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ERRATUM

# Knot Floer homology detects fibred knots

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An important step in [5] uses JSJ theory [3, 4] to deduce some topological information about the knot complement when the knot Floer homology is monic, see [5, Sect. 6]. The version of JSJ theory cited there is from [1]. However, as pointed out by Kronheimer, the definition of "product regions" in [1] is not the one we want. In this note, we will provide the necessary background on JSJ theory following [3]. Some arguments in [5] will then be modified.

We first briefly explain the mistake in [5]. In [5, Sect. 6], we need a submanifold of $M$, such that every product annulus or product disk can be homotoped into this submanifold. The existence of such submanifold is well-known in JSJ theory, but the version of JSJ theory cited from [1] does not provide such existence result. In fact, the definition of "product regions" there [1, Definition 3.1] requires that every component of the product region contains a component of the suture. This condition is very restrictive and was ignored by the author in [5].

In this note we will use the standard JSJ theory to get the existence of such submanifold (called the characteristic pair), and prove that a large part of this submanifold is a product submanifold. This will be sufficient for our purpose.

**Definition 1** An *$n$-manifold pair* is a pair $(M, T)$ where $M$ is an $n$-manifold and $T$ is an $(n-1)$-manifold contained in $\partial M$. A 3-manifold pair $(M, T)$ is *irreducible* if $M$ is irreducible and $T$ is incompressible. An irreducible 3-manifold pair $(M, T)$ is *Haken* if each component of $M$ contains an incompressible surface.

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**Definition 2** [3, p. 10] A compact 3-manifold pair $(\mathcal{S}, \mathcal{T})$ is called an *$I$-pair* if $\mathcal{S}$ is an $I$-bundle over a compact surface, and $\mathcal{T}$ is the corresponding $\partial I$-bundle. A compact 3-manifold pair $(\mathcal{S}, \mathcal{T})$ is called an *$S^1$-pair* if $\mathcal{S}$ is a Seifert fibred manifold and $\mathcal{T}$ is a union of Seifert fibres in some Seifert fibration. A *Seifert pair* is a compact 3-manifold pair $(\mathcal{S}, \mathcal{T})$, each component of which is an $I$-pair or an $S^1$-pair.

**Definition 3** [3, p. 138] A *characteristic pair* for a compact, irreducible 3-manifold pair $(M, T)$ is a perfectly-embedded Seifert pair $(\Sigma, \Phi) \subset (M, \mathrm{int}(T))$ such that if $f$ is any essential, nondegenerate map of an arbitrary Seifert pair $(\mathcal{S}, \mathcal{T})$ into $(M, T)$, $f$ is homotopic, as a map of pairs, to a map $f'$ such that $f'(\mathcal{S}) \subset \Sigma$ and $f'(\mathcal{T}) \subset \Phi$.

The definition of a perfectly-embedded pair can be found in [3, p. 4]. We note that the definition requires that $\Sigma \cap \partial M = \Phi$, so $\Sigma$ is disjoint from $\partial M - T$.

The main result in JSJ theory is the following theorem due to Jaco–Shalen [3, p. 138] and Johannson [4].

**Theorem 4** (Characteristic Pair Theorem) *Every Haken 3-manifold pair $(M, T)$ has a characteristic pair. This characteristic pair is unique up to ambient isotopy relative to $(\partial M - \mathrm{int}(T))$.*

**Definition 5** Let $(M, \gamma)$ be a sutured manifold. A 3-manifold pair $(P, Q) \subset (M, R(\gamma))$ is a *product pair* if $P = F \times [0, 1]$, $Q = F \times \{0, 1\}$ for some compact surface $F$, and $F \times 0 \subset R_-(\gamma)$, $F \times 1 \subset R_+(\gamma)$. We also require that $P \cap A = \emptyset$ or $A$ for any annular component $A$ of $\gamma$. A product pair is *gapless* if no component of its exterior is a product pair.

**Definition 6** Suppose $(M, \gamma)$ is a taut sutured manifold, $(\Sigma, \Phi)$ is the characteristic pair for $(M, R(\gamma))$. The *characteristic product pair* for $M$ is the union of all components of $(\Sigma, \Phi)$ which are product pairs. A *maximal product pair* for $M$ is a gapless product pair $(\mathcal{P}, \mathcal{Q})$ such that it contains the characteristic product pair, and if $(\mathcal{P}', \mathcal{Q}') \supset (\mathcal{P}, \mathcal{Q})$ is another gapless product pair, then there is an ambient isotopy relative to $\gamma$ that takes $(\mathcal{P}', \mathcal{Q}')$ to $(\mathcal{P}, \mathcal{Q})$.

Although the uniqueness of maximal product pairs is not guaranteed by the definition, the existence is obvious. In fact, if there is an infinite ascending chain of gapless product pairs

$$(\mathcal{P}_0, \mathcal{Q}_0) \subset \cdots \subset (\mathcal{P}_i, \mathcal{Q}_i) \subset (\mathcal{P}_{i+1}, \mathcal{Q}_{i+1}) \subset \cdots,$$

such that $(\mathcal{P}_i, \mathcal{Q}_i) \neq (\mathcal{P}_{i+1}, \mathcal{Q}_{i+1})$ up to ambient isotopy relative to $\gamma$, then we get a contradiction by Haken's Finiteness Theorem [2, Theorem III.20].

The exterior of a maximal product pair is also a sutured manifold. By definition the exterior does not contain essential product annuli or essential product disks.

Now we are ready to modify the arguments in [5]. The next theorem is a reformulation of [5, Theorem 6.2]. The proof is not changed though.

**Theorem 6.2′** *Suppose $(M, \gamma)$ is an irreducible balanced sutured manifold, $\gamma$ has only one component, and $(M, \gamma)$ is vertically prime. Let $\mathcal{E}$ be the subgroup of $H_1(M)$ spanned by the first homologies of product annuli in $M$. If $\widehat{HFS}(M, \gamma) \cong \mathbb{Z}$, then $\mathcal{E} = H_1(M)$.*

**Corollary 7** *In the last theorem, suppose* $(\Pi, \Psi)$ *is the characteristic product pair for* $M$, *then the map*

$$i_*\colon H_1(\Pi) \to H_1(M)$$

*is surjective.*

*Proof* We recall that such an $M$ is a homology product [5, Proposition 3.1].

Suppose $(\Sigma, \Phi)$ is the characteristic pair for $(M, R(\gamma))$, then any product annulus can be homotoped into $(\Sigma, \Phi)$ without crossing $\gamma$. Let $\Phi_+ = (\Phi \cap R_+(\gamma)) \subset \mathrm{int}(R_+(\gamma))$. Theorem 6.2′ implies that the map $H_1(\Phi_+) \to H_1(R_+(\gamma))$ is surjective, so $\partial\Phi_+$ consists of separating circles in $R_+(\gamma)$. If a component $(\mathcal{S}, \mathcal{T})$ of $(\Sigma, \Phi)$ is an $S^1$-pair, then $\mathcal{T} \cap R_+(\gamma)$ consists of annuli by definition. We conclude that each annulus is null-homologous in $H_1(R_+(\gamma))$.

Suppose a product annulus $A$ contributes to $H_1(M)$ nontrivially, and it can be homotoped into a component $(\sigma, \varphi)$ of $(\Sigma, \Phi)$. Given the result from the last paragraph, this $(\sigma, \varphi)$ cannot be an $S^1$-pair. It is neither a twisted $I$-bundle since the two components of $\partial A$ are contained in different components of $R(\gamma)$. So $(\sigma, \varphi)$ must be a trivial $I$-bundle, and the two components of $\varphi$ lie in different components of $R(\gamma)$. In other words, $(\sigma, \varphi)$ is a product pair. Now our desired result follows from Theorem 6.2′. □

The following proof of the main theorem in [5] is only slightly changed. Basically we use "maximal product pair" here instead of the wrong notion "characteristic product region" in [5].

*Proof of* [5, Theorem 1.1] Suppose $(M, \gamma)$ is the sutured manifold obtained by cutting open $Y - \mathrm{int}(\mathrm{Nd}(K))$ along $F$, $(\mathcal{P}, \mathcal{Q})$ is a maximal product pair for $M$. We need to show that $M$ is a product. By [5, Proposition 3.1], $M$ is a homology product. Moreover, by [5, Theorem 4.1], we can assume $M$ is vertically prime.

If $M$ is not a product, then $M - \mathcal{P}$ is nonempty. Thus there exist some product annuli in $(M, \gamma)$, which split off $\mathcal{P}$ from $M$. Let $(M', \gamma')$ be the remaining sutured manifold. By definition $(\mathcal{P}, \mathcal{Q})$ contains the characteristic product pair for $M$. Corollary 7 then implies that the map $H_1(\mathcal{P}) \to H_1(M)$ is surjective. So $R_\pm(\gamma')$ are planar surfaces, and $M' \cap \mathcal{P}$ consists of separating product annuli in $M$. Since we assume that $M$ is vertically prime, $M'$ must be connected. (See the first paragraph in the proof of [5, Theorem 5.1].) Moreover, $M'$ is also vertically prime. By [5, Theorem 5.1], $\widehat{HFS}(M', \gamma') \cong \mathbb{Z}$.

We add some product 1-handles to $M'$ to get a new sutured manifold $(M'', \gamma'')$ with $\gamma''$ connected. By [5, Proposition 2.9], $\widehat{HFS}(M'', \gamma'') \cong \mathbb{Z}$. It is easy to see that $M''$ is also vertically prime. [5, Proposition 3.1] shows that $M''$ is a homology product.

Let $H$ be one of the product 1-handles added to $M'$ such that $H$ connects two different components of $\gamma'$. By Theorem 6.2′, there is at least one product annulus $A$ in $M''$, such that $A$ cannot be homotoped to be disjoint from the cocore of $H$. Isotope $A$ if necessary, we find that at least one component of $A \cap M'$ is an essential product disk in $M'$, a contradiction to the assumption that $(\mathcal{P}, \mathcal{Q})$ is a maximal product pair. □

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## References

1. Cooper, D., Long, D.: Virtually Haken Dehn-filling. J. Differ. Geom. **52**(1), 173–187 (1999)
2. Jaco, W.: Lectures on Three-manifold Topology. CBMS Regional Conference Series in Mathematics, vol. 43. American Mathematical Society, Providence (1980)
3. Jaco, W., Shalen, P.: Seifert fibered spaces in 3-manifolds, Mem. Am. Math. Soc. **21**(220) (1979)
4. Johannson, K.: Homotopy Equivalences of 3-manifolds with Boundaries. Lecture Notes in Mathematics, vol. 761. Springer, Berlin (1979)
5. Ni, Y.: Knot Floer homology detects fibred knots. Invent. Math. **170**(3), 577–608 (2007)