton of Ae_2, and on the right we exhibit one of the two skeleta of the module M of that example. The graphs of the three uniserial modules M_i coincide with graphs of their skeleta.

$$\begin{tikzcd}
2 \arrow[d, dash, "\alpha"'] \arrow[dr, dash, "\rho"] & & & 3 \arrow[d, dash, "\beta"'] & 4 \arrow[d, dash, "\gamma"'] & 6 \\
3 \arrow[d, dash, "\beta"'] & 6 & & 5 & 3 \arrow[d, dash, "\beta"'] & \bullet \\
5 \arrow[d, dash, "\epsilon"'] & & & & 5 \arrow[d, dash, "\delta"'] & \\
2 & & & & 4 &
\end{tikzcd}$$

3. Structure of the Λ-modules of finite projective dimension: first installment

We continue to let Λ denote a truncated path algebra. The pivotal result of this section is Theorem 3.1, which exhibits a strong finiteness property of the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. This property is responsible for the first set of structure results which we assemble in Corollary 3.3. Moreover, Theorem 3.1 will readily yield contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ in Λ-mod and Λ-Mod, respectively (Sect. 4). To describe this finiteness property, we let

ε be the sum of the idempotents corresponding to the non-precyclic vertices of Q

and observe that, for any Λ-module M, the subspace εM is actually a Λ-submodule. The Λ-module εM always has finite projective dimension as all of its composition factors are non-precyclic. The next theorem will show that the functor

$$F = \Lambda/\varepsilon J \otimes_\Lambda - : \Lambda\text{-Mod} \to \Lambda\text{-Mod}, \quad M \mapsto M/\varepsilon J M$$

takes the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ to a category of finite representation type, namely to the full subcategory of Λ-Mod, whose objects are direct sums of copies of the local modules $\mathcal{A}_i = \Lambda e_i/\varepsilon J e_i$, for $1 \le i \le n$. Clearly, each $\mathcal{A}_i$ has finite projective dimension, and equality $\mathcal{A}_i = S_i$ holds precisely when the corresponding vertex e_i is non-precyclic. Moreover, each $\mathcal{A}_i$ has a tree graph (relative to the top element $e_i + \varepsilon J e_i$) and is thus determined up to isomorphism by this graph.

Our proof of the next theorem rests on the prerequisites we have established in the previous section.

Theorem 3.1. *Let M be any Λ-module, not necessarily finitely generated. Then the following conditions are equivalent:*

(i) $\operatorname{p dim} M < \infty$.

(ii) *$F(M)$ is a direct sum of copies of the local modules $\mathcal{A}_1, \dots, \mathcal{A}_n$.*

(iii) *Given a projective cover $f : P \to M$, all simple composition factors of* $\operatorname{Ker}(f)$ *are non-precyclic.*

Remark. Thus, each object M of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ is an extension of the $\varepsilon\Lambda\varepsilon$-module εM by a direct sum of copies of the (indecomposable projective) $(1-\varepsilon)\Lambda(1-\varepsilon)$-modules $\mathcal{A}_i$ corresponding to the precyclic vertices e_i, where all the mentioned modules have a canonical left Λ-structure, since $\varepsilon\Lambda\varepsilon = \Lambda\varepsilon$. Conversely, every such extension belongs to $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. In short, if we arrange the simple modules so that $S_1, \dots, S_m$ are precyclic and $S_{m+1}, \dots, S_n$ non-precyclic, we obtain $\varepsilon = e_{m+1} + \cdots + e_n$ and find

$$\mathcal{P}^{<\infty}(\Lambda\text{-Mod}) = \mathrm{Ext}^1_\Lambda\,(\mathrm{Add}(\mathcal{A}_1, \dots, \mathcal{A}_m), \varepsilon\Lambda\varepsilon\text{-}\mathrm{Mod}),$$

and

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \mathrm{Ext}^1_\Lambda\,(\mathrm{add}(\mathcal{A}_1, \dots, \mathcal{A}_m)\,,\ \varepsilon\Lambda\varepsilon\text{-}\mathrm{mod}).$$

Proof of Theorem 3.1. To verify "(i) $\Longrightarrow$ (ii)", suppose that M is a nonzero module of finite projective dimension, and let $f : P \to M$ be a projective cover. Then also $\mathrm{p\,dim}(M/\varepsilon JM) < \infty$, and f induces an epimorphism $\overline{f} : P/\varepsilon JP \to M/\varepsilon JM$ with kernel contained in $JP/\varepsilon JP$. If $\overline{f}$ were not an isomorphism, Lemma 2.2 would yield a skeleton σ of $M/\varepsilon JM$ with a σ-critical path q ending in a precyclic vertex: Indeed, any skeleton σ of $M/\varepsilon JM$ would then be properly contained in a skeleton σ^+ of $P/\varepsilon JP$ in P. Any path q of minimal length in $\sigma^+ \setminus \sigma$ would be σ-critical, and, since all paths in the latter set difference have positive length and $\mathrm{Ker}(\overline{f}) \subseteq JP/\varepsilon JP$ has only precyclic composition factors, q would be as required.

By Theorem 2.3, this would force a direct summand isomorphic to Λq into the syzygy of $M/\varepsilon JM$, which, in light of Observation 2.4, would contradict finiteness of the projective dimension of $M/\varepsilon JM$. Thus $\overline{f}$ is an isomorphism $P/\varepsilon JP \cong M/\varepsilon JM$. Since the former quotient is a direct sum of copies of the $\mathcal{A}_i$, so is the latter.

To prove "(ii) $\Longrightarrow$ (iii)", suppose that $F(M) \cong \bigoplus_{1\le i\le n} \mathcal{A}_i^{r_i}$ with $r_i \ge 0$, and let $f : P \to M$ be a projective cover. Then $M/JM \cong F(M)/JF(M) \cong \bigoplus_{1\le i\le n} S_i^{r_i}$, which shows that P is isomorphic to $\bigoplus_{1\le i\le n}(\Lambda e_i)^{r_i}$. Thus

$$P/\varepsilon JP \cong \bigoplus_{1\le i\le n} \mathcal{A}_i^{r_i} \cong M/\varepsilon JM$$

by condition (ii). It follows that $\mathrm{Ker}(f) \subseteq \varepsilon JP$ as claimed.

The implication "(iii) $\Longrightarrow$ (i)" is straightforward. □

Theorem 3.1 moreover shows the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ to have an unusual closure property under certain types of subobjects. We call a submodule U of a module M *top-stably embedded in M*, in case $U \cap JM = JU$.

Corollary 3.2. *The category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ is closed under top-stably embedded submodules.*

Proof. Suppose M in Λ-Mod has finite projective dimension and $U \subseteq M$ is a top-stably embedded submodule. Let $f_1 : P_1 \to U$ be a projective cover of U. By top-embeddedness, we can extend f_1 to a projective cover $f : P = P_1 \oplus P_2 \to M$.

The argument for Theorem 3.1 now guarantees that the induced map $\overline{f} : P/\varepsilon JP \to M/\varepsilon JM$ is an isomorphism, whence $\overline{f_1}$ is an isomorphism $P_1/\varepsilon JP_1 \to U/\varepsilon JU$. Consequently, $U/\varepsilon JU$ is a direct sum of copies of the $\mathcal{A}_i$, which shows $\operatorname{p\,dim} U < \infty$. □

For any set Ψ of finitely generated left Λ-modules, we denote by $\operatorname{filt}(\Psi)$ the full subcategory of Λ-mod having as objects those modules X that have a finite filtration $X_0 = 0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$ with consecutive factors $X_i/X_{i-1} \in \Psi$. By $\operatorname{Filt}(\Psi)$ we denote the analogous subcategory obtained by waiving the requirement that the filtrations with factors in Ψ be finite, replacing the natural number r by any ordinal number.

Again let $e_1, \dots, e_m$ be the precyclic vertices of the quiver Q of Λ, and $e_{m+1}, \dots, e_n$ the non-precyclic ones; that is, the idempotent ε that gives rise to the functor F above equals $\sum_{m+1 \le i \le n} e_i$. We record the information provided by Theorem 3.1 in slightly different form for future reference.

Corollary 3.3. Structural information on the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ **and** $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$.

A. Simple objects and composition series. *The set of simple objects in* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is*

$$\Psi = \{\mathcal{A}_1, \dots, \mathcal{A}_n\} = \{\mathcal{A}_1, \dots, \mathcal{A}_m, S_{m+1}, \dots, S_n\}.$$

Moreover,

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \operatorname{filt}(\Psi) \text{ and } \mathcal{P}^{<\infty}(\Lambda\text{-Mod}) = \operatorname{Filt}(\Psi).$$

Given $M \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod})$*, the cardinal multiplicities of the simple* $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$*-composition factors of* M *(with respect to an ordinal-indexed composition series) are isomorphism invariants of* M*.*

B. Separation of precyclic and non-precyclic portions in the modules of finite projective dimension. *Any object* $M \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ *has a unique largest submodule* $U(M) = \varepsilon M$ *with the property that all simple composition factors of* $U(M)$ *are among* $S_{m+1}, \dots, S_n$*, and* $M/U(M)$ *is a direct sum of copies of the remaining simple objects in* $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$*, namely* $\mathcal{A}_1, \dots, \mathcal{A}_m$*. In particular, the simple composition factors of* $U(M)$ *in* $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ *and* Λ-Mod *coincide, and hence so do the composition lengths in the two categories, provided that* M *is finitely generated.*

C. Separation in the indecomposable projective objects Λe_i**.** *For* $i \le m$*, we have* $\Lambda e_i/U(\Lambda e_i) \cong \mathcal{A}_i$*, and given a non-precyclic simple* S_j*, its multiplicity as a composition factor of* $U(\Lambda e_i)$ *is equal to the number of paths of lengths* $\le L$ *from* e_i *to* e_j*. For* $i \ge m+1$*, we have* $U(\Lambda e_i) = \Lambda e_i$*.*

D. Finitistic dimensions. 1. $\operatorname{findim} \Lambda = 1$. $\operatorname{Findim} \Lambda = \max\{\operatorname{p\,dim} \mathcal{A}_i, p\ dim S_j \mid i \le m,\ j \ge m+1\}$.

Proof. Let M be any object in $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. As was pointed out above, $U(M) = \varepsilon M$ is then a Λ-submodule of M, which clearly has a filtration with consecutive factors among $S_{m+1}, \dots, S_n$. By Theorem 3.1, $M/U(M)$ is a direct sum of copies of $\mathcal{A}_1, \dots, \mathcal{A}_m$, and thus, evidently, $M/U(M)$ has a filtration with factors among the $\mathcal{A}_i$, $i \le m$. The remaining assertions are easy consequences. □

Corollary 3.3 will be supplemented in Corollary 5.4, where the indecomposable (relative) injective objects of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ will be identified. The combined information will prove helpful in Sect. 8, towards exploiting a duality relating the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ of modules of finite projective dimension over the strongly tilted algebra $\widetilde{\Lambda}$ to the more directly accessible category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$.

Remark 3.4. Standard stratification of truncated path algebras. Filtration categories similar to those above have been studied extensively in the context of quasi-hereditary and standardly stratified algebras (see, e.g., [1,9,18]). In fact, the set $\Psi = \{\mathcal{A}_1, \ldots, \mathcal{A}_n\} = \{\mathcal{A}_1, \ldots, \mathcal{A}_m, S_{m+1}, \ldots, S_n\}$ in the preceding corollary can be viewed as the set of *standard* Λ-modules relative to a pre-order on the set of isomorphism classes of simple Λ-modules. This pre-order is defined by specifying $S_i \preceq S_j$ if either e_i is precyclic or there exists a path from e_i to e_j in Q, and we write $S_i \prec S_j$ if $S_i \preceq S_j$ and $S_j \npreceq S_i$. It is then easy to see that $\mathcal{A}_i$ coincides with the standard module Δ_i—defined to be the unique highest-dimensional quotient of P_i having only composition factors $S_j \preceq S_i$. Under this preorder on the simples, the algebra Λ is standardly stratified in the sense of Cline–Parshall–Scott [8]: Indeed, each projective Λe_i has a filtration with top factor isomorphic to Δ_i and remaining factors isomorphic to Δ_j for $S_i \prec S_j$; moreover, the kernel of the canonical epimorphism $\Delta_i \to S_i$ has only simple composition factors S_j with $S_j \preceq S_i$. The C-P-S standardly stratified algebras have been further studied by Frisk [12], and some of our results in Sect. 6 can also be obtained as applications of the theory he develops. We continue this discussion in Remark 6.2.

4. Contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$

Theorem 4.1. *For every truncated path algebra* Λ*, the category* $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ *is contravariantly finite in* Λ-mod*, and* $\mathcal{A}_i$ *is a minimal (right)* $\mathcal{P}^{<\infty}$*-approximation of* S_i *for* $1 \le i \le n$*. In particular, the minimal approximations of the simple modules are local, and hence indecomposable.*

Proof. As we pointed out in Sect. 2, contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ follows from the existence of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the S_i. Let $\phi : \mathcal{A}_i \to S_i$ be the canonical epimorphism, sending the coset $e_i + \varepsilon J e_i$ to $e_i + J e_i$. To see that ϕ is a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i, let M be any object in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and f a nonzero map in $\mathrm{Hom}_\Lambda(M, S_i)$. Clearly, f factors through the canonical epimorphism π from M to $M/\varepsilon J M$; say $f = f' \circ \pi$. By Theorem 3.1, the latter factor module is isomorphic to a direct sum of $\mathcal{A}_j$'s. The map f', being an epimorphism onto S_i, clearly factors through a copy of $\mathcal{A}_i$ in any such decomposition, which guarantees that f factors through ϕ. Minimality of ϕ as a $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i follows from the indecomposability of $\mathcal{A}_i$. □

It is easy to describe the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of arbitrary finite dimensional Λ-modules in terms of their projective covers. In fact, the description extends to the infinite dimensional case, thus providing us with minimal $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$-approximations of arbitrary objects in Λ-Mod.

Theorem 4.2. *Let* $M \in \Lambda$-Mod, *say* $M \cong P/C$, *where* P *is projective and* $C \subseteq JP$. *If we identify* M *with* P/C, *the canonical map* $\phi : P/\varepsilon C \to P/C$ *is a minimal (right)* $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$*-approximation of* M.

In particular, $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ *is contravariantly finite in* Λ-Mod.

Proof. We note that $P/\varepsilon C$ has finite projective dimension. To see that ϕ is a $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$-approximation of P/C, let $f : N \to P/C$ be any homomorphism with source $N \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. We clearly do not lose generality in identifying N with a quotient Q/D, where Q is projective and $D \subseteq JQ$. To ascertain that f factors through ϕ, we consider the following diagram:

$$\begin{tikzcd}
P \arrow[r, "\mathrm{can}"] & P/\varepsilon C \arrow[r, "\phi"] & P/C \\
 & & Q/D \arrow[u, "f"'] \arrow[ul, dashed, "f''"] \\
 & & Q \arrow[u, "\mathrm{can}"'] \arrow[uull, dashed, "f'"]
\end{tikzcd}$$

Here the map f' making the larger triangle commute exists due to the fact that $\phi \circ \mathrm{can}$ is a surjection. To see that f' induces a map f'' rendering the smaller triangle commutative, apply Theorem 3.1 to find $\varepsilon D = D$. Hence the inclusion $D \subseteq \mathrm{Ker}(f \circ \mathrm{can})$ implies $f'(D) \subseteq \varepsilon\, \mathrm{Ker}(\phi \circ \mathrm{can}) = \varepsilon C$, and we obtain f'' as desired.

To check minimality of ϕ, let g be an endomorphism of $P/\varepsilon C$ with $\phi \circ g = \phi$. Then $g = \mathrm{id} + g'$, where the image of g' is contained in $C/\varepsilon C \subseteq J(P/\varepsilon C)$. In particular, g' is nilpotent, and hence g is invertible. □

Remark concerning skeleta of minimal approximations. The description of the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations in Theorem 4.2 has a simple interpretation in terms of skeleta (introduced in Definition 2.1). Given $M = P/C$ as above and a distinguished sequence of top elements $(z_r)_{r \in R}$ of P, any skeleton σ of M is contained in the (unique) skeleton σ' of P with respect to these top elements; here σ' consists of *all* paths in P, that is, of all nonzero elements pz_r where p is a path of length $\leq L$ in Q. We enlarge σ to obtain a skeleton σ'' of the $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation $P/\varepsilon C$ of M by adding paths from σ' as follows: The new paths are simply those paths in $\sigma' \setminus \sigma$ that end in precyclic vertices; clearly these induce a basis for $C/\varepsilon C$.

The remark on skeleta makes it straightforward to compute the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of any module $M \in \Lambda$-mod from a minimal projective presentation of M. We illustrate this in Sect. 5.

Let A be any finite dimensional algebra for the moment. As we pointed out in Sect. 2, contravariant finiteness of $\mathcal{P}^{<\infty}(A\text{-mod})$ implies that the objects of $\mathcal{P}^{<\infty}(A\text{-mod})$ can be described as the direct summands of the modules N with the following property: N has a finite filtration whose consecutive factors are among the minimal $\mathcal{P}^{<\infty}(A\text{-mod})$-approximations of the simple left A-modules. In general, this description of the objects in $\mathcal{P}^{<\infty}$(A-mod) cannot be sharpened so as to allow omitting the step of taking direct summands of suitably filtered modules. Indeed, the category of modules with filtrations as described need not be closed under direct summands. However, for truncated path algebras, it is. This was already recorded in Corollary 3.3.

4.1. A generalization of Theorem 4.1 via a result of Smalø

After establishing contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ for truncated path algebras Λ, we noticed that Theorem 4.1 permits a modest generalization by way of a theorem of Smalø. In particular, this approach yields an alternate proof for contravariant finiteness of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ when Λ is a truncated path algebra. In the following A will be an arbitrary Artin algebra.

Theorem 4.3. *[19] Consider the triangular matrix ring*

$$A = \begin{pmatrix} \Delta & 0 \\ M & \Gamma \end{pmatrix},$$

where Γ and Δ are Artin algebras and ${}_\Gamma M_\Delta$ is a bimodule such that $\operatorname{p\,dim}{}_\Gamma M < \infty$. *Then $\mathcal{P}^{<\infty}(A\text{-mod})$ is contravariantly finite in A-mod if and only if $\mathcal{P}^{<\infty}(\Delta\text{-mod})$ and $\mathcal{P}^{<\infty}(\Gamma\text{-mod})$ are contravariantly finite subcategories of Δ-mod and Γ-mod, respectively.*

Guided by Sect. 3, we apply this theorem to the following scenario.

Corollary 4.4. *Assume that $\varepsilon \in A$ is an idempotent such that the following hold:*

(i) *$\varepsilon A = A\varepsilon A$;*
(ii) $\operatorname{p\,dim}_A(A\varepsilon/J\varepsilon) < \infty$ *(in view of (i), this is equivalent to* $\operatorname{gl\,dim} \varepsilon A\varepsilon < \infty$*);*
(iii) $\mathcal{P}^{<\infty}(A/A\varepsilon A\text{-mod})$ *is contravariantly finite in* $A/A\varepsilon A$-mod.

Then $\mathcal{P}^{<\infty}$(A-mod) *is contravariantly finite in* A-mod.

Proof. Condition (i), being equivalent to $A\varepsilon = \varepsilon A\varepsilon$, implies that A is isomorphic to the triangular matrix ring

$$\begin{pmatrix} A/A\varepsilon A & 0 \\ \varepsilon A(1-\varepsilon) & \varepsilon A\varepsilon \end{pmatrix}.$$

Clearly, the condition that the corner ring $\varepsilon A\varepsilon$ has finite global dimension ensures that the remaining hypotheses of the theorem are satisfied. □

To derive Theorem 4.1 from Corollary 4.4, we observe: When $A = \Lambda$ is a truncated path algebra and ε is the sum of the non-precyclic primitive idempotents, the quotient $\Gamma := \Lambda/\Lambda\varepsilon\Lambda$ is again a truncated path algebra. Its quiver is given by the full subquiver of the quiver of Λ on the precyclic vertices. Consequently, any Γ-module of finite projective dimension is projective, i.e., $\mathcal{P}^{<\infty}(\Gamma\text{- mod}) = \operatorname{add}({}_\Gamma\Gamma)$. It follows that $\mathcal{P}^{<\infty}(\Gamma\text{- mod})$ is contravariantly finite in Γ-mod, with projective covers serving as $\mathcal{P}^{<\infty}(\Gamma\text{- mod})$-approximations. Note, moreover, that the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of a precyclic simple S_i can be identified with the minimal $\mathcal{P}^{<\infty}(\Gamma\text{-}\operatorname{mod})$-approximation of S_i, which coincides with its projective cover $\Gamma e_i = \Lambda e_i/\varepsilon\Lambda e_i = \mathcal{A}_i$. Smalø's proof of Theorem 4.3 makes use of the observation that $\mathcal{P}^{<\infty}(A\text{-}\operatorname{mod})$ can be identified with $\operatorname{Ext}^1_A(\mathcal{P}^{<\infty}(\Gamma\text{-}\operatorname{mod}), \mathcal{P}^{<\infty}(\Delta\text{-}\operatorname{mod}))$. Specializing to our setting, we hereby recover the description of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ given in the remark after Theorem 3.1:

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) = \operatorname{Ext}^1_\Lambda\,(\operatorname{add}(\Lambda/\varepsilon\Lambda), \varepsilon\Lambda\varepsilon\text{-}\operatorname{mod}).$$

In fact, Theorem 4.3 generalizes to infinite dimensional modules over a triangular matrix ring A, since $\mathcal{P}^{<\infty}(A\text{-}\operatorname{Mod})$ can be identified with $\operatorname{Ext}^1_A(\mathcal{P}^{<\infty}(\Gamma\text{-}\operatorname{Mod}), \mathcal{P}^{<\infty}(\Delta\text{-}\operatorname{Mod}))$. Hence, in the case of a truncated path algebra Λ, Smalø's arguments will also yield an alternative proof of contravariant finiteness of the big category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. The key observations providing the link are as follows. For a truncated path algebra Λ, the categories $\mathcal{P}^{<\infty}(\Lambda/\varepsilon\Lambda\text{-}\operatorname{Mod}) = \operatorname{Add}(\Lambda/\varepsilon\Lambda)$ and $\mathcal{P}^{<\infty}(\varepsilon\Lambda\varepsilon\text{-}\operatorname{Mod}) = \varepsilon\Lambda\varepsilon\text{-}\operatorname{Mod}$ are contravariantly finite in the (big) module categories $\Lambda/\varepsilon\Lambda\text{-}\operatorname{Mod}$ and $\varepsilon\Lambda\varepsilon\text{-}\operatorname{Mod}$, respectively.

5. Strong tilting modules over truncated path algebras

We start with some remarks that hold for arbitrary Artin algebras A.

In [4], Auslander and Reiten showed that any contravariantly finite resolving subcategory $\mathcal{C}$ of A-mod which is contained in $\mathcal{P}^{<\infty}$(A-mod) gives rise to a basic tilting module which is uniquely determined up to isomorphism by $\mathcal{C}$. (Following their terminology, we call a module T “basic” if its endomorphism ring is basic in the usual sense.) In the special case where the subcategory $\mathcal{C}$ equals $\mathcal{P}^{<\infty}$(A-mod), they call this tilting module *strong*. As noted by Happel and Unger (see [13]), existence provided, the strong tilting module plays a distinguished role in the partially ordered set of all basic tilting objects in A-mod: it is the unique minimal element, the regular left A-module occupying the opposite end of the spectrum. On the other hand, we do not know of interesting concrete instances, where the strong tilting module is completely understood, beyond the situation of finite global dimension of A; in this extreme case the strong tilting module is just the minimal injective cogenerator.

The purpose of this section is to explore the strong tilting module T over an arbitrary truncated path algebra; in light of Sect. 4, existence is guaranteed. We will see that constructibility of the minimal $\mathcal{P}^{<\infty}$-approximations in Λ-mod in this situation allows us to pin down T in terms of the quiver Q and the Loewy length L of Λ. From these data one can then (with some mild effort) compute the corresponding tilted algebra $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$—that is, determine quiver and relations for $\widetilde{\Lambda}$.

According to [4], the basic strong tilting module corresponding to a contravariantly finite resolving subcategory $\mathcal{C}$ of A-mod which is contained in $\mathcal{P}^{<\infty}$(A-mod), is characterized by the following property: It is the direct sum of the indecomposable Ext-injective objects of $\mathcal{C}$, one from each isomorphism class. The following lemma provides a source of Ext-injectives in any contravariantly finite resolving subcategory $\mathcal{C}$ of A-mod. It is well known [5], but we include a short proof for the convenience of the reader. Yet, this source does not yield *all* Ext-injectives in $\mathcal{C}$, in general, not even in case $\mathcal{C}$ equals the category $\mathcal{P}^{<\infty}$(A-mod) over a left serial string algebra A, as Example 5.2 will show.

Recall that a subcategory $\mathcal{C}$ of A-mod is called *resolving* if it contains the finitely generated projective left A-modules and is closed under extensions and kernels of epimorphisms.

Lemma 5.1. *Let $\mathcal{C}$ be a contravariantly finite resolving subcategory of A-*mod*, where A is an arbitrary finite dimensional algebra. Then the minimal $\mathcal{C}$-approximation of any finitely generated injective left A-module is Ext-injective in $\mathcal{C}$.*

Proof. Suppose E is a finitely generated injective left A-module, $f : B \to E$ a minimal $\mathcal{C}$-approximation of E, and $0 \to B \xrightarrow{g} X \to Y \to 0$ a short exact sequence in $\mathcal{C}$. Then f factors through g since E is injective, that is, $f = hg$ for a suitable homomorphism $h : X \to E$. Moreover, our hypothesis on f implies $h = fj$ for some $j : X \to B$, because X belongs to $\mathcal{C}$. Thus $f = fjg$, and since f is right minimal, jg is an isomorphism. This yields splitness of g. □

In the following example, $\mathcal{P}^{<\infty}$(A-mod) is contravariantly finite in A-mod, but the corresponding basic strong tilting module fails to be a direct summand of the minimal $\mathcal{P}^{<\infty}$(A-mod)-approximation of the minimal injective cogenerator of A-mod.

Example 5.2. Let $A = KQ/I$, where Q is the quiver

$$1 \xrightarrow{\ \alpha\ } 2 \xrightarrow{\ \beta\ } 3 \circlearrowright \gamma$$

and $I \subseteq KQ$ is generated by $\gamma\beta\alpha$ and γ^2. Then A is left serial, and hence $\mathcal{P}^{<\infty}$(A-mod) is contravariantly finite by [7] (it is easy to verify contravariant finiteness directly in this example). The minimal $\mathcal{P}^{<\infty}$(A-mod)-approximation of the minimal cogenerator equals $B = (Ae_1)^3 \oplus Ae_2$, and thus provides only two isomorphism classes of indecomposable Ext-injectives in $\mathcal{P}^{<\infty}$(A-mod). In particular, B fails to be a tilting module.

Next we will see that, by contrast, all Ext-injective objects in $\mathcal{P}^{<\infty}$(A-mod) are obtained as in Lemma 5.1, provided that $A = \Lambda$ is a truncated path algebra. This fact will lead to the announced description of the basic strong tilting object in Λ-mod.

For the remainder of the section, we again focus on a truncated path algebra $\Lambda = KQ/I$ with vertices $e_1, \dots, e_n$, and let $\mathcal{A}_i$ be the minimal $\mathcal{P}^{<\infty}$(Λ-mod)-approximation of the simple left Λ-module S_i, for $1 \le i \le n$, as described in Sect. 4. Moreover, we denote by $E(S_i)$ the injective envelope of S_i and by $\mathcal{B}_i$ the minimal $\mathcal{P}^{<\infty}$(Λ-mod)-approximation of $E(S_i)$. As will be illustrated in the sequel, not only the $\mathcal{A}_i$, but also the $\mathcal{B}_i$ can be explicitly determined from Q and L by way of Theorem 4.2. Consequently, Theorem 5.3 will permit us to construct the basic strong tilting module $T \in \Lambda$-mod from these data.

Theorem 5.3. *Let $S_1, \dots, S_m$ be the precyclic simple modules, and $S_{m+1}, \dots, S_n$ the non-precyclic ones. As before, denote by $\mathcal{A}_i$ the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i, and by $\mathcal{B}_i$ the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $E(S_i)$, for $1 \le i \le n$. Then the basic strong tilting module in Λ-mod is the direct sum*

$$T = \bigoplus_{1 \le i \le m} \mathcal{A}_i \;\oplus \bigoplus_{m+1 \le i \le n} \mathcal{B}_i .$$

- *Concerning the first subsum: The categories* $\operatorname{add}\left(\bigoplus_{1 \le i \le m} \mathcal{A}_i\right)$ *and* $\operatorname{add}\left(\bigoplus_{1 \le i \le m} \mathcal{B}_i\right)$ *coincide; that is,*

$$\bigoplus_{1 \le i \le m} \mathcal{B}_i \cong \bigoplus_{1 \le i \le m} \mathcal{A}_i^{t_i},$$

  for suitable exponents $t_i \ge 1$. This direct sum has only precyclic simple composition factors.
- *Concerning the second subsum, $\bigoplus_{m+1 \le i \le n} \mathcal{B}_i$: Suppose $i \ge m+1$. Then $\mathcal{B}_i$ is indecomposable, has the same top as $E(S_i)$, and has exactly one simple composition factor isomorphic to S_i, namely the socle of $E(S_i)$. Moreover, every submodule of $\mathcal{B}_i$ that is not contained in $J\mathcal{B}_i$ contains this copy of S_i. As for the other simple composition factors of $\mathcal{B}_i$ in the category $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$: We have $U(\mathcal{B}_i) = \varepsilon\mathcal{B}_i = \varepsilon E(S_i)$, and $\mathcal{B}_i/U(\mathcal{B}_i) \cong \bigoplus_{j \le m} \mathcal{A}_j^{k_{ij}}$, where k_{ij} is the multiplicity of S_j as a direct summand of the top $E(S_i)/JE(S_i) \cong \mathcal{B}_i/J\mathcal{B}_i$.*

Crucial notation: Summands of the strong basic tilting module and primitive idempotents in the corresponding tilted algebra. In the following, we will write the strong basic tilting module $T \in \Lambda$-mod in the form

$$T = \bigoplus_{1 \le i \le n} T_i,$$

where $T_i = \mathcal{A}_i$ for each precyclic vertex e_i, and $T_i = \mathcal{B}_i$ if e_i is not precyclic. In other words, T_i is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i if e_i is precyclic, and T_i is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of the injective envelope $E(S_i)$ otherwise. Moreover, for each $i \in \{1 \dots, n\}$, we denote by $\widetilde{e}_i$ the canonical projection relative to this decomposition, followed by the embedding into T, that is, $\widetilde{e}_i : T \to T_i \hookrightarrow T$. This yields primitive idempotents $\widetilde{e}_1, \dots, \widetilde{e}_n$ in the tilted algebra $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ which are in obvious one-to-one correspondence with the e_i.

Proof of Theorem 5.3 The description of the basic strong tilting module T in the first claim will follow from Lemma 5.1 once the two remaining claims have been established. Indeed, the second assertion entails that the Ext-injective object $\bigoplus_{1 \le i \le n} \mathcal{B}_i$ of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ has n pairwise non-isomorphic indecomposable direct summands, and the final assertion guarantees that each of them occurs precisely once in the direct sum displayed in the first claim.

First let $i \le m$. Clearly, $E(S_i)$ has only precyclic simple composition factors in this case, whence the top of $E(S_i)$ is a direct sum of copies of $S_1, \dots, S_m$.

Consequently, the projective cover P of $E(S_i)$ is a direct sum $\bigoplus_{1\le j\le m}(\Lambda e_j)^{m_{ij}}$ for suitable $m_{ij} \geq 0$. If $E(S_i) \cong P/C$, then $\varepsilon C = \varepsilon P$. Now apply Theorem 4.2 to obtain $\mathcal{B}_i = P/\varepsilon C = \bigoplus_{1\le j\le m} A_j^{m_{ij}}$ as claimed. Clearly, each of $\mathcal{A}_1, \dots, A_m$ arises as a direct summand of some $\mathcal{B}_j$ for $j \leq m$.

Next let $i \geq m+1$, and set $\mathcal{B} = \mathcal{B}_i$. In proving the description of $\mathcal{B}$ given in the last part of the theorem, we observe that the projective cover of $E(S_i)$ can be described as follows: Let $(p_r)_{r\in R}$ be the different paths of length $\leq L$ which end in the vertex e_i and are maximal with these two properties; by maximality we mean that the inequality $\text{length}(p_r) < L$ occurs only in case p_r starts in a source of Q. If $e(r)$ is the starting point of p_r, then the projective cover of $E(S_i)$ is $P = \bigoplus_{r\in R}\Lambda z_r$ with $\Lambda z_r \cong \Lambda e(r)$. It is clearly harmless to identify $E(S_i)$ with a factor module P/C, where $C \subseteq JP$; thus $E(S_i)$ has a sequence of top elements $x_r = z_r + C$ normed by $e(r)$, respectively. We use Theorem 4.2 once again to find $\mathcal{B} = P/\varepsilon C$. In particular, we find that the socle S_i of $E(S_i)$ is contained in the socle of $\mathcal{B}$, and is, in fact, the only non-precyclic simple summand in $\text{soc}(\mathcal{B})$. More precisely, we obtain: $S_i = Kp_r(z_r + \varepsilon C)$ for each $r \in R$, and $p_s(z_r + \varepsilon C) = 0$ for $s \neq r$. Moreover, any top element of $\mathcal{B}$ has the form $z = z' + z''$ where z' is a nontrivial K-linear combination of the residue classes $z_r + \varepsilon C$, and z'' belongs to $J\mathcal{B}$. For all but the last of the assertions concerning $\mathcal{B}$, it suffices to show that S_i is contained in any cyclic submodule of $\mathcal{B}$ which is generated by a top element z. To verify this containment, we note that, by construction, $p_r z'' = 0$ for all r, since either $\text{length}(p_r) = L$ or else p_r starts in a source of Q. This yields $Kp_r z = S_i$, for any index r for which $z_r + \varepsilon C$ makes a nontrivial appearance in z', and thus proves the auxiliary statement. The final assertion is an immediate consequence of Theorems 3.1 and 4.2. □

Theorem 5.3 allows for multiple occurrences of the $\mathcal{A}_j$, for $j \leq m$, in the direct sum $\bigoplus_{1\le i\le m}\mathcal{B}_i$. Multiplicities larger than 1 are a common occurrence in fact. For the truncated path algebra Λ_2 in Example 5.6 below, for instance, $m = 3$ and $\bigoplus_{1\le i\le 3}\mathcal{B}_i = \mathcal{A}_1^3 \oplus \mathcal{A}_2 \oplus \mathcal{A}_3$.

Next we extend the description of the injective objects of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ to the big category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ of not necessarily finite dimensional modules of finite projective dimension.

Corollary 5.4. Information on the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, second installment. *Again, let $T = \bigoplus_{1\le i\le n} T_i$ be the strong basic tilting module in* Λ-mod, *where $T_i = \mathcal{A}_i$ for $1 \leq i \leq m$, and $T_i = \mathcal{B}_i$ is a minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $E(S_i)$ for $i \geq m+1$. Then the full subcategory of injective objects of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ is equal to* $\text{Add}(T)$.

The indecomposable injective objects $T_1, \dots, T_m$ are simple in the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$. For $i \geq m+1$, the non-precyclic submodule $U(T_i)$ of T_i and the precyclic factor module $T_i/U(T_i)$ of T_i (in the sense of Corollary 3.3 B), both objects of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, are described in the final part of Theorem 5.3*; in particular, T_i fails to be simple in the category $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$, unless S_i is injective.*

Proof. Since T is a strong tilting module, the injective objects of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ coincide with the objects of $\text{add}(T)$, and, by Theorem 5.3, T is a cogenerator for $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$. We deduce that T is even a cogenerator for the category

$\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$: Indeed, any left Λ-module M of finite projective dimension is a directed union of its top-stably embedded finitely generated submodules $M_r, r \in R$, all of which belong to $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ by Corollary 3.2. Hence M embeds into a direct limit $\varinjlim E_r$, where $E_r \in \operatorname{add}(T)$ contains M_r as a submodule. But by [16, Observation 3.1], the latter direct limit belongs to $\operatorname{Add}(T)$, because T is Σ-pure injective. Since $\operatorname{Add}(T)$ is evidently closed under direct summands, this shows that all injective objects of $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ belong to $\operatorname{Add}(T)$.

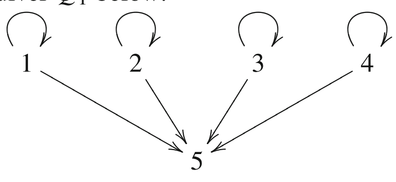
For the converse, let $M \in \mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ and choose M_r in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ for $r \in R$, as in the preceding paragraph. Then $\operatorname{Ext}^1_\Lambda\left(M, T^{(X)}\right)$ is an inverse limit of the spaces $\operatorname{Ext}^1_\Lambda\left(M_r, T^{(X)}\right) \cong \operatorname{Ext}^1_\Lambda\left(M_r, T\right)^{(X)}$, all of which are zero. This shows $T^{(X)}$ to be injective in $\mathcal{P}^{<\infty}(\Lambda\text{-Mod})$ for any index set X. □

While the structure of the summands $T_1, \ldots, T_m$ of the strong basic tilting module $T \in \Lambda$-mod is transparent, the structure of the remaining summands $T_{m+1}, \ldots, T_n$ is somewhat harder to visualize from the formal description. The labeled and layered graphs of the T_i (in the sense of Sect. 2) permit us to understand the structure of the T_i in any given example at a glance. By means of Theorem 4.2, these graphs can be readily obtained from graphs of the $E(S_i)$, the latter being obvious. We leave the easy combinatorial proof of the following remark to the reader.

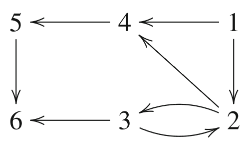
Remark 5.5. For each $i \leq n$, the indecomposable direct summand T_i of the basic strong tilting module has a (unique, up to isomorphism) layered and labeled graph without closed edge paths; in other words, this graph is a tree. Conversely, T_i is uniquely determined, up to isomorphism, by this graph.

Instead of spelling out the easy algorithm for constructing these graphs, we will illustrate the procedure with two examples. In particular, we will see: Whenever e_i is a non-precyclic vertex, the layered graph of T_i may be visualized as a daddy longlegs. The body is represented by the socle S_i of $E(S_i)$, and the legs, usually ramified, are in one-to-one correspondence with the simple summands in the top of $E(S_i)$. In the upcoming example, we do not label the arrows in our quivers, and accordingly omit labels on the edges of the graphs representing modules; as the considered quivers have no double arrows, this omission is harmless. The second of the two specific situations exhibited will be revisited in Sect. 9.

Example 5.6. Let Λ_1 be the truncated path algebra of Loewy length 2 based on the quiver Q_1 below:

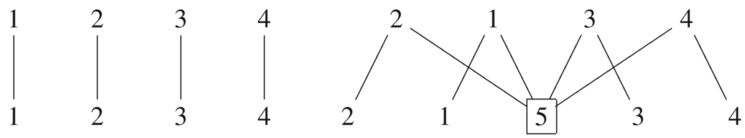


Q_1



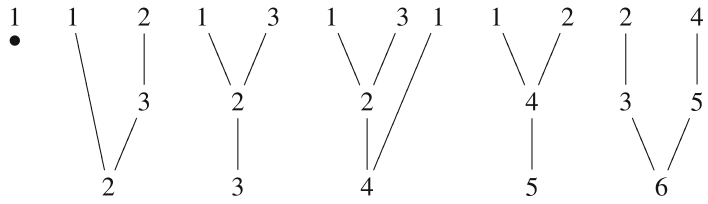
Q_2

The basic strong tilting module in Λ_1-mod has the following layered graph:

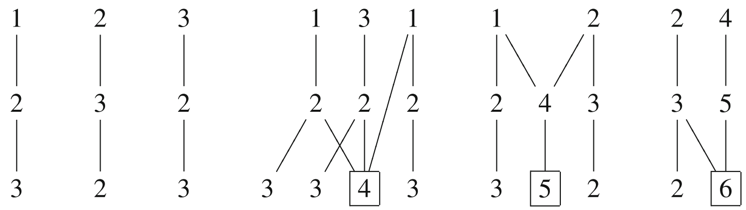


Here the first four trees (from left to right) represent the summands $E(S_i) = T_i = \mathcal{A}_i$ corresponding to the precyclic simple modules $S_1, \dots, S_4$. The last represents T_5, the direct summand corresponding to the non-precyclic simple module S_5; that is, T_5 is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $E(S_5)$. In the graph, the socle of $E(S_5)$—the legless body of the spider—is highlighted.

Now we consider the truncated path algebra Λ_2 of Loewy length 3 based on the second of the above quivers, Q_2. The indecomposable injective left Λ_2-modules have graphs:



Using Theorem 5.3, we obtain the following graph for the basic strong tilting module in Λ_2-mod.



Here the first three trees represent the direct summands corresponding to the precyclic simples S_1, S_2, S_3. The remaining three trees depict the direct summands corresponding to the non-precyclic simples S_4, S_5, S_6; they are the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of $E(S_4)$, $E(S_5)$, and $E(S_6)$, respectively, degenerate specimens of spiders in this example. Again the bodies of these spiders are highlighted.

6. Filtrations for the categories ${}_\Lambda T^{\perp}$ and ${}^{\perp}({}_{\widetilde{\Lambda}} DT)$

In this short section, we develop some structural information on the objects in the categories of the title. Such perpendicular categories appear naturally in tilting theory, and are pivotal in transferring information between Λ-mod and $\widetilde{\Lambda}$-mod. In our situation: Since ${}_\Lambda T$ is strong, we know that ${}_\Lambda T^{\perp} = \mathcal{P}^{<\infty}(\Lambda\text{-mod})^{\perp}$ (this is essentially the definition of a strong tilting module from [4], but see also [2]). Our results concerning filtrations of the objects in the targeted subcategories of Λ-mod and $\widetilde{\Lambda}$-mod parallel the characterization of the objects of $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ in terms of Ψ-filtrations, in Corollary 3.3.

We keep the notational conventions of Sect. 5, labeling the precyclic simple left Λ-modules $S_1, \dots, S_m$. As we saw in Theorem 5.3, the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximations of the modules in the set

$$\Theta = \{S_1, \dots, S_m\} \cup \{E(S_{m+1}), \dots, E(S_n)\}$$

are precisely the indecomposable direct summands T_i of ${}_\Lambda T$. It turns out that the set Θ also has a strong impact on the structure of the objects in the equivalent subcategories ${}_\Lambda T^\perp$ of Λ-mod and ${}^\perp({}_{\widetilde{\Lambda}} DT)$ of $\widetilde{\Lambda}$-mod. We write $\mathrm{Hom}_\Lambda(T, \Theta)$ for the set of $\widetilde{\Lambda}$-modules $\mathrm{Hom}_\Lambda(T, X)$ with $X \in \Theta$, and refer back to Corollary 3.3 for further notation.

Theorem 6.1. *We have an equality of subcategories* ${}_\Lambda T^\perp = \mathrm{filt}(\Theta)$. *Moreover, any Λ-module X in ${}_\Lambda T^\perp$ has a unique largest submodule $V(X)$ with only precyclic composition factors, and the quotient $X/V(X)$ is a direct sum of copies of suitable injectives among the $E(S_i)$ for $i \geq m+1$.*

Proof. The second statement will follow from our proof of the first. Clearly each $E(S_j)$, for $m+1 \leq j \leq n$, belongs to ${}_\Lambda T^\perp$. Next, we observe that $M \in {}_\Lambda T^\perp$ whenever $\varepsilon M = 0$. To see this, we apply the functor $\Lambda/\varepsilon\Lambda \otimes_\Lambda -$ to an extension $0 \to M \longrightarrow X \longrightarrow T \to 0$. This functor is exact since $\Lambda/\varepsilon\Lambda \cong (1-\varepsilon)\Lambda_\Lambda$ is a projective right Λ-module. Since $T/\varepsilon T$ is projective over $\Lambda/\varepsilon\Lambda$ (see the remarks following Corollary 4.4), we obtain a splitting $X/\varepsilon X \to M$, which yields a splitting of the original extension upon composition with the canonical map $X \to X/\varepsilon X$. In particular, $S_i \in {}_\Lambda T^\perp$ for each $1 \leq i \leq m$. Since ${}_\Lambda T^\perp$ is extension-closed, we have $\mathrm{filt}(\Theta) \subseteq {}_\Lambda T^\perp$.

For the reverse inclusion, suppose $X \in {}_\Lambda T^\perp$, and let $Y = V(X)$ be the largest submodule of X contained in $\mathrm{filt}(S_1, \dots, S_m)$. The above shows that $Y \in {}_\Lambda T^\perp$, and thus $Z = X/Y \in {}_\Lambda T^\perp$ as well, since ${}_\Lambda T^\perp$ is coresolving. Moreover, maximality of Y ensures that $\mathrm{soc}\, Z$ has no precyclic composition factors, i.e., $\mathrm{soc}\, Z = \varepsilon(\mathrm{soc}\, Z)$. We claim that Z must be injective. Consider the injective envelope of Z

$$0 \to Z \longrightarrow E(Z) \longrightarrow W \to 0.$$

For each $1 \leq i \leq m$, we clearly have $\mathrm{Hom}_\Lambda(T_i, Z) = \mathrm{Hom}_\Lambda(T_i, E(Z)) = 0$ since such T_i have only precyclic composition factors. By hypothesis, $\mathrm{Ext}^1_\Lambda(T_i, Z) = 0$, and it follows that $\mathrm{Hom}_\Lambda(T_i, W) = 0$. Since each precyclic simple occurs in the top of some T_i with $1 \leq i \leq m$, we must have $\mathrm{soc}\, W = \varepsilon(\mathrm{soc}\, W)$. We now consider the pullback of the above extension along the inclusion $\mathrm{soc}\, W \to W$. Since $\mathrm{soc}\, W \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $Z \in {}_\Lambda T^\perp = \mathcal{P}^{<\infty}(\Lambda\text{-mod})^\perp$, the pullback sequence splits. Hence the inclusion $\mathrm{soc}\, W \to W$ factors through $g : E(Z) \to W$. However, $g(\mathrm{soc}\, E(Z)) = g(\mathrm{soc}\, Z) = 0$ implies that $\mathrm{soc}\, W = 0$, and hence that $W = 0$. This shows that $Z \in \mathrm{add}(E_{m+1} \oplus \cdots \oplus E_n)$, and therefore $X \in \mathrm{filt}(\Theta)$. □

Theorem 6.1 moreover yields dual filtrations for the modules in the subcategory ${}^{\perp}({}_{\widetilde{\Lambda}}DT)$ of $\widetilde{\Lambda}$-mod: Indeed, due to Miyashita [17, 1.15], the adjoint pair of functors

$$\big(T \otimes_{\widetilde{\Lambda}} -, \operatorname{Hom}_{\Lambda}(T, -)\big)$$

induces inverse equivalences

$${}^{\perp}({}_{\widetilde{\Lambda}}DT) \longleftrightarrow ({}_{\Lambda}T)^{\perp}.$$

Consequently,

$${}^{\perp}({}_{\widetilde{\Lambda}}DT) = \operatorname{Hom}_{\Lambda}(T, {}_{\Lambda}T^{\perp}) = \operatorname{filt}\,(\operatorname{Hom}_{\Lambda}(T, \Theta)).$$

In particular, the regular left $\widetilde{\Lambda}$-module $\widetilde{\Lambda} = \operatorname{Hom}_{\Lambda}(T, T)$ has a $\operatorname{Hom}_{\Lambda}(T, \Theta)$-filtration. For information about the right regular structure of $\widetilde{\Lambda}$ (in a restricted situation), we refer to Theorem 8.2.

Remark 6.2. Standard stratification of truncated path algebras. As noted in Remark 3.4, any truncated path algebra Λ is standardly stratified with respect to the pre-order

$$S_i \preceq S_j \;\Leftrightarrow\; i \text{ is precyclic or there is a path from } i \text{ to } j \text{ in } Q.$$

Corollary 3.3 then asserts that $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ coincides with the category of Λ-modules which are filtered by the standard modules. Hence, the tilting module corresponding to this filtration category by [4]—usually called the *characteristic* tilting module of Λ—coincides with the strong tilting module ${}_{\Lambda}T$. Furthermore, the modules in Θ can be viewed as the *proper costandard* Λ-modules $\overline{\nabla}_i$, where we define $\overline{\nabla}_i$ to be the maximal submodule of $E(S_i)$ for which $\overline{\nabla}_i/\operatorname{soc}\overline{\nabla}_i$ has no composition factors $S_j \succeq S_i$. With these observations, Theorem 6.1 becomes a consequence of the familiar formula $\operatorname{filt}(\overline{\nabla}) = \operatorname{filt}\Delta^{\perp}$, which was proved first for quasi-hereditary algebras in [18], and recently extended to standardly stratified algebras by Frisk [12]. Moreover, Theorem 21 of [12] describes the subcategory $\operatorname{Hom}_{\Lambda}(T, \Theta)$ as the subcategory of $\widetilde{\Lambda}$-modules that are filtered by the *proper standard* left $\widetilde{\Lambda}$-modules ${}_{\widetilde{\Lambda}}\overline{\Delta}_i$, defined dually to the proper costandard modules. The fact that ${}_{\widetilde{\Lambda}}\widetilde{\Lambda}$ has a suitable filtration with factors among these modules, as pointed out above, then corresponds to $\widetilde{\Lambda}$ being right standardly stratified (with respect to the preorder opposite to $\preceq$).

7. Dualities induced by strong tilting modules and quivers without precyclic sources

The primary purpose of this section is a characterization of the truncated path algebras Λ with the property that the strong tilting module ${}_{\Lambda}T$ is also a *strong* tilting module over $\widetilde{\Lambda} = \operatorname{End}_{\Lambda}(T)^{\mathrm{op}}$. (Our convention that, for $f, g \in \operatorname{End}_{\Lambda}(T)$, the product $f \circ g$ stands for "first apply g, then f" makes T a right $\widetilde{\Lambda}$-module.) As is to be expected, this situation allows for particularly effective transfer of information between the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. We begin with a review of known categorical connections induced by tilting and deduce an alternate argument for a general characterization of strongness in a tilting module.

For an arbitrary tilting module ${}_AT_B$ with $B = \mathrm{End}_A(T)^{\mathrm{op}}$, one not only obtains equivalent pairs of subcategories of A-mod and B-mod, respectively, but also partial dualities A-mod $\leftrightarrow$ mod-B. In fact, it follows from [17, Theorem 3.5] that the functors $\mathrm{Hom}_A(-,T)$ and $\mathrm{Hom}_B(-,T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\text{A-mod}) \cap {}^{\perp}({}_AT) \quad \longleftrightarrow \quad \mathcal{P}^{<\infty}(\text{mod-B}) \cap {}^{\perp}(T_B). \tag{‡}$$

Consequently, if $X \in \mathcal{P}^{<\infty}(\text{A-mod}) \cap {}^{\perp}({}_AT)$, applying $\mathrm{Hom}_A(-,T)$ to a minimal projective resolution of X yields an exact $\mathrm{add}(T_B)$-coresolution of finite length for $\mathrm{Hom}_A(X,T)$. Thus we see that the subcategories linked by the above duality coincide with the subcategories $\mathrm{fcog}({}_AT)$ and $\mathrm{fcog}(T_B)$, respectively, consisting of the modules with finite $\mathrm{add}(T)$-coresolutions, that is, admitting exact sequences of the form

$$0 \to X \to T_1 \to \cdots \to T_s \to 0$$

with $T_i \in \mathrm{add}(T)$. On the other hand, by [4, Theorem 5.5b], ${}_AT$ is strong if and only if $\mathcal{P}^{<\infty}(\text{A-mod}) = \mathrm{fcog}({}_AT)$. Hence, strongness of the tilting module ${}_AT$ amounts to the equality

$$\mathcal{P}^{<\infty}(\text{A-mod}) \cap {}^{\perp}({}_AT) = \mathcal{P}^{<\infty}(\text{A-mod});$$

symmetrically, T_B is strong if and only if $\mathcal{P}^{<\infty}(\text{mod-B}) \cap {}^{\perp}(T_B) = \mathcal{P}^{<\infty}(\text{mod-B})$. Miyashita's duality (‡) thus specializes to a duality

$$\mathcal{P}^{<\infty}(\text{A-mod}) \longleftrightarrow \mathcal{P}^{<\infty}(\text{mod-B})$$

in case the tilting bimodule ${}_AT_B$ is strong on both sides. This latter fact, a compelling motivation for exploring strongness of ${}_AT_B$, can also be found in [4, Proposition 6.6], where the dual for strong cotilting modules is stated. The following convenient criterion for strongness is stated in dual form in [4, Proposition 6.5], where it is attributed to Auslander and Green [3]. We supply a short alternate proof based on Miyashita's duality.

Proposition 7.1. [3] *Let ${}_AT_B$ be a tilting module. Then T_B is a strong tilting module in* mod-B *if and only if all simple left A-modules embed into* $\mathrm{soc}({}_AT)$.

Proof. For convenience, we set $\mathcal{X} = \mathrm{fcog}({}_AT)$ and $\mathcal{Y} = \mathrm{fcog}(T_B)$. Suppose that T_B is strong and that $\mathrm{Hom}_A(S,T) = 0$ for a simple A-module S. We shall obtain a contradiction by showing that $S = 0$. Since $\mathcal{X}$ is contravariantly finite and resolving, we can find an exact sequence $0 \to K \longrightarrow X_1 \longrightarrow X_0 \longrightarrow S \to 0$ with $X_i \in \mathcal{X}$. Applying $\mathrm{Hom}_A(-,T)$, we obtain an exact sequence $0 \to \mathrm{Hom}_A(X_0,T) \longrightarrow \mathrm{Hom}_A(X_1,T) \longrightarrow Y \to 0$ in mod-B with $\mathrm{Hom}_A(X_i,T) \in \mathcal{Y}$. Since T_B is strong, $\mathcal{Y} = \mathcal{P}^{<\infty}(\text{mod-B})$, and thus $Y \in \mathcal{Y}$. It follows that $Y \cong \mathrm{Hom}_A(K,T)$, and applying $\mathrm{Hom}_B(-,T)$ gives a short exact sequence $0 \to K \longrightarrow X_1 \longrightarrow X_0 \to 0$, implying that $S = 0$.

Conversely, suppose that $\operatorname{Hom}_A(S, T) \neq 0$ for all simple A-modules S. Thus there exist nonzero maps from any nonzero A-module X to T. Now suppose $\operatorname{p\,dim}_B Y < \infty$. Since $\mathcal{Y}$ contains the projective B-modules, we may assume that $\Omega Y \in \mathcal{Y}$. Hence we have an exact sequence $0 \to \operatorname{Hom}_A(X, T) \longrightarrow \operatorname{Hom}_A(T_0, T) \longrightarrow Y \to 0$ for some $X \in \mathcal{X}$, $T_0 \in \operatorname{add}({}_A T)$ and $f : T_0 \to X$. Left-exactness of $\operatorname{Hom}_A(-, T)$ now implies that $\operatorname{Hom}_A(\operatorname{Coker}(f), T) = 0$, and hence $\operatorname{Coker}(f) = 0$. Since $\mathcal{X}$ is resolving, $\operatorname{Ker}(f) \in \mathcal{X}$, and it follows that $Y \cong \operatorname{Hom}_A(\operatorname{Ker}(f), T) \in \mathcal{Y}$. Hence $\mathcal{Y} = \mathcal{P}^{<\infty}(\text{mod-B})$, and T_B is strong. □

For the remainder of this section, we return to a truncated path algebra $\Lambda = KQ/I$ with basic strong tilting module ${}_\Lambda T$. In this case, Proposition 7.1 translates into a straightforward criterion for the quiver Q equivalent to strongness of $T_{\widetilde{\Lambda}}$ as a tilting object in mod-$\widetilde{\Lambda}$. By the preceding general discussion, the equivalence of conditions (1) and (3) in Corollary 7.2 below holds whenever ${}_A T$ is strong in A-mod, not only in case $A = \Lambda$ is a truncated path algebra. We re-emphasize the equivalence in our specialized context for easy reference in the upcoming applications.

Corollary 7.2. *Let Λ be a truncated path algebra with basic strong tilting module $T \in \Lambda$-mod, and $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{op}$. Then the following statements are equivalent:*

(1) *T is a strong tilting module in* mod-$\widetilde{\Lambda}$.
(2) *The quiver Q of Λ has no precyclic source.*
(3) *The functors $\operatorname{Hom}_\Lambda(-, T)$ and $\operatorname{Hom}_{\widetilde{\Lambda}}(-, T)$ induce dualities between the categories $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.*

Proof. (1) $\Longleftrightarrow$ (2): We refer to the description of T given in Theorem 5.3. In light of this theorem, every nonprecyclic simple S_i occurs in $\operatorname{soc}_\Lambda(T_i)$, and hence $\bigoplus_{i \geq m+1} S_i$ always embeds into $\operatorname{soc}_\Lambda(T)$. On the other hand, a precyclic simple S_i occurs in $\operatorname{soc}_\Lambda(T)$ precisely when the corresponding vertex e_i is the endpoint of a path of length L in Q. That this be satisfied for all precyclic vertices is tantamount to the requirement that all precyclic vertices be postcyclic, that is, to non-existence of a precyclic source.

(3) $\Longleftrightarrow$ (1) was already justified in the more general scenario considered ahead of Proposition 7.1. □

Corollary 7.2 implies in particular that $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is contravariantly finite in mod-$\widetilde{\Lambda}$, whenever the quiver Q of Λ is without precyclic source. We conjecture that, more strongly, $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ is always contravariantly finite in mod-$\widetilde{\Lambda}$ for a truncated path algebra Λ, irrespective of the shape of the underlying quiver Q.

8. The categories $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ and $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ in the Λ-$\widetilde{\Lambda}$-symmetric situation

Throughout this section, we assume that the quiver Q has no precyclic source. By Corollary 7.2, this places us in the "Λ-$\widetilde{\Lambda}$-symmetric situation" of the section title.

Let Λ and $T = \bigoplus_{1 \le i \le n} T_i \in \Lambda$-mod be as before; in particular, $T_i = \mathcal{A}_i = \Lambda e_i / \varepsilon J e_i$ for $i \le m$, and T_i is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of $E(S_i)$ for $i \ge m+1$ (cf. Sect. 5). Again, $e_1, \dots, e_m$ are the precyclic vertices in the Gabriel quiver Q of Λ, and $e_{m+1}, \dots, e_n$ the non-precyclic ones. Our choice of corresponding primitive idempotents $\widetilde{e}_1, \dots, \widetilde{e}_n$ in the strongly tilted algebra $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ was introduced after the statement of Theorem 5.3. Moreover, we denote by $\widetilde{J}$ the Jacobson radical of $\widetilde{\Lambda}$, and by $\widetilde{S}_i = \widetilde{e}_i \widetilde{\Lambda} / \widetilde{e}_i \widetilde{J}$ the simple right $\widetilde{\Lambda}$-modules corresponding to the $\widetilde{e}_i$.

By our blanket assumption and Corollary 7.2, the functors $\operatorname{Hom}_\Lambda(-, T)$ and $\operatorname{Hom}_{\widetilde{\Lambda}}(-, T)$ induce inverse dualities

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \leftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}).$$

Just as in the case of the truncated path algebra Λ, the homology of the tilted algebra $\widetilde{\Lambda}$ is therefore in turn governed by a bicoloring of the vertices $\widetilde{e}_i$ of its quiver $\widetilde{Q}$. This bicoloring of the $\widetilde{e}_i$ is lined up, via $\operatorname{Hom}_\Lambda(T, -)$, with the one that stems from the placement of the e_i relative to oriented cycles in Q. By way of caution, we point out that it does not carry over to a symmetric placement of the $\widetilde{e}_i$ within $\widetilde{Q}$ in general: Indeed, an idempotent $\widetilde{e}_i$ corresponding to a non-precyclic vertex e_i may lie on an oriented cycle of $\widetilde{Q}$; see Example 9.1.

Since both $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ are contravariantly finite (in Λ-mod and mod-$\widetilde{\Lambda}$, respectively), we know the little finitistic dimensions on the pertinent sides of Λ and $\widetilde{\Lambda}$ to coincide with the big finitistic dimensions. Combining Corollary 7.2 with [10], we thus obtain:

Proposition 8.1. *Suppose Λ is a truncated path algebra based on a quiver without precyclic source, and let T be the corresponding basic strong Λ-$\widetilde{\Lambda}$ tilting bimodule. Then*

$$\text{l. Findim}\,\Lambda = \text{l. findim}\,\Lambda = \operatorname{p\,dim}_\Lambda T = \operatorname{p\,dim}_{\widetilde{\Lambda}} T = \text{r. findim}\,\widetilde{\Lambda} = \text{r. Findim}\,\widetilde{\Lambda}.$$

Theorem 8.5 below will, moreover, permit us to express the right finitistic dimensions of $\widetilde{\Lambda}$ in terms of the simple modules $\widetilde{S}_{m+1}, \dots, \widetilde{S}_n$ alone. Namely,

$$\text{r. findim}\,\widetilde{\Lambda} = \max\{\operatorname{p\,dim} \widetilde{S}_j \mid m+1 \le j \le n\}.$$

This follows from the description of composition series in the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. This last equality actually simplifies the situation encountered for the left finitistic dimensions of Λ; indeed, either of the equalities $\text{l. findim}\,\Lambda = \max\{\operatorname{p\,dim} S_j \mid m+1 \le j \le n\}$ and $\text{l. findim}\,\Lambda = 1 + \max\{\operatorname{p\,dim} S_j \mid m+1 \le j \le n\}$ can be realized for suitable truncated path algebras; see [11].

In view of the duality of Corollary 7.2, the next theorem follows readily from the mirror symmetric information provided by Corollaries 3.3 and 5.4. The reference to precyclic and non-precyclic portions of modules in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ in the upcoming results refers to the cycle structure of the quiver Q, not to that of $\widetilde{Q}$. Paralleling the definition of the key idempotent ε in Λ, we introduce the idempotent

$$\widetilde{\epsilon} = \sum_{m+1 \le i \le n} \widetilde{e}_i$$

in $\widetilde{\Lambda}$.

Theorem 8.2. The category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. *We continue to assume that the quiver Q has no precyclic source.*

A. Simple objects and composition series in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$**.** *The simple objects of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ are precisely the right $\widetilde{\Lambda}$-modules*

$\widetilde{e}_i\widetilde{\Lambda} = \mathrm{Hom}_\Lambda(T_i, T)$ *for* $1 \le i \le m$ *and* $\widetilde{S}_i = \mathrm{Hom}_\Lambda(S_i, T)$ *for* $m+1 \le i \le n$. *Moreover,* $\widetilde{e}_i\widetilde{J} \neq 0$ *for* $i \le m$.

In particular, a simple right $\widetilde{\Lambda}$-module $\widetilde{S}_i$ has finite projective dimension if and only if $i \ge m+1$.

B. Heredity property: Separation of precyclic and non-precyclic portions in the objects of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$**.** *Each module $\widetilde{M}$ in $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ has a unique subobject $U(\widetilde{M})$ maximal with respect to being a direct sum of copies of the projective modules $\widetilde{e}_i\widetilde{\Lambda}$ with $i \le m$. All simple composition factors of $\widetilde{M}/U(\widetilde{M})$ are among $\widetilde{S}_{m+1}, \dots, \widetilde{S}_n$.*

In particular, $U(\widetilde{M})$ equals $\widetilde{M}(1-\widetilde{\epsilon})\widetilde{\Lambda}$, the submodule of $\widetilde{M}$ generated by all elements x with $x\widetilde{e}_i = x$ for some $i \le m$, and this module is projective. Moreover, if $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\widetilde{M} \cong \mathrm{Hom}_\Lambda(M, T)$, the composition length of $\widetilde{M}/U(\widetilde{M})$ in mod-$\widetilde{\Lambda}$ *coincides with that of the Λ-module $U(M)$ of* Corollary 3.3.

C. Separation in the indecomposable projective objects $\widetilde{e}_i\widetilde{\Lambda} = \mathrm{Hom}_\Lambda(T_i, T)$**.** *For $i \le m$, the submodule $U(\widetilde{e}_i\widetilde{\Lambda})$ of Part B equals $\widetilde{e}_i\widetilde{\Lambda}$. Now suppose $i \ge m+1$. In this case, $U(\widetilde{e}_i\widetilde{\Lambda}) \cong \bigoplus_{j \le m}(\widetilde{e}_j\widetilde{\Lambda})^{k_{ij}}$, where k_{ij} is the multiplicity of S_j in T_i/JT_i; thus, k_{ij} equals the number of those paths of length L in Q, which start in the precyclic vertex e_j and end in e_i.*

D. The indecomposable injective objects of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$**.** *These are the $\widetilde{\Lambda}$-modules $\widetilde{E}_i = \mathrm{Hom}_\Lambda(\Lambda e_i, T) \cong e_i T_{\widetilde{\Lambda}}$. Each $\widetilde{E}_i$ has a unique simple subobject in the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. This subobject is $\widetilde{e}_i\widetilde{\Lambda}$ in case $i \le m$, and equals $\widetilde{S}_i$ in case $i \ge m+1$. In particular, $U(\widetilde{E}_i) = \widetilde{e}_i\widetilde{\Lambda}$ for $i \le m$, and $U(\widetilde{E}_i) = 0$ for $i \ge m+1$.*

Proof. In light of the duality of Corollary 7.2, it is routine to translate the assertions of Corollaries 3.3 and 5.4 into statements concerning $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$. Only the claim that $\widetilde{e}_i\widetilde{J} \neq 0$ for $i \le m$ (under A) requires further backing. Since e_i is a precyclic vertex and Q is free of precyclic sources, e_i is also postcyclic. In particular, there exists an arrow from e_j to e_i for a suitable index j, possibly equal to i. Since e_j is clearly again precyclic, we have $j \le m$. This, in turn, places a composition factor S_i into JT_j, and thus gives rise to a nonzero map in $\mathrm{Hom}_\Lambda(T_i, T_j)$ which fails to be an isomorphism. □

Some of the mirror-symmetric structure statements for objects in $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ can be pushed beyond the categorical level as follows: Namely, if $M \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$ and $\widetilde{M} = \mathrm{Hom}_\Lambda(M, T)$, then $\varepsilon M = M$ is equivalent to $\widetilde{M}\widetilde{\epsilon} = \widetilde{M}$. More generally, the equality $\big(\widetilde{M}/U(\widetilde{M})\big)\,\widetilde{\epsilon} = \widetilde{M}/U(\widetilde{M})$ is tantamount to $\varepsilon\,U(M) = U(M)$. On the other hand, such non-categorical dual statements are not consistently available: While $\varepsilon\,(M/U(M)) = 0$, we find $U(\widetilde{M})\widetilde{\epsilon} \neq 0$, in general. In particular, while $\varepsilon\mathcal{A}_i = 0$ for $i \le m$, the corresponding projective $\widetilde{\Lambda}$-module $\widetilde{\mathcal{A}_i} \cong \widetilde{e}_i\widetilde{\Lambda}$ typically has composition factors of finite projective dimension in its radical (see Example 9.1).

This latter fact is contrasted by the following "non-heredity" condition for the $\widetilde{e}_i\widetilde{\Lambda}$ with $i \le m$.

Corollary 8.3. *Whenever $\widetilde{S}_i$ has infinite projective dimension, that is, when $i \le m$ in our ordering of the vertices, the projective module $\widetilde{e}_i\widetilde{\Lambda}$ has no proper nonzero submodule of finite projective dimension.*

We now turn our attention to the big category $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. Essentially, all of the structure results above carry over, but require additional argumentation, as we are leaving the stage of the duality $\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \leftrightarrow \mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$.

Proposition 8.4. *Let $\widetilde{M} \in \mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ be such that all simple summands of $\widetilde{M}/\widetilde{M}\widetilde{J}$ in* Mod-$\widetilde{\Lambda}$ *have infinite projective dimension. Then $\widetilde{M}$ is projective, that is*

$$\widetilde{M} \cong \bigoplus_{i \le m} \left(\widetilde{e}_i\widetilde{\Lambda}\right)^{(\tau_i)}$$

for suitable cardinal numbers τ_i.

Proof. We first observe that the hypothesis on $\widetilde{M}/\widetilde{M}\widetilde{J}$ is tantamount to the equality $\widetilde{M} = \widetilde{M}(1-\widetilde{\epsilon})\widetilde{\Lambda}$. So, if $\widetilde{M}$ is finitely generated, then $\widetilde{M} = U(\widetilde{M})$ is projective and has the required format by the preceding theorem.

Now we drop the extra hypothesis on $\widetilde{M}$. From [16, Theorem 4.4], it follows that $\widetilde{M}$ is a direct limit of a directed system of finitely generated $\widetilde{\Lambda}$-modules of finite projective dimension; let $\widetilde{M}_r$, $r \in R$, be the members of such a system. Since $\widetilde{M} = \widetilde{M}(1-\widetilde{\epsilon})\widetilde{\Lambda}$, it is harmless to assume that also $\widetilde{M}_r = \widetilde{M}_r(1-\widetilde{\epsilon})\widetilde{\Lambda}$ for all $r \in R$. As we already saw, this ensures that all of the $\widetilde{M}_r$ are projective of the correct format and, flatness being the same as projectivity over a finite dimensional algebra, we find that their direct limit, $\widetilde{M} = \varinjlim \widetilde{M}_r$, is projective as well. It is clear that only projectives with tops in $\{\widetilde{S}_1, \dots, \widetilde{S}_m\}$ will occur as indecomposable direct summands of $\widetilde{M}$. □

Theorem 8.5. The category $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. *We continue to assume that the quiver Q has no precyclic source.*

A. Composition series. *Each object $\widetilde{M} \in \mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ has an ordinal-indexed composition series with consecutive factors among the relative simple objects $\widetilde{e}_i\widetilde{\Lambda}$ for $i \le m$ and $\widetilde{S}_i$ for $i \ge m+1$. The cardinalities in which these factors occur are isomorphism invariants of $\widetilde{M}$.*

B. Heredity property: Separation of precyclic and non-precyclic portions in the objects of $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. *Every object $\widetilde{M}$ in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda}))$ has a unique submodule $U(\widetilde{M})$ which is maximal with respect to being a direct sum of copies of the projectives $\widetilde{e}_i\widetilde{\Lambda}$ with $i \le m$, and all simple composition factors of $\widetilde{M}/U(\widetilde{M})$ are among $\widetilde{S}_{m+1}, \dots, \widetilde{S}_n$.*

In particular, $U(\widetilde{M}) = \widetilde{M}(1-\widetilde{\epsilon})\widetilde{\Lambda}$ is generated by all the elements $x \in \widetilde{M}$ with $x\widetilde{e}_i = x$ for some $i \le m$, and this module is projective.

C. The injective objects of the category $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. *The injective objects of $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ coincide with the objects in* $\text{Add}(\bigoplus_{1\le i\le n} \widetilde{E}_i) = \text{Add}(T_{\widetilde{\Lambda}})$.

Proof. We first prove part B. Let $\widetilde{M} \in \mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. Then $U(\widetilde{M}) = \widetilde{M}(1-\widetilde{\epsilon})\widetilde{\Lambda}$ clearly again belongs to $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$. By Proposition 8.4, we infer that $U(\widetilde{M})$ is a direct sum of copies of suitable $\widetilde{e}_i\widetilde{\Lambda}$ with $i \le m$. By construction $\big(\widetilde{M}/U(\widetilde{M})\big)\widetilde{\epsilon} = \widetilde{M}/U(\widetilde{M})$, whence all simple composition factors of this quotient are among $\widetilde{S}_{m+1}, \dots, \widetilde{S}_n$. The claim regarding composition series in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ follows.

Part C can be established in analogy with Corollary 5.4, given that any $\widetilde{M} \in \mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ is a directed union of submodules $\widetilde{M}_r \in \mathcal{P}^{<\infty}(\Lambda\text{-mod})$. $\square$

Remark 8.6. Standard stratification of truncated path algebras. If Q has precyclic sources, we can still make use of the tilting duality (‡) displayed in Sect. 7 to obtain a duality

$$\mathcal{P}^{<\infty}(\Lambda\text{-mod}) \leftrightarrow \mathcal{P}^{<\infty}(\text{mod -}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}}).$$

Hence, the conclusions of Theorem 8.2 are still valid for the category $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda}) \cap {}^{\perp}(T_{\widetilde{\Lambda}})$. Although this category is a proper subcategory of $\mathcal{P}^{<\infty}(\text{mod-}\widetilde{\Lambda})$ when Q has a precyclic source, it always contains the projective right $\widetilde{\Lambda}$-modules. In particular, we see that $\widetilde{\Lambda}_{\widetilde{\Lambda}}$ has a filtration with factors belonging to the set

$$\tilde{\Psi} := \text{Hom}_{\Lambda}(\Psi, T) = \left\{\widetilde{e}_1\widetilde{\Lambda}, \dots, \widetilde{e}_m\widetilde{\Lambda}, \widetilde{S}_{m+1}, \dots, \widetilde{S}_n\right\},$$

with Ψ as in Corollary 3.3. Since Ψ is the set of standard Λ-modules, it follows from [12, Proposition 23] that $\tilde{\Psi}$ consists precisely of the standard right modules of $\widetilde{\Lambda}$ relative to the opposite of the pre-order on the simple Λ-modules: $\widetilde{S}_i \preceq \widetilde{S}_j \Leftrightarrow S_j \preceq S_i$. That is to say, the module $\text{Hom}_{\Lambda}(\mathcal{A}_i, T)$ is the largest quotient of $\widetilde{e}_i\widetilde{\Lambda}$ all of whose composition factors $\widetilde{S}_j$ satisfy $\widetilde{S}_j \preceq \widetilde{S}_i$. As we observed earlier, $\widetilde{\Lambda}$ is right standardly stratified, and thus each projective $\widetilde{e}_i\widetilde{\Lambda}$ has a filtration with top factor isomorphic to $\text{Hom}_{\Lambda}(\mathcal{A}_i, T)$ and subsequent factors isomorphic to $\text{Hom}_{\Lambda}(\mathcal{A}_j, T)$ for j with $S_j \prec S_i$.

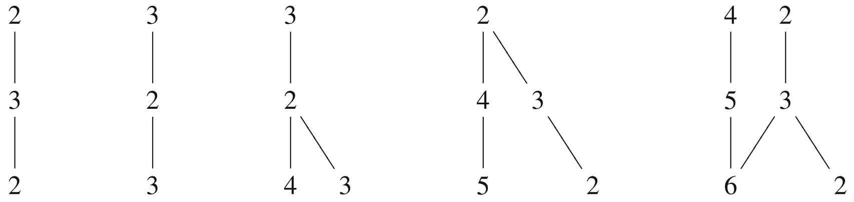
9. Examples

We first give an example based on a quiver Q without precyclic source.

Example 9.1. Let Q be the quiver

$$\begin{tikzcd} 5 \arrow[d] & 4 \arrow[l] & \\ 6 & 3 \arrow[l] \arrow[r, bend right] & 2 \arrow[ul] \arrow[l, bend right] \end{tikzcd}$$

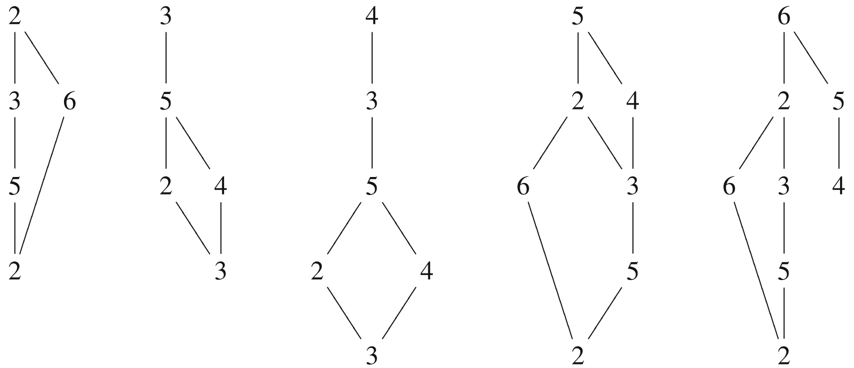
and Λ the truncated path algebra KQ/I, where I is generated by all paths of length 3. The strong basic tilting module $T = \bigoplus_{2 \le i \le 6} T_i$ is determined up to isomorphism by its graph, namely the following forest:



T_2 T_3 T_4 T_5 T_6

As in Example 5.6(2), one obtains T as follows: First one identifies the cycle-bound vertices of Q, namely e_2 and e_3 in this case, whence $T_i = \mathcal{A}_i$ is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of S_i for $i = 2, 3$, and T_i is the minimal $\mathcal{P}^{<\infty}(\Lambda\text{-mod})$-approximation of the injective envelope $E(S_i)$ for $i = 4, 5, 6$.

We will give some detail on the computation of quiver and relations of $\widetilde{\Lambda} = \operatorname{End}_\Lambda(T)^{\mathrm{op}}$ in the more challenging Example 9.2, and leave the present case as an exercise. In the present example, $\operatorname{End}_\Lambda(T)$ coincides with the algebra A presented in Example 2.5. Note that the Gabriel quiver of A—it is displayed in 2.5—contains no multiple edges, whence we may suppress the labeling of the edges of the graphs of the indecomposable projective right $\widetilde{\Lambda}$-modules (= indecomposable projective left $\operatorname{End}_\Lambda(T)$-modules) without losing information. They are as follows:

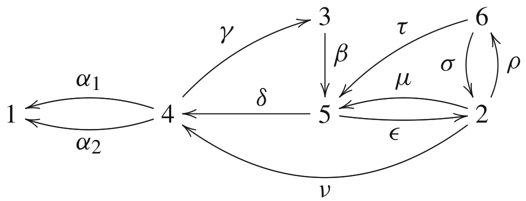


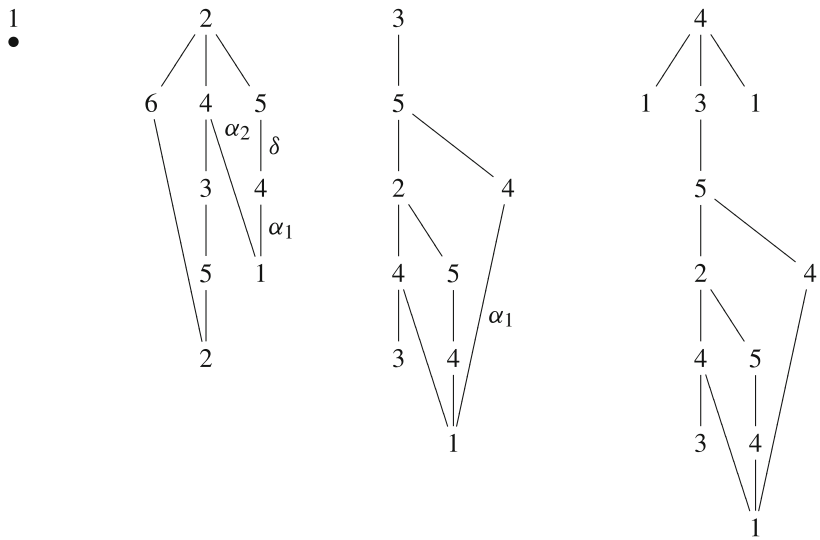
Since the quiver Q has no precyclic source, the bimodule ${}_\Lambda T_{\widetilde{\Lambda}}$ is strong on both sides (Sect. 7), and Theorem 8.5 provides us with information on the big category $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$: The simple objects are $\widetilde{e}_i\widetilde{\Lambda}$ for $i = 2, 3$ and $\widetilde{S}_i = \widetilde{e}_i\widetilde{\Lambda}/\widetilde{e}_i\widetilde{J}$ for $i = 4, 5, 6$. Moreover, the objects in $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ are precisely the modules $\widetilde{M}$ with the following structure: each has a unique largest submodule $U(\widetilde{M})$ which is a direct sum of copies of $\widetilde{e}_i\widetilde{\Lambda}$, $i = 2, 3$, and $\widetilde{M}/U(\widetilde{M})$ has only composition factors among $\widetilde{S}_4, \widetilde{S}_5, \widetilde{S}_6$. In particular, we glean from the above graphs that $U(\widetilde{e}_4\widetilde{\Lambda}) \cong \widetilde{e}_3\widetilde{\Lambda}$ and $\widetilde{e}_4\widetilde{\Lambda}/U(\widetilde{e}_4\widetilde{\Lambda}) \cong \widetilde{S}_4$; similarly, $U(\widetilde{e}_5\widetilde{\Lambda}) \cong \widetilde{e}_2\widetilde{\Lambda}$ and $\widetilde{e}_5\widetilde{\Lambda}/U(\widetilde{e}_5\widetilde{\Lambda})$ is the two-dimensional uniserial module with top $\widetilde{S}_5$ and socle $\widetilde{S}_4$; finally, $U(\widetilde{e}_6\widetilde{\Lambda}) \cong \widetilde{e}_2\widetilde{\Lambda}$ and $\widetilde{e}_6\widetilde{\Lambda}/U(\widetilde{e}_6\widetilde{\Lambda})$ is the three-dimensional uniserial module with radical layering $\big(\widetilde{S}_6, \widetilde{S}_5, \widetilde{S}_4\big)$. The injective objects of $\mathcal{P}^{<\infty}(\text{Mod-}\widetilde{\Lambda})$ are the direct sums of $\widetilde{E}_i = \operatorname{Hom}_\Lambda(T, T_i)$ for $2 \le i \le 6$. The general theory moreover tells us that, for $i = 2, 3$, the module $\widetilde{E}_i$ is an essential extension of $U(\widetilde{E}_i) = \widetilde{e}_i\widetilde{\Lambda}$, and for $i = 4, 5, 6$, the only composition factors of $\widetilde{E}_i$ are among $\widetilde{S}_j$, $4 \le j \le 6$.

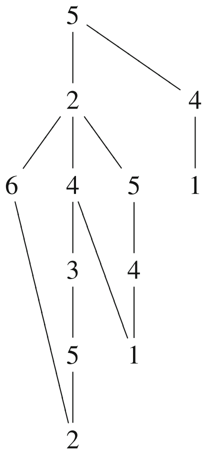
Our final example has a single precyclic source. In fact, the quiver of the truncated path algebra we will consider next results from that of Example 9.1 by the addition of a single source labeled 1. As we will see, this destroys the duality encountered in the previous example. In particular, the simple left Λ-modules of finite projective dimension are no longer in one-to-one correspondence with the simple right $\widetilde{\Lambda}$-modules of finite projective dimension. The more intricate duality theory that governs the general situation will be explored in a sequel to this paper.

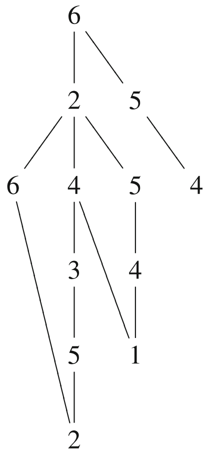
Example 9.2. This time, Λ is the truncated path algebra of Loewy length 3 based on the quiver $Q = Q_2$ of Example 5.6 for which we already computed the basic strong tilting module $T = \bigoplus_{1 \le i \le 6} T_i$. Again we exhibit the quiver of $\widetilde{\Lambda} = \mathrm{End}_\Lambda(T)^{\mathrm{op}}$, followed by graphs of the indecomposable projective right $\widetilde{\Lambda}$-modules.

For ease of reading, we display the opposite of $\widetilde{Q}$, namely, the quiver of $\mathrm{End}_\Lambda(T)$. Subsequently, we present the graphs of the indecomposable projective right $\widetilde{\Lambda}$-modules (= indecomposable projective left $\mathrm{End}_\Lambda(T)$-modules). We label edges only where the two arrows from vertex 4 to vertex 1 might otherwise lead to ambiguities.



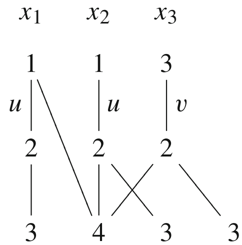






Note that the simple module $\widetilde{S}_1$ corresponding to the precyclic vertex e_1 of Q is projective, that is, its graph consists only of the vertex 1. Generators for $\widetilde{I}$ such that $\widetilde{\Lambda} \cong K\widetilde{Q}/\widetilde{I}$ can be read off the graphs of the indecomposable projective right modules above, as $\widetilde{I}$ can be generated by monomial relations and binomial relations of the form $p - q$, where p and q are paths in $\widetilde{Q}$.

For our (partial) glossary, we refer the reader back to the graph of the basic strong tilting module $T = \bigoplus_{1 \le i \le 6} T_i$ which was displayed in Example 5.6(2). For $i = 1, 2$, let $\alpha_i : T_4 \to T_1$ be the epimorphism sending the top element x_i of T_4 shown in the graph below to the top of T_1, and sending the x_j for $j \neq i$ to zero. Then α_1, α_2 are clearly K-linearly independent modulo $\widetilde{e}_4\widetilde{J}^2\widetilde{e}_1$; thus they yield two arrows $\widetilde{e}_4 \to \widetilde{e}_1$ in the quiver of $\mathrm{End}_\Lambda(T)$. Next, $\beta \in \mathrm{Hom}_\Lambda(T_3, T_5)$ is chosen so as to send a top element of T_3 to an element in $e_3JT_5 \setminus J^2T_5$; any such choice lies outside $\widetilde{e}_3\widetilde{J}^2\widetilde{e}_5$. The map $\gamma \in \mathrm{Hom}_\Lambda(T_4, T_3)$ is the obvious epimorphism with kernel $\Lambda x_1 + \Lambda x_2$. We content ourselves with spelling out a few not quite so obvious additional choices of arrows. Namely, we have two K-linearly independent homomorphisms $T_2 \to T_4$: one of them is ν, sending a top element of T_2 to $ux_2 - vx_3$, where



is the graph of T_4, relative to suitable top elements x_1, x_2, x_3, normed by e_1, e_1, and e_3, respectively, such that the socle of $\Lambda(ux_2 - vx_3)$ equals S_3. One checks that ν does not belong to $\widetilde{e}_2\widetilde{J}^2\widetilde{e}_4$. On the other hand, the map in $\mathrm{Hom}_\Lambda(T_2, T_4)$ that sends a top element of T_2 to ux_1 factors through T_5 by way of an obvious choice of arrow $\mu : T_2 \to T_5$ and the map $\delta : T_5 \to T_4$ specified below. Consequently, only ν qualifies as an arrow from $\widetilde{e}_2$ to $\widetilde{e}_4$. We have two linearly independent maps δ and

δ' in $\mathrm{Hom}_\Lambda(T_5, T_4)$, both of which belong to $(\widetilde{e}_5 \widetilde{J} \widetilde{e}_4) \setminus (\widetilde{e}_5 \widetilde{J}^2 \widetilde{e}_4)$: Say δ sends a top element $y = e_2 y$ of T_5 to ux_2, and δ' sends y to vx_3. These assignments pin down δ and δ' up to nonzero scalar factors; in particular, δ maps T_5 onto a submodule of codimension 1 of the module $\Lambda x_1 + \Lambda x_2 \subset T_4$ with graph

$$\begin{tikzcd}[arrows=dash]
1 \arrow[d] \arrow[ddr] & 1 \arrow[d] & \\
2 \arrow[d] & 2 \arrow[d] \arrow[dr] & \\
3 & 4 & 3
\end{tikzcd}$$

and δ' maps T_5 onto a submodule of codimension 1 of $\Lambda x_1 + \Lambda x_3$. Then the difference $\delta - \delta'$ equals $\epsilon \nu \in \widetilde{e}_5 \widetilde{J}^2 \widetilde{e}_4$, where $\epsilon \in \mathrm{Hom}_\Lambda(T_5, T_2)$ is a suitable epimorphism sending y to a top element of T_2. So $\widetilde{Q}$ contains only a single arrow from $\widetilde{e}_5$ to $\widetilde{e}_4$; we picked the map δ for the above graphs (as opposed to the alternate choice, δ').

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