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ERRATUM

Erratum to: The Tate conjecture for $K3$ surfaces over finite fields

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1 Statement of Theorem 4

In the statement of Theorem 4, it should be assumed that the second and third Betti cohomology groups of Y have no p-torsion. This is implicitly used in 2.2 to compute the second crystalline cohomology group of X as well as its Hodge number $h^{2,0}$. Furthermore, the Beauville-Bogomolov form should be assumed to be even so as to apply Borcherds' results—this is satisfied in all known cases.

Corollary 5 and 6 still hold: the second and third cohomology group of a variety of $K3^{[n]}$ type are torsion-free as can be easily checked by a direct computation, see e.g. [3, Proposition 2.5].

2 Correction of a mistake in Proposition 25

As pointed out to us by Olivier Benoist, there is an error in the proof of Proposition 25 of the original article. This has consequences in Corollary 29

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of the original article. We indicate how to fix accordingly the contents of Sect. 5 of the paper and leave the main theorem and its consequences unchanged. The beautiful survey [2] briefly discusses how to work around the mistake and still recover part of the results.

In order to explain where the mistake arises, let us keep the notation of Proposition 25. Looking at the beginning of page 140, Lemma 5.12 in [5] only shows the equality $\mathcal{O}(MD')_{U_k} = \overline{\kappa}^*(\lambda_{\mathcal{A}}^N)_{U_k}$ between restrictions of the line bundles to the special fiber of U. As a consequence, the ampleness of $M\overline{D}_k$ is proved only when $U_k = \overline{T}_k$, i.e., when $\overline{T}$ is smooth over W.

As we will explain below, it is possible to ensure that $\overline{T}$ is indeed smooth over W in the context of Proposition 25. However, the proof of the main theorem relies on an inductive argument contained in the proof of Proposition 28, which makes it necessary to ensure the smoothness of auxiliary varieties in the inductive process. As a consequence, we will need to modify the order of the inductive process.

Before explaining the actual details, let us explain the strategy of the proof and the—mostly technical—differences. Consider the family $\pi : \mathcal{X} \to T$, together with the distinguished point 0 on the special fiber of π. We want to prove that cohomology classes of divisors on $\mathcal{X}_0$, which is assumed to be supersingular, form a space of rank b, where b is the second Betti number of X. Let C_0 be the connected component of the supersingular locus of T passing through 0. Then the dimension of C is $b-3-E((b-1)/2)$, where E denotes the integer part function. In particular, it is positive. The key idea of the proof—which remains unchanged in this erratum—is to show that, while C_0 might not be proper over k, it is possible to construct a partial compactification $\overline{T}$ of T in a modular way so that the closure C of C_0 in $\overline{T}$ is proper, thus allowing us to apply Borcherds' positivity results as in [5]—see the remark at the end of 4.1.

Crucial to the argument above is the fact that while C is not the parameter space for a smooth proper deformation of $\mathcal{X}_0$, it still carries enough cohomological information, in the form here of a $K3$ crystal. In this erratum, due to the inductive nature of the proof, we will be led to consider subvarieties of $\overline{T}$ that might be supported entirely on the boundary $\overline{T} \setminus T$. In particular, it might happen that the deformation of $\mathcal{X}_0$ does not extend to any single point of these subvarieties. However, as in the main step of the proof, this does not matter as the crystal associated to the second cohomology group of $\mathcal{X}_0$ does extend.

We start by giving a more precise definition of the partial compactification $\overline{T}$ of T—this is a variation on the construction of 4.1. Let $\overline{T}$ be the normalization of the closure of the image of T in $\mathcal{A}_{g,d',n}$. Let $\psi : \mathcal{A} \to \overline{T}$ be the polarized abelian scheme obtained by pulling back the universal abelian variety on $\mathcal{A}_{g,d',n}$. Let $H_{\mathcal{A}}$ be the crystal over $\overline{T}_k$ induced be the first relative cohomology group of $\mathcal{A}$, and let E be the crystal $\mathcal{H}om(H_{\mathcal{A}}, H_{\mathcal{A}})$ over $\overline{T}_k$. The properties of T that we need are the following.

Proposition 2.1 *The scheme $\overline{T}$ is smooth over W. The second relative primitive crystalline cohomology of $\mathcal{X}$ over T_k extends to a $K3$ crystal H over $\overline{T}_k$, and the Kuga–Satake relation of Proposition 13 extends to an embedding of crystals*

$$H \hookrightarrow E.$$

The supersingular locus of $\overline{T}$, defined as in Definition 19, is proper.

Proof Since the polarization of $\mathcal{X}$ is prime to the residual characteristic p by assumption, this is a consequence of the main theorem of Kisin's paper [4]. Section 4 of [6] provides further details—see also the discussion of [7, Section 3].

The properness of the supersingular locus of $\overline{T}$ is proved in Proposition 20. □

Proposition 2.2 *There exists a lattice Λ of rank $b-5$ such that $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate and the intersection $\overline{Z}_\Lambda \cap C$ has positive dimension.*

Note that C has dimension $b-3-E((b-1)/2)$, and $\overline{Z}_\Lambda$ has codimension $b-6$ in $\overline{T}$, so it is not obvious in general that the intersection of C and $\overline{Z}_\Lambda$ be non-empty, let alone positive-dimensional. The proposition below refines the relevant dimension estimates.

Proposition 2.3 *Let Λ be a lattice of rank $r \leq b-5$. Then any component of the supersingular locus of $\overline{Z}_\Lambda$ has codimension at most $E(\frac{b-r}{2})$.*

Proof Let h be an integer. Any component of the locus in $(\overline{Z}_\Lambda)_k$ above which the crystal H has height at least $h+1$ has codimension at most h. Indeed, as a set, it is locally defined by h equations [8, Proposition 11]. Furthermore, by [1, Theorem 0.1], at any point of finite height h of $(\overline{Z}_\Lambda)_k$, we have

$$2h \leq b-r. \tag{2.1}$$

This shows that the codimension of any component of the supersingular locus of $(\overline{Z}_\Lambda)_k$ is at most $E(\frac{b-r}{2})$. □

Corollary 2.4 *Let Λ be a lattice of rank $r \leq b-5$. Then any component of the supersingular locus of $\overline{Z}_\Lambda$ has positive dimension.*

Proof The dimension of $(\overline{Z}_\Lambda)_k$ is $b-2-r$ and $r \leq b-5$. Since $b-2-r > E(\frac{b-r}{2})$ if $r \leq b-5$, the claim is proved.

Proof of Proposition 2.2 We show by induction on r that for all $r \leq b-5$, there exists a lattice Λ of rank r such that $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate and $\overline{Z}_\Lambda$ intersects C along a subscheme of positive dimension.

When $r = 1$, the result simply expresses that C is positive-dimensional and the polarization on $\mathcal{X} \to T$ is prime to p. To pass from r to $r + 1$, let Λ_r be a lattice of rank r satisfying the result. Then, up to normalization, $\overline{Z}_\Lambda$ is smooth over W by Kisin's result [4] again.

As a consequence, as in the paper, we can apply the proof of [5, Theorem 3.1]—this time working in the normalization of $\overline{Z}_{\Lambda_r}$—to find a lattice Λ of rank $r + 1$, containing Λ_r, such that $\Lambda \otimes \mathbb{Z}/p\mathbb{Z}$ is non-degenerate and such that $\overline{Z}_\Lambda \cap (C \cap \overline{Z}_{\Lambda_r}) = \overline{Z}_\Lambda \cap C$ is not empty.

If $r + 1 \leq b - 5$, the dimension of any component of $\overline{Z}_\Lambda \cap C$ is positive, which concludes the proof. □

Let Λ be a lattice of rank $b-5$ as in Proposition 2.2. Let $\overline{Z}^0_\Lambda$ be an irreducible component of $\overline{Z}_\Lambda$. Applying [5, Theorem 3.1] as in the proof above to $\overline{Z}^0_\Lambda$, we can find two lattices of rank $b - 4$ containing Λ, denoted by Λ_1 and Λ_2, satisfying the following conditions:

(1) $\Lambda_1 \otimes \mathbb{Z}/p\mathbb{Z}$ and $\Lambda_2 \otimes \mathbb{Z}/p\mathbb{Z}$ are both non-degenerate;
(2) there is no isomorphism $\Lambda_1 \otimes \mathbb{Q} \to \Lambda_2 \otimes \mathbb{Q}$ inducing the identity on Λ;
(3) the intersections $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_1} \cap C$ and $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_2} \cap C$ are both non-empty.

If one of the intersections $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_1} \cap C$ and $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_2} \cap C$ is positive-dimensional,[1] the same argument as above shows that we can find a lattice Λ_3 of rank $b-3$ such that $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_3} \cap C$ is non-empty. In that case, Sect. 5.2 allows us to conclude the proof of Theorem 4. As a consequence, let us assume that the intersections $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_1} \cap C$ and $\overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_2} \cap C$ are both zero-dimensional.

For $i = 1, 2$, let us write $\overline{Z}^0_{\Lambda_i} = \overline{Z}^0_\Lambda \cap \overline{Z}_{\Lambda_i}$. Since we assumed that $\overline{Z}^0_{\Lambda_i} \cap C$ is zero-dimensional, and since $\overline{Z}^0_{\Lambda_i}$ has relative dimension 2 over W, the locus of points of height 2 in the special fiber of $\overline{Z}^0_{\Lambda_i}$ has dimension 1—in particular, it is not empty. This follows as before from [8, Proposition 11], proving that, as a set, the locus of points of height at least $h + 1$ is locally defined by one equation inside the locus of points of height at least h.

Let v be an element of Λ. Then Proposition 22 shows that there exists a positive integer N—which is a large enough power of p—such that the line spanned by Nv extends as a trivial subcrystal of H over C. Since Λ has finite rank, we can find N independently of v.

Let c_i be a closed point of the intersection $\overline{Z}^0_{\Lambda_i} \cap C$. By construction, $N\Lambda$ extends as a trivial subcrystal of H over C, and Λ_i embeds as a subcrystal of the crystal H_i lying above c_i in a way that is compatible with the inclusion $\Lambda \subset \Lambda_i$.

[1] It is possible to prove that this actually never happens.

Up to enlarging N, we can assume that $N\Lambda_1$ extends as a trivial subcrystal of H over C. The crystal H_2 contains both $N\Lambda_1$ and Λ_2, so H_2 contains a trivial subcrystal of rank $b-3$, containing Λ_2. The deformation space of that subcrystal has codimension at most 1 in $\overline{Z}^0_{\Lambda_2}$. We claim that this deformation space is flat over W. Assume by contradiction that it contains the special fiber of $\overline{Z}^0_{\Lambda_2}$. Our assumption ensures in particular that it contains a point of finite height 2, which contradicts inequality (2.1) with $h = 2$ and $r = b - 3$. This contradiction shows that c_2 lies in some $\overline{Z}_{\Lambda'}$, where Λ' has rank $b-3$. Section 5.2 allows us to conclude the proof of Theorem 4.

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