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Zeros of the i.i.d. Gaussian Laurent Series on an Annulus: Weighted Szegő Kernels and Permanental-Determinantal Point Processes

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Abstract: On an annulus $\mathbb{A}_q := \{z \in \mathbb{C} : q < |z| < 1\}$ with a fixed $q \in (0, 1)$, we study a Gaussian analytic function (GAF) and its zero set which defines a point process on $\mathbb{A}_q$ called the zero point process of the GAF. The GAF is defined by the i.i.d. Gaussian Laurent series such that the covariance kernel parameterized by $r > 0$ is identified with the weighted Szegő kernel of $\mathbb{A}_q$ with the weight parameter r studied by McCullough and Shen. The GAF and the zero point process are rotationally invariant and have a symmetry associated with the q-inversion of coordinate $z \leftrightarrow q/z$ and the parameter change $r \leftrightarrow q^2/r$. When $r = q$ they are invariant under conformal transformations which preserve $\mathbb{A}_q$. Conditioning the GAF by adding zeros, new GAFs are induced such that the covariance kernels are also given by the weighted Szegő kernel of McCullough and Shen but the weight parameter r is changed depending on the added zeros. We also prove that the zero point process of the GAF provides a permanental-determinantal point process (PDPP) in which each correlation function is expressed by a permanent multiplied by a determinant. Dependence on r of the unfolded 2-correlation function of the PDPP is studied. If we take the limit $q \to 0$, a simpler but still non-trivial PDPP is obtained on the unit disk $\mathbb{D}$. We observe that the limit PDPP indexed by $r \in (0, \infty)$ can be regarded as an interpolation between the determinantal point process (DPP) on $\mathbb{D}$ studied by Peres and Virág ($r \to 0$) and that DPP of Peres and Virág with a deterministic zero added at the origin ($r \to \infty$).

1. Introduction and Main Results

1.1. Weighted Szegő kernel and GAF on an annulus. For a domain $D \subset \mathbb{C}$, let X be a random variable on a probability space which takes values in the space of analytic functions on D. If $(X(z_1), \dots, X(z_n))$ follows a mean zero complex Gaussian distribution for every $n \in \mathbb{N}$ and every $z_1, \dots, z_n \in D$, X is said to be a *Gaussian analytic function* (GAF) [35]. In the present paper the zero set of X is regarded as a point process on D denoted by a nonnegative-integer-valued Radon measure $\mathcal{Z}_X = \sum_{z \in D : X(z)=0} \delta_z$, and it is simply called the *zero point process* of the GAF. Zero point processes of GAFs have been extensively studied in quantum and statistical physics as solvable models of quantum chaotic systems and interacting particle systems [10,11,17,27,32,48,49]. Many important characterizations of their probability laws have been reported in probability theory [9,24,35,54,64,71,75].

A typical example of GAF is provided by the i.i.d. Gaussian power series defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$: Let $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and $\{\zeta_n\}_{n\in\mathbb{N}_0}$ be i.i.d. standard complex Gaussian random variables with density $e^{-|z|^2}/\pi$ and consider a random power series,

$$X_{\mathbb{D}}(z) = \sum_{n=0}^{\infty} \zeta_n z^n, \tag{1.1}$$

which defines an analytic function on $\mathbb{D}$ a.s. This gives a GAF on $\mathbb{D}$ with a covariance kernel

$$\mathbf{E}[X_{\mathbb{D}}(z)\overline{X_{\mathbb{D}}(w)}] = \frac{1}{1 - z\overline{w}} =: S_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D}. \tag{1.2}$$

This kernel is identified with the reproducing kernel of the Hardy space $H^2(\mathbb{D})$ called the *Szegő kernel* of $\mathbb{D}$ [1,7,8,60]. Peres and Virág [64] proved that $\mathcal{Z}_{X_{\mathbb{D}}}$ is a *determinantal point process* (DPP) such that the correlation kernel is given by $S_{\mathbb{D}}(z, w)^2 = (1-z\overline{w})^{-2}$, $z, w \in \mathbb{D}$ with respect to the reference measure $\lambda = m/\pi$. Here m represents the Lebesgue measure on $\mathbb{C}$; $m(dz) := dxdy$, $z = x + \sqrt{-1}y \in \mathbb{C}$. (See Theorem 2.11 in Sect. 2.7 below). This correlation kernel is identified with the reproducing kernel of the Bergman space on $\mathbb{D}$, which is called the *Bergman kernel* of $\mathbb{D}$ and denoted here by $K_{\mathbb{D}}(z, w)$, $z, w \in \mathbb{D}$ [1,7,8,33,60]. Thus the study of Peres and Virág on $X_{\mathbb{D}}$ and $\mathcal{Z}_{X_{\mathbb{D}}}$ is associated with the following relationship between kernels on $\mathbb{D}$ [64],

$$\mathbf{E}[X_{\mathbb{D}}(z)\overline{X_{\mathbb{D}}(w)}]^2 = S_{\mathbb{D}}(z, w)^2 = K_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D}. \tag{1.3}$$

(A brief review of reproducing kernels will be given in Sect. 2.1.)

Let $q \in (0, 1)$ be a fixed number and we consider the annulus $\mathbb{A}_q := \{z \in \mathbb{C} : q < |z| < 1\}$. In the present paper we will report the fact that, if we consider a GAF given by the i.i.d. Gaussian Laurent series $X_{\mathbb{A}_q}$ on $\mathbb{A}_q$, we will observe interesting new phenomena related with $X_{\mathbb{A}_q}$ and its zero point process $\mathcal{Z}_{X_{\mathbb{A}_q}}$. The present results are reduced to those by Peres and Virág [64] in the limit $q \to 0$. Conversely, the point processes associated with $X_{\mathbb{D}}$ are extended to those associated with $X_{\mathbb{A}_q}$ in this paper. The obtained new point processes can be regarded as *elliptic extensions* of the previous ones, since expressions for the former given by polynomials and rational functions of arguments are replaced by those of the theta functions with the arguments and the nome $p = q^2$ for the latter [29,39,45,65,67,77,78,81]. Moreover, we will introduce another parameter $r > 0$ in addition to q, and one-parameter families of GAFs, $\{X^r_{\mathbb{A}_q} : r > 0\}$ and zero point processes, $\{\mathcal{Z}_{X^r_{\mathbb{A}_q}} : r > 0\}$ will be constructed on $\mathbb{A}_q$. Here put $X_{\mathbb{A}_q} := X^q_{\mathbb{A}_q}$ and $\mathcal{Z}_{X_{\mathbb{A}_q}} := \mathcal{Z}_{X^q_{\mathbb{A}_q}}$. Construction of a model on an annulus will serve as a solid starting point for arguing general theory on multiply connected domains. Even if the models are different, studies in this direction provide useful hints for us to proceed the generalization [5,14–16,26,30,31,37,41,63,66,83].

Consider the Hilbert space of analytic functions on $\mathbb{A}_q$ equipped with the inner product

$$\langle f, g\rangle_{H_r^2(\mathbb{A}_q)} = \frac{1}{2\pi}\int_{\gamma_1\cup\gamma_q} f(z)\overline{g(z)}\sigma_r(dz), \quad f, g \in H_r^2(\mathbb{A}_q)$$

with

$$\sigma_r(dz) = \begin{cases} d\phi, & \text{if } z \in \gamma_1 := \{e^{\sqrt{-1}\phi} : \phi \in [0, 2\pi)\}, \\ r d\phi, & \text{if } z \in \gamma_q := \{q e^{\sqrt{-1}\phi} : \phi \in [0, 2\pi)\}, \end{cases}$$

which we write as $H_r^2(\mathbb{A}_q)$. A complete orthonormal system (CONS) of $H_r^2(\mathbb{A}_q)$ is given by $\{e_n^{(q,r)}\}_{n\in\mathbb{Z}}$ with

$$e_n^{(q,r)}(z) = \frac{z^n}{\sqrt{1 + rq^{2n}}}, \quad z \in \mathbb{A}_q, \quad n \in \mathbb{Z},$$

and the reproducing kernel is given by [56]

$$S_{\mathbb{A}_q}(z, w; r) = \sum_{n\in\mathbb{Z}} e_n^{(q,r)}(z)\overline{e_n^{(q,r)}(w)} = \sum_{n=-\infty}^{\infty} \frac{(z\overline{w})^n}{1 + rq^{2n}}. \tag{1.4}$$

This infinite series converges absolutely for $z, w \in \mathbb{A}_q$. When $r = q$, this Hilbert function space is known as the Hardy space on $\mathbb{A}_q$ denoted by $H^2(\mathbb{A}_q)$ and the reproducing kernel $S_{\mathbb{A}_q}(\cdot, \cdot) := S_{\mathbb{A}_q}(\cdot, \cdot; q)$ is called the Szegő kernel of $\mathbb{A}_q$ [60,68]. The kernel (1.4) with a parameter $r > 0$ is considered as a *weighted Szegő kernel* of $\mathbb{A}_q$ [61] and $H_r^2(\mathbb{A}_q)$ is the *reproducing kernel Hilbert space* (RKHS) [3] with respect to $S_{\mathbb{A}_q}(\cdot, \cdot; r)$ [56,57]. We call r the *weight parameter* in this paper. We note that (1.4) implies that $S_{\mathbb{A}_q}(z, z; r)$ is a monotonically decreasing function of the weight parameter $r \in (0, \infty)$ for each fixed $z \in \mathbb{A}_q$.

Associated with $H_r^2(\mathbb{A}_q)$, we consider the Gaussian Laurent series

$$X_{\mathbb{A}_q}^r(z) := \sum_{n\in\mathbb{Z}} \zeta_n e_n^{(q,r)}(z) = \sum_{n=-\infty}^{\infty} \zeta_n \frac{z^n}{\sqrt{1 + rq^{2n}}}, \tag{1.5}$$

where $\{\zeta_n\}_{n\in\mathbb{Z}}$ are i.i.d. standard complex Gaussian random variables with density $e^{-|z|^2}/\pi$. Since $\lim_{n\to\infty} |\zeta_n|^{1/n} = 1$ a.s., we apply the Cauchy–Hadamard criterion to the positive and negative powers of $X_{\mathbb{A}_q}^r(z)$ separately to conclude that this random Laurent series converges a.s. whenever $z \in \mathbb{A}_q$. Moreover, since the distribution ζ_n is symmetric, both of γ_1 and γ_q are natural boundaries [38, p.40]. Hence $X_{\mathbb{A}_q}^r$ provides a GAF on $\mathbb{A}_q$ whose covariance kernel is given by the weighted Szegő kernel of $\mathbb{A}_q$,

$$\mathbf{E}[X_{\mathbb{A}_q}^r(z)\overline{X_{\mathbb{A}_q}^r(w)}] = S_{\mathbb{A}_q}(z, w; r), \quad z, w \in \mathbb{A}_q,$$

and the zero point process is denoted by $\mathcal{Z}_{X_{\mathbb{A}_q}^r} := \sum_{z\in\mathbb{A}_q : X_{\mathbb{A}_q}^r(z)=0} \delta_z$. In particular, we write $X_{\mathbb{A}_q}(z) := X_{\mathbb{A}_q}^q(z)$, $z \in \mathbb{A}_q$ and $\mathcal{Z}_{X_{\mathbb{A}_q}} := \mathcal{Z}_{X_{\mathbb{A}_q}^q}$ as mentioned above.

We recall *Schottky's theorem* (see, for instance, [4]): The group of conformal (i.e., angle-preserving one-to-one) transformations from $\mathbb{A}_q$ to itself is generated by the rotations and the q-inversions $T_q(z) := q/z$. The invariance of the present GAF and its zero point process under rotation is obvious. Using the properties of $S_{\mathbb{A}_q}$, we can prove the following.

Proposition 1.1. (i) *The GAF $X^r_{\mathbb{A}_q}$ given by (1.5) has the (q,r)-inversion symmetry in the sense that*

$$\left\{(T_q'(z))^{1/2} X^r_{\mathbb{A}_q}(T_q(z))\right\} \stackrel{\mathrm{d}}{=} \left\{\sqrt{\frac{q}{r}} X^{q^2/r}_{\mathbb{A}_q}(z)\right\}, \quad z \in \mathbb{A}_q,$$

where $T_q'(z) := \frac{dT_q}{dz}(z) = -q/z^2$.

(ii) *For $\mathcal{Z}_{X^r_{\mathbb{A}_q}} = \sum_i \delta_{Z_i}$, let $T_q^* \mathcal{Z}_{X^r_{\mathbb{A}_q}} := \sum_i \delta_{T_q^{-1}(Z_i)}$. Then $T_q^* \mathcal{Z}_{X^r_{\mathbb{A}_q}} \stackrel{\mathrm{d}}{=} \mathcal{Z}_{X^{q^2/r}_{\mathbb{A}_q}}$.*

(iii) *In particular, when $r = q$, the GAF $X_{\mathbb{A}_q}$ is invariant under conformal transformations which preserve $\mathbb{A}_q$, and so is its zero point process $\mathcal{Z}_{X_{\mathbb{A}_q}}$.*

This result should be compared with the conformal invariance of the DPP of Peres and Virág on $\mathbb{D}$ stated as Proposition 2.12 in Sect. 2.7 below. The proof of Proposition 1.1 is given in Sect. 3.1.

Remark 1. Note that $(T_q'(z))^{1/2} = \sqrt{-1}q^{1/2}/z$ is single valued and non-vanishing in $\mathbb{A}_q$, and so is $(T_q'(z))^{L/2}$ if $L \in \mathbb{N}$. By the calculation given in Sect. 3.1, we have the equality,

$$(T_q'(z))^{L/2}\overline{(T_q'(w))^{L/2}} S_{\mathbb{A}_q}(T_q(z), T_q(w); r)^L = \left(\frac{q}{r}\right)^L S_{\mathbb{A}_q}(z, w; q^2/r)^L.$$

We define $X^{r,(L)}_{\mathbb{A}_q}$ as the centered GAF with the covariance kernel $S_{\mathbb{A}_q}(z, w; r)^L$ on $\mathbb{A}_q$, $L \in \mathbb{N}$. Then it is rotationally invariant and having the (q,r)-inversion symmetry in the sense

$$\left\{(T_q'(z))^{L/2} X^{r,(L)}_{\mathbb{A}_q}(T_q(z))\right\} \stackrel{\mathrm{d}}{=} \left\{\left(\frac{q}{r}\right)^{L/2} X^{q^2/r,(L)}_{\mathbb{A}_q}(z)\right\}, \quad z \in \mathbb{A}_q.$$

This implies that the zero point process of $X^{r,(L)}_{\mathbb{A}_q}$ is also rotationally invariant and symmetric under the (q,r)-inversion. In particular, the GAF $X^{(L)}_{\mathbb{A}_q} := X^{q,(L)}_{\mathbb{A}_q}$ and its zero point process are invariant under conformal transformations which preserve $\mathbb{A}_q$. By definition $X^{(1)}_{\mathbb{A}_q} = X_{\mathbb{A}_q}$ given by (1.5) with $r = q$. The formula (C.1) and Proposition C.2 in Appendix C imply that $X^{(2)}_{\mathbb{A}_q}$ is realized by $X^{(2)}_{\mathbb{A}_q}(z) = \sum_{n\in\mathbb{Z}} \zeta_n c^{(2)}_n z^n$, $z \in \mathbb{A}_q$, where $c^{(2)}_{-1} = \sqrt{a - 1/(2\log q)}$ with $a = a(q)$ given by (C.9), $c^{(2)}_n = \sqrt{(n+1)/(1-q^{2(n+1)})}$, $n \in \mathbb{Z} \setminus \{-1\}$, and $\{\zeta_n\}_{n\in\mathbb{Z}}$ are i.i.d. standard complex Gaussian random variables with density $e^{-|z|^2}/\pi$. We do not know explicit expressions for the Gaussian Laurent series of $X^{(L)}_{\mathbb{A}_q}$ for $L = 3, 4, \dots$, but it is expected that $\lim_{q\to 0} X^{(L)}_{\mathbb{A}_q}(z) \stackrel{\mathrm{d}}{=} X^{(L)}_{\mathbb{D}}(z) := \sum_{n\in\mathbb{N}_0} \zeta_n \frac{\sqrt{L(L+1)\cdots(L+n-1)}}{\sqrt{n!}} z^n$, $z \in \mathbb{D}$, and $X^{(1)}_{\mathbb{A}_q}$ and $X^{(2)}_{\mathbb{A}_q}$ given above indeed satisfy such limit transitions. Here $\{X^{(L)}_{\mathbb{D}} : L > 0\}$ is the family of GAFs on $\mathbb{D}$ studied in [35, Sections 2.3 and 5.4] which are invariant under conformal transformations mapping $\mathbb{D}$ to itself.

Let $\theta(\cdot) := \theta(\cdot; q^2)$ be the theta function, whose definition and basic properties are given in Sect. 2.2. Following the standard way [29,67], we put $\theta(z_1, \ldots, z_n) := \prod_{i=1}^{n} \theta(z_i)$. Then (1.4) is expressed as [56]

$$S_{\mathbb{A}_q}(z, w; r) = \frac{q_0^2 \theta(-rz\overline{w})}{\theta(-r, z\overline{w})}, \quad z, w \in \mathbb{A}_q \tag{1.6}$$

with $q_0 := \prod_{n=1}^{\infty}(1 - q^{2n})$, as proved in Sect. 2.3.

Remark 2. Consider an operator $(U_q f)(z) := f(q^2 z)$ acting on holomorphic functions f on $\mathbb{C}^{\times}$. For $n \in \mathbb{N}$, Rosengren and Schlosser [67] called f an A_{n-1} theta function of norm $a \in \mathbb{C}^{\times}$ if

$$(U_q f)(z) = \frac{(-1)^n}{az^n} f(z).$$

It is shown that f is an A_{n-1} theta function of norm a if and only if there exist C, $b_1, \ldots, b_n$ such that $\prod_{\ell=1}^{n} b_\ell = a$ and $f(z) = C\theta(b_1 z, \ldots, b_n z)$ [67, Lemma 3.2]. In the following, given n points $z_1, \ldots, z_n \in \mathbb{A}_q$, we will evaluate the weighted Szegő kernel at these points. In this case, the weight parameter r for $H_r^2(\mathbb{A}_q)$ can be related to a norm for A_{n-1} theta functions as explained below. Put $a = -r \prod_{\ell=1}^{n} \overline{z_\ell}$ and let $\Theta_j^{(n,a)}(z) := C\theta(-rz\overline{z_j}) \prod_{1 \le \ell \le n, \ell \ne j} \theta(z\overline{z_\ell})$, $z \in \mathbb{A}_q$, $j = 1, \ldots, n$. Then $\{\Theta_j^{(n,a)}(z)\}_{j=1}^{n}$ form a basis of the n-dimensional space of the A_{n-1} theta functions of norm a. If we choose $C = q_0^2/\theta(-r)$, then evaluations of the weighted Szegő kernel at the n points are expressed as $S_{\mathbb{A}_q}(z_i, z_j; r) = \Theta_j^{(n,a)}(z_i) / \prod_{\ell=1}^{n} \theta(z_i \overline{z_\ell})$, $i = 1, \ldots, n$. Multivariate extensions of such elliptic function spaces were studied in [78].

1.2. McCullough–Shen formula for the conditional Szegő kernel. For any non-empty set D, given a positive definite kernel $k(z, w)$ on $D \times D$, we can define a centered Gaussian process on D, X_D, such that the covariance kernel is given by $\mathbf{E}[X_D(z)\overline{X_D(w)}] = k(z, w)$, $z, w \in D$. The kernel k induces RKHS $\mathcal{H}_k$ realized as a function space having k as the reproducing kernel [3]. Now we define a *conditional kernel*

$$k^{\alpha}(z, w) = k(z, w) - \frac{k(z, \alpha)k(\alpha, w)}{k(\alpha, \alpha)}, \quad z, w \in D, \tag{1.7}$$

for $\alpha \in D$ such that $k(\alpha, \alpha) > 0$. Then, k^{α} is a reproducing kernel for the Hilbert subspace $\mathcal{H}_k^{\alpha} := \{f \in \mathcal{H}_k : f(\alpha) = 0\}$. The corresponding centered Gaussian process on D whose covariance kernel is given by k^{α} is equal in law to X_D given that $X_D(\alpha) = 0$.

We can verify that if $D \subsetneq \mathbb{C}$ is a simply connected domain with $\mathcal{C}^{\infty}$ smooth boundary and the Szegő kernel S_D can be defined on it, Riemann's mapping theorem implies the equality [2,6]

$$S_D^{\alpha}(z, w) = S_D(z, w) h_{\alpha}(z) \overline{h_{\alpha}(w)}, \quad z, w, \alpha \in D, \tag{1.8}$$

where h_{α} is the Riemann mapping function; the unique conformal map from D to $\mathbb{D}$ satisfying $h_{\alpha}(\alpha) = 0$ and $h'_{\alpha}(\alpha) > 0$. Actually (1.8) is equivalent with (2.7) derived

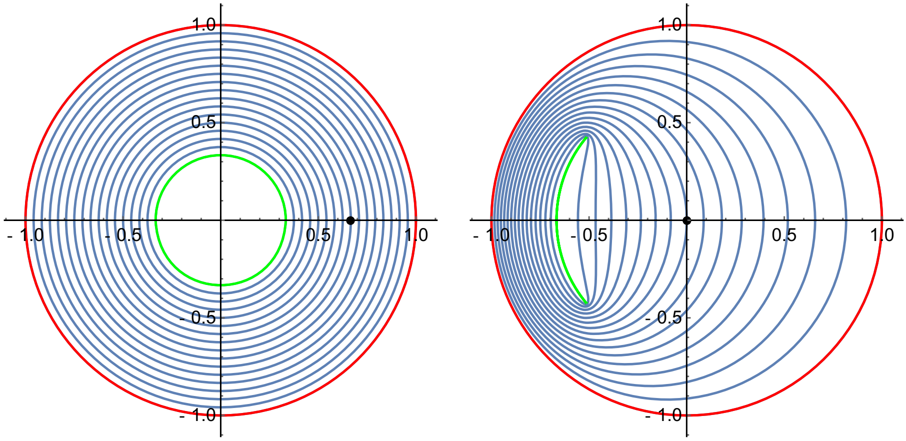

Fig. 1. Conformal map $h_\alpha^q : \mathbb{A}_q \to \mathbb{D} \setminus$ {a circular slit} is illustrated for $q = 1/3$ and $\alpha = 2/3$. The point $\alpha = 2/3$ in $\mathbb{A}_{1/3}$ is mapped to the origin. The outer boundary γ_1 of $\mathbb{A}_{1/3}$ (denoted by a red circle) is mapped to a unit circle (a red circle) making the boundary of $\mathbb{D}$. The inner boundary $\gamma_{1/3}$ of $\mathbb{A}_{1/3}$ (a green circle) is mapped to a circular slit (denoted by a green arc) which is a part of the circle with radius $\alpha = 2/3$, where the map is two-to-one except the two points on $\gamma_{1/3}$ mapped to the two edges of the circular slit

from Riemann's mapping theorem in Sect. 2.1 below. In particular, when $D = \mathbb{D}$, h_α is the Möbius transformation $\mathbb{D} \to \mathbb{D}$ sending α to the origin,

$$h_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z} = z \frac{1 - \alpha/z}{1 - \overline{\alpha} z}, \quad z, \alpha \in \mathbb{D}. \tag{1.9}$$

Since the theta function $\theta(z)$ can be regarded as an elliptic extension of $1 - z$ as suggested by the formula $\lim_{q \to 0} \theta(z; q^2) = 1 - z$ given by (2.16) below, we can think of the following function as an elliptic extension of (1.9);

$$h_\alpha^q(z) := z \frac{\theta(\alpha/z)}{\theta(\overline{\alpha} z)} = -\alpha \frac{\theta(z/\alpha)}{\theta(z\overline{\alpha})}, \quad z, \alpha \in \mathbb{A}_q. \tag{1.10}$$

We can prove that h_α^q is identified with a conformal map from $\mathbb{A}_q$ to the unit disk with a circular slit in it, in which $\alpha \in \mathbb{A}_q$ is sent to the origin [56]. See Fig. 1 and Lemma 2.9 in Sect. 2.6. McCullough and Shen proved the equality

$$S_{\mathbb{A}_q}^\alpha(z, w; r) = S_{\mathbb{A}_q}(z, w; r|\alpha|^2) h_\alpha^q(z) \overline{h_\alpha^q(w)}, \quad z, w, \alpha \in \mathbb{A}_q, \tag{1.11}$$

as an extension of (1.8) [56]. See Sect. 2.6 below for a direct proof of this equality by Weierstrass' addition formula of the theta function (2.18). Up to the factor $h_\alpha^q(z)\overline{h_\alpha^q(w)}$ the conditional kernel $S_{\mathbb{A}_q}^\alpha(z, w; r)$ remains the weighted Szegő kernel, but the weight parameter should be changed from r to $r|\alpha|^2$.

Following (1.7), conditional kernels $k^{\alpha_1, \dots, \alpha_n}$ are inductively defined as

$$k^{\alpha_1, \dots, \alpha_n}(z, w) = (k^{\alpha_1, \dots, \alpha_{n-1}})^{\alpha_n}(z, w), \quad z, w, \alpha_1, \dots, \alpha_n \in D, \quad n = 2, 3, \dots. \tag{1.12}$$

The kernels $k^{\alpha_1, \dots, \alpha_n}$, $n = 2, 3, \dots$, will construct Hilbert subspaces $\mathcal{H}_k^{\alpha_1, \dots, \alpha_n} := \{f \in \mathcal{H}_k : f(\alpha_1) = \dots = f(\alpha_n) = 0\}$.

For $n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{A}_q$, define

$$\gamma^q_{\{\alpha_\ell\}_{\ell=1}^n}(z) := \prod_{\ell=1}^{n} h^q_{\alpha_\ell}(z), \quad z \in \mathbb{A}_q. \tag{1.13}$$

Then the McCullough and Shen formula (1.11) [56] is generalized as

$$S^{\alpha_1,\dots,\alpha_n}_{\mathbb{A}_q}(z, w; r) = S_{\mathbb{A}_q}\Big(z, w; r \prod_{\ell=1}^{n} |\alpha_\ell|^2\Big) \gamma^q_{\{\alpha_\ell\}_{\ell=1}^n}(z) \overline{\gamma^q_{\{\alpha_\ell\}_{\ell=1}^n}(w)}, \quad z, w \in \mathbb{A}_q, \tag{1.14}$$

for $n \in \mathbb{N}, \alpha_1, \dots, \alpha_n \in \mathbb{A}_q$. We can give probabilistic interpretations of the above facts as follows.

Proposition 1.2. *For any $\alpha_1, \dots, \alpha_n \in \mathbb{A}_q$, $n \in \mathbb{N}$, the following hold.*

(i) *The following equality is established,*

$$\begin{aligned} &\{X^r_{\mathbb{A}_q}(z) : z \in \mathbb{A}_q\} \text{ given } \{X^r_{\mathbb{A}_q}(\alpha_1) = \dots = X^r_{\mathbb{A}_q}(\alpha_n) = 0\} \\ &\stackrel{\mathrm{d}}{=} \left\{ \gamma^q_{\{\alpha_\ell\}_{\ell=1}^n}(z) X^{r \prod_{\ell=1}^n |\alpha_\ell|^2}_{\mathbb{A}_q}(z) : z \in \mathbb{A}_q \right\}. \end{aligned}$$

(ii) *Let $\mathcal{Z}^{\alpha_1,\dots,\alpha_n}_{X^r_{\mathbb{A}_q}}$ denote the zero point process of the GAF $X^r_{\mathbb{A}_q}(z)$ given $\{X^r_{\mathbb{A}_q}(\alpha_1) = \dots = X^r_{\mathbb{A}_q}(\alpha_n) = 0\}$. Then, $\mathcal{Z}^{\alpha_1,\dots,\alpha_n}_{X^r_{\mathbb{A}_q}} \stackrel{\mathrm{d}}{=} \mathcal{Z}_{X^{r \prod_{\ell=1}^n |\alpha_\ell|^2}_{\mathbb{A}_q}} + \sum_{i=1}^n \delta_{\alpha_i}$.*

Remark 3. For the GAF on $\mathbb{D}$ studied by Peres and Virág [64], $\{X_{\mathbb{D}}(z) : z \in \mathbb{D}\}$ given $\{X_{\mathbb{D}}(\alpha) = 0\}$ is equal in law to $\{h_\alpha(z) X_{\mathbb{D}}(z) : z \in \mathbb{D}\}$, $\forall \alpha \in \mathbb{D}$, where h_α is given by (1.9), and then, in the notation used in Proposition 1.2, $\mathcal{Z}^\alpha_{X_{\mathbb{D}}} \stackrel{\mathrm{d}}{=} \mathcal{Z}_{X_{\mathbb{D}}} + \delta_\alpha$, $\forall \alpha \in \mathbb{D}$. Hence, no new GAF nor new zero point process appear by conditioning of zeros. For the present GAF on $\mathbb{A}_q$, however, conditioning of zeros induces new GAFs and new zero point processes as shown by Proposition 1.2. Actually, by (1.4) the covariance of the induced GAF $X^{r \prod_{\ell=1}^n |\alpha_\ell|^2}_{\mathbb{A}_q}$ is expressed by $S_{\mathbb{A}_q}(z, w; r \prod_{\ell=1}^n |\alpha_\ell|^2) = \sum_{n=-\infty}^{\infty} (z\overline{w})^n / (1 + r \prod_{\ell=1}^n |\alpha_\ell|^2 q^{2n})$. Since $q < |\alpha_\ell| < 1$, as increasing the number of conditioning zeros, the variance of induced GAF monotonically increases, in which the increment is a decreasing function of $|\alpha_\ell| \in (q, 1)$.

1.3. Correlation functions of the zero point process. We introduce the following notation. For an $n \times n$ matrix $M = (m_{ij})_{1 \le i, j \le n}$,

$$\operatorname{perdet} M = \operatorname*{perdet}_{1 \le i, j \le n} [m_{ij}] := \operatorname{per} M \det M, \tag{1.15}$$

that is, perdet M denotes per M multiplied by det M. Note that perdet is a special case of *hyperdeterminants* introduced by Gegenbauer following Cayley (see [25,51,53] and references therein). If M is a positive semidefinite hermitian matrix, then per $M \ge$ det $M \ge 0$ [52, Section II.4] [59, Theorem 4.2], and hence perdet $M \ge 0$ by the definition (1.15).

The following will be proved in Sect. 3.2.

Theorem 1.3. *Consider the zero point process $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ on $\mathbb{A}_q$. Then, it is a permanental-determinantal point process (PDPP) in the sense that it has correlation functions $\{\rho^n_{\mathbb{A}_q}\}_{n\in\mathbb{N}}$ given by*

$$\rho^n_{\mathbb{A}_q}(z_1,\dots,z_n;r)=\frac{\theta(-r)}{\theta(-r\prod_{k=1}^n|z_k|^4)}\operatorname*{perdet}_{1\le i,j\le n}\left[S_{\mathbb{A}_q}\Big(z_i,z_j;r\prod_{\ell=1}^n|z_\ell|^2\Big)\right] \tag{1.16}$$

for every $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{A}_q$ with respect to m/π.

In Appendix A we rewrite this theorem using the notion of hyperdeterminants (Theorem A.2).

Remark 4. (i) The PDPP with correlation functions (1.16) turns out to be a *simple point process*, i.e., there is no multiple point a.s., due to the existence of two-point correlation function with respect to the Lebesgue measure m/π [40, Lemma 2.7]. (ii) Using the explicit expression (1.16) together with the Frobenius determinantal formula (3.3), we can verify that for every $n\in\mathbb{N}$, the n-point correlation $\rho^n_{\mathbb{A}_q}(z_1,\dots,z_n)>0$ if all coordinates $z_1,\dots,z_n\in\mathbb{A}_q$ are different from each other, and that $\rho^n_{\mathbb{A}_q}(z_1,\dots,z_n)=0$ if some of $z_1,\dots,z_n$ coincide; e.g., $z_i=z_j$, $i\neq j$, by the determinantal factor in perdet (1.15).

Remark 5. The determinantal point processes (DPPs) and the permanental point processes (PPPs) have the n-correlation functions of the forms

$$\rho^n_{\mathrm{DPP}}(z_1,\dots,z_n)=\det_{1\le i,j\le n}[K(z_i,z_j)],\quad \rho^n_{\mathrm{PPP}}(z_1,\dots,z_n)=\operatorname*{per}_{1\le i,j\le n}[K(z_i,z_j)],$$

respectively (cf. [35,55,73]). Due to Hadamard's inequality for the determinant [52, Section II.4] and Lieb's inequality for the permanent [50], we have

$$\rho^2_{\mathrm{DPP}}(z_1,z_2)\le\rho^1_{\mathrm{DPP}}(z_1)\rho^1_{\mathrm{DPP}}(z_2),\quad \rho^2_{\mathrm{PPP}}(z_1,z_2)\ge\rho^1_{\mathrm{PPP}}(z_1)\rho^1_{\mathrm{PPP}}(z_2),$$

in other words, the unfolded 2-correlation functions are ≤ 1 or ≥ 1, respectively (see Sect. 1.4). These correlation inequalities suggest a repulsive nature (negative correlation) for DPPs and an attractive nature (positive correlation) for PPPs. Some related topics are discussed in [70]. Since perdet is considered to have intermediate nature between determinant and permanent, PDPPs are expected to exhibit both repulsive and attractive characters, depending on the position of two points z_1 and z_2. For example, Remark 4 (ii) shows the repulsive nature inherited from the DPP side. The two-sidedness of the present PDPP will be clearly described in Theorem 1.6 given below.

Remark 6. Since correlation functions are transformed as in Lemma 2.10 given in Sect. 2.7, Proposition 1.1 (ii) is rephrased using correlation functions as

$$\rho^n_{\mathbb{A}_q}(T_q(z_1),\dots,T_q(z_n);r)\prod_{\ell=1}^n|T_q'(z_\ell)|^2=\rho^n_{\mathbb{A}_q}(z_1,\dots,z_n;q^2/r) \tag{1.17}$$

for any $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{A}_q$, where $T_q(z)=q/z$ and $|T_q'(z)|^2=q^2/|z|^4$. In the correlation functions $\{\rho^n_{\mathbb{A}_q}\}_{n\in\mathbb{N}}$ given by Theorem 1.3, we see an *inductive structure* such that the functional form of the permanental-determinantal correlation kernel

$S_{\mathbb{A}_q}(\cdot, \cdot; r \prod_{\ell=1}^n |z_\ell|^2)$ is depending on the points $\{z_1, \dots, z_n\}$, which we intend to measure by $\rho_{\mathbb{A}_q}^n$, via the weight parameter $r \prod_{\ell=1}^n |z_\ell|^2$. This is due to the inductive structure of the induced GAFs generated in conditioning of zeros as mentioned in Remark 3. In addition, the reference measure m/π is also weighted by $\theta(-r)/\theta(-r \prod_{k=1}^n |z_k|^4)$. As demonstrated by a direct proof of (1.17) given in Sect. 3.3, such a *hierarchical structure* of correlation functions and reference measures is necessary to realize the (q, r)-inversion symmetry (1.17) (and the invariance under conformal transformations preserving $\mathbb{A}_q$ when $r = q$).

Remark 7. The nonexistence of zero in $\mathbb{D}$ of $S_{\mathbb{D}}(\cdot, \alpha)$, $\alpha \in \mathbb{D}$ and the uniqueness of zero in $\mathbb{A}_q$ of $S_{\mathbb{A}_q}(\cdot, \alpha)$, $\alpha \in \mathbb{A}_q$ are concluded from a general consideration (see, for instance, [7, Chapter 27]). Define

$$\widehat{\alpha} := -\frac{q}{\overline{\alpha}}, \quad \alpha \in \mathbb{A}_q. \tag{1.18}$$

The fact $S_{\mathbb{A}_q}(\widehat{\alpha}, \alpha) = 0$, $\alpha \in \mathbb{A}_q$ was proved as Theorem 1 in [79] by direct calculation, for which a simpler proof will be given below (Lemma 2.3) using theta functions (1.6). For the GAF $X_{\mathbb{D}}$ studied by Peres and Virág [64], all points in $\mathbb{D}$ are correlated, while the GAF $X_{\mathbb{A}_q}$ has a pair structure of independent points $\{\{\alpha, \widehat{\alpha}\} : \alpha \in \mathbb{A}_q\}$ (Proposition 2.4). As a special case of (1.14), we have

$$S_{\mathbb{A}_q}^{\alpha,\widehat{\alpha}}(z, w) = S_{\mathbb{A}_q}(z, w) f_\alpha^q(z) \overline{f_\alpha^q(w)}, \quad z, w \in \mathbb{A}_q$$

with

$$\begin{aligned} f_\alpha^q(z) &:= \frac{1}{z} h_\alpha^q(z) h_{\widehat{\alpha}}^q(z) \\ &= z \frac{\theta(-qz\overline{\alpha}, \alpha/z)}{\theta(-qz/\alpha, \overline{\alpha}z)} = -\alpha \frac{\theta(-qz\overline{\alpha}, z/\alpha)}{\theta(-qz/\alpha, z\overline{\alpha})}. \end{aligned}$$

We notice that f_α^q is identified with the *Ahlfors map* from $\mathbb{A}_q$ to $\mathbb{D}$, that is, it is holomorphic and gives the two-to-one map from $\mathbb{A}_q$ to $\mathbb{D}$ satisfying $f_\alpha^q(\alpha) = f_\alpha^q(\widehat{\alpha}) = 0$. The Ahlfors map has been extensively studied (see, for instance, [7, Chapter 13]), and the above explicit expression using theta functions will be useful. We can verify that if we especially consider the $2n$-correlation of n-pairs $\{\{z_i, \widehat{z_i}\}\}_{i=1}^n$ of points in the zero point process $\mathcal{Z}_{X_{\mathbb{A}_q}}$, the hierarchical structure mentioned above vanishes and the formula (1.16) of Theorem 1.3 is simplified as

$$\rho_{\mathbb{A}_q}^{2n}(z_1, \widehat{z_1}, \dots, z_n, \widehat{z_n}; q) = \operatorname*{perdet}_{1 \le i,j \le n} \begin{bmatrix} S_{\mathbb{A}_q}(z_i, z_j) & S_{\mathbb{A}_q}(z_i, \widehat{z_j}) \\ S_{\mathbb{A}_q}(\widehat{z_i}, z_j) & S_{\mathbb{A}_q}(\widehat{z_i}, \widehat{z_j}) \end{bmatrix}$$

for any $n \in \mathbb{N}$. In other words, we need the hierarchical structure of correlation functions and reference measure in order to describe the probability distributions of general configurations of the zero point process $\mathcal{Z}_{X_{\mathbb{A}_q}^r}$.

The density of zeros on $\mathbb{A}_q$ with respect to m/π is given by

$$\rho_{\mathbb{A}_q}^1(z; r) = \frac{\theta(-r)}{\theta(-r|z|^4)} S_{\mathbb{A}_q}(z, z; r|z|^2)^2 = \frac{q_0^4 \theta(-r, -r|z|^4)}{\theta(-r|z|^2, |z|^2)^2}, \quad z \in \mathbb{A}_q, \tag{1.19}$$

which is always positive. Since $\rho^1_{\mathbb{A}_q}(z;r)$ depends only on the modulus of the coordinate $|z| \in (q,1)$, the PDPP is rotationally invariant. As shown by (2.20–2.22) in Sect. 2.2, in the interval $x \in (-\infty, 0)$, $\theta(x)$ is positive and strictly convex with $\lim_{x\downarrow -\infty} \theta(x) = \lim_{x\uparrow 0} \theta(x) = +\infty$, while in the interval $x \in (q^2, 1)$, $\theta(x)$ is positive and strictly concave with $\theta(x) \sim q_0^2(x-q^2)/q^2$ as $x \downarrow q^2$ and $\theta(x) \sim q_0^2(1-x)$ as $x \uparrow 1$. Therefore, the density shows divergence both at the inner and outer boundaries as

$$\rho^1_{\mathbb{A}_q}(z;r) \sim \begin{cases} \dfrac{q^2}{(|z|^2-q^2)^2}, & |z| \downarrow q, \\ \dfrac{1}{(1-|z|^2)^2}, & |z| \uparrow 1, \end{cases} \tag{1.20}$$

which is independent of r and implies $\mathbf{E}[\mathcal{Z}_{X^r_{\mathbb{A}_q}}(\mathbb{A}_q)] = \infty$. If M is a 2×2 matrix, we see that perdet $M = \det(M \circ M)$, where $M \circ M$ denotes the Hadamard product of M, i.e., entrywise multiplication, $(M\circ M)_{ij} = M_{ij}M_{ij}$. Then the two-point correlation is expressed by a single determinant as

$$\rho^2_{\mathbb{A}_q}(z_1,z_2;r) = \frac{\theta(-r)}{\theta(-r|z_1|^4|z_2|^4)} \det_{1\le i,j\le 2}\Big[S_{\mathbb{A}_q}(z_i,z_j;r|z_1|^2|z_2|^2)^2\Big], \quad z_1,z_2 \in \mathbb{A}_q. \tag{1.21}$$

The above GAF and the PDPP induce the following limiting cases. With fixed $r>0$ we take the limit $q \to 0$. By the reason explained in Remark 8 below, in this limiting procedure, we should consider the point processes $\{\mathcal{Z}_{X^r_{\mathbb{A}_q}} : q > 0\}$ to be defined on the punctured unit disk $\mathbb{D}^\times := \{z \in \mathbb{C} : 0 < |z| < 1\}$ instead of $\mathbb{D}$. Although the limit point process is given on $\mathbb{D}^\times$ by definition, it can be naturally viewed as a point process defined on $\mathbb{D}$, which we will introduce below. Let $H^2_r(\mathbb{D})$ be the Hardy space on $\mathbb{D}$ with the weight parameter $r>0$, whose inner product is given by

$$\langle f,g\rangle_{H^2_r(\mathbb{D})} = \frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi + rf(0)g(0), \quad f,g \in H^2_r(\mathbb{D}).$$

The reproducing kernel of $H^2_r(\mathbb{D})$ is given by

$$\begin{aligned} S_{\mathbb{D}}(z,w;r) &= \sum_{n=0}^\infty e_n^{(0,r)}(z)\overline{e_n^{(0,r)}(w)} = \frac{1}{1+r} + \sum_{n=1}^\infty (z\overline{w})^n \\ &= \frac{1+rz\overline{w}}{(1+r)(1-z\overline{w})}, \quad z,w \in \mathbb{D}. \end{aligned} \tag{1.22}$$

The GAF associated with $H^2_r(\mathbb{D})$ is then defined by

$$X^r_{\mathbb{D}}(z) = \frac{\zeta_0}{\sqrt{1+r}} + \sum_{n=1}^\infty \zeta_n z^n, \quad z \in \mathbb{D} \tag{1.23}$$

so that the covariance kernel is given by $\mathbf{E}[X^r_{\mathbb{D}}(z)\overline{X^r_{\mathbb{D}}(w)}] = S_{\mathbb{D}}(z,w;r)$, $z,w \in \mathbb{D}$. For the conditional GAF given a zero at $\alpha \in \mathbb{D}$, the covariance kernel is given by

$$S^\alpha_{\mathbb{D}}(z,w;r) = S_{\mathbb{D}}(z,w;r|\alpha|^2)h_\alpha(z)\overline{h_\alpha(w)}, \quad z,w,\alpha \in \mathbb{D},$$

where the replacement of the weight parameter r by $r|\alpha|^2$ should be done, even though the factor $h_\alpha(z)$ is simply given by the Möbius transformation (1.9).

For the zero point process Theorem 1.3 is reduced to the following by the formula $\lim_{q\to 0}\theta(z;q^2) = 1-z$.

Corollary 1.4. *Assume that $r > 0$. Then $\mathcal{Z}_{X^r_{\mathbb{D}}}$ is a PDPP on $\mathbb{D}$ with the correlation functions*

$$\rho^n_{\mathbb{D}}(z_1,\dots,z_n;r) = \frac{1+r}{1+r\prod_{k=1}^n |z_k|^4} \operatorname*{perdet}_{1\le i,j\le n}\left[S_{\mathbb{D}}\Big(z_i,z_j;r\prod_{\ell=1}^n |z_\ell|^2\Big)\right] \quad (1.24)$$

for every $n\in\mathbb{N}$ and $z_1,\dots,z_n\in\mathbb{D}$ with respect to m/π. In particular, the density of zeros on $\mathbb{D}$ is given by

$$\rho^1_{\mathbb{D}}(z;r) = \frac{(1+r)(1+r|z|^4)}{(1+r|z|^2)^2(1-|z|^2)^2}, \quad z\in\mathbb{D}. \quad (1.25)$$

As r increases the first term in (1.23), which gives the value of the GAF at the origin, decreases and hence the variance at the origin, $\mathbf{E}[|X^r_{\mathbb{D}}(0)|^2] = S_{\mathbb{D}}(0,0;r) = (1+r)^{-1}$ decreases monotonically. As a result the density of zeros in the vicinity of the origin increases as r increases. Actually we see that $\rho^1_{\mathbb{D}}(0;r) = 1+r$.

Remark 8. The asymptotics (1.20) show that the density of zeros of $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ diverges at the inner boundary $\gamma_q = \{z : |z| = q\}$ for each $q > 0$ while the density of $\mathcal{Z}_{X^r_{\mathbb{D}}}$ is finite at the origin as in (1.25). Therefore infinitely many zeros near the inner boundary γ_q seem to vanish in the limit as $q\to 0$. This is the reason why we regard the base space of $\{\mathcal{Z}_{X^r_{\mathbb{A}_q}} : q > 0\}$ and the limit point process $\mathcal{Z}_{X^r_{\mathbb{D}}}$ as $\mathbb{D}^\times$ instead of $\mathbb{D}$ as mentioned before. (See Sect. 2.7 for the general formulation of point processes.) Indeed, in the vague topology, with which we equip a configuration space, we cannot see configurations outside each compact set, hence infinitely many zeros are not observed on each compact set in $\mathbb{D}^\times$ (not $\mathbb{D}$) for any sufficiently small $q > 0$ depending on the compact set that we take.

We note that if we take the further limit $r\to 0$ in (1.22), we obtain the Szegő kernel of $\mathbb{D}$ given by (1.2). Since the matrix $(S_{\mathbb{D}}(z_i,z_j)^{-1})_{1\le i,j\le n} = (1-z_i\overline{z_j})_{1\le i,j\le n}$ has rank 2, the following equality called *Borchardt's identity* holds (see Theorem 3.2 in [59], Theorem 5.1.5 in [35]),

$$\operatorname*{perdet}_{1\le i,j\le n}\left[(1-z_i\overline{z_j})^{-1}\right] = \det_{1\le i,j\le n}\left[(1-z_i\overline{z_j})^{-2}\right]. \quad (1.26)$$

By the relation (1.3), the $r\to 0$ limit of $\mathcal{Z}_{X^r_{\mathbb{D}}}$ is identified with the DPP on $\mathbb{D}$, $\mathcal{Z}_{X_{\mathbb{D}}}$, studied by Peres and Virág [64], whose correlation functions are given by

$$\rho^n_{\mathbb{D},\mathrm{PV}}(z_1,\dots,z_n) = \det_{1\le i,j\le n}[K_{\mathbb{D}}(z_i,z_j)], \quad n\in\mathbb{N}, \quad z_1,\dots,z_n\in\mathbb{D},$$

with respect to m/π (see Sect. 2.7 below).

Remark 9. We see from (1.22) that $\lim_{r\to\infty} S_{\mathbb{D}}(z,w;r) = (1-z\overline{w})^{-1} - 1$, $z, w \in \mathbb{D}$, which can be identified with the conditional kernel given a zero at the origin; $S^0_{\mathbb{D}}(z,w) = S_{\mathbb{D}}(z,w) - S_{\mathbb{D}}(z,0)S_{\mathbb{D}}(0,w)/S_{\mathbb{D}}(0,0)$ for $S_{\mathbb{D}}(z,0) \equiv 1$. In this limit we can use Borchardt's identity again, since the rank of the matrix $(S_{\mathbb{D}}(z_i,z_j;\infty)^{-1})_{1\le i,j\le n} = (z_i^{-1}z_j^{-1}-1)_{1\le i,j\le n}$ is two. Then, thanks to the proper limit of the prefactor of perdet in (1.24) when $z_k \in \mathbb{D}^\times$ for all $k = 1, 2, \dots, n$; $\lim_{r\to\infty}(1+r)/(1+r\prod_{k=1}^n |z_k|^4) = \prod_{k=1}^n |z_k|^{-4}$, we can verify that $\lim_{r\to\infty} \rho^n_{\mathbb{D}}(z_1,\dots,z_n;r) = \rho^n_{\mathbb{D},\mathrm{PV}}(z_1,\dots,z_n)$ for every $n \in \mathbb{N}$, and every $z_1,\dots,z_n \in \mathbb{D}^\times$. On the other hand, taking (1.23) into account, we have $X^\infty_{\mathbb{D}}(z) = z\sum_{n=1}^\infty \zeta_n z^{n-1} \overset{\mathrm{d}}{=} zX_{\mathbb{D}}(z)$, from which, we can see that as $r \to \infty$, $\mathcal{Z}_{X^r_{\mathbb{D}}}$ converges to $\mathcal{Z}_{X^\infty_{\mathbb{D}}} \overset{\mathrm{d}}{=} \mathcal{Z}_{X_{\mathbb{D}}} + \delta_0$; that is, the DPP of Peres and Virág with a deterministic zero added at the origin. This is consistent with the fact that $\rho^1_{\mathbb{D}}(0;r) = 1+r$ diverges as $r \to \infty$. Since $\mathcal{Z}_{X^0_{\mathbb{D}}} := \lim_{r\to 0}\mathcal{Z}_{X^r_{\mathbb{D}}} \overset{\mathrm{d}}{=} \mathcal{Z}_{X_{\mathbb{D}}}$ as mentioned above, the one-parameter family of PDPPs $\{\mathcal{Z}_{X^r_{\mathbb{D}}} : r \in (0,\infty)\}$ can be regarded as an interpolation between the DPP of Peres and Virág and that DPP with a deterministic zero added at the origin.

1.4. Unfolded 2-correlation functions. By the determinantal factor in perdet (1.15) the PDPP shall be negatively correlated when distances of points are short in the domain $\mathbb{A}_q$. The effect of the permanental part [55,73] in perdet will appear in long distances. Contrary to such a general consideration for the PDPP, if we take the double limit, $q \to 0$ and then $r \to 0$, Borchardt's identity (1.26) becomes applicable and the zero point process is reduced to the DPP studied by Peres and Virág [64]. In addition to this fact, the two-point correlation of the PDPP can be generally expressed using a single determinant as explained in the sentence above (1.21). We have to notice the point, however, that the weight parameter $r|z|^2$ of $S_{\mathbb{A}_q}$ for the density (1.19) is replaced by $r|z_1|^2|z_2|^2$ for the two-point correlation (1.21), and the prefactor $\theta(-r)/\theta(-r|z|^4)$ of $S^2_{\mathbb{A}_q}$ for $\rho^1_{\mathbb{A}_q}$ is changed to $\theta(-r)/\theta(-r|z_1|^4|z_2|^4)$ for $\rho^2_{\mathbb{A}_q}$. Here we show that due to such alterations our PDPP can not be reduced to any DPP in general and it has indeed both of negative and positive correlations depending on the distance of points and the values of parameters. In order to clarify this fact, we study the two-point correlation function normalized by the product of one-point functions,

$$g_{\mathbb{A}_q}(z,w;r) := \frac{\rho^2_{\mathbb{A}_q}(z,w;r)}{\rho^1_{\mathbb{A}_q}(z;r)\rho^1_{\mathbb{A}_q}(w;r)}, \quad (z,w) \in \mathbb{A}_q^2, \tag{1.27}$$

where $\rho^1_{\mathbb{A}_q}$ and $\rho^2_{\mathbb{A}_q}$ are explicitly given by (1.19) and (1.21), respectively. This function is simply called an intensity ratio in [64], but here we call it an *unfolded 2-correlation function* following a terminology used in random matrix theory [27]. We will prove the following: (i) When $0 < q < 1$, in the short distance, the correlation is generally repulsive in common with DPPs (Proposition 1.5). (ii) There exists a critical value $r_0 = r_0(q) \in (q,1)$ for each $q \in (0,1)$ such that if $r \in (r_0,1)$ positive correlation emerges between zeros when the distance between them is large enough within $\mathbb{A}_q$ (Theorem 1.6). (iii) The limits $g_{\mathbb{D}}(z,w;r) := \lim_{q\to 0} g_{\mathbb{A}_q}(z,w;r)$, $z, w \in \mathbb{D}^\times$ and $r_{\mathrm{c}} := \lim_{q\to 0} r_0(q)$ are well-defined, and r_{c} is positive. When $r \in [0,r_{\mathrm{c}})$ all positive correlations vanish in $g_{\mathbb{D}}(z,w;r)$ (Proposition 1.7), while when $r \in (r_{\mathrm{c}},\infty)$ positive correlations can survive (Remark 13 given at the end of Sect. 3). In addition to these rigorous results, we will report the numerical results for $q \in (0,1)$: In intermediate distances between zeros, positive correlations are observed at any value of $r \in (q,1)$, but the distance-dependence of correlations shows two distinct patterns depending on the value of r, whether $r \in (q, r_0)$ or $r \in (r_0, 1)$ (Fig. 2).

It should be noted that the (q,r)-inversion symmetry (1.17) implies the equality (see the second assertion of Lemma 2.10 given below),

$$g_{\mathbb{A}_q}(q/z, q/w; q^2/r) = g_{\mathbb{A}_q}(z, w; r), \quad (z,w) \in \mathbb{A}_q^2. \tag{1.28}$$

Provided that the moduli of coordinates $|z|, |w|$ are fixed, we can verify that the unfolded 2-correlation function takes a minimum (resp. maximum) when $\arg w = \arg z$ (resp. $\arg w = -\arg z$) (Lemma 3.3 in Sect. 3.4.1). We consider these two extreme cases. By putting $w = x, z = q/x \in (\sqrt{q}, 1)$ we define the function

$$G^{\wedge}_{\mathbb{A}_q}(x; r) = g_{\mathbb{A}_q}(q/x, x; r), \quad x \in (\sqrt{q}, 1) \tag{1.29}$$

in order to characterize the short distance behavior of correlation, and by putting $w = -z = x \in (q, 1)$ we define the function

$$G^{\vee}_{\mathbb{A}_q}(x; r) = g_{\mathbb{A}_q}(-x, x; r), \quad x \in (q, 1) \tag{1.30}$$

in order to characterize the long distance behavior of correlation.

Since the PDPP is rotationally symmetric, $G^{\wedge}_{\mathbb{A}_q}(x; r)$ shows correlation between two points on *any* line passing through the origin located in the *same* side with respect to the inner circle γ_q of $\mathbb{A}_q$. The Euclidean distance between these two points is $(x^2 - q)/x$ and it becomes zero as $x \to \sqrt{q}$. We can see the power law with index $\beta = 2$ in the short distance correlation as follows, which is the common feature with DPPs [27].

Proposition 1.5. *As* $x \to \sqrt{q}$, $G^{\wedge}_{\mathbb{A}_q}(x; r) \sim c(r)(x - \sqrt{q})^{\beta}$ *with* $\beta = 2$, *where* $c(r) = (8q_0^4 r^3 \theta(-qr)^6)/(q^2\theta(q)^2\theta(-r)^6) > 0$.

Proof is given in Sect. 3.4.2.

The function $G^{\vee}_{\mathbb{A}_q}(x; r)$ shows correlation between two points on a line passing through the origin, which are located in the *opposite* sides with respect to γ_q and have the Euclidean distance $2x$. Long-distance behavior of the PDPP will be characterized by this function in the limit $x \to 1$ in $\mathbb{A}_q$ (see Remark 10 given below). In this limit the correlation decays as $G^{\vee}_{\mathbb{A}_q}(x; r) \to 1$. We find that the decay obeys the power law with a fixed index $\eta = 4$, but the sign of the coefficient changes at a special value of r for each $q \in (0, 1)$. Given (q, r), define τ_q and ϕ_{-r} by

$$q = e^{\sqrt{-1}\pi\tau_q}, \quad -r = e^{\sqrt{-1}\phi_{-r}},$$

and consider the Weierstrass $\wp$-function $\wp(\phi_{-r}) = \wp(\phi_{-r}; \tau_q)$ given by (2.38) in Sect. 2.4 below. The functions of q, $e_i = e_i(q)$, $i = 1, 2, 3$ and $g_2 = g_2(q)$ are defined by (2.39) and (2.41).

Theorem 1.6. (i) *For* $r > 0$,

$$G^{\vee}_{\mathbb{A}_q}(x; r) \sim 1 + \kappa(r)(1 - x^2)^{\eta} \quad \text{as } x \uparrow 1, \tag{1.31}$$

and

$$G^{\vee}_{\mathbb{A}_q}(x;r) \sim 1 + \frac{\kappa(r)}{q^8}(x^2 - q^2)^{\eta} \quad \textit{as } x \downarrow q, \tag{1.32}$$

with $\eta = 4$*, where*

$$\kappa(r) = \kappa(r;q) := 5\wp(\phi_{-r})^2 + 2e_1\wp(\phi_{-r}) - (e_1^2 + g_2/2). \tag{1.33}$$

The coefficient κ *has the reciprocity property, periodicity property, and their combination,*

$$\kappa(1/r) = \kappa(r), \quad \kappa(q^2 r) = \kappa(r), \quad \kappa(q^2/r) = \kappa(r). \tag{1.34}$$

Hence for the parameter space $\{(q,r) : q \in [0,1], r > 0\}$*, a fundamental cell is given by* $\Omega := \{(q,r) : q \in (0,1), q \le r \le 1\}$*.*

(ii) *It is enough to describe* $\kappa(r)$ *in* Ω*. Let*

$$\wp_+ = \wp_+(q) := -\frac{e_1}{5} + \frac{1}{10}\sqrt{24e_1^2 + 10g_2}. \tag{1.35}$$

Then $e_1 > \wp_+ > e_2 > e_3$*, and*

$$r_0 = r_0(q) := \exp\left[-\frac{1}{2}\int_{\wp_+}^{e_1} \frac{ds}{\sqrt{(e_1 - s)(s - e_2)(s - e_3)}}\right] \tag{1.36}$$

satisfies the inequalities,

$$q < r_0(q) < 1, \quad q \in (0,1). \tag{1.37}$$

The coefficient $\kappa(r)$ *in (1.31) and (1.32) changes its sign at* $r = r_0$ *as follows;* $\kappa(r) < 0$ *if* $r \in (q, r_0)$*, and* $\kappa(r) > 0$ *if* $r \in (r_0, 1)$*.*

(iii) *The curve* $\{r = r_0(q) : q \in (0,1)\} \subset \Omega$ *satisfies the following;*

(a) $$r_{\rm c} := \lim_{q \to 0} r_0(q) = \frac{1 - \frac{\sqrt{4-\sqrt{6}}}{\sqrt{5}}}{1 + \frac{\sqrt{4-\sqrt{6}}}{\sqrt{5}}} = 2\sqrt{6} - 3 - 2\sqrt{8 - 3\sqrt{6}} = 0.2846303639\cdots,$$

(b) $r_0(q) \sim r_{\rm c} + cq^2$ *as* $q \to 0$

with $$c = \frac{8}{3}\left[-72 + 22\sqrt{6} + 3(4\sqrt{6} - 1)\sqrt{8 - 3\sqrt{6}}\right] = 8.515307593\cdots,$$

(c) $r_0(q) \sim 1 - \dfrac{1}{2}(1 - q)$ *as* $q \to 1$*.*

The proof is given in Sects. 3.4.3–3.4.5.

Remark 10. For $s > 0$, define a horizontal slit $[-s + \sqrt{-1}, s + \sqrt{-1}]$ in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im} z > 0\}$ and consider a doubly connected domain $D(s) := \mathbb{H} \setminus [-s + \sqrt{-1}, s + \sqrt{-1}]$. Such a domain is called the *chordal standard domain* with connectivity $n = 2$ in [5] (see also Chapter VII in [60]). As briefly explained in Appendix B, the conformal map from $\mathbb{A}_q$ to $D(s)$ is given by

$$H_q(z) = -2\left\{\zeta(-\sqrt{-1}\log z) + \sqrt{-1}(\eta_1/\pi)\log z\right\}, \quad z \in \mathbb{A}_q, \tag{1.38}$$

where the Weierstrass ζ-function and its special value η_1 are defined in Sect. 2.4 below. This conformal map is chosen so that the boundary points are mapped as

$$H_q(-1) = 0, \quad H_q(1) = \infty, \quad H_q(\pm\sqrt{-1}q) = \mp s + \sqrt{-1}, \quad H_q(\pm q) = \sqrt{-1}.$$

The $x \to 1$ limit for a pair of points $-x$ and x on $\mathbb{A}_q \cap \mathbb{R}$ is hence regarded as the pull-back of an infinite-distance limit of two points on $\mathbb{H} \cap \sqrt{-1}\mathbb{R}$. In the $q \to 0$ limit, H_q is reduced to the Cayley transformation from $\mathbb{D}$ to $\mathbb{H}$, $H_0(z) = -\sqrt{-1}(z+1)/(z-1)$, such that $H_0(-1) = 0$, $H_0(1) = \infty$ and $H_0(0) = \sqrt{-1}$.

Theorem 1.6 implies that if $r \in (r_0, 1)$, $G^{\vee}_{\mathbb{A}_q}(x; r) > 1$ when x is closed to q or 1. Appearance of such positive correlations proves that the present PDPP $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ is indeed unable to be identified with any DPP.

Let $g_{\mathbb{D}}(z, w; r) = \rho^2_{\mathbb{D}}(z, w; r)/\rho^1_{\mathbb{D}}(z; r)\rho^1_{\mathbb{D}}(w; r)$, $(z, w) \in \mathbb{D}^2$. The asymptotic (1.31) holds for $G^{\vee}_{\mathbb{D}}(x; r) := g_{\mathbb{D}}(-x, x; r)$ with $\kappa_0(r) := \lim_{q\to 0} \kappa(r; q)$, which has the reciprocity property $\kappa_0(1/r) = \kappa_0(r)$ (see (3.21) in Sect. 3.4.5). When $r \in (r_c, 1/r_c)$, $\kappa_0(r) > 0$ and hence $G^{\vee}_{\mathbb{D}}(x; r) > 1$ for $x \lesssim 1$, which indicates appearance of attractive interaction at large intervals in $\mathbb{D}^{\times}$. When $r \in [0, r_c)$ or $r \in (1/r_c, \infty)$, negative $\kappa_0(r)$ implies $G^{\vee}_{\mathbb{D}}(x; r) < 1$ even for $x \lesssim 1$. Moreover, we can prove the following.

Proposition 1.7. *If $r \in [0, r_c)$, then $g_{\mathbb{D}}(z, w; r) < 1$, $\forall (z, w) \in \mathbb{D}^2$.*

The proof is given in Sect. 3.5. We should note that this statement does not hold for $r \in (1/r_c, \infty)$, since we can verify that $g_{\mathbb{D}}(z, w; r)$ can exceed 1 when $r > 1$ (see Remark 13 in Sect. 3.5). Therefore, we say that there are two phases for the PDPP $\mathcal{Z}_{X^r_{\mathbb{D}}}$ in the following sense:

(i) Repulsive phase: when $r \in [0, r_c)$, all pairs of zeros are negatively correlated.
(ii) Partially attractive phase: when $r \in (r_c, \infty)$, positive correlations emerge between zeros.

When $q \in (0, 1)$, however, the repulsive phase seems to disappear and positive correlations can be observed at any value of $r > 0$. Figure 2 shows numerical plots of $G^{\vee}_{\mathbb{A}_q}(x; r)$ for $q = 0.1$ with $r_0(0.1) = 0.348\cdots$. The red solid (resp. blue dashed) curve shows the pair correlation for $r = 0.2$ (resp. $r = 0.6$). Since $r = 0.2 < r_0(0.1)$ the red solid curve tends to be less than unity in the vicinity of edges $x = q$ and $x = 1$ as shown in the insets (following Theorem 1.6 (i) and (ii)), but shows two local maxima greater than unity and then has a unique minimum < 1 at the point near $\sqrt{q} = 0.316\cdots$. On the other hand, the blue dashed curve with $r = 0.6 > r_0(0.1)$ tends to be greater than unity in the vicinity of edges as shown in the insets (following Theorem 1.6 (i) and (ii)), but shows two local minima < 1 and then has a unique maximum > 1 at the point near $\sqrt{q} = 0.316\cdots$. As demonstrated by these plots, the change of sign of $\kappa(r)$ at $r = r_0 \in (q, 1)$ seems to convert a global pattern of correlations. Figure 3 shows a numerical plot of the curve $r = r_0(q)$, $q \in (0, 1)$ in the fundamental cell Ω of the parameter space. Detailed characterization of correlations (not only pair correlations but also $\rho^n_{\mathbb{A}_q}$, $n \geq 3$) in PDPPs will be a future problem.

The paper is organized as follows. In Sect. 2 we give preliminaries, which include a brief review of reproducing kernels, conditional Szegő kernels, and a general treatment of point processes including DPPs. There we also give definitions and basic properties of special functions used to represent and analyze GAFs and their zero point processes on an annulus. Section 3 is devoted to proofs of theorems. Concluding remarks are given in Sect. 4. Appendices will provide additional information related to the present study.

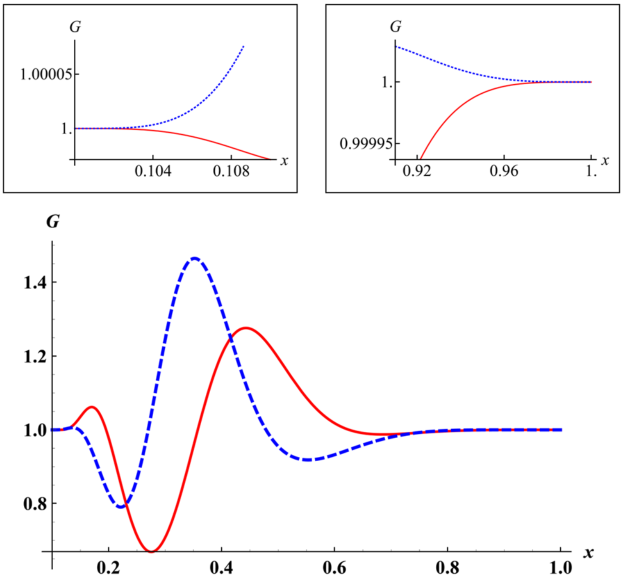

Fig. 2. Numerical plots of $G^{\vee}_{\mathbb{A}_q}(x; r)$ with $q = 0.1$ are given in the interval $(q, 1)$ for $r = 0.2$ (red solid curve) and $r = 0.6$ (blue dashed curve). Note that $0.2 < r_0(0.1) = 0.348 \cdots < 0.6$. Then following Theorem 1.6 (i) and (ii), the red solid curve (resp. blue dashed curve) approaches to the unity from below (resp. from above) as $x \to q = 0.1$ (see the upper left inset) and as $x \to 1$ (see the upper right inset). In the intermediate values of x, the red solid curve shows two local maxima greater than unity and a unique minimum < 1 at the point near $\sqrt{q} = 0.316 \cdots$, while the blue dashed curve has two local minima < 1 and a unique maximum > 1 at the point near $\sqrt{q} = 0.316 \cdots$. The global pattern of correlations is converted when the sign of $\kappa(r)$ is changed

2. Preliminaries

2.1. Reproducing kernels. A *Hilbert function space* is a Hilbert space $\mathcal{H}$ of functions on a domain D in $\mathbb{C}^d$ equipped with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ such that evaluation at each point of D is a continuous functional on $\mathcal{H}$. Therefore, for each point $w \in D$, there is an element of $\mathcal{H}$, which is called the reproducing kernel at w and denote by k_w, with the property $\langle f, k_w \rangle_{\mathcal{H}} = f(w), \forall f \in \mathcal{H}$. Because $k_w \in \mathcal{H}$, it is itself a function on D, $k_w(z) = \langle k_w, k_z \rangle_{\mathcal{H}}$. We write

$$k_{\mathcal{H}}(z, w) := k_w(z) = \langle k_w, k_z \rangle_{\mathcal{H}}$$

and call it the *reproducing kernel* for $\mathcal{H}$. By definition, it is hermitian; $\overline{k_{\mathcal{H}}(z, w)} = k_{\mathcal{H}}(w, z), z, w \in D$. If $\mathcal{H}$ is a holomorphic Hilbert function space, then $k_{\mathcal{H}}$ is holomorphic in the first variable and anti-holomorphic in the second. We see that $k_{\mathcal{H}}(z, w)$ is a positive semi-definite kernel: for any $n \in \mathbb{N} := \{1, 2, \ldots\}$, for any points $z_i \in D$ and $\xi_i \in \mathbb{C}, i = 1, 2, \ldots, n$,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} k_{\mathcal{H}}(z_i, z_j) \xi_i \overline{\xi_j} = \left\| \sum_{i=1}^{n} \xi_i k_{\mathcal{H}}(z_i, \cdot) \right\|_{\mathcal{H}}^2 \geq 0. \tag{2.1}$$

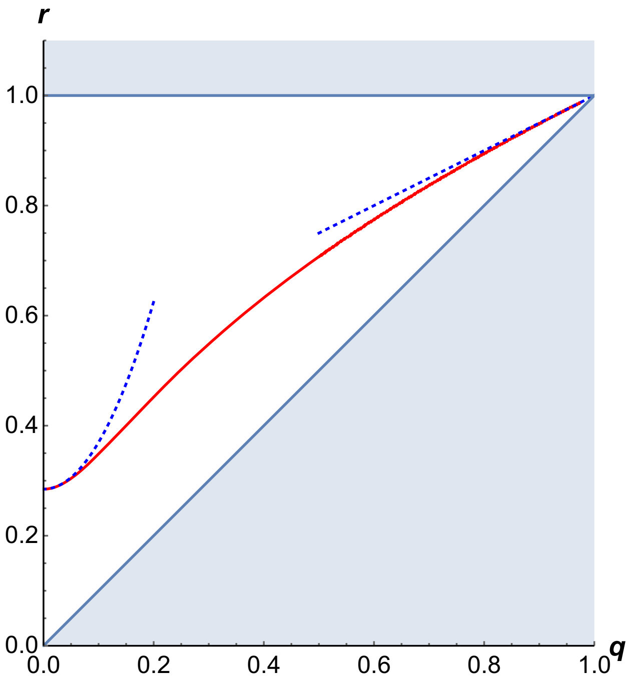

Fig. 3. The curve $r = r_0(q)$ given by (1.36) in Theorem 1.6 (ii) is numerically plotted (in red) in the fundamental cell Ω in the parameter space, which is located between the diagonal line $r = q$ (shown by a blue line) and the horizontal line $r = 1$ satisfying (1.37). The parabolic curve $r_c + cq^2$ given by (iii) (b) and the line $1 - (1 - q)/2$ by (iii) (c) are also dotted, which approximate $r = r_0(q)$ well for $q \gtrsim 0$ and $q \lesssim 1$, respectively

Let $\{e_n : n \in \mathcal{I}\}$ be any CONS for $\mathcal{H}$, where $\mathcal{I}$ is an index set. Then one can prove that the reproducing kernel for $\mathcal{H}$ is written in the form

$$k_{\mathcal{H}}(z, w) = \sum_{n \in \mathcal{I}} e_n(z)\overline{e_n(w)}. \tag{2.2}$$

We note that the positive definiteness of the kernel (2.1) is equivalent with the situation such that, for any points $z_i \in D, i \in \mathbb{N}$, the matrix $(k_{\mathcal{H}}(z_i, z_j))_{1 \leq i, j \leq n}$ has a nonnegative determinant, $\det_{1 \leq i, j \leq n}[k_{\mathcal{H}}(z_i, z_j)] \geq 0$, for any $n \in \mathbb{N}$.

Here we show two examples of holomorphic Hilbert function spaces, the *Bergman space* and the *Hardy space*, for a unit disk $\mathbb{D}$ and the domains which are conformally transformed from $\mathbb{D}$ [1,7,8,33,60].

The Bergman space on $\mathbb{D}$, denoted by $L^2_{\mathrm{B}}(\mathbb{D})$, is the Hilbert space of holomorphic functions on $\mathbb{D}$ which are square-integrable with respect to the Lebesgue measure on $\mathbb{C}$ [8]. The inner product for $L^2_{\mathrm{B}}(\mathbb{D})$ is given by

$$\langle f, g \rangle_{L^2_{\mathrm{B}}(\mathbb{D})} := \frac{1}{\pi} \int_{\mathbb{D}} f(z)\overline{g(z)} m(dz) = \sum_{n=0}^{\infty} \frac{\widehat{f}(n)\overline{\widehat{g}(n)}}{n+1},$$

where the nth Taylor coefficient of f at 0 is denoted by $\widehat{f}(n)$; $f(z) = \sum_{n=0}^{\infty} \widehat{f}(n) z^n$. Let $\widetilde{e_n}(z) := \sqrt{n+1} z^n, n \in \mathbb{N}_0$. Then $\{\widetilde{e_n}(z)\}_{n \in \mathbb{N}_0}$ form a CONS for $L^2_{\mathrm{B}}(\mathbb{D})$ and the reproducing kernel (2.2) is given by

$$K_{\mathbb{D}}(z,w) := k_{L^2_{\mathrm{B}}(\mathbb{D})}(z,w) = \sum_{n\in\mathbb{N}_0}(n+1)(z\overline{w})^n = \frac{1}{(1-z\overline{w})^2}, \quad z,w\in\mathbb{D}. \tag{2.3}$$

This kernel is called the *Bergman kernel* of $\mathbb{D}$.

The Hardy space on $\mathbb{D}$, $H^2(\mathbb{D})$, consists of holomorphic functions on $\mathbb{D}$ such that the Taylor coefficients form a square-summable series;

$$\|f\|^2_{H^2(\mathbb{D})} := \sum_{n\in\mathbb{N}_0}|\widehat{f}(n)|^2 < \infty, \quad f\in H^2(\mathbb{D}).$$

For every $f\in H^2(\mathbb{D})$, the non-tangential limit $\lim_{r\uparrow 1} f(re^{\sqrt{-1}\phi})$ exists a.e. by Fatou's theorem and we write it as $f(e^{\sqrt{-1}\phi})$. It is known that $f(e^{\sqrt{-1}\phi})\in L^2(\partial\mathbb{D})$ [1]. Then one can prove that the inner product of $H^2(\mathbb{D})$ is given by the following three different ways [1],

$$\langle f,g\rangle_{H^2(\mathbb{D})} = \begin{cases} \displaystyle\sum_{n\in\mathbb{N}_0}\widehat{f}(n)\overline{\widehat{g}(n)}, \\ \displaystyle\lim_{r\uparrow 1}\frac{1}{2\pi}\int_0^{2\pi} f(re^{\sqrt{-1}\phi})\overline{g(re^{\sqrt{-1}\phi})}d\phi, \\ \displaystyle\frac{1}{2\pi}\int_0^{2\pi} f(e^{\sqrt{-1}\phi})\overline{g(e^{\sqrt{-1}\phi})}d\phi, \end{cases} \quad f,g\in H^2(\mathbb{D}), \tag{2.4}$$

with $\|f\|^2_{H^2(\mathbb{D})} = \langle f,f\rangle_{H^2(\mathbb{D})}$. Let σ be the measure on the boundary of $\mathbb{D}$ which is the usual arc length measure. Then the last expression of the inner product (2.4) is written as $\langle f,g\rangle_{H^2(\mathbb{D})} = (1/2\pi)\int_{\gamma_1} f(z)\overline{g(z)}\sigma(dz)$, where γ_1 is a unit circle $\{e^{\sqrt{-1}\phi} : \phi\in[0,2\pi)\}$ giving the boundary of $\mathbb{D}$. If we set $e_n(z) := e_n^{(0,0)}(z) = z^n, n\in\mathbb{N}_0$, then $\{e_n(z)\}_{n\in\mathbb{N}_0}$ form CONS for $H^2(\mathbb{D})$. The reproducing kernel (2.2) is given by

$$S_{\mathbb{D}}(z,w) := k_{H^2(\mathbb{D})}(z,w) = \sum_{n\in\mathbb{N}_0}(z\overline{w})^n = \frac{1}{1-z\overline{w}}, \quad z,w\in\mathbb{D}, \tag{2.5}$$

which is called the *Szegő kernel* of $\mathbb{D}$.

Let $f : D\to\widetilde{D}$ be a conformal transformation between two bounded domains $D,\widetilde{D}\subsetneq\mathbb{C}$ with $\mathcal{C}^\infty$ smooth boundary. We find an argument in Chapter 12 of [7] concluding that the derivative of the transformation f denoted by f' has a single valued square root on D. We let $\sqrt{f'(z)}$ denote one of the square roots of f'. The Szegő kernel and the Bergman kernel are then transformed by f as

$$\begin{aligned} S_D(z,w) &= \sqrt{f'(z)}\overline{\sqrt{f'(w)}}S_{\widetilde{D}}(f(z),f(w)), \\ K_D(z,w) &= |f'(z)||f'(w)|K_{\widetilde{D}}(f(z),f(w)), \quad z,w\in D. \end{aligned} \tag{2.6}$$

See Chapters 12 and 16 of [7]. Consider the special case in which $D \subsetneq \mathbb{C}$ is a simply connected domain with $\mathcal{C}^\infty$ smooth boundary and $\widetilde{D} = \mathbb{D}$. For each $\alpha \in D$, Riemann's mapping theorem gives a unique conformal transformation [2];

$$h_\alpha : D \to \mathbb{D} \quad \text{conformal such that } h_\alpha(\alpha) = 0,\ h'_\alpha(\alpha) > 0.$$

Such h_α is called the *Riemann mapping function*. By (2.5), the first equation in (2.6) gives the following formula [6],

$$S_D(z, w) = \frac{S_D(z, \alpha)\overline{S_D(w, \alpha)}}{S_D(\alpha, \alpha)} \frac{1}{1 - h_\alpha(z)\overline{h_\alpha(w)}}, \quad z, w, \alpha \in D. \tag{2.7}$$

Similarly, we have

$$K_D(z, w) = \frac{S_D(z, \alpha)^2\overline{S_D(w, \alpha)^2}}{S_D(\alpha, \alpha)^2} \frac{1}{(1 - h_\alpha(z)\overline{h_\alpha(w)})^2}, \quad z, w, \alpha \in D. \tag{2.8}$$

Hence the following relationship is established,

$$S_D(z, w)^2 = K_D(z, w), \quad z, w \in D. \tag{2.9}$$

Although the Szegő kernel could be eliminated from the right-hand sides of (2.7) and (2.8) by noting that $h'_\alpha(z) = S_D(z, \alpha)^2/S_D(\alpha, \alpha)$, the formula (2.7) and the relation (2.9) played important roles in the study by Peres and Virág [64]. As a matter of fact, (2.7) is equivalent with (1.8) and the combination of (2.9) and (1.26) gives (1.3).

2.2. Theta function θ. Assume that $p \in \mathbb{C}$ is a fixed number such that $0 < |p| < 1$. We use the following standard notation [29,45,67],

$$(a; p)_n := \prod_{i=0}^{n-1}(1 - ap^i), \qquad (a; p)_\infty := \prod_{i=0}^{\infty}(1 - ap^i),$$
$$(a_1, \dots, a_k; p)_\infty := (a_1; p)_\infty \cdots (a_k; p)_\infty. \tag{2.10}$$

The *theta function* with argument z and nome p is defined by

$$\theta(z; p) := (z, p/z; p)_\infty. \tag{2.11}$$

We often use the shorthand notation $\theta(z_1, \dots, z_n; p) := \prod_{i=1}^{n} \theta(z_i; p)$.

As a function of z, the theta function $\theta(z; p)$ is holomorphic in $\mathbb{C}^\times$ and has single zeros precisely at p^i, $i \in \mathbb{Z}$, that is,

$$\{z \in \mathbb{C}^\times : \theta(z; p) = 0\} = \{p^i : i \in \mathbb{Z}\}. \tag{2.12}$$

We will use the inversion formula

$$\theta(1/z; p) = -\frac{1}{z}\theta(z; p) \tag{2.13}$$

and the quasi-periodicity property

$$\theta(pz; p) = -\frac{1}{z}\theta(z; p) \tag{2.14}$$

of the theta function. By comparing (2.13) and (2.14) and performing the transformation $z \mapsto 1/z$, we immediately see the periodicity property,

$$\theta(p/z; p) = \theta(z; p). \tag{2.15}$$

By Jacobi's triple product identity (see, for instance, [29, Section 1.6]), we have the Laurent expansion

$$\theta(z; p) = \frac{1}{(p; p)_\infty} \sum_{n \in \mathbb{Z}} (-1)^n p^{\binom{n}{2}} z^n.$$

One can show that [62, Chapter 20]

$$\lim_{p \to 0} \theta(z; p) = 1 - z, \tag{2.16}$$

$$\theta'(1; p) := \frac{\partial \theta(z; p)}{\partial z}\Big|_{z=1} = -(p; p)_\infty^2. \tag{2.17}$$

The theta function satisfies the following *Weierstrass' addition formula* [44],

$$\theta(xy, x/y, uv, u/v; p) - \theta(xv, x/v, uy, u/y; p) = \frac{u}{y} \theta(yv, y/v, xu, x/u; p). \tag{2.18}$$

When p is real and $p \in (0, 1)$, we see that

$$\overline{\theta(z; p)} = \theta(\overline{z}; p). \tag{2.19}$$

In this case the definition (2.11) with (2.10) implies that

$$\left.\begin{array}{l} \theta(x; p) > 0,\ x \in (p^{2i+1}, p^{2i}) \\ \theta(x; p) = 0,\ x = p^i \\ \theta(x; p) < 0,\ x \in (p^{2i}, p^{2i-1}) \end{array}\right\} \quad i \in \mathbb{Z},$$
$$\theta(x; p) > 0, \quad x \in (-\infty, 0). \tag{2.20}$$

Moreover, we can prove the following: In the interval $x \in (-\infty, 0)$, $\theta(x) := \theta(x; p)$ is strictly convex with

$$\min_{x \in (-\infty, 0)} \theta(x) = \theta(-\sqrt{p}) = \prod_{n=1}^{\infty} (1 + p^{n-1/2})^2 > 0, \tag{2.21}$$

and $\lim_{x \downarrow -\infty} \theta(x) = \lim_{x \uparrow 0} \theta(x) = +\infty$, and in the interval $x \in (p, 1)$, $\theta(x)$ is strictly concave with

$$\max_{x \in (p, 1)} \theta(x) = \theta(\sqrt{p}) = \prod_{n=1}^{\infty} (1 - p^{n-1/2})^2, \tag{2.22}$$

$\theta(x) \sim (p; p)_\infty^2 (x - p)/p$ as $x \downarrow p$, and $\theta(x) \sim (p; p)_\infty^2 (1 - x)$ as $x \uparrow 1$, where (2.14) and (2.17) were used.

2.3. Ramanujan ρ_1-function, Jordan–Kronecker function and weighted Szegő kernel of $\mathbb{A}_q$. Assume that $q \in (0, 1)$. Consider the so-called *Ramanujan ρ_1-function* [19,80] defined by

$$\rho_1(z) = \rho_1(z; q) = \frac{1}{2} + \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{z^n}{1 - q^{2n}} \tag{2.23}$$

with $q^2 < |z| < 1$. As a generalization of ρ_1 the following function has been studied in [19,56,80],

$$f^{\mathrm{JK}}(z, a) = f^{\mathrm{JK}}(z, a; q) := \sum_{n \in \mathbb{Z}} \frac{z^n}{1 - aq^{2n}}, \tag{2.24}$$

with $q^2 < |z| < 1$, $a \notin \{q^{2i} : i \in \mathbb{Z}\}$, which is called the *Jordan–Kronecker function* (see [80, p.59] and [82, pp.70-71]).

Proposition 2.1. *Assume that $r > 0$. Then the weighted Szegő kernel of $\mathbb{A}_q$ (1.4) is expressed by the Jordan–Kronecker function (2.24) as*

$$S_{\mathbb{A}_q}(z, w; r) = f^{\mathrm{JK}}(z\overline{w}, -r), \quad z, w \in \mathbb{A}_q. \tag{2.25}$$

In particular, the Szegő kernel of $\mathbb{A}_q$ is given by $S_{\mathbb{A}_q}(z, w) = f^{\mathrm{JK}}(z\overline{w}, -q)$, $z, w \in \mathbb{A}_q$.

The *bilateral basic hypergeometric series* in base p with one numerator parameter a and one denominator parameter b is defined by [29]

$$ {}_1\psi_1(a; b; p, z) = {}_1\psi_1\left[\begin{matrix} a \\ b \end{matrix}; p, z\right] := \sum_{n \in \mathbb{Z}} \frac{(a; p)_n}{(b; p)_n} z^n, \quad |b/a| < |z| < 1.$$

The Jordan–Kronecker function (2.24) is a special case of the ${}_1\psi_1$ function [19,80];

$$f^{\mathrm{JK}}(z, a; q) = \frac{1}{1 - a} {}_1\psi_1(a; aq^2; q^2, z).$$

The following equality is known as *Ramanujan's ${}_1\psi_1$ summation formula* [19,29,80],

$$\sum_{n \in \mathbb{Z}} \frac{(a; p)_n}{(b; p)_n} z^n = \frac{(az, p/(az), p, b/a; p)_\infty}{(z, b/(az), b, p/a; p)_\infty}, \quad |b/a| < |z| < 1.$$

Combining the above two equalities with an appropriate change of variables, we obtain [19,80]

$$f^{\mathrm{JK}}(z, a) = f^{\mathrm{JK}}(z, a; q) = \frac{(az, q^2/(az), q^2, q^2; q^2)_\infty}{(z, q^2/z, a, q^2/a; q^2)_\infty} = \frac{q_0^2 \theta(za; q^2)}{\theta(z, a; q^2)}, \tag{2.26}$$

where $q_0 := \prod_{n \in \mathbb{N}} (1 - q^{2n}) = (q^2; q^2)_\infty$. Note that $\theta(z; q^2)$ is a holomorphic function of z in $\mathbb{C}^\times$. Hence relying on (2.26), for every fixed a in $\mathbb{C}^\times \setminus \{q^{2i} : i \in \mathbb{Z}\}$, $f^{\mathrm{JK}}(\cdot, a)$ can be analytically continued to $\mathbb{C}^\times \setminus \{q^{2i} : i \in \mathbb{Z}\}$. The poles are located exactly at the zeros of $\theta(z; q^2)$ appearing in the denominator; $\{q^{2i} : i \in \mathbb{Z}\}$. The following symmetries of f^{JK} are readily verified by (2.26) using (2.13) and (2.14) [19,80].

$$f^{\mathrm{JK}}(z, a) = f^{\mathrm{JK}}(a, z), \tag{2.27}$$

$$f^{\mathrm{JK}}(z,a) = -f^{\mathrm{JK}}(z^{-1},a^{-1}), \tag{2.28}$$

$$f^{\mathrm{JK}}(z,a) = zf^{\mathrm{JK}}(z,aq^2) = af^{\mathrm{JK}}(zq^2,a). \tag{2.29}$$

As shown in Chapter 3 in [80], (2.24) is rewritten as

$$f^{\mathrm{JK}}(z,a) = \frac{1-za}{(1-z)(1-a)} + \sum_{n=1}^{\infty} q^{2n^2} z^n a^n \left(1 + \frac{zq^{2n}}{1-zq^{2n}} + \frac{aq^{2n}}{1-aq^{2n}}\right)$$
$$- \sum_{n=1}^{\infty} q^{2n^2} z^{-n} a^{-n} \left(1 + \frac{z^{-1}q^{2n}}{1-z^{-1}q^{2n}} + \frac{a^{-1}q^{2n}}{1-a^{-1}q^{2n}}\right),$$

which is completely symmetric in z and a and valid for $z, a \notin \{q^{2i} : i \in \mathbb{Z}\}$. The equalities (2.27)–(2.29) are proved also using this expression [80].

From now on, we assume that $p = q^2$ and hence $\theta(\cdot)$ means $\theta(\cdot; q^2)$ in the following. We replace z by $z\overline{w}$ and a by $-r$ in (2.26). Then Proposition 2.1 implies the following.

Proposition 2.2 (McCullough and Shen [56]). *For* $r > 0$

$$S_{\mathbb{A}_q}(z,w;r) = \frac{q_0^2 \theta(-rz\overline{w})}{\theta(-r, z\overline{w})}, \quad z, w \in \mathbb{A}_q. \tag{2.30}$$

In particular,

$$S_{\mathbb{A}_q}(z,w) = S_{\mathbb{A}_q}(z,w;q) = \frac{q_0^2 \theta(-qz\overline{w})}{\theta(-q, z\overline{w})}, \quad z, w \in \mathbb{A}_q. \tag{2.31}$$

Since $\theta(\cdot)$ is holomorphic in the punctured complex plane $\mathbb{C}^{\times} := \{z \in \mathbb{C} : |z| > 0\}$, by the expression (2.30), $S_{\mathbb{A}_q}(z,w;r)$ can be analytically continued to $\mathbb{C}^{\times}$ as an analytic function of z, r and an anti-analytic function of w. Actually the inversion formula (2.13) and the quasi-periodicity property (2.14) of the theta function given in Sect. 2.2 imply the following functional equations,

$$\text{(i)} \quad S_{\mathbb{A}_q}(q^2 z, w; r) = -\frac{1}{r} S_{\mathbb{A}_q}(z,w;r),$$
$$\text{(ii)} \quad S_{\mathbb{A}_q}(1/z, w; r) = -S_{\mathbb{A}_q}(z, 1/w; 1/r),$$
$$\text{(iii)} \quad S_{\mathbb{A}_q}(z, w; q^2 r) = \frac{1}{z\overline{w}} S_{\mathbb{A}_q}(z,w;r). \tag{2.32}$$

Then the following is easily verified.

Lemma 2.3. *Assume that* $\alpha \in \mathbb{A}_q$. *Then* $S_{\mathbb{A}_q}(z,\alpha;r)$ *has zeros at* $z = -q^{2i}/(\overline{\alpha}r)$, $i \in \mathbb{Z}$ *in* $\mathbb{C}^{\times}$. *In particular,* $S_{\mathbb{A}_q}(z,\alpha)$ *has a unique zero in* $\mathbb{A}_q$ *at* $z = \widehat{\alpha}$ *given by (1.18).*

Proof. Since θ is holomorphic in $\mathbb{C}^{\times}$, the expression (2.30) implies that $S_{\mathbb{A}_q}(z,\alpha;r)$ is meromorphic in $\mathbb{C}^{\times}$. By (2.12), $S_{\mathbb{A}_q}(z,\alpha;r)$ vanishes in $\mathbb{C}^{\times}$ only if $-z\overline{\alpha}r = q^{2i}$, $i \in \mathbb{Z}$. By assumption $|\alpha| \in (q,1)$. Hence, when $r = q$, $|-q^{2i}/(\overline{\alpha}r)| = q^{2i-1}/|\alpha| \in (q,1)$, if and only if $i = 1$. □

The second assertion of Lemma 2.3 gives the following probabilistic statement.

Proposition 2.4. *For each* $\alpha \in \mathbb{A}_q$, $X_{\mathbb{A}_q}(\alpha)$ *and* $X_{\mathbb{A}_q}(\widehat{\alpha})$ *are mutually independent.*

2.4. Weierstrass elliptic functions and other functions. Here we show useful relations between the theta function, Ramanujan ρ_1-function, Jordan–Kronecker function, and *Weierstrass elliptic functions*.

Assume that ω_1 and ω_3 are complex numbers such that if we set $\tau = \omega_3/\omega_1$, then $\mathrm{Im}\tau > 0$. The lattice $\mathbb{L}(\omega_1, \omega_3)$ on $\mathbb{C}$ with lattice generators $2\omega_1$ and $2\omega_3$ is given by

$$\mathbb{L} = \mathbb{L}(\omega_1, \omega_3) := \{2m\omega_1 + 2n\omega_3 : (m, n) \in \mathbb{Z}^2\}.$$

The *Weierstrass* $\wp$*-function* and ζ*-function* are defined by

$$\begin{aligned}\wp(\phi) = \wp(\phi|2\omega_1, 2\omega_3) &:= \frac{1}{\phi^2} + \sum_{v\in\mathbb{L}(\omega_1,\omega_3)\setminus\{0\}} \left[\frac{1}{(\phi - v)^2} - \frac{1}{v^2}\right],\\ \zeta(\phi) = \zeta(\phi|2\omega_1, 2\omega_3) &:= \frac{1}{\phi} + \sum_{v\in\mathbb{L}(\omega_1,\omega_3)\setminus\{0\}} \left[\frac{1}{\phi - v} + \frac{1}{v} + \frac{\phi}{v^2}\right].\end{aligned} \tag{2.33}$$

(See, for instance, Chapter 23 in [62].) We put $\omega_2 = -(\omega_1+\omega_3)$. By the definition (2.33) we see that $\wp(\phi)$ is even and $\zeta(\phi)$ is odd with respect to ϕ, and $\wp(\phi)$ is an elliptic function (i.e., a doubly periodic meromorphic function in $\mathbb{C}$); $\wp(\phi + 2\omega_\nu) = \wp(\phi)$, $\nu = 1, 2, 3$. We note that $\wp'(\omega_\nu) = 0$, $\nu = 1, 2, 3$, $\wp(\phi) = -\zeta'(\phi)$, and $\zeta(\phi + 2\omega_\nu) = \zeta(\phi) + 2\eta_\nu$ where $\eta_\nu := \zeta(\omega_\nu)$, $\nu = 1, 2, 3$. In the present paper we consider the following setting;

$$\begin{aligned}&\omega_1 = \pi, \quad \frac{\omega_3}{\omega_1} = \tau_q, \quad \text{and}\\ &q = e^{\sqrt{-1}\pi\tau_q} \in (0, 1) \iff \tau_q = -\sqrt{-1}\frac{\log q}{\pi} \in \sqrt{-1}\mathbb{R}_{>0}.\end{aligned} \tag{2.34}$$

In the terminology of [29, page 304], when we regard $p := q^2$ as the nome of the theta function, τ_q shall be called the *nome modular parameter*, and when we regard $q = p^{1/2} =: e^{2\sqrt{-1}\pi\sigma_q}$ as the *base* of q-special functions, τ_q will be the twice of the *base modular parameter* σ_q. In this setting, the $\wp$-function is considered as a function of an argument ϕ and the modular parameter τ_q though q. Then we have the following expansions,

$$\begin{aligned}\wp(\phi) = \wp(\phi; \tau_q) &= -\frac{1}{12} + 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} + \frac{1}{4}\frac{1}{\sin^2(\phi/2)} - 2\sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos(n\phi)\\ &= -\frac{1}{12} + 2\sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} - \sum_{n=-\infty}^{\infty} \frac{e^{\sqrt{-1}\phi}q^{2n}}{(1 - e^{\sqrt{-1}\phi}q^{2n})^2}.\end{aligned} \tag{2.35}$$

We use the notation

$$z = e^{\sqrt{-1}\phi_z} \iff \phi_z = -\sqrt{-1}\log z. \tag{2.36}$$

Then $\phi_{zw} = \phi_z + \phi_w$, $\phi_{z^{-1}} = -\phi_z$, and $\phi_{q^2} = 2\omega_3$ modulo $2\pi\mathbb{Z}$. Hence the evenness and the periodicity property of $\wp$ are written as

$$\wp(-\phi_z) = \wp(\phi_{z^{-1}}) = \wp(\phi_z), \quad \wp(\phi_{q^2 z}) = \wp(\phi_z). \tag{2.37}$$

The expansion (2.35) is written as

$$\begin{aligned}\wp(\phi_z) = \wp(\phi_z;\tau_q) &= -\frac{1}{12} - \frac{z}{(1-z)^2} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - \sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\left(z^n + \frac{1}{z^n}\right) \\ &= -\frac{1}{12} - \frac{z}{(1-z)^2} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - \sum_{n=1}^{\infty}\frac{zq^{2n}}{(1-zq^{2n})^2} - \sum_{n=1}^{\infty}\frac{z^{-1}q^{2n}}{(1-z^{-1}q^{2n})^2}.\end{aligned} \tag{2.38}$$

The special values of $\wp$ are denoted by

$$\begin{aligned} e_1 = e_1(q) &:= \wp(\pi) = \wp(\phi_{-1};\tau_q) \\ &= \frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}, \\ e_2 = e_2(q) &:= \wp(\pi+\pi\tau_q) = \wp(\phi_{-q};\tau_q) \\ &= -\frac{1}{12} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} + 2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1+q^{2n-1})^2}, \\ e_3 = e_3(q) &:= \wp(\pi\tau_q) = \wp(\phi_q;\tau_q) \\ &= -\frac{1}{12} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} - 2\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(1-q^{2n-1})^2}. \end{aligned} \tag{2.39}$$

We see that

$$e_1 + e_2 + e_3 = 0, \tag{2.40}$$

and define

$$\begin{aligned} g_2 = g_2(q) &:= 2(e_1^2 + e_2^2 + e_3^2) = -4(e_2e_3 + e_3e_1 + e_1e_2) > 0, \\ g_3 = g_3(q) &:= 4e_1e_2e_3 = \frac{4}{3}(e_1^3 + e_2^3 + e_3^3). \end{aligned} \tag{2.41}$$

The *imaginary transformation* of $\wp$ is given by [80, p.31], $\wp(\phi;\tau_q) = \tau_q^{-2}\wp(\phi/\tau_q; -1/\tau_q)$. Hence (2.35) is written as

$$\begin{aligned}\wp(\phi) = \wp(\phi;\tau_q) = \frac{1}{|\tau_q|^2}\Bigg[&\frac{1}{12} + \frac{1}{4}\frac{1}{\sinh^2(\phi/(2|\tau_q|))} \\ &- 2\sum_{n=1}^{\infty}\frac{e^{-2n\pi/|\tau_q|}}{(1-e^{-2n\pi/|\tau_q|})^2} + 2\sum_{n=1}^{\infty}\frac{ne^{-2n\pi/|\tau_q|}}{1-e^{-2n\pi/|\tau_q|}}\cosh(n\phi/|\tau_q|)\Bigg],\end{aligned} \tag{2.42}$$

where we used the relation $\tau_q = \sqrt{-1}|\tau_q|$ which is valid in the present setting (2.34).

It can be verified that $\wp$ satisfies the following differential equations [62, Chapter 23]),

$$\begin{aligned}\wp'(\phi)^2 &= 4\wp(\phi)^2 - g_2\wp(\phi) - g_3 \\ &= 4(\wp(\phi)-e_1)(\wp(\phi)-e_2)(\wp(\phi)-e_3),\end{aligned} \tag{2.43}$$

$$\wp''(\phi) = 6\wp(\phi)^2 - \frac{g_2}{2}. \tag{2.44}$$

When $q \in (0, 1)$, $e_1, e_2, e_3 \in \mathbb{R}$ and the following inequalities hold ([58, Section 2.8]),

$$e_3 < e_2 < e_1. \tag{2.45}$$

From (2.43), we see that $\wp$ inverts the incomplete elliptic integral [47,58]. Under the setting (2.34), we will use the following special result [62, (23.6.31)] (see Section 6.12 of [47]); if $e_2 \le x \le e_1$, then $\wp^{-1}(x) \in [\omega_1, \omega_1 + \omega_3] := \{\pi + \sqrt{-1}y : 0 \le y \le \pi|\tau_q|\}$ and

$$y = \frac{1}{2}\int_x^{e_1} \frac{ds}{\sqrt{(e_1 - s)(s - e_2)(s - e_3)}}. \tag{2.46}$$

We introduce the Euler operator

$$\mathcal{D}_z = z\frac{\partial}{\partial z}. \tag{2.47}$$

If we use the notation (2.36), then $\mathcal{D}_z = -\sqrt{-1}\partial/\partial\phi_z$.

Lemma 2.5. *Under the notation (2.36), the following equalities hold,*

$$f^{\mathrm{JK}}(z, a) f^{\mathrm{JK}}(z, b) = \mathcal{D}_z f^{\mathrm{JK}}(z, ab) + (\rho_1(a) + \rho_1(b)) f^{\mathrm{JK}}(z, ab), \tag{2.48}$$

$$f^{\mathrm{JK}}(z, a) f^{\mathrm{JK}}(z, a^{-1}) = \mathcal{D}_z \rho_1(z) - \mathcal{D}_a \rho_1(a), \tag{2.49}$$

$$\mathcal{D}_z \rho_1(z) = -\sqrt{-1}\frac{d}{d\phi_z}\rho_1(z) = -\wp(\phi_z) + \frac{P}{12}, \tag{2.50}$$

$$f^{\mathrm{JK}}(z, a) f^{\mathrm{JK}}(z, a^{-1}) = \wp(\phi_a) - \wp(\phi_z), \tag{2.51}$$

$$f^{\mathrm{JK}}(z, -1)^2 = e_1 - \wp(\phi_z), \tag{2.52}$$

where

$$P = P(q) = 1 - 24\sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2} = \frac{12}{\pi}\eta_1(q) = 1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}}.$$

The equality (2.48) is called the *fundamental multiplicative identity of the Jordan–Kronecker function* in [19,80]. The equality (2.49) is obtained by taking the limit $b \to 1/a$ in (2.48) [19]. The derivation of (2.50) is also found in [19]. Combination of (2.49) and (2.50) gives (2.51). The equality (2.52) is a special case of (2.51) with $a = -1$ where the definition of e_1 is used.

We set

$$a_n(z) := \mathcal{D}_z^n \log\theta(z), \quad n \in \mathbb{N}. \tag{2.53}$$

Lemma 2.6. *The following equalities hold,*

$$a_1(z) = \frac{1}{2} - \rho_1(z), \quad a_2(z) = \wp(\phi_z) - \frac{P}{12},$$
$$a_3(z) = -\sqrt{-1}\wp'(\phi_z), \quad a_4(z) = -\wp''(\phi_z).$$

Proof. For $a_1(z)$ we have

$$a_1(z) = z\frac{\theta'(z)}{\theta(z)} = -\frac{z}{1-z} - \sum_{n=1}^{\infty}\left(\frac{zq^{2n}}{1-zq^{2n}} - \frac{z^{-1}q^{2n}}{1-z^{-1}q^{2n}}\right)$$
$$= -\frac{z}{1-z} - \left\{\rho_1(z) - \frac{1+z}{2(1-z)}\right\} = \frac{1}{2} - \rho_1(z). \tag{2.54}$$

For $a_2(z)$ use (2.50) in Lemma 2.5. Use $\mathcal{D}_z = -\sqrt{-1}\partial/\partial\phi_z$ for $a_3(z)$ and $a_4(z)$. □

Lemma 2.7. *The following equalities holds,*

(i) $\lim_{z\to 1}\left(a_1(z) + \frac{z}{1-z}\right) = 0,$

(ii) $\gamma_2 := \lim_{z\to 1}\left\{a_2(z) + \frac{z}{(1-z)^2}\right\} = -2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2} = \frac{P-1}{12},$

(iii) $a_1(-1) = \frac{1}{2},$

(iv) $a_2(-1) = \frac{1}{4} + 2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}.$

Proof. We notice (2.54) and

$$a_2(z) = -\frac{z}{(1-z)^2} - \sum_{n=1}^{\infty}\left\{\frac{zq^{2n}}{(1-zq^{2n})^2} + \frac{z^{-1}q^{2n}}{(1-z^{-1}q^{2n})^2}\right\}.$$

The formulas (i)–(iv) are all obtained from these equalities. □

We note that the following is the case,

$$\theta'(-1) = -\theta(-1)/2 \iff a_1(-1) = 1/2 \iff \rho_1(-1) = 0.$$

2.5. $q \to 0$ *limits and asymptotics in* $q \to 1$. By the definition (1.4), the following are readily confirmed;

$$\lim_{q\to 0} S_{\mathbb{A}_q}(z, w) = S_{\mathbb{D}}(z, w),$$
$$\lim_{q\to 0} S_{\mathbb{A}_q}(z, w; r) = \frac{1 + rz\overline{w}}{(1+r)(1-z\overline{w})} = \frac{1}{1-z\overline{w}} - \frac{r}{1+r} =: S_{\mathbb{D}}(z, w; r),$$
$$\lim_{r\to 0} S_{\mathbb{D}}(z, w; r) = S_{\mathbb{D}}(z, w).$$

Notice that if we use the expressions (2.30) and (2.31) in Proposition 2.2, their $q \to 0$ limits are immediately obtained by (2.16) with $p = q^2$.

By (2.35) and (2.38), we see the following,

$$\lim_{q\to 0}\wp(\phi; \tau_q) = -\frac{1}{12} + \frac{1}{4\sin^2(\phi/2)},$$

$$\lim_{q\to 0} \wp(\phi_z;\tau_q) = -\frac{1}{12} - \frac{z}{(1-z)^2} = -\frac{1+10z+z^2}{12(1-z)^2}. \tag{2.55}$$

Similarly, (2.39) and (2.41) give

$$e_1(0) = 1/6, \quad e_2(0) = e_3(0) = -1/12, \quad g_2(0) = 1/12, \quad g_3(0) = 1/216. \tag{2.56}$$

In the present setting (2.34), $q \to 1 \iff |\tau_q| \to 0$. For $\mathrm{Re}\,\phi \in (0, 2\pi)$, (2.42) gives the following asymptotics in $|\tau_q| \to 0$,

$$\begin{aligned}\wp(\phi,\tau_q) &\sim (1/12 + e^{-\phi/|\tau_q|} + e^{-(2\pi-\phi)/|\tau_q|})/|\tau_q|^2,\\ e_1 &\sim (1/12 + 2e^{-\pi/|\tau_q|})/|\tau_q|^2, \quad e_2 \sim (1/12 - 2e^{-\pi/|\tau_q|})/|\tau_q|^2.\end{aligned} \tag{2.57}$$

By (2.40), the above implies

$$e_3 = -(e_1+e_2) \sim -1/(6|\tau_q|^2), \quad g_2 \sim (1 + 4e^{-\pi/|\tau_q|})/|\tau_q|^2. \tag{2.58}$$

2.6. Conditional weighted Szegő kernels. For $r > 0$, define

$$S^{\alpha}_{\mathbb{A}_q}(z,w;r) := S_{\mathbb{A}_q}(z,w;r) - \frac{S_{\mathbb{A}_q}(z,\alpha;r)S_{\mathbb{A}_q}(\alpha,w;r)}{S_{\mathbb{A}_q}(\alpha,\alpha;r)}, \quad z,w,\alpha \in \mathbb{A}_q. \tag{2.59}$$

We put $S^{\alpha}_{\mathbb{A}_q}(z,w;r) = f(z,w;r,\alpha)h^q_\alpha(z)\overline{h^q_\alpha(w)}$ assuming $\overline{f(w,z;r,\alpha)} = f(z,w;r,\alpha)$ and here we intend to determine f. By the definition of the conditional kernel (1.7), we can verify that $S^{\alpha}_{\mathbb{A}_q}$ satisfies the same functional equations with (2.32) (i) and (iii); $S^{\alpha}_{\mathbb{A}_q}(q^2z,w;r) = -(1/r)S^{\alpha}_{\mathbb{A}_q}(z,w;r)$, $S^{\alpha}_{\mathbb{A}_q}(z,w;q^2r) = (1/z\overline{w})S^{\alpha}_{\mathbb{A}_q}(z,w;r)$, but in the equation corresponding to (2.32) (ii) the conditioning parameter α should be also inverted as $S^{\alpha}_{\mathbb{A}_q}(1/z,w;r) = -S^{1/\alpha}_{\mathbb{A}_q}(z,1/w;1/r)$. Moreover (2.32) (i) implies $S^{q^2\alpha}_{\mathbb{A}_q}(z,w;r) = S^{\alpha}_{\mathbb{A}_q}(z,w;r)$. On the other hand, (1.10) gives $h^q_\alpha(q^2z) = |\alpha|^2 h^q_\alpha(z)$, $h^q_{q^2\alpha}(z) = z^2(\overline{\alpha}/\alpha)h^q_\alpha(z)$, and $h^q_\alpha(1/z) = (\alpha/\overline{\alpha})h^q_{1/\alpha}(z)$. Hence f should satisfy the functional equations

$$\begin{aligned}&\text{(i)} \quad f(q^2z,w;r,\alpha) = -\frac{1}{r|\alpha|^2}f(z,w;r,\alpha),\\ &\text{(ii)} \quad f(1/z,w;r,\alpha) = -f(z,1/w;1/r,1/\alpha),\\ &\text{(iii)} \quad f(z,w;q^2r,\alpha) = \frac{1}{z\overline{w}}f(z,w;r,\alpha),\\ &\text{(iv)} \quad f(z,w;r,q^2\alpha) = \frac{1}{(z\overline{w})^2}f(z,w;r,\alpha).\end{aligned}$$

Comparing them with (2.32), it is easy to verify that if $f(z,w;r,\alpha) = S_{\mathbb{A}_q}(z,w;r|\alpha|^2)$, these functional equations are satisfied. The above observation implies the equality (1.11). Actually, McCullough and Shen proved the following.

Proposition 2.8 (McCullough and Shen [56]). *The equality (1.11) holds with (1.10).*

McCullough and Shen proved the above by preparing an auxiliary lemma. Here we give a direct proof from Weierstrass' addition formula (2.18).

Proof. We put (2.59) with (2.30) and (1.10) to (1.11), then the equality is expressed by theta functions. After multiplying both sides by the common denominator, we see that the equality (1.11) is equivalent to the following,

$$\begin{aligned}&\theta(-rz\overline{w}, -r|\alpha|^2, \overline{\alpha}z, \alpha\overline{w}) - \theta(-r\overline{\alpha}z, -r\alpha\overline{w}, z\overline{w}, |\alpha|^2)\\ &\quad = z\overline{w}\theta(-rz\overline{w}|\alpha|^2, \alpha z^{-1}, \overline{\alpha}\,\overline{w}^{-1}, -r).\end{aligned} \tag{2.60}$$

Now we change the variables from $\{z, \overline{w}, \alpha, r\}$ to $\{x, y, u, v\}$ as $\overline{\alpha}z = x/y$, $\alpha\overline{w} = u/v$, $z\overline{w} = x/v$, $|\alpha|^2 = u/y$, and $r = -yv$. Then the left-hand side of (2.60) becomes $\theta(xy, x/y, uv, u/v) - \theta(xv, x/v, uy, u/y)$, and the right-hand side becomes $(x/v)\theta(yv, (y/v)^{-1}, xu, (x/u)^{-1})$ which is equal to $(u/y)\theta(yv, y/v, xu, x/u)$ by (2.13). Hence Weierstrass' addition formula (2.18) proves the equality (2.60). The proof is complete. □

We can prove the following.

Lemma 2.9. *For* $\alpha \in \mathbb{A}_q$,

(i) $h^q_\alpha(\alpha) = 0$,

(ii) $0 < |h^q_\alpha(z)| < 1 \quad \forall z \in \mathbb{A}_q \setminus \{\alpha\}$,

(iii) $|h^q_\alpha(z)| = \begin{cases} 1, & \text{if } z \in \gamma_1 := \{z \in \mathbb{C} : |z| = 1\},\\ |\alpha|, & \text{if } z \in \gamma_q := \{z \in \mathbb{C} : |z| = q\},\end{cases}$

(iv) $h^{q\prime}_\alpha(\alpha) = -\dfrac{\theta'(1)}{\theta(|\alpha|^2)} = \dfrac{q_0^2}{\theta(|\alpha|^2)} > 0$,

(v) $\lim_{q\to 0} h^q_\alpha(z) = \dfrac{z-\alpha}{1-z\overline{\alpha}}$.

Proof. When $w = z$, (2.59) gives $S^\alpha_{\mathbb{A}_q}(z, z; r) = S_{\mathbb{A}_q}(z, z; r) - |S_{\mathbb{A}_q}(z, \alpha; r)|^2/S_{\mathbb{A}_q}(\alpha, \alpha; r) \geq 0$, $z \in \mathbb{A}_q$, which implies $0 \leq S^\alpha_{\mathbb{A}_q}(z, z; r)/S_{\mathbb{A}_q}(z, z; r) \leq 1$, $z \in \mathbb{A}_q$. As noted just after (1.4), $S_{\mathbb{A}_q}(z, z; r)$ is monotonically decreasing in $r > 0$. Then, by (1.11), $S^\alpha_{\mathbb{A}_q}(z, z; r) = S_{\mathbb{A}_q}(z, z; r|\alpha|^2)|h^q_\alpha(z)|^2 > S_{\mathbb{A}_q}(z, z; r)|h^q_\alpha(z)|^2$, if $|\alpha| < 1$. Hence it is proved that $|h^q_\alpha(z)| < S^\alpha_{\mathbb{A}_q}(z, z; r)/S_{\mathbb{A}_q}(z, z; r) \leq 1$, $z \in \mathbb{A}_q$. By the explicit expression (1.10) and by basic properties of the theta function given in Sect. 2.2, provided $z \in \overline{\mathbb{A}_q} := \mathbb{A}_q \cup \gamma_1 \cup \gamma_q$, it is verified that $h^q_\alpha(z) = 0$ if and only if $z = \alpha$, and $|h^q_\alpha(z)| = 1$ if and only if $z \in \gamma_1$. Using (2.14) and (2.19), we can show that

$$h^q_\alpha(qe^{\sqrt{-1}\phi}) = qe^{\sqrt{-1}\phi}\frac{\theta(\alpha q^{-1}e^{-\sqrt{-1}\phi})}{\theta(q^2\overline{\alpha}q^{-1}e^{\sqrt{-1}\phi})} = -\overline{\alpha}e^{2\sqrt{-1}\phi}\frac{\overline{\theta(\overline{\alpha}q^{-1}e^{\sqrt{-1}\phi})}}{\theta(\overline{\alpha}q^{-1}e^{\sqrt{-1}\phi})}, \quad \phi \in [0, 2\pi).$$

Then (i)–(iii) are proved. If we apply (i) and (2.17) to the derivative of (1.10) with respect to z, then (iv) is obtained. Applying (2.16) to (1.10) proves (v). The proof is complete. □

Since $h^q_\alpha(\cdot)$, $\alpha \in \mathbb{A}_q$ is holomorphic in $\mathbb{A}_q$, (i)–(iii) of Lemma 2.9 implies that h^q_α gives a conformal map from $\mathbb{A}_q$ to $\mathbb{D} \setminus \{\text{a circular slit}\}$ as shown by Fig. 1. In addition (v) of Lemma 2.9 means $\lim_{q\to 0} h^q_\alpha(z) = h_\alpha(z)$, where $h_\alpha(z)$ is the Riemann mapping function associated to α in $\mathbb{D}$ given by a Möbius transformation (1.9).

Remark 11. The present function $h_\alpha^q(z)$ is closely related with the *Blaschke factor* defined on page 17 in [68] for an annulus $\mathbb{A}_{q^{1/2},q^{-1/2}} := \{z \in \mathbb{C} : q^{1/2} < |z| < q^{-1/2}\}$, whose explicit expression using the theta functions was given on pp. 386–388 in [20]. These two functions are, however, different from each other. Let $\widehat{h_\alpha^q}(z)$ denote the Blaschke factor for the annulus $\mathbb{A}_q$, which is appropriately transformed from the function given in [20] for $\mathbb{A}_{q^{1/2},q^{-1/2}}$. We found that

$$\widehat{h_\alpha^q}(z) = z^{-\log\alpha/\log q} h_\alpha^q(z).$$

Also for the Blaschke factor $\widehat{h_\alpha^q}(z)$, (i), (ii), and (v) in Lemma 2.9 are satisfied, but instead of (iii), we have $|\widehat{h_\alpha^q}(z)| = 1$ if and only if $z \in \gamma_1 \cup \gamma_q$ for $z \in \overline{\mathbb{A}_q}$. Moreover, $\widehat{h_\alpha^q}$ is not univalent in $\mathbb{A}_q$ and is branched.

2.7. Correlation functions of point processes and the DPP of Peres and Virág on $\mathbb{D}$. A point process is formulated as follows. Let S be a base space, which is locally compact Hausdorff space with a countable base, and λ be a Radon measure on S. The configuration space of a point process on S is given by the set of nonnegative-integer-valued Radon measures;

$$\mathrm{Conf}(S) = \left\{ \xi = \sum_i \delta_{x_i} : x_i \in S, \xi(\Lambda) < \infty \text{ for all bounded set } \Lambda \subset S \right\}.$$

$\mathrm{Conf}(S)$ is equipped with the topological Borel σ-fields with respect to the vague topology. A point process on S is a $\mathrm{Conf}(S)$-valued random variable $\Xi = \Xi(\cdot)$. If $\Xi(\{x\}) \in \{0, 1\}$ for any point $z \in S$, then the point process is said to be *simple*. Assume that $\Lambda_i, i = 1, \ldots, m, m \in \mathbb{N}$ are disjoint bounded sets in S and $k_i \in \mathbb{N}_0, i = 1, \ldots, m$ satisfy $\sum_{i=1}^m k_i = n \in \mathbb{N}_0$. A symmetric measure λ^n on S^n is called the n-th *correlation measure*, if it satisfies

$$\mathbf{E}\left[\prod_{i=1}^m \frac{\Xi(\Lambda_i)!}{(\Xi(\Lambda_i) - k_i)!}\right] = \lambda^n(\Lambda_1^{k_1} \times \cdots \times \Lambda_m^{k_m}),$$

where when $\Xi(\Lambda_i) - k_i < 0$, we interpret $\Xi(\Lambda_i)!/(\Xi(\Lambda_i) - k_i)! = 0$. If λ^n is absolutely continuous with respect to the n-product measure $\lambda^{\otimes n}$, the Radon–Nikodym derivative $\rho^n(x_1, \ldots, x_n)$ is called the *n-point correlation function* with respect to the reference measure λ; that is, $\lambda^n(dx_1 \cdots dx_n) = \rho^n(x_1, \ldots, x_n)\lambda^{\otimes n}(dx_1 \cdots dx_n)$.

Consider the case in which S is given by a domain $\widetilde{D} \subset \mathbb{C}$ and $\Xi = \sum_i \delta_{Z_i}$ is a point process on $\widetilde{D}$ associated with the correlation functions $\{\rho_{\widetilde{D}}^n\}_{n\in\mathbb{N}}$. Here we assume that the reference measure λ is given by the Lebesgue measure m on $\mathbb{C}$ multiplied by a constant for simplicity (e.g., $\lambda = m/\pi$). For a one-to-one measurable transformation $F : D \to \widetilde{D}$, $D \subset \mathbb{C}$, we write the pull-back of the point process from $\widetilde{D}$ to D as $F^*\Xi := \sum_i \delta_{F^{-1}(Z_i)}$. We assume that F is analytic and $F'(z) = dF(z)/dz, z \in D$ is well-defined. By definition the following is derived.

Lemma 2.10. *The point process $F^*\Xi$ on D has correlation functions $\{\rho_D^n\}_{n\in\mathbb{N}}$ with respect to λ given by*

$$\rho_D^n(z_1, \ldots, z_n) = \rho_{\widetilde{D}}^n(F(z_1), \ldots, F(z_n)) \prod_{i=1}^n |F'(z_i)|^2, \quad n \in \mathbb{N}, \quad z_1, \ldots, z_n \in D.$$

The unfolded 2-correlation function (1.27) is hence invariant under transformation,

$$g_D(z_1, z_2) = g_{\widetilde{D}}(F(z_1), F(z_2)), \quad z_1, z_2 \in D.$$

For a point process $\Xi = \sum_i \delta_{Z_i}$ on $D \subset \mathbb{C}$, assume that there is a measurable function $K_D^{\det} : D \times D \to \mathbb{C}$ such that the correlation functions are given by the determinants of $K_D^{\det}$; that is,

$$\rho_D^n(z_1, \ldots, z_n) = \det_{1 \le i,j \le n} [K_D^{\det}(z_i, z_j)] \quad \text{for every } n \in \mathbb{N} \text{ and } z_1, \ldots, z_n \in D$$

with respect to λ. Then Ξ is said to be a *determinantal point process* (DPP) on D with the *correlation kernel* $K_D^{\det}$. For a one-to-one measurable transformation $F : D \to \widetilde{D}$, $D \subset \mathbb{C}$ with a bounded derivative F', $F^*\Xi$ is also a DPP on D such that the correlation kernel with respect to λ is given by

$$K_D^{\det}(z, w) := |F'(z)||F'(w)|K_{\widetilde{D}}^{\det}(F(z), F(w)), \quad z, w \in D. \tag{2.61}$$

See [35,42,43,72–74,76] for general construction and basic properties of determinantal point processes.

The zero point process $\mathcal{Z}_{X_\mathbb{D}}$ of the GAF $X_\mathbb{D}$ defined by (1.1) has the unit circle $\partial\mathbb{D}$ as a natural boundary and $\mathcal{Z}_{X_\mathbb{D}}(\mathbb{D}) = \infty$ a.s. For this zero process, Peres and Virág [64] showed the following remarkable result.

Theorem 2.11 *(*Peres and Virág [64]*).* *$\mathcal{Z}_{X_\mathbb{D}}$ is a DPP on $\mathbb{D}$ such that the correlation kernel with respect to m/π is given by the Bergman kernel $K_\mathbb{D}$ of $\mathbb{D}$ given by (2.3).*

The distribution of $\mathcal{Z}_{X_\mathbb{D}}$ is invariant under Möbius transformations that preserve $\mathbb{D}$ [64,75]. This invariance is a special case of the following, which can be proved using the conformal transformations of the Szegő kernel and the Bergman kernel given by (2.6) [35,64].

Proposition 2.12 *(*Peres and Virág [64]*).* *Let $\widetilde{D} \subsetneq \mathbb{C}$ be a simply connected domain with $\mathcal{C}^\infty$ boundary. Then there is a GAF $X_{\widetilde{D}}$ with covariance kernel $\mathbf{E}[X_{\widetilde{D}}(z)\overline{X_{\widetilde{D}}(w)}] = S_{\widetilde{D}}(z, w)$, $z, w \in \widetilde{D}$, where $S_{\widetilde{D}}$ denotes the Szegő kernel of $\widetilde{D}$. The zero point process $\mathcal{Z}_{X_{\widetilde{D}}}$ is the DPP such that the correlation kernel is given by the Bergman kernel $K_{\widetilde{D}}$ of $\widetilde{D}$. This DPP is conformally invariant in the following sense. If $D \subsetneq \mathbb{C}$ is another simply connected domain with $\mathcal{C}^\infty$ boundary, and $f : D \to \widetilde{D}$ is a conformal transformation, then $f^*\mathcal{Z}_{X_{\widetilde{D}}}$ has the same distribution as $\mathcal{Z}_{X_D}$. In other words, $f^*\mathcal{Z}_{X_{\widetilde{D}}}$ is a DPP such that the correlation kernel (2.61) with $K_{\widetilde{D}}^{\det} = K_{\widetilde{D}}$ is equal to the Bergman kernel K_D of D.*

3. Proofs

3.1. Proof of Proposition 1.1. Use the expression (2.25) of $S_{\mathbb{A}_q}(z, w; r)$ in Proposition 2.1. Using (2.28) and (2.29), we can show that

$$\begin{aligned}\sqrt{T_q'(z)}\overline{\sqrt{T_q'(w)}} S_{\mathbb{A}_q}(T_q(z), T_q(w); r) &= \sqrt{(-q)/z^2}\overline{\sqrt{(-q)/w^2}} f^{\mathrm{JK}}(q^2/z\overline{w}, -r) \\ &= \frac{q}{z\overline{w}} f^{\mathrm{JK}}(q^2/z\overline{w}, -r) = -\frac{q}{z\overline{w}} f^{\mathrm{JK}}(z\overline{w}/q^2, -1/r)\end{aligned}$$

$$= \frac{q}{rz\overline{w}} f^{\mathrm{JK}}(z\overline{w}, -1/r) = \frac{q}{r} f^{\mathrm{JK}}(z\overline{w}, -q^2/r) = \frac{q}{r} S_{\mathbb{A}_q}(z, w; q^2/r).$$

In particular, when $r = q$,

$$\sqrt{T_q'(z)}\sqrt{\overline{T_q'(w)}} S_{\mathbb{A}_q}(T_q(z), T_q(w); q) = \sqrt{T_q'(z)}\sqrt{\overline{T_q'(w)}} S_{\mathbb{A}_q}(T_q(z), T_q(w)) = S_{\mathbb{A}_q}(z, w),$$

which implies the invariance of the GAF $X_{\mathbb{A}_q}$ under conformal transformations preserving $\mathbb{A}_q$ by Schottkey's theorem [4].

3.2. Proof of Theorem 1.3. We recall a general formula for correlation functions of zero point process of a GAF, which is found in [64], but here we use a slightly different expression given by Proposition 6.1 of [71]. Let $\partial_z \partial_{\overline{w}} := \frac{\partial^2}{\partial z \partial \overline{w}}$.

Proposition 3.1. *The correlation functions of $\mathcal{Z}_{X_D}$ of the GAF X_D on $D \subsetneq \mathbb{C}$ with covariance kernel $S_D(z, w)$ are given by*

$$\rho_D^n(z_1, \dots, z_n) = \frac{\operatorname*{per}_{1 \le i, j \le n} \left[(\partial_z \partial_{\overline{w}} S_D^{z_1, \dots, z_n})(z_i, z_j) \right]}{\det_{1 \le i, j \le n} \left[S_D(z_i, z_j) \right]}, \quad n \in \mathbb{N}, \quad z_1, \dots, z_n \in D,$$

with respect to m/π, whenever $\det_{1 \le i, j \le n}[S_D(z_i, z_j)] > 0$. Here the conditional kernels are defined by (1.7) and (1.12).

Here we abbreviate $\gamma^q_{\{z_\ell\}_{\ell=1}^n}$ given by (1.13) to γ_n^q. Then (1.14) gives $S_{\mathbb{A}_q}^{z_1, \dots, z_n}(z, w; r) = S_{\mathbb{A}_q}(z, w; r \prod_{\ell=1}^n |z_\ell|^2) \gamma_n^q(z) \overline{\gamma_n^q(w)}$ for $z, w, z_1, \dots, z_n \in \mathbb{A}_q$. By Lemma 2.9 (i), this formula gives

$$(\partial_z \partial_{\overline{w}} S_{\mathbb{A}_q}^{z_1, \dots, z_n})(z_i, z_j; r) = S_{\mathbb{A}_q}\Big(z_i, z_j; r \prod_{\ell=1}^n |z_\ell|^2\Big) \gamma_n^{q\prime}(z_i) \overline{\gamma_n^{q\prime}(z_j)}.$$

Therefore, Proposition 3.1 gives now

$$\rho_{\mathbb{A}_q}^n(z_1, \dots, z_n; r) = \frac{\operatorname*{per}_{1 \le i, j \le n} \left[S_{\mathbb{A}_q}\left(z_i, z_j; r \prod_{\ell=1}^n |z_\ell|^2\right) \right] \prod_{k=1}^n |\gamma_n^{q\prime}(z_k)|^2}{\det_{1 \le i, j \le n} \left[S_{\mathbb{A}_q}(z_i, z_j; r) \right]}. \quad (3.1)$$

By (1.10) and Lemma 2.9 (i) and (iv), we see that

$$\prod_{i=1}^n |\gamma_n^{q\prime}(z_i)|^2 = \prod_{i=1}^n \Big| \Big(\prod_{1 \le j \le n, j \ne i} h_{z_j}^q(z_i) \Big) h_{z_i}^{q\prime}(z_i) \Big|^2 = \prod_{i=1}^n \Big| \Big(\prod_{1 \le j \le n, j \ne i} z_i \frac{\theta(z_j/z_i)}{\theta(\overline{z_j} z_i)} \Big) \frac{q_0^2}{\theta(|z_i|^2)} \Big|^2$$
$$= \left| \frac{q_0^{2n} \prod_{1 \le i < j \le n} z_i \theta(z_j/z_i) \cdot \prod_{1 \le i' < j' \le n} z_{j'} \theta(z_{i'}/z_{j'})}{\prod_{i=1}^n \prod_{j=1}^n \theta(z_i \overline{z_j})} \right|^2.$$

By (2.13), $z_i \theta(z_j/z_i) = z_i(-z_j/z_i)\theta(z_i/z_j) = -z_j \theta(z_i/z_j)$. Hence this is written as

$$\prod_{i=1}^n |\gamma_n^{q\prime}(z_i)|^2 = q_0^{4n} \left| \frac{(-1)^{n(n-1)/2} \big(\prod_{1 \le i < j \le n} z_j \theta(z_i/z_j) \big)^2}{\prod_{i=1}^n \prod_{j=1}^n \theta(z_i \overline{z_j})} \right|^2$$

$$= q_0^{4n} \left(\frac{\prod_{1 \le i < j \le n} |z_j|^2 \theta(z_i/z_j, \overline{z_i}/\overline{z_j})}{\prod_{i=1}^n \prod_{j=1}^n \theta(z_i \overline{z_j})} \right)^2. \tag{3.2}$$

The following identity is known as an elliptic extension of Cauchy's evaluation of determinant due to Frobenius (see Theorem 1.1 in [39], Theorem 66 in [45], Corollary 4.7 in [67], and references therein),

$$\det_{1 \le i,j \le n} \left[\frac{\theta(t x_i a_j)}{\theta(t, x_i a_j)} \right] = \frac{\theta(t \prod_{k=1}^n x_k a_k)}{\theta(t)} \frac{\prod_{1 \le i < j \le n} x_j a_j \theta(x_i/x_j, a_i/a_j)}{\prod_{i=1}^n \prod_{j=1}^n \theta(x_i a_j)}.$$

By (2.30) in Proposition 2.2, we have

$$q_0^{2n} \frac{\prod_{1 \le i < j \le n} |z_j|^2 \theta(z_i/z_j, \overline{z_i}/\overline{z_j})}{\prod_{i=1}^n \prod_{j=1}^n \theta(z_i \overline{z_j})} = \frac{\theta(-s)}{\theta(-s \prod_{\ell=1}^n |z_\ell|^2)} \det_{1 \le i,j \le n} \left[S_{\mathbb{A}_q}(z_i, z_j; s) \right], \quad \forall s > 0. \tag{3.3}$$

Then (3.2) is written as

$$\begin{aligned} \prod_{i=1}^n |\gamma_n^{q'}(z_i)|^2 &= \frac{\theta(-r)}{\theta(-r \prod_{\ell=1}^n |z_\ell|^2)} \det_{1 \le i,j \le n} [S_{\mathbb{A}_q}(z_i, z_j; r)] \\ &\quad \times \frac{\theta(-r \prod_{\ell=1}^n |z_\ell|^2)}{\theta(-r \prod_{\ell=1}^n |z_\ell|^4)} \det_{1 \le i,j \le n} \left[S_{\mathbb{A}_q}\left(z_i, z_j; r \prod_{\ell=1}^n |z_\ell|^2\right) \right] \\ &= \frac{\theta(-r)}{\theta(-r \prod_{\ell=1}^n |z_\ell|^4)} \det_{1 \le i,j \le n} [S_{\mathbb{A}_q}(z_i, z_j; r)] \det_{1 \le i,j \le n} \left[S_{\mathbb{A}_q}\left(z_i, z_j; r \prod_{\ell=1}^n |z_\ell|^2\right) \right]. \end{aligned}$$

Applying the above to (3.1), the correlation functions in Theorem 1.3 are obtained.

3.3. Direct proof of the (q, r)-inversion symmetry of correlation functions. The following is a corollary of Proposition 1.1 (ii) and (iii). Here we give a direct proof using the explicit formulas for correlation functions given in Theorem 1.3.

Corollary 3.2. *For every $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{A}_q$,*

$$\rho_{\mathbb{A}_q}^n(T_q(z_1), \ldots, T_q(z_n); r) \prod_{\ell=1}^n |T_q'(z_\ell)|^2 = \rho_{\mathbb{A}_q}^n(z_1, \ldots, z_n; q^2/r). \tag{3.4}$$

In particular, $\rho_{\mathbb{A}_q}^n(T_q(z_1), \ldots, T_q(z_n); q) \prod_{\ell=1}^n |T_q'(z_\ell)|^2 = \rho_{\mathbb{A}_q}^n(z_1, \ldots, z_n; q)$, for $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in \mathbb{A}_q$.

Proof. We calculate $\rho_{\mathbb{A}_q}^n(T_q(z_1), \ldots, T_q(z_n); r)$ for $\rho_{\mathbb{A}_q}^n$ given by (1.16) in Theorem 1.3. By (2.25) in Proposition 2.1,

$$\begin{aligned} S_{\mathbb{A}_q}\left(T_q(z), T_q(w); r \prod_{\ell=1}^n |T_q(z_\ell)|^2\right) &= S_{\mathbb{A}_q}\left(q z^{-1}, q \overline{w}^{-1}; q^{2n} r \prod_{\ell=1}^n |z_\ell|^{-2}\right) \\ &= f^{\mathrm{JK}}\left(q^2 (z\overline{w})^{-1}, -q^{2n} r \prod_{\ell=1}^n |z_\ell|^{-2}\right) = -f^{\mathrm{JK}}\left(q^{-2} z\overline{w}, -q^{-2n} r^{-1} \prod_{\ell=1}^n |z_\ell|^2\right), \end{aligned}$$

where we used (2.28) at the last equation. If we apply the equality between the leftmost side and the rightmost side in (2.29), we see that the above is equal to $q^{-2n}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2$ $f^{\mathrm{JK}}(z\overline{w}, -q^{-2n}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2)$. Then we apply the first equality in (2.29) $n+1$ times and obtain

$$S_{\mathbb{A}_q}\Big(T_q(z), T_q(w); r\prod_{\ell=1}^{n}|T_q(z_\ell)|^2\Big) = q^{-2n}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2(z\overline{w})^{n+1}f^{\mathrm{JK}}\Big(z\overline{w}, -q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2\Big)$$
$$= q^{-2n}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2(z\overline{w})^{n+1}S_{\mathbb{A}_q}\Big(z, w; q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2\Big). \quad (3.5)$$

Here we note that by definition (1.15) of perdet, the multilinearity of permanent and determinant implies the equality

$$\operatorname*{perdet}_{1\le i,j\le n}\Big[ab_ic_jm_{ij}\Big] = a^{2n}\prod_{k=1}^{n}b_k^2c_k^2\cdot\operatorname*{perdet}_{1\le i,j\le n}[m_{ij}].$$

Then by (3.5), we have

$$\operatorname*{perdet}_{1\le i,j\le n}\Big[S_{\mathbb{A}_q}\Big(T_q(z_i), T_q(z_j); r\prod_{\ell=1}^{n}|T_q(z_\ell)|^2\Big)\Big]$$
$$= \operatorname*{perdet}_{1\le i,j\le n}\Big[q^{-2n}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2(z_i\overline{z}_j)^{n+1}S_{\mathbb{A}_q}\Big(z_i, z_j; q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2\Big)\Big]$$
$$= q^{-4n^2}r^{-2n}\prod_{\ell=1}^{n}|z_\ell|^{4(2n+1)}\operatorname*{perdet}_{1\le i,j\le n}\Big[S_{\mathbb{A}_q}\Big(z_i, z_j; q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^2\Big)\Big]. \quad (3.6)$$

Now we consider the prefactor of perdet in (1.16). By (2.15), $\theta(-r) = \theta(-q^2/r)$. On the other hand,

$$\theta\Big(-r\prod_{\ell=1}^{n}|T_q(z_\ell)|^4\Big) = \theta\Big(-rq^{4n}\prod_{\ell=1}^{n}|z_\ell|^{-4}\Big) = \theta\Big(-q^{-2(2n-1)}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^4\Big).$$

If we apply (2.14) once, then we find that the above is equal to $q^{-2(2n-1)}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^4$ $\theta(-q^{-2(2n-2)}r^{-1}\prod_{\ell=1}^{n}|z_\ell|^4)$. We apply (2.14) $2n-1$ more times. Then the above turns to be equal to $q^{-2\sum_{i=1}^{2n-1}i}\ r^{-2n}\prod_{\ell=1}^{n}|z_\ell|^{8n}\ \theta(-q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^4)$. Then we have the equality

$$\frac{\theta(-r)}{\theta\Big(-r\prod_{\ell=1}^{n}|T_q(z_\ell)|^4\Big)} = q^{2n(2n-1)}r^{2n}\prod_{\ell=1}^{n}|z_\ell|^{-8n}\frac{\theta(-q^2/r)}{\theta\Big(-q^2r^{-1}\prod_{\ell=1}^{n}|z_\ell|^4\Big)}. \quad (3.7)$$

Combining the results (3.6) and (3.7), we have

$$\rho_{\mathbb{A}_q}^n(T_q(z_1), \ldots, T_q(z_n); r) = \rho_{\mathbb{A}_q}^n(z_1, \ldots, z_n; q^2/r)q^{-2n}\prod_{\ell=1}^{n}|z_\ell|^4.$$

Since $|T_q'(z)|^2 = q^2/|z|^4$, (3.4) is proved. □

3.4. Proofs of Proposition 1.5 and Theorem 1.6.

3.4.1. Upper and lower bounds of unfolded 2-correlation function By (1.19) and (1.21), the unfolded 2-correlation function (1.27) is explicitly written as follows,

$$g_{\mathbb{A}_q}(z,w;r) = \frac{\theta(-r|z|^2, -r|w|^2, -r|z|^4|w|^2, -r|z|^2|w|^4)^2}{\theta(-r, -r|z|^4, -r|w|^4, -r|z|^4|w|^4)\theta(-r|z|^2|w|^2)^4} \times \left[1 - \left\{\frac{\theta(|z|^2, |w|^2)}{\theta(-r|z|^4|w|^2, -r|z|^2|w|^4)}\right\}^2 \frac{|\theta(-rz\overline{w}|z|^2|w|^2)|^4}{|\theta(z\overline{w})|^4}\right], \quad (3.8)$$

with $z, w \in \mathbb{A}_q$. Using (2.12) and (2.14), it is readily verified that

$$g_{\mathbb{A}_q}(1,z;r) = g_{\mathbb{A}_q}(z,1;r) = g_{\mathbb{A}_q}(q,z;r) = g_{\mathbb{A}_q}(z,q;r) = 1, \quad z \in \mathbb{A}_q.$$

Lemma 3.3. *If we set* $a = |z|, b = |w|, a, b \in (q, 1)$*, then*

$$g_{\mathbb{A}_q}(a,b;r) \le g_{\mathbb{A}_q}(z,w;r) \le g_{\mathbb{A}_q}(-a,b;r), \quad z,w \in \mathbb{A}_q,$$

where

$$g_{\mathbb{A}_q}(\pm a,b;r) = \frac{b^2\theta(\pm a/b, -ra^2, -rb^2)^2}{\theta(-r, -ra^4, -rb^4)\theta(\pm ab)^4\theta(-ra^2b^2)^3} \times \left[\theta(-ra^4b^2, -ra^2b^4)\theta(\pm ab)^2 + \theta(a^2, b^2)\theta(\mp ra^3b^3)^2\right]. \quad (3.9)$$

First we show the following lemma.

Lemma 3.4. *Let* $\alpha, \beta > 0$ *with* $\alpha \notin \{q^{2i} : i \in \mathbb{Z}\}$*. Then the function* $|\theta(-\beta e^{i\varphi})/\theta(\alpha e^{i\varphi})|^2$ *on* $\varphi \in [0, 2\pi)$ *attains its maximum at* $\varphi = 0$ *and its minimum at* $\varphi = \pi$.

Proof. Set $f(x;\alpha,\beta) = (1+2\beta x + x^2)/(1-2\alpha x + x^2)$ for $x \in [-1,1]$. Then,

$$\left|\frac{\theta(-\beta e^{i\varphi})}{\theta(\alpha e^{i\varphi})}\right|^2 = \prod_{n=0}^{\infty} f(\cos\varphi; \alpha q^{2n}, \beta q^{2n}) \prod_{m=0}^{\infty} f(\cos\varphi; \alpha^{-1}q^{2(m+1)}, \beta^{-1}q^{2(m+1)})$$

Since $\partial f(x;\alpha,\beta)/\partial x = 2(1+\alpha\beta)(\alpha+\beta)/(1-2\alpha x+\alpha^2)^2 \ge 0$, f attains its maximum (resp. minimum) at $x = 1$ (resp. $x = -1$). Hence the assertion follows. □

Now we proceed to the proof of Lemma 3.3.

Proof. We set $z = ae^{\sqrt{-1}\varphi_z}$, $w = be^{\sqrt{-1}\varphi_w}$, $a, b \in (q,1)$, $\varphi_z, \varphi_w \in [0, 2\pi)$. Then we can see that (3.8) depends on the angles φ_z, φ_w only through the factor $|\theta(-rz\overline{w}|z|^2|w|^2)/\theta(z\overline{w})|^4$, and we have $|\theta(-rz\overline{w}|z|^2|w|^2)/\theta(z\overline{w})|^2 = |\theta(-ra^3b^3 e^{\sqrt{-1}(\varphi_z-\varphi_w)})/\theta(abe^{\sqrt{-1}(\varphi_z-\varphi_w)})|^2$. We can conclude that $\theta(-ra^3b^3)^2/\theta(ab)^2 \ge \left|\theta(-rz\overline{w}|z|^2|w|^2)/\theta(z\overline{w})\right|^2 \ge \theta(ra^3b^3)^2/\theta(-ab)^2$ from Lemma 3.4, and the inequalities are proved. If we use Weierstrass' addition formula (2.18) by setting $x = r^{1/2}a^{5/2}b^{3/2}$, $y = -r^{1/2}a^{3/2}b^{1/2}$, $u = -r^{1/2}a^{3/2}b^{5/2}$, and $v = r^{1/2}a^{1/2}b^{3/2}$, then we obtain $\theta(-ra^4b^2, -ra^2b^4)\theta(-ab)^2 - \theta(a^2,b^2)\theta(ra^3b^3)^2 = b^2\theta(-a/b)^2\theta(-ra^2b^2, -ra^4b^4)$. Using this equality and the one obtained by replacing a by $-a$, it is easy to obtain (3.9). The proof is complete. □

3.4.2. Proof of Proposition 1.5 By the definition (1.29), if we use (2.13)–(2.15), we can derive the following from (3.9),

$$G^{\wedge}_{\mathbb{A}_q}(x;r) = \frac{r^2\theta(qx^2)^2\theta(-rx^2,-r^{-1}x^2)^3}{x^2\theta(q)^2\theta(-r)^4\theta(-rx^4,-r^{-1}x^4)}\left[1+\frac{\theta(-rq,x^2)^2}{\theta(q)^2\theta(-rx^2,-r^{-1}x^2)}\right], \quad x \in (\sqrt{q},1).$$

Since $x^2 \sim q\{1+2q^{-1/2}(x-\sqrt{q})\}$ when $x \sim \sqrt{q}$, $\theta(qx^2) \sim \theta(q^2\{1+2q^{-1/2}(x-\sqrt{q})\}) \sim -\theta(1+2q^{-1/2}(x-\sqrt{q}))$, where (2.14) was used. Then

$$\theta(qx^2) \sim -\theta'(1)\cdot 2q^{-1/2}(x-\sqrt{q}) = 2q_0^2q^{-1/2}(x-\sqrt{q}) \quad \text{as } x \to \sqrt{q}.$$

where (2.17) was used. Hence $\theta(qx^2)^2 \sim (4q_0^4/q)(x-\sqrt{q})^2$ as $x \to \sqrt{q}$, and $G^{\wedge}_{\mathbb{A}_q}(x;r) \asymp (x-\sqrt{q})^2$ as $x \to \sqrt{q}$. Using (2.13)–(2.15), we can show that $\theta(-r^{-1}q) = \theta(-rq)$, $\theta(-r^{-1}q^2) = \theta(-r)$, $\theta(-rq^2) = r^{-1}\theta(-r)$. Then the coefficient is determined as given by $c(r)$.

3.4.3. Proof of Theorem 1.6 (i) Replacing x by $\sqrt{c}$ in (1.30), here we consider $\widetilde{G}(c) = \widetilde{G}(c;r,q) := G^{\vee}_{\mathbb{A}_q}(\sqrt{c};r)$, $c \in (q^2,1)$. From (3.9) in Lemma 3.3, we have

$$\widetilde{G}(c) = \frac{c\theta(-rc)^4\theta(-1,-rc^3)^2}{\theta(-r)\theta(-c)^2\theta(-rc^2)^5}\left[1+\frac{\theta(c,rc^3)^2}{\theta(-c,-rc^3)^2}\right]. \tag{3.10}$$

It is easy to see that $\widetilde{G}(1) = 1$. Here we will prove the following.

Proposition 3.5.

$$\widetilde{G}'(1) = \widetilde{G}''(1) = \widetilde{G}'''(1) = 0, \tag{3.11}$$

$$\widetilde{G}^{(4)}(1) = \widetilde{G}^{(4)}(1;r,q) = 12(10\wp(\phi_{-r})^2 + 4e_1\wp(\phi_{-r}) - 2e_1^2 - g_2). \tag{3.12}$$

This proposition implies (1.31) with (1.33), since $\kappa(r) = \widetilde{G}^{(4)}(1)/4!$. By (1.28), we have the equality $G^{\vee}_{\mathbb{A}_q}(x;r) = G^{\vee}_{\mathbb{A}_q}(q/x;q^2/r)$. Since $x \to q$ is equivalent with $q/x \to 1$, (1.31) implies

$$G_{\mathbb{A}_q}(q/x;q^2/r) \sim 1+\kappa(q^2/r)(1-(q/x)^2)^4 \quad \text{as } x \downarrow q.$$

Therefore, once Proposition 3.5 is proved and hence (1.33) is verified, then the equalities (1.34) are immediately concluded from (2.37). With the third equality in (1.34), the above proves (1.32). If $r > 1$, then $1/r < 1$, on the other hand if $0 < r < q$, given $q \in (0,1)$, then $q^2/r > q$. Hence by the first and the third equalities in (1.34) the values of $\kappa(r)$ in the parameter space outside of Ω can be determined by those in Ω. By the three equalities in (1.34), the structure in Ω described by Proposition 1.6 (ii) and (iii) is repeatedly mapped into the parameter space outside of Ω.

Now we proceed to the proof of Proposition 3.5. First we decompose $\widetilde{G}(c)$ given by (3.10) as

$$\widetilde{G}(c) = I(c) + J(c) = I(c) + \beta_r^2(c-1)^2I(c)K(c),$$

with

$$I(c) = \frac{c\theta(-rc)^4\theta(-1,-rc^3)^2}{\theta(-r)\theta(-c)^2\theta(-rc^2)^5}, \qquad \beta_r = \frac{\theta'(1)\theta(r)}{\theta(-1,-r)},$$

$$K(c) = \left(\frac{\theta(c)}{(c-1)\theta'(1)}\right)^2 \frac{\theta(-1, -r, rc^3)^2}{\theta(-c, -rc^3, r)^2}, \tag{3.13}$$

The following is easily verified, where $\mathcal{D}_z$ denotes the Euler operator (2.47).

Lemma 3.6. *Suppose that f is a $\mathcal{C}^\infty$-function and $f(1) = 1$, then*

$$\mathcal{D}_z \log f(z)|_{z=1} = f'(1),$$
$$\mathcal{D}_z^2 \log f(z)|_{z=1} = f''(1) + f'(1) - f'(1)^2.$$

If, in addition, $f'(1) = 0$, then

$$\mathcal{D}_z^2 \log f(z)|_{z=1} = f''(1),$$
$$\mathcal{D}_z^3 \log f(z)|_{z=1} = f'''(1) + 3f''(1),$$
$$\mathcal{D}_z^4 \log f(z)|_{z=1} = f^{(4)}(1) + 6f'''(1) + 7f''(1) - 3f''(1)^2.$$

Recall that $a_n(z), n \in \mathbb{N}$ are defined by (2.53) in Sect. 2.4.

Proposition 3.7. (i) $\mathcal{D}_z^n\big(\log\theta(\alpha z^k)\big) = k^n a_n(\alpha z^k)$, (ii) $\mathcal{D}_z\big(a_n(\alpha z^k)\big) = k a_{n+1}(\alpha z^k)$.

This proposition is a corollary of the following lemma.

Lemma 3.8. *Suppose that f is a $\mathcal{C}^\infty$-function. Let $F_n(w) := \mathcal{D}_w^n \log f(w), n \in \mathbb{N}$. Then for $k, n \in \mathbb{N}$ and a constant α, $\mathcal{D}_z^n\big(\log f(\alpha z^k)\big) = k^n F_n(\alpha z^k)$.*

Proof. It suffices to show the equality $\mathcal{D}_z\big(F_n(\alpha z^k)\big) = kF_{n+1}(\alpha z^k)$. Indeed,

$$\mathcal{D}_z\big(F_n(\alpha z^k)\big) = z \cdot \frac{d}{dw}F_n(w)\Big|_{w=\alpha z^k} \cdot \alpha k z^{k-1} = k \cdot \left(w\frac{d}{dw}F_n(w)\right)\Big|_{w=\alpha z^k} = kF_{n+1}(\alpha z^k).$$

Then the proof is complete. □

Lemma 3.9. $\beta_r^2 = a_2(-1) - a_2(-r)$.

Proof. First we note that by (2.25) in Proposition 2.1, (2.30) in Proposition 2.2 and (2.17), $\beta_r = -S_{\mathbb{A}_q}(-1, 1; r) = -f^{\mathrm{JK}}(-1, -r)$. Then by (2.52) in Lemma 2.5 in Sect. 2.4

$$\beta_r^2 = e_1 - \wp(\phi_{-r}) = \wp(\pi) - \wp(\phi_{-r}) = \wp(\phi_{-1}) - \wp(\phi_{-r}). \tag{3.14}$$

Hence the formula for $a_2(z)$ in Lemma 2.6 in Sect. 2.4 proves the statement. □

By the definition (3.13), it is easy to see that

$$I(1) = K(1) = 1, \quad J(1) = J'(1) = 0, \quad J''(1) = 2\beta_r^2. \tag{3.15}$$

In what follows, we will use Proposition 3.7 (i) repeatedly. Using Lemma 3.6 with $I(1) = 1$ and Proposition 3.7 (i),

$$\begin{aligned}
I'(1) &= \mathcal{D}_c \log I(c)\Big|_{c=1} \\
&= 1 + 4a_1(-rc) + 2 \cdot 3a_1(-rc^3) - 2a_1(-c) - 5 \cdot 2a_1(-rc^2)\Big|_{c=1} \\
&= 1 - 2a_1(-1) = 0,
\end{aligned}$$

where Lemma 2.7 (iii) in Sect. 2.4 was used at the last equality. Therefore, $\widetilde{G}'(1) = I'(1) + J'(1) = 0$.

From now on, we use the notation $A_n(r) := a_n(r) - a_n(-r)$, $n \in \mathbb{N}$. Using Lemma 3.6 with $K(1) = 1$, Proposition 3.7 (i), and Lemma 2.7 (i), (iii) in Sect. 2.4, we see that

$$\begin{aligned} K'(1) &= \mathcal{D}_c \log K(c)\Big|_{c=1} \\ &= 2\{a_1(c) + c/(1-c)\} + 2\cdot 3a_1(rc^3) - 2a_1(-c) - 2\cdot 3a_1(-rc^3)\Big|_{c=1} \\ &= 6a_1(r) - 1 - 6a_1(-r) = -1 + 6A_1(r). \end{aligned} \tag{3.16}$$

Using Lemma 3.6 with $I(1) = 1$, $I'(1) = 0$ and Proposition 3.7 (i),

$$\begin{aligned} I''(1) &= \mathcal{D}_c^2 \log I(c)\Big|_{c=1} \\ &= 4a_2(-rc) + 2\cdot 3^2 a_2(-rc^3) - 2a_2(-c) - 5\cdot 2^2 a_2(-rc^2)\Big|_{c=1} \\ &= 2(a_2(-r) - a_2(-1)) = -2\beta_r^2, \end{aligned} \tag{3.17}$$

where we used Lemma 3.9 at the last equality. Therefore, by (3.15), we obtain $\widetilde{G}''(1) = I''(1) + J''(1) = 0$.

Lemma 3.10. $\mathcal{D}_r \beta_r = \beta_r A_1(r)$. *Moreover,* $\lim_{r\to 1} \beta_r A_1(r) = (\theta'(1)/\theta(-1))^2$.

Proof. We observe that $\mathcal{D}_r \log \beta_r = \mathcal{D}_r \log \theta(r) - \mathcal{D}_r \log \theta(-r) = a_1(r) - a_1(-r) = A_1(r)$. On the other hand, $\mathcal{D}_r \log \beta_r = \mathcal{D}_r \beta_r / \beta_r$. Hence we obtain the first assertion. Note that

$$\lim_{r\to 1} \frac{\beta_r}{r-1} = \lim_{r\to 1} \frac{\theta'(1)}{\theta(-1)\theta(-r)} \frac{\theta(r)}{r-1} = \left(\frac{\theta'(1)}{\theta(-1)}\right)^2. \tag{3.18}$$

From Lemma 2.7 (i) and (iii) in Sect. 2.4, we see that $(r-1)A_1(r) = 1 + \mathrm{O}(r-1)$ as $r \to 1$ and the second assertion is also proved. □

Lemma 3.11. $a_3(-r) = -2\beta_r^2 A_1(r)$. *In particular,* $a_3(-1) = 0$.

Proof. We apply $\mathcal{D}_r$ to both sides of the identity of Lemma 3.9. From Lemma 3.10, we have the left-hand side $\mathcal{D}_r \beta_r^2 = 2\beta_r \cdot \mathcal{D}_r \beta_r = 2\beta_r^2 A_1(r)$, which is equal to the right-hand side $-\mathcal{D}_r a_2(-r) = -a_3(-r)$. The second assertion is obtained using the second assertion of Lemma 3.10 and the fact that $\beta_1 = 0$. □

Using Lemma 3.6 with $I(1) = 1$, $I'(1) = 0$, Proposition 3.7 (i), and Lemma 3.11, we see that

$$\begin{aligned} I'''(1) + 3I''(1) &= \mathcal{D}_c^3 \log I(c)\Big|_{c=1} \\ &= 4a_3(-rc) + 2\cdot 3^3 a_3(-rc^3) - 2a_3(-c) - 5\cdot 2^3 a_3(-rc^2)\Big|_{c=1} = 18a_3(-r). \end{aligned}$$

With (3.17) we have $I'''(1) = 18a_3(-r) + 6\beta_r^2$. By the Leibnitz rule, we see that

$$\begin{aligned} J'''(1) &= 3\frac{d}{dc}(\beta_r^2 I(c)K(c))\Big|_{c=1} \cdot 2 = 6\beta_r^2 (I'(1)K(1) + I(1)K'(1)) = 6\beta_r^2 K'(1) \\ &= -6\beta_r^2 + 36\beta_r^2 A_1(r) = -6\beta_r^2 - 18a_3(-r). \end{aligned}$$

Here we used the fact $I'(1) = 0$, (3.16) and Lemma 3.11. Therefore, we have $\widetilde{G}'''(1) = I'''(1) + J'''(1) = 0$. The proof of (3.11) is complete now.

Then we begin to prove (3.12). Using Lemma 3.6 with $I(1) = 1$, $I'(1) = 0$, Proposition 3.7 (i), and Lemma 3.11,

$$\begin{aligned}
I^{(4)}(1) + 6I'''(1) + 7I''(1) - 3I''(1)^2 &= \mathcal{D}_c^4 \log I(c)\Big|_{c=1} \\
&= 4a_4(-rc) + 2 \cdot 3^4 a_4(-rc^3) - 2a_4(-c) - 5 \cdot 2^4 a_4(-rc^2)\Big|_{c=1} \\
&= 86a_4(-r) - 2a_4(-1).
\end{aligned}$$

Therefore, $I^{(4)}(1) = 86a_4(-r) - 2a_4(-1) - 108a_3(-r) - 22\beta_r^2 + 12\beta_r^4$.

Lemma 3.12. $a_4(-r) = -2\beta_r^2(2A_1(r)^2 + A_2(r))$. *In particular,* $a_4(-1) = -2(\theta'(1)/\theta(-1))^4$.

Proof. Applying $\mathcal{D}_r$ to both sides of the first assertion of Lemma 3.11 together with Proposition 3.7 (ii) yields the first assertion. The second assertion follows from (3.18) and the facts that $(r-1)A_1(r) = 1 + \mathrm{O}(r-1)$ and $(r-1)^2 A_2(r) = -1 + \mathrm{O}(r-1)^2$ as $r \to 1$, which are verified by Lemma 2.7 (i)–(iv) in Sect. 2.4. □

By the Leibnitz rule,

$$\begin{aligned}
J^{(4)}(1) &= \beta_r^2 \Big(\frac{4!}{2!2!0!} I''(1)K(1) + \frac{4!}{2!1!1!} I'(1)K'(1) + \frac{4!}{2!0!2!} I(1)K''(1) \Big) \cdot 2 \\
&= 2\beta_r^2 \left(-12\beta_r^2 + 6K''(1) \right),
\end{aligned}$$

where we used the fact $I'(1) = 0$, (3.16) and (3.17). From (3.16), we have

$$\begin{aligned}
\mathcal{D}_c^2 \log K(c)\Big|_{c=1} &= 2\{a_2(c) + c/(c-1)^2\} + 2 \cdot 3^2 a_2(rc^3) - 2a_2(-c) - 2 \cdot 3^2 a_2(-rc^3)\Big|_{c=1} \\
&= 2(\gamma_2 - a_2(-1)) + 18A_2(r),
\end{aligned}$$

where Lemma 2.7 (ii) in Sect. 2.4 was used. Using Lemma 3.6 with $K(1) = 1$ and nonzero $K'(1)$ given by (3.16), we obtain

$$\begin{aligned}
K''(1) &= K'(1)^2 - K'(1) + \mathcal{D}_c^2 \log K(c)\Big|_{c=1} \\
&= (-1 + 6A_1(r))(-2 + 6A_1(r)) + 2(\gamma_2 - a_2(-1)) + 18A_2(r) \\
&= 2 - 18A_1(r) + 36A_1(r)^2 + 2(\gamma_2 - a_2(-1)) + 18A_2(r).
\end{aligned}$$

Hence we have

$$\begin{aligned}
J^{(4)}(1) &= -24\beta_r^4 + 24\beta_r^2 + 108 \cdot 2\beta_r^2(2A_1(r)^2 + A_2(r)) - 108 \cdot 2\beta_r^2 A_1(r) + 24\beta_r^2(\gamma_2 - a_2(-1)) \\
&= -24\beta_r^4 + 24\beta_r^2 - 108a_4(-r) + 108a_3(-r) + 24\beta_r^2(\gamma_2 - a_2(-1)),
\end{aligned}$$

where Lemmas 3.11 and 3.12 were used. Therefore,

$$\begin{aligned}
\widetilde{G}^{(4)}(1) &= I^{(4)}(1) + J^{(4)}(1) \\
&= -22a_4(-r) - 12\beta_r^4 + 24\beta_r^2(\gamma_2 - a_2(-1) + 1/12) - 2a_4(-1).
\end{aligned}$$

Now we use the equality $a_4(-r) = -\wp''(\phi_{-r})$ given by Lemma 2.6 in Sect. 2.4 and (3.14). We also note that we can verify the equality $\gamma_2 - a_2(-1) + 1/12 = -e_1$ from Lemma 2.7 (ii), (iv) and (2.39) in Sect. 2.4. Then the above is written as

$$\begin{aligned}\widetilde{G}^{(4)}(1) &= 22\wp''(\phi_{-r}) - 12(\wp(\phi_{-r}) - e_1)^2 + 24(\wp(\phi_{-r}) - e_1)e_1 + 2\wp''(\pi)\\ &= 22\wp''(\phi_{-r}) - 12\wp(\phi_{-r})^2 + 48e_1\wp(\phi_{-r}) - 36e_1^2 + 2\wp''(\pi).\end{aligned}$$

Finally we use the differential equation (2.44) of $\wp$. Then (3.12) is obtained. Proposition 3.5 is hence proved and the proof of Theorem 1.6 (i) is complete.

3.4.4. Proof of Theorem 1.6 (ii) By the definition and the properties of $\wp$ explained in Sect. 2.4, the following is proved for $q \in (0, 1)$.

Lemma 3.13. *For $r \in (q, 1)$, $\wp(\phi_{-r})$ is a monotonically increasing function of r.*

By (2.40) and (2.41), we see that $\kappa(r)$ given by (1.33) is written as follows,

$$\kappa(r) = 2(\wp(\phi_{-r}) - e_2)(\wp(\phi_{-r}) - e_3) + 6(\wp(\phi_{-r}) + e_1)(\wp(\phi_{-r}) - e_1).$$

Hence $\kappa(1) = 2(e_1 - e_2)(e_1 - e_3)$ and $\kappa(q) = 6(e_3 + e_1)(e_3 - e_1)$. Then by the inequalities (2.45), we can conclude that $\kappa(1) > 0$ and $\kappa(q) < 0$. By (1.33), we have $\kappa(r) = 5(\wp(\phi_{-r}) - \wp_+)(\wp(\phi_{-r}) - \wp_-)$ with the roots $\wp_\pm = \wp_\pm(q) = (-2e_1 \pm \sqrt{24e_1^2 + 10g_2})/10$ satisfying $\wp_- < 0 < \wp_+$. Since monotonicity is guaranteed by Lemma 3.13 for $r \in (q, 1)$, r_0 is the unique zero of κ in the interval $(q, 1)$. This is determined by

$$\wp(\phi_{-r_0}) = \wp_+, \tag{3.19}$$

which is equivalent to

$$\phi_{-r_0} = \wp^{-1}(\wp_+) \quad \Longleftrightarrow \quad r_0 = -e^{\sqrt{-1}\wp^{-1}(\wp_+)} = e^{\sqrt{-1}(-\pi + \wp^{-1}(\wp_+))}. \tag{3.20}$$

Using (2.40), (2.41), and (2.45), we can verify by (1.35) that $e_3 < e_2 < \wp_+ < e_1$. Hence (2.46) implies

$$-\pi + \wp^{-1}(\wp_+) = \frac{\sqrt{-1}}{2}\int_{\wp_+}^{e_1} \frac{ds}{\sqrt{(e_1 - s)(s - e_2)(s - e_3)}}$$

and (3.20) gives (1.36). The proofs of (1.37) and the assertion mentioned below it are complete.

3.4.5. Proof of Theorem 1.6 (iii) In the limit $q \to 0$, we have (2.56) and (1.35) gives $\wp_+(0) = (-2 + 3\sqrt{6})/60$. The integral appearing in (1.36) is then reduced to

$$\frac{1}{2}\int_{\wp_+(0)}^{1/6} \frac{ds}{(s + 1/12)\sqrt{1/6 - s}} = -\log\frac{1 - 2\sqrt{1/6 - \wp_+(0)}}{1 + 2\sqrt{1/6 - \wp_+(0)}} = -\log\frac{1 - \frac{\sqrt{4-\sqrt{6}}}{\sqrt{5}}}{1 + \frac{\sqrt{4-\sqrt{6}}}{\sqrt{5}}}.$$

Hence the first expression for r_c in (a) is obtained.

Remark 12. If we apply (2.55) and (2.56) in Sect. 2.5 to (1.33), then we have

$$\kappa_0(r) := \lim_{q\to 0} \kappa(r;q) = -\frac{r^4 + 12r^3 - 58r^2 + 12r + 1}{16(1+r)^4}$$
$$= -\frac{(r+r^{-1})^2 + 12(r+r^{-1}) - 60}{16(r^{1/2}+r^{-1/2})^4}. \tag{3.21}$$

Since we have assumed $r_c \in (0,1)$, $r_c + r_c^{-1} \in (2,\infty)$. Then we see that $r = r_c$ satisfies the equation

$$r + r^{-1} = 2(2\sqrt{6}-3) \iff r^2 - 2(2\sqrt{6}-3)r + 1 = 0.$$

The above quadratic equation has two positive solutions which are reciprocal to each other. The second expression for r_c in (a) is the smaller one of them.

From (2.35) and (2.39), we have

$$\wp(\phi_{-r}) = -1/12 + r/(1+r)^2 + \{2 + (r+r^{-1})\}q^2$$
$$+ \{6 + (r+r^{-1}) - 2(r^2+r^{-2})\}q^4 + O(q^6),$$
$$e_1 = 1/6 + 4q^2 + 4q^4 + O(q^6), \quad e_2 = -1/12 + 2q - 2q^2 + 8q^3 - 2q^4 + O(q^5),$$
$$e_3 = -1/12 - 2q - 2q^2 - 8q^3 - 2q^4 + O(q^5), \quad g_2 = 1/12 + 20q^2 + 180q^4 + O(q^6).$$

Then the equation (3.19) is expanded in the variable q as

$$-1/12 + (r_0^{1/2} + r_0^{-1/2})^{-2} + \{2 + (r_0 + r_0^{-1})\}q^2 + \{6 + (r_0 + r_0^{-1}) - 2(r_0^2 + r_0^{-2})\}q^4$$
$$= -(2 - 3\sqrt{6})/60 - 2(6 - 29\sqrt{6})q^2/15 - 2(18 + 2533\sqrt{6})q^4/45 + O(q^6).$$

Put $r_0 = r_c + c_1 q + c_2 q^2 + O(q^3)$ and use the value of r_c given by (a). Then we have $c_1 = 0$ and the assertion (b) is proved.

For (c) we consider the asymptotics of the equation (3.19). By (2.57) and (2.58) we have $(1/12 + e^{-\phi_{-r_0(q)}/|\tau_q|} + e^{-(2\pi-\phi_{-r_0(q)})/|\tau_q|})/|\tau_q|^2 \sim 1/(12|\tau_q|^2)$ in $|\tau_q| \to 0$. This is satisfied if and only if $e^{-\phi_{-r_0(q)}/|\tau_q|} + e^{-(2\pi-\phi_{-r_0(q)})/|\tau_q|} = 0$, that is, $\cos((\pi - \phi_{-r_0}(q))/\tau_q) = 0$. Under the setting (2.34) with $r \in (0,1)$, this is realized by

$$\pi - \phi_{-r_0(q)} = -\pi\tau_q/2 \iff r_0(q) = -e^{\sqrt{-1}\phi_{-r_0}(q)} = e^{\sqrt{-1}\pi\tau_q/2} = q^{1/2}.$$

Since $q^{1/2} = (1-(1-q))^{1/2} \sim 1 - (1-q)/2$ as $q \to 1$, (c) is proved.

Hence the proof of Theorem 1.6 (iii) is complete.

3.5. Proof of Proposition 1.7. By taking the $q \to 0$ limit in Lemma 3.3, the following is obtained.

Lemma 3.14. *Assume that $r > 0$. If we set $a = |z|, b = |w|$, $a, b \in (0,1]$, then $g_{\mathbb{D}}(z,w;r) \le g_{\mathbb{D}}(-a,b;r)$, where*

$$g_{\mathbb{D}}(-a,b;r) = \frac{(a+b)^2(1+ra^2)^2(1+rb^2)^2}{(1+ab)^4(1+r)(1+ra^4)(1+rb^4)(1+ra^2b^2)^3}$$
$$\times \Big\{a^6b^6(2 - a^2 + 2ab - b^2 + 2a^2b^2)r^2$$

$$+ a^2b^2(a^2 - 2ab + 4a^3b + b^2 + a^4b^2 + 4ab^3 - 2a^3b^3 + a^2b^4)r$$
$$+ (2 - a^2 + 2ab - b^2 + 2a^2b^2)\Big\}.$$

From now on we will assume $r \in (0, 1]$. It is easy to see that $g_{\mathbb{D}}(-a, b; r) = g_{\mathbb{D}}(a, -b; r)$, and $g_{\mathbb{D}}(-a, 1; r) = g_{\mathbb{D}}(-1, b; r) = 1, a, b \in (0, 1]$. We define a function $D(a, b; r)$ by

$$\frac{\partial g_{\mathbb{D}}(-a, b; r)}{\partial a} = \frac{4a^7b^4r^{5/2}D(a, b; r)(1-a)(1+a)(1-b)^2(1+b)^2(a+b)(1+ra^2)(1+rb^2)^2}{(1+ab)^5(1+r)(1+ra^4)^2(1+ra^2b^2)^4(1+rb^4)}.$$

The above implies that if $D(a, b; r) \geq 0$ for $r \in (0, r_c)$, $\forall (a, b) \in (0, 1]^2$, then Proposition 1.7 is proved.

We can prove the following.

Lemma 3.15. *Let* $p(x) := x + 1/x$ *and* $\widetilde{D}(a, b; s) = p(a^7b^4s^5) + 13p(a^3b^2s^3) - 46p(a^4b^2s)$. *Then* $D(a, b; r) \geq \widetilde{D}(a, b; r^{1/2})$, $\forall (a, b, r) \in (0, 1]^3$.

Proof. A tedious but direct computation shows that

$$\begin{aligned} D(a, b; r) &= p(a^7b^4r^{5/2}) \\ &+ \big\{p(a^3b^2r^{3/2}) + 5p(a^5b^2r^{3/2}) - p(a^2b^3r^{3/2}) + 3p(a^4b^3r^{3/2}) \\ &+ 2p(a^6b^3r^{3/2}) + 3p(a^5b^4r^{3/2})\big\} \\ &- \big\{10p(ar^{1/2}) + 5p(a^3r^{1/2}) + 2p(ab^{-2}r^{1/2}) + 3p(b^{-1}r^{1/2}) + 9p(a^2b^{-1}r^{1/2}) \\ &+ 5p(br^{1/2}) + 10p(a^2br^{1/2}) + p(a^4br^{1/2}) + 2p(ab^2r^{1/2}) - p(ab^4r^{1/2})\big\}. \end{aligned} \tag{3.22}$$

We note that $p(x)$ is decreasing on $(0, 1]$ and $p(x) = p(x^{-1})$. By the monotonicity of $p(x)$, the following inequalities are guaranteed,

$$3p(a^4b^3r^{3/2}) \geq p(a^2b^3r^{3/2}) + 2p(a^3b^2r^{3/2}), \tag{3.23}$$
$$2p(ab^2r^{1/2}) \leq p(ab^4r^{1/2}) + p(a^4b^2r^{1/2}), \tag{3.24}$$
$$\max\{p(ab^{-2}r^{1/2}),\ p(b^{-1}r^{1/2}),\ p(a^2b^{-1}r^{1/2})\} \leq p(a^4b^2r^{1/2}). \tag{3.25}$$

For (3.22) we apply (3.23) in the first braces and do (3.24) and (3.25) in the second braces. Then the desired inequality readily follows. □

Now we prove the following.

Lemma 3.16. *Let*

$$m(s) := \inf_{(a,b,u)\in(0,1]\times(0,1]\times(0,s]} \widetilde{D}(a, b; u),$$

and $s_c := r_c^{1/2}$. *Then,* $m(s)$ *attains its minimum at* $(1, 1, s)$ *and* $m(s) \geq 0$ *if and only if* $0 < s \leq s_c$.

Proof. We fix $s \in (0, 1]$. For $x \in (0, 1]$, we consider the curve C_x defined by $a^2b = x$, or equivalently by $b = x/a^2$. We note that

$$(0, 1]^2 = \bigcup_{x\in(0,1]} \{(a, b) \in (0, 1]^2 : a^2b = x,\ x^{1/2} \leq a \leq 1\}.$$

On the curve C_x, we can write $\widetilde{D}(a, a^{-2}x; s) = p(a^{-1}x^4s^5) + 13p(a^{-1}x^2s^3) - 46p(x^2s)$, $x^{1/2} \leq a \leq 1$. Since $p'(x) = 1 - x^{-2} \leq 0$ for $x \in (0, 1]$, we have

$$\frac{\partial}{\partial a}\widetilde{D}(a, a^{-2}x; s) = p'(a^{-1}x^4s^5)(-a^{-2}x^4s^5) + 13p'(a^{-1}x^2s^3)(-a^{-2}x^2s^3) \geq 0,$$

and hence $\widetilde{D}(a, a^{-2}x; s)$ attains its minimum at $a = x^{1/2}$ and $b = 1$. Therefore, for $s \in (0, 1]$,

$$\inf_{(a,b)\in(0,1]\times(0,1]} \widetilde{D}(a, b; s) = \inf_{x\in(0,1]} \widetilde{D}(x^{1/2}, 1; s) = \inf_{a\in(0,1]} \widetilde{D}(a, 1; s). \quad (3.26)$$

For $(a, u) \in (0, 1]\times(0, s]$, we consider $\widetilde{D}(a, 1; u) = p(a^7u^5)+13p(a^3u^3)-46p(a^4u)$. For $y \in (0, s]$, we consider the curve C'_y defined by $a^4u = y$ or equivalently, by $a = (y/u)^{1/4}$. Note that

$$(0, 1] \times (0, s] = \bigcup_{y\in(0,s]} \{(a, u) \in (0, 1] \times (0, s] : a^4u = y,\ \ y \leq u \leq s\}.$$

Then, on the curve C'_y, we can write $\widetilde{D}((y/u)^{1/4}, 1; u) = p(y^{7/4}u^{13/4})+13p(y^{3/4}u^{9/4})-46p(y)$, $y \leq u \leq s$. Since $(\partial/\partial u)\widetilde{D}((y/u)^{1/4}, 1; u) \leq 0$, we conclude that $\widetilde{D}((y/u)^{1/4}, 1; u)$ attains its minimum at $u = s$, and hence, from (3.26), we have

$$m(s) = \inf_{(a,u)\in(0,1]\times(0,s]} \widetilde{D}(a, 1; u) = \inf_{y\in(0,s]} \widetilde{D}((y/s)^{1/4}, 1; s) = \inf_{a\in(0,1]} \widetilde{D}(a, 1; s). \quad (3.27)$$

It suffices to show $m(s) \geq 0$ if $s \leq s_c$. Since $xp'(x) = q(x) := x - 1/x$, we can verify easily that

$$\begin{aligned} a\frac{\partial}{\partial a}\widetilde{D}(a, 1; s) &= 7q(a^7s^5) + 13 \cdot 3q(a^3s^3) - 46 \cdot 4q(a^4s) \\ &= \frac{1}{a^7s^5}(7a^{14}s^{10} - 7 + 39a^{10}s^8 - 39a^4s^2 - 184a^{11}s^6 + 184a^3s^4) \\ &=: \frac{1}{a^7s^5}\delta(a, s). \end{aligned}$$

For $a, s \in (0, 1]$, we see that

$$\begin{aligned} \delta(a, s) &\leq 7s^{10} + 39s^8 + 184a^2s^4 - 39a^4s^2 - 7 \\ &= 7s^{10} + 39s^8 - 39s^2\left(a^2 - \frac{92}{39}s^2\right)^2 + \frac{92^2}{39}s^6 - 7 \leq 7s^{10} + 39s^8 + \frac{92^2}{39}s^6 - 7. \end{aligned}$$

Since the last function of s is increasing in $(0, 1]$ and it takes a negative value at $s = 11/20$, we have $\delta(a, s) < 0$ for $(a, s) \in (0, 1] \times (0, 11/20]$. Therefore, $\widetilde{D}(a, 1, s)$ is decreasing in a for $s \in (0, 11/20]$, which together with (3.27) implies

$$m(s) = \widetilde{D}(1, 1, s) = \frac{1 + s^2}{s^5}(s^8 + 12s^6 - 58s^4 + 12s^2 + 1).$$

Here we note Remark 12 given in Sect. 3.4.5. Consequently, $m(s) \geq m(s_c) = 0$ for $s \in (0, s_c]$ as $s_c = r_c^{1/2} = 0.533\cdots \leq 11/20$. $\square$

Remark 13. We see that

$$g_{\mathbb{D}}(-r^{-1/4}, r^{-1/4}; r) = \frac{6 + r + r^{-1}}{4(r^{1/2} + r^{-1/2})} =: \widetilde{g}(r).$$

It is readily verified that $\widetilde{g}(1) = 1$ and $d\widetilde{g}(r)/dr = (r-1)^3/\{8r^{3/2}(r+1)^2\} \geq 0, r \geq 1$. Then, $\widetilde{g}(r) > 1$ for any $r > 1$. Since $1/r_c = 3.51 \cdots > 1$, the PDPP $\mathcal{Z}_{X_{\mathbb{D}}^r}$ is still in the partially attractive phase although $\kappa_0(r)$ becomes negative when $r \in (1/r_c, \infty)$ due to the symmetry $r \leftrightarrow 1/r$ built in (3.21).

4. Concluding Remarks

Peres and Virág proved a relationship between the Szegő Kernel $S_{\mathbb{D}}$ and the Bergman kernel $K_{\mathbb{D}}$ in the context of probability theory: A GAF is defined so that its covariance kernel is given by $S_{\mathbb{D}}$. Then the zero point process $\mathcal{Z}_{X_{\mathbb{D}}}$ is proved to be a DPP for which the correlation kernel is given by $K_{\mathbb{D}}$. The background of their work is explained in the monograph [35], in which we find that the *Edelman–Kostlan formula* [24] gives the density of $\mathcal{Z}_{X_{\mathbb{D}}}$ with respect to m/π as

$$\rho^1_{\mathbb{D},\mathrm{PV}}(z) = \frac{1}{4}\Delta \log S_{\mathbb{D}}(z, z), \quad z \in \mathbb{D},$$

where $\Delta := 4\partial_z \partial_{\overline{z}}$. Moreover, we have the equality

$$K_{\mathbb{D}}(z, w) = \partial_z \partial_{\overline{w}} \log S_{\mathbb{D}}(z, w) = S_{\mathbb{D}}(z, w)^2, \quad z, w \in \mathbb{D}. \tag{4.1}$$

On the other hand, as explained above (2.9), for the kernels on simply connected domain $D \subsetneq \mathbb{C}$, the equality

$$S_D(z, w)^2 = K_D(z, w), \quad z, w \in D, \tag{4.2}$$

is established.

In the present paper, we have reported our work to generalize the above to a family of GAFs and their zero point processes on the annulus $\mathbb{A}_q$. By comparing the expression (1.19) for the density obtained from Theorem 1.3 with (C.4) in Proposition C.1 in Appendix C.2 given below, we can recover the Edelman–Kostlan formula as follows,

$$\rho^1_{\mathbb{A}_q}(z; r) = \frac{\theta(-r)}{\theta(-r|z|^4)} S_{\mathbb{A}_q}(z, z; r|z|^2)^2 = \frac{1}{4}\Delta \log S_{\mathbb{A}_q}(z, z; r), \quad z \in \mathbb{A}_q.$$

However, (4.1) does not hold for the weighted Szegő kernel for $H^2_r(\mathbb{A}_q)$. As shown by (C.3), the second log-derivative of $S_{\mathbb{A}_q}(z, w; r)$ cannot be expressed by $S_{\mathbb{A}_q}(z, w; r)$ itself but a new function $S_{\mathbb{A}_q}(z, w; rz\overline{w})$ should be introduced. In addition the proportionality between the square of the Szegő kernel and the Bergman kernel (4.2) is no longer valid for the point processes on $\mathbb{A}_q$ as shown in Proposition C.2 in Appendix C.3.

The Borchardt identity plays an essential role in the proof of Peres and Virág, which is written as

$$\operatorname*{perdet}_{1 \leq i,j \leq n} \Big[S_{\mathbb{D}}(z_i, z_j) \Big] = \det_{1 \leq i,j \leq n} \Big[S_{\mathbb{D}}(z_i, z_j)^2 \Big], \quad n \in \mathbb{N}, \quad z_1, \ldots, z_n \in \mathbb{D}.$$

Since the n-point correlation function $\rho^n_{\mathbb{D},\mathrm{PV}}(z_1, \ldots, z_n)$ of $\mathcal{Z}_{X_{\mathbb{D}}}$ is given by the left-hand side, $\forall n \in \mathbb{N}$, this equality proves that $\mathcal{Z}_{X_{\mathbb{D}}}$ is a DPP. For $S_{\mathbb{A}_q}$ the corresponding equality does not hold. We have proved, however, that all correlation functions of our two-parameter family of zero point processes $\{\mathcal{Z}_{X^r_{\mathbb{A}_q}} : q \in (0,1), r > 0\}$ on $\mathbb{A}_q$ can be expressed using perdet defined by (1.15) and we stated that they are permanental-determinantal point processes (PDPPs).

We would like to place an emphasis on the fact that the present paper is not an incomplete work nor just replacing determinants by perdet's. The essentially new points, which are not found in the previous works [35,64], are the following:

(i) Even if we start from the GAF whose covariance kernel is given by the original Szegő kernel $S_{\mathbb{A}_q}(\cdot,\cdot) = S_{\mathbb{A}_q}(\cdot,\cdot;q)$ on $\mathbb{A}_q$, the full description of conditioning with zeros needs a series of new covariance kernels.

(ii) The covariance kernels of the induced GAFs generated by conditioning of zeros are identified with the *weighted* Szegő kernel $S_{\mathbb{A}_q}(\cdot,\cdot;\alpha)$ studied by McCullough and Shen [56]. In the present study, the weight parameter α plays an essential role, since it is determined by $\alpha = r\prod_{\ell=1}^n |z_\ell|^2$ and represents a geometrical information of the zeros in $\mathbb{A}_q$ $\{z_1,\dots,z_n\}$, $n \in \mathbb{N}$ put in the conditioning.

(iii) Corresponding to such an inductive structure of conditional GAFs, the correlation kernel of our PDPP of $\mathcal{Z}_{\mathbb{A}^r_q}$, $r > 0$ is given by $S_{\mathbb{A}_q}(\cdot,\cdot;\alpha)$ with $\alpha = r\prod_{\ell=1}^n |z_\ell|^2$ in order to give the correlation function for the points $\{z_1,\dots,z_n\}$; $\rho^n_{\mathbb{A}_q}(z_1,\dots,z_n;r)$. In addition, the n-product measure of the Lebesgue measure on $\mathbb{C}$ divided by π, $(m/\pi)^{\otimes n}$, should be weighted by $\theta(-r)/\theta(-r\prod_{k=1}^n |z_k|^4)$ to properly provide $\rho^n_{\mathbb{A}_q}(\cdot;r)$.

(iv) The parameter r also plays an important role to describe the symmetry of the GAF and its zero point process under the transformation which we call the (q,r)-inversion,

$$(z,r) \longleftrightarrow \left(\frac{q}{z}, \frac{q^2}{r}\right) \in \mathbb{A}_q \times (0,\infty). \tag{4.3}$$

And if we adjust $r = q$ the GAF and its zero point process become invariant under conformal transformations which preserve $\mathbb{A}_q$.

Inapplicability of the Borchardt identity to our zero poin processes $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ causes interesting behaviors of them as *interacting particle systems*. We have proved that the short-range interaction between points is repulsive with index $\beta = 2$ in a similar way to the usual DPP, but attractive interaction is also observed in $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$. The index for decay of correlations is given by $\eta = 4$. We found that there is a special value $r = r_0(q) \in (q,1)$ for each $q \in (0,1)$ at which the coefficient of the power-law decay of correlations changes its sign. We have studied the zero point process obtained in the limit $q \to 0$, which has a parameter $r \in [0,\infty)$. In this PDPP $\mathcal{Z}_{X^r_{\mathbb{D}}}$, $r_c := r_0(0)$ can be regarded as the critical value separating two phases in the sense that if $r \in [0, r_c)$ the zero point process is completely repulsive, while if $r \in (r_c,\infty)$ attractive interaction emerges depending on distances between points.

There are many open problems, since such PDPPs have not been studied so far. Here we list out some of them.

(1) We prove that the GAF $X^r_{\mathbb{A}_q}$ and its zero point process $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ have the rotational invariance and the (q,r)-inversion symmetry, and when $r = q$, they are invariant under conformal transformations which preserve $\mathbb{A}_q$ (Proposition 1.1). We claimed in

Remark 1 that $X^r_{\mathbb{A}_q}$ can be extended to a one-parameter family of GAFs $\{X^{r,(L)}_{\mathbb{A}_q} : L \in \mathbb{N}\}$ having the rotational invariance and the (q, r)-inversion symmetry and this family is an extension of $\{X^{(L)}_{\mathbb{D}} : L \in \mathbb{N}\}$ studied in [35, Sections 2.3 and 5.4] in the sense that $\lim_{q\to 0} X^{r,(L)}_{\mathbb{A}_q}|_{r=q} \stackrel{\mathrm{d}}{=} X^{(L)}_{\mathbb{D}}$, $L \in \mathbb{N}$. In [35, Section 2.5], it is argued that $\{X^{(L)}_{\mathbb{D}} : L > 0\}$ is the only GAFs, up to multiplication by deterministic non-vanishing analytic functions, whose zeros are isometry-invariant under the conformal transformations preserving $\mathbb{D}$. This assertion is proved by the fact that the zero point process of the GAF is completely determined by its first correlation function. Therefore, the "canonicality" of the GAFs $\{X^{(L)}_{\mathbb{D}} : L > 0\}$ is guaranteed by the uniqueness, up to multiplicative constant, of the density function with respect to $m(dz)/\pi$, $\rho^1_{\mathbb{D},\mathrm{PV}}(z) = 1/(1-|z|^2)^2$, which is invariant under the Möbius transformations preserving $\mathbb{D}$. We have found, however, that the density function with parameter $r > 0$, $\varrho(z; r)$, $z \in \mathbb{A}_q$ is not uniquely determined to be $\rho^1_{\mathbb{A}_q}(z; r)$ as (1.19) by the requirement that it is rotationally invariant and having the (q, r)-inversion symmetry. For example, we have the three-parameter ($\alpha_1 > 1-\alpha_2, \alpha_2 > 0, \alpha_3 \in \mathbb{R}$) family of density functions,

$$\varrho(z; r; \alpha_1, \alpha_2, \alpha_3) = \frac{\theta(-r)^{\alpha_3}}{\theta(-|z|^{2\alpha_1} r^{\alpha_2})} f^{\mathrm{JK}}(|z|^2, -|z|^{2\beta_1} r^{\beta_2})^2$$

with $\beta_1 = (\alpha_1 - \alpha_2)(\alpha_1 + \alpha_2 - 1)/4 + 1/2$, $\beta_2 = \alpha_2(\alpha_1 + \alpha_2 - 1)/2$, which satisfy the above requirement of symmetry. We see that $\varrho(z; r; 2, 1, 1) = \rho^1_{\mathbb{A}_q}(z; r)$ and $\lim_{q\to 0} \varrho(z; q; \alpha_1, \alpha_2, \alpha_3) = \rho^1_{\mathbb{D},\mathrm{PV}}(z)$. The present study of the GAFs on $\mathbb{A}_q$ and their zero point processes will be generalized in the future.

(2) As shown by (1.20), the asymptotics of the density of zeros $\rho^1_{\mathbb{A}_q}(z) \sim (1-|z|^2)^{-2}$ with respect to $m(dz)/\pi$ in the vicinity of the outer boundary of $\mathbb{A}_q$ can be identified with the metric in the hyperbolic plane called the *Poincaré disk model* (see, for instance, [18,34]). The zero point process $\mathcal{Z}_{X_{\mathbb{D}}}$ of Peres and Virág can be regarded as a uniform DPP on the Poincaré disk model [13,22,64]. Is there any meaningful geometrical space in which the present zero point process $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ seems to be uniform? As mentioned in Remark 3, conditioning with zeros does not induce any new GAF on $\mathbb{D}$ [64], but it does on $\mathbb{A}_q$. Is it possible to give some geometrical explanation for such a new phenomenon appearing in replacing $\mathbb{D}$ by $\mathbb{A}_q$ reported in the present paper?

(3) As mentioned above and in Theorem 1.6 (i), we have found power-law decays of unfolded 2-correlation functions to the unity with an index $\eta = 4$. Although the coefficient of this power-law changes depending on q and r, the index $\eta = 4$ seems to be universal in the PDPPs $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$, $\mathcal{Z}_{X^r_{\mathbb{D}}}$ and the DPP of Peres and Virág $\mathcal{Z}_{X_{\mathbb{D}}}$ (except the PDPPs at $r = r_0(q) \in (q, 1)$, $q \in [0, 1)$). The present proof of Theorem 1.6 (i) relied on brute force calculations showing vanishing of derivatives up to the third order. Simpler proof is required. In the metric of a proper hyperbolic space, the decay of correlation will be exponential. In such a representation, what is the meaning of the 'universal value' of η?

(4) As mentioned in Remark 9, the simplified PDPPs $\{\mathcal{Z}_{X^r_{\mathbb{D}}} : r \in (0, \infty)\}$ can be regarded as an interpolation between the DPP of Peres and Virág $\mathcal{Z}_{X_{\mathbb{D}}}$ and its deterministic perturbation at the origin $\mathcal{Z}_{X_{\mathbb{D}}} + \delta_0$. The first approximation of the perturbation of the deterministic zero near $r = \infty$ is given by $\frac{-1}{\sqrt{1+r}}\zeta_0/\zeta_1$ by solving

the approximated linear equation $\frac{\zeta_0}{\sqrt{1+r}} + \zeta_1 z = 0$. Here the ratio ζ_0/ζ_1 is distributed according to the push-forward of the uniform distribution on the unit sphere by the stereographic projection (see Krishnapur [46] for the matrix generalization). Can we trace such a flow of zeros in $\{\mathcal{Z}_{X_{\mathbb{D}}^r} : r \in (0, \infty)\}$ more precisely?

(5) The simplified PDPP $\mathcal{Z}_{X_{\mathbb{D}}^r}$ was introduced as a $q \to 0$ limit of the PDPP $\mathcal{Z}_{\mathbb{X}_{\mathbb{A}_q}^r}$ in this paper. On the other hand, as shown by (1.23) the GAF $X_{\mathbb{D}}^r$ can be obtained from the GAF $X_{\mathbb{D}}$ of Peres and Virág by adding a one-parameter ($r > 0$) perturbation on a single term. Can we explain the hierarchical structures of these GAF and PDPP on $\mathbb{D}$ and the existence of the critical value r_c for correlations of $\mathcal{Z}_{X_{\mathbb{D}}^r}$ apart from all gadgets related to elliptic functions? Can we expect any interesting phenomenon at $r = r_c$?

(6) As mentioned at the end of Section 1.3, the zero point process of the GAF $X_{\mathbb{D}}$ studied by Peres and Virág [64] is the DPP $\mathcal{Z}_{X_{\mathbb{D}}}$, whose correlation kernel is given by $K_{\mathbb{D}}(z, w) = S_{\mathbb{D}}(z, w)^2 = 1/(1 - z\overline{w})^2$, $z, w \in \mathbb{D}$ with respect to the reference measure m/π. Krishnapur [46] introduced a one-parameter ($\ell \in \mathbb{N}$) extension of DPPs $\{\mathcal{Z}_{X_{\mathbb{D}}}^{(\ell)} : \ell \in \mathbb{N}\}$, whose correlation kernels are given by $K_{\mathbb{D}}^{(\ell)}(z, w) = 1/(1 - z\overline{w})^{\ell+1}$ with respect to the reference measure $\ell(1 - |z|^2)^{\ell-1} m/\pi$ on $\mathbb{D}$. He proved that $\mathcal{Z}_{\mathbb{D}}^{(\ell)}$ is realized as the zeros of $\det[\sum_{n\in\mathbb{N}} \mathsf{G}_n z^n]$, where $\{\mathsf{G}_n\}_{n\in\mathbb{N}}$ are i.i.d. complex Ginibre random matrices of size $\ell \in \mathbb{N}$. A similar extension of the present PDPP $\mathcal{Z}_{\mathbb{A}_q^r}$ on $\mathbb{A}_q$ will be challenging.

(7) For $0 < t \le 1$, let $\mathbb{D}_t := \{z \in \mathbb{C} : |z| < t\}$. The CLT for the number of points $\mathcal{Z}_{X_{\mathbb{D}}}(\mathbb{D}_t)$ as $t \to 1$ can be easily shown, since $\mathcal{Z}_{X_{\mathbb{D}}}$ is a DPP and then $\mathcal{Z}_{X_{\mathbb{D}}}(\mathbb{D}_t)$ can be expressed as a sum of independent Bernoulli random variables [64, Corollary 3 (iii)] [69]. For $0 < q \le s < t \le 1$, let $\mathbb{A}_{s,t} := \{z \in \mathbb{C} : s < |z| < t\}$. It would also be expected that the CLT holds for $\left(\mathcal{Z}_{\mathbb{A}_q}(\mathbb{A}_{s,\sqrt{q}}), \mathcal{Z}_{\mathbb{A}_q}(\mathbb{A}_{\sqrt{q},t})\right)$ as $s \to q$ and $t \to 1$ simultaneously in some sense. Is there a useful expression for those random variables as above and can we prove the CLTs for them?

(8) In the present paper, we have tried to characterize the density functions and the unfolded 2-correlation functions of the PDPPs. As demonstrated by Fig.2, change of global structure is observed at $r = r_c$ for the unfolded 2-correlation function. Precise description of such a topological change is required. More detailed quantitative study would be also interesting. For example, we can show that $G_{\mathbb{A}_q}^{\vee}(x; r)$ plotted in Fig.2 attains its maximum at $x = \sqrt{q}$ when $r = 1$ and the value is given by $G_{\mathbb{A}_q}^{\vee}(\sqrt{q}; 1) = (q_2(q)^8 + q_3(q)^8)/(16 q q_1(q)^8) > 1$, where $q_1(q) := \prod_{n=1}^{\infty}(1 + q^{2n})$, $q_2(q) := \prod_{n=1}^{\infty}(1 + q^{2n-1})$, and $q_3(q) := \prod_{n=1}^{\infty}(1 - q^{2n-1})$. How about the local minima? Moreover, systematic study on three-point and higher-order correlations will be needed to obtain a better understanding of differences between PDPPs and DPPs.

(9) Matsumoto and one of the present authors [54] studied the *real* GAF on a plane and proved that the zeros of the real GAF provide a Pfaffian point process (PfPP). There a Pfaffian–Hafnian analogue of Borchardt's identity was used [36]. Is it meaningful to consider the Pfaffian–Hafnian analogues of PDPPs? Systematic study on the comparison among DPPs, PfPPs, permanental PPs, Hafnian PPs, PDPPs, and Hafnian–Pfaffian PPs will be a challenging future problem.

(10) The symmetry of the present GAF and its zero point process under the (q, r)-inversion (4.3) mentioned above and the pairing of uncorrelated points in the GAF

$X_{\mathbb{A}_q}$ shown by Proposition 2.4 suggest that the inner boundary γ_q plays essentially the same role as the outer boundary γ_1. As an extension of the Riemann mapping function for a simply connected domain $D \subsetneq \mathbb{C}$, a function mapping a multiply connected domain to the unit disk is called the *Ahlfors map* [7,79]. (See also Remark 7 again.) Could we use such Ahlfors maps to construct and analyze GAFs and their zero point processes on general multiply connected domains?

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A. Hyperdeterminantal Point Processes

Recall that determinant and permanent are defined for an $n \times n$ matrix (a 2nd order tensor on an n-dimensional space) $M = (m_{i_1 i_2})_{1 \leq i_1, i_2 \leq n}$ as

$$\det M = \det_{1 \leq i_1, i_2 \leq n} [m_{i_1 i_2}] := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) \prod_{\ell=1}^{n} m_{\ell \sigma(\ell)} = \frac{1}{n!} \sum_{(\sigma_1, \sigma_2) \in \mathfrak{S}_n^2} \mathrm{sgn}(\sigma_1) \mathrm{sgn}(\sigma_2) \prod_{\ell=1}^{n} m_{\sigma_1(\ell) \sigma_2(\ell)},$$
$$\mathrm{per}\, M = \mathrm{per}_{1 \leq i_1, i_2 \leq n} [m_{i_1 i_2}] := \sum_{\sigma \in \mathfrak{S}_n} \prod_{\ell=1}^{n} m_{\ell \sigma(\ell)} = \frac{1}{n!} \sum_{(\sigma_1, \sigma_2) \in \mathfrak{S}_n^2} \prod_{\ell=1}^{n} m_{\sigma_1(\ell) \sigma_2(\ell)}, \tag{A.1}$$

where $\mathfrak{S}_n$ denotes the symmetric group of order n. The notion of determinant has been extended as follows. Cayley's first hyperdeterminant is defined for a k-th order tensor (hypermatrix) on an n-dimensional space $M = (m_{i_1 \ldots i_k})_{1 \leq i_1, \ldots, i_k \leq n}$ as

$$\mathrm{Det}\, M = \mathrm{Det}_{1 \leq i_1, \ldots, i_k \leq n} [m_{i_1 \ldots i_k}] := \frac{1}{n!} \sum_{(\sigma_1, \ldots, \sigma_k) \in \mathfrak{S}_n^k} \prod_{i=1}^{k} \mathrm{sgn}(\sigma_i) \prod_{\ell=1}^{n} m_{\sigma_1(\ell) \ldots \sigma_k(\ell)}. \tag{A.2}$$

It is straightforward to see that $\mathrm{Det}\, M = 0$ if k is odd. Gegenbauer generalized (A.2) to the case where some of the indices are non-alternated. If $\mathcal{I}$ denotes a subset of $\{1, \ldots, k\}$, one has

$$\mathrm{Det}_{\mathcal{I}} M = \mathrm{Det}_{\mathcal{I}\, 1 \leq i_1, \ldots, i_k \leq n} [m_{i_1 \ldots i_k}] := \frac{1}{n!} \sum_{(\sigma_1, \ldots, \sigma_k) \in \mathfrak{S}_n^k} \prod_{i \in \mathcal{I}} \mathrm{sgn}(\sigma_i) \prod_{\ell=1}^{n} m_{\sigma_1(\ell) \ldots \sigma_k(\ell)}. \tag{A.3}$$

These extensions of the determinant are called *hyperdeterminants*. See [25,51,53] and references therein.

Lemma A.1. *Let $A = (a_{i_1 i_2})$ and $B = (b_{i_1 i_2})$ be $n \times n$ matrices. Then* $\mathrm{per}\, A \det B = \mathrm{Det}_{\{2,3\}}\, C$, *where $C = (c_{i_1 i_2 i_3})$ is the $n \times n \times n$ hypermatrix with the entries*

$$c_{i_1 i_2 i_3} = a_{i_2 i_1} b_{i_2 i_3}, \quad i_1, i_2, i_3 \in \{1, \dots, n\}. \tag{A.4}$$

In particular, $\mathrm{perdet}\, M = \mathrm{Det}_{\{2,3\}}[m_{i_2 i_1} m_{i_2 i_3}]$, *where* $\mathrm{perdet}\, M$ *is defined by (1.15).*

Proof. By the definition (A.1),

$$\begin{aligned}
\mathrm{per}\, A \det B &= \sum_{\tau_1 \in \mathfrak{S}_n} \prod_{i=1}^n a_{i\tau_1(i)} \sum_{\tau_2 \in \mathfrak{S}_n} \mathrm{sgn}(\tau_2) \prod_{j=1}^n b_{j\tau_2(j)} = \sum_{\tau_1 \in \mathfrak{S}_n} \sum_{\tau_2 \in \mathfrak{S}_n} \mathrm{sgn}(\tau_2) \prod_{i=1}^n a_{i\tau_1(i)} b_{i\tau_2(i)} \\
&= \frac{1}{n!} \sum_{\sigma_1 \in \mathfrak{S}_n} \sum_{\sigma_2 \in \mathfrak{S}_n} \sum_{\sigma_3 \in \mathfrak{S}_n} \mathrm{sgn}(\sigma_1^{-1} \circ \sigma_3) \prod_{i=1}^n a_{i\, \sigma_1^{-1} \circ \sigma_2(i)} b_{i\, \sigma_1^{-1} \circ \sigma_3(i)} \\
&= \frac{1}{n!} \sum_{\sigma_1 \in \mathfrak{S}_n} \sum_{\sigma_2 \in \mathfrak{S}_n} \sum_{\sigma_3 \in \mathfrak{S}_n} \mathrm{sgn}(\sigma_1) \mathrm{sgn}(\sigma_3) \prod_{i=1}^n a_{\sigma_1(i)\sigma_2(i)} b_{\sigma_1(i)\sigma_3(i)}.
\end{aligned}$$

We change the symbols of permutations as $\sigma_1 \to \rho_2, \sigma_2 \to \rho_1, \sigma_3 \to \rho_3$. Then the above is written as $(1/n!) \sum_{\rho_1 \in \mathfrak{S}_n} \sum_{\rho_2 \in \mathfrak{S}_n} \sum_{\rho_3 \in \mathfrak{S}_n} \mathrm{sgn}(\rho_2) \mathrm{sgn}(\rho_3) \prod_{i=1}^n a_{\rho_2(i)\rho_1(i)} b_{\rho_2(i)\rho_3(i)}$. Hence if we assume (A.4), then this is written as $(1/n!) \sum_{(\sigma_1, \sigma_2, \sigma_3) \in \mathfrak{S}_n^3} \prod_{i \in \{2,3\}} \mathrm{sgn}(\sigma_i) \prod_{j=1}^n c_{\sigma_1(j)\sigma_2(j)\sigma_3(j)}$. By the definition (A.3), the proof is complete. □

Theorem 1.3 of the present paper can be written in the following way.

Theorem A.2. $\mathcal{Z}_{X^r_{\mathbb{A}_q}}$ *is a hyperdeterminantal point process (hDPP) in the sense that it has correlation functions expressed by hyperdeterminants as*

$$\begin{aligned}
\rho^n_{\mathbb{A}_q}(z_1, \dots, z_n; r) &= \frac{\theta(-r)}{\theta(-r \prod_{k=1}^n |z_k|^4)} \\
&\times \mathop{\mathrm{Det}_{\{2,3\}}}_{1 \le i_1, i_2, i_3 \le n} \left[S_{\mathbb{A}_q}\Big(z_{i_2}, z_{i_1}; r \prod_{\ell=1}^n |z_\ell|^2\Big) S_{\mathbb{A}_q}\Big(z_{i_2}, z_{i_3}; r \prod_{\ell=1}^n |z_\ell|^2\Big) \right],
\end{aligned}$$

for every $n \in \mathbb{N}$ and $z_1, \dots, z_n \in \mathbb{A}_q$ with respect to m/π.

B. Conformal Map from $\mathbb{A}_q$ to $D(s)$

A general Schwarz–Christoffel formula for conformal maps from $\mathbb{A}_q$ to a doubly connected domain is given as Eq.(1) in [21] and on page 68 in [23]. We can read that a conformal map from $\mathbb{A}_q$ to a chordal standard domain $D(s)$, $s > 0$ is given in the form

$$f(z) = C \int_{-1}^{z} \frac{\theta(-\sqrt{-1}qu, \sqrt{-1}qu)}{\theta(u)} du,$$

where C is a parameter. We can show that the integral is transformed into an integral of the Weierstrass $\wp$-function and hence the map is expressed by the ζ-function. A result is given by (1.38) in Remark 10. We note that the obtained function H_q is related to the Villat kernel $\mathcal{K}$ (see, for instance, [28]),

$$\mathcal{K}(z) = \mathcal{K}(z; q) := \sum_{n \in \mathbb{Z}} \frac{1 + q^{2n} z}{1 - q^{2n} z} = \frac{1 + z}{1 - z} + 2 \sum_{n=1}^{\infty} \left(\frac{q^{2n}}{q^{2n} - z} + \frac{q^{2n} z}{1 - q^{2n} z} \right), \quad z \in \mathbb{A}_q,$$

by a simple relation $H_q(z) = \sqrt{-1}\mathcal{K}(z)$, $z \in \mathbb{A}_q$. Moreover, we can verify the equality $\mathcal{K}(z) = 2\rho_1(z)$, $z \in \mathbb{A}_q$.

C. Bergman Kernel and Szegő Kernel of an Annulus

C.1. $K_{\mathbb{A}_q}$ expressed by Weierstrass $\wp$-function. A CONS for the Bergman space on $\mathbb{A}_q$ is given by $\{\widetilde{e}_n^{(q)}(z)\}_{n\in\mathbb{Z}}$ where we set

$$\widetilde{e}_n^{(q)}(z) = \begin{cases} \sqrt{\dfrac{n+1}{1-q^{2(n+1)}}} z^n, & n \in \mathbb{Z} \setminus \{-1\}, \\ \sqrt{\dfrac{1}{-2\log q}} z^{-1}, & n = -1. \end{cases}$$

The Bergman kernel of $\mathbb{A}_q$ is then given by

$$\begin{aligned} K_{\mathbb{A}_q}(z, w) &:= k_{L^2_{\mathrm{B}}(\mathbb{A}_q)}(z, w) = \sum_{n\in\mathbb{Z}} \widetilde{e}_n^{(q)}(z)\overline{\widetilde{e}_n^{(q)}(w)} \\ &= -\frac{1}{2\log q}\frac{1}{z\overline{w}} + \frac{1}{z\overline{w}} \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{n}{1-q^{2n}}(z\overline{w})^n, \quad z, w \in \mathbb{A}_q. \end{aligned} \tag{C.1}$$

Using (2.35) and the notation (2.36), we can verify that this kernel is expressed using the Weierstrass $\wp$-function (2.35) as [8]

$$K_{\mathbb{A}_q}(z, w) = -\frac{1}{2\log q}\frac{1}{z\overline{w}} - \frac{1}{z\overline{w}}\Big(\wp(\phi_{z\overline{w}}) + \frac{P}{12}\Big), \quad z, w \in \mathbb{A}_q. \tag{C.2}$$

C.2. Second log-derivatives of $S_{\mathbb{A}_q}$. Here we prove the following.

Proposition C.1. *For $r > 0$, the following equality holds,*

$$\partial_z \partial_{\overline{w}} \log S_{\mathbb{A}_q}(z, w; r) = \frac{\theta(-r)}{\theta(-r(z\overline{w})^2)} S_{\mathbb{A}_q}(z, w; rz\overline{w})^2, \quad z, w \in \mathbb{A}_q. \tag{C.3}$$

In particular,

$$\Delta \log S_{\mathbb{A}_q}(z, z; r) = 4\frac{\theta(-r)}{\theta(-r|z|^4)} S_{\mathbb{A}_q}(z, z; r|z|^2)^2, \quad z \in \mathbb{A}_q. \tag{C.4}$$

Proof. Let $\vartheta_1(\xi) := \sqrt{-1}q^{1/4}q_0 e^{-\sqrt{-1}\xi}\theta(e^{2\sqrt{-1}\xi})$ [29, (11.2.2)]. This is one of the well-known four kinds of *Jacobi theta functions* $\vartheta_i(\xi)$, $i = 0, 1, 2, 3$. (See [29, Section 1.6] and [62, Section 20.5].) Using ϑ_1, (2.30) in Proposition 2.2 is written as

$$S_{\mathbb{A}_q}(z, w; r) = \frac{\sqrt{-1}\vartheta_1'(0)\vartheta_1(\phi_{-rz\overline{w}}/2)}{2\vartheta_1(\phi_{-r}/2)\vartheta_1(\phi_{z\overline{w}}/2)},$$

where the notation (2.36) has been used. This gives

$$\partial_z \partial_{\overline{w}} \log S_{\mathbb{A}_q}(z, w; r) = -\Big(\partial_\xi^2 \log \vartheta_1(\xi)\Big|_{\xi=\phi_{-rz\overline{w}}/2} - \partial_\xi^2 \log \vartheta_1(\xi)\Big|_{\xi=\phi_{z\overline{w}}/2}\Big)/(4z\overline{w}).$$

We use the equality $\wp(2\omega_1 z/\pi) = (\pi/(2\omega_1))\{\vartheta_1'''(0)/(3\vartheta_1'(0)) - \partial_z^2 \log \vartheta_1(z)\}$ (see Eq. (23.6.14) in [62]). In the setting (2.34) we have

$$\partial_z \partial_{\overline{w}} \log S_{\mathbb{A}_q}(z, w; r) = \big(\wp(\phi_{-rz\overline{w}}) - \wp(\phi_{z\overline{w}})\big)/(z\overline{w}). \tag{C.5}$$

Now we use (2.51) in Lemma 2.5 given in Sect. 2.4 [19]. Combining with (2.25) in Proposition 2.1, (C.5) gives

$$\partial_z \partial_{\overline{w}} \log S_{\mathbb{A}_q}(z, w; r) = f^{\mathrm{JK}}(z\overline{w}, -rz\overline{w}) f^{\mathrm{JK}}(z\overline{w}, -(rz\overline{w})^{-1})/(z\overline{w})$$
$$= S_{\mathbb{A}_q}(z, w; rz\overline{w}) S_{\mathbb{A}_q}(z, w; (rz\overline{w})^{-1})/(z\overline{w}). \quad \text{(C.6)}$$

The expression (2.25) of $S_{\mathbb{A}_q}(\cdot, \cdot; r)$ in Proposition 2.1 gives

$$S_{\mathbb{A}_q}(z, w; (rz\overline{w})^{-1}) = f^{\mathrm{JK}}(z\overline{w}, -(rz\overline{w})^{-1}) = -f^{\mathrm{JK}}((z\overline{w})^{-1}, -rz\overline{w})$$
$$= -S_{\mathbb{A}_q}(z^{-1}, w^{-1}; rz\overline{w}), \quad \text{(C.7)}$$

where (2.28) was used. On the other hand, the expression (2.30) of $S_{\mathbb{A}_q}(\cdot, \cdot; r)$ in Proposition 2.2 gives $S_{\mathbb{A}_q}(z, w; rz\overline{w}) = q_0^2 \theta(-r(z\overline{w})^2)/\theta(-rz\overline{w}, z\overline{w})$, and

$$S_{\mathbb{A}_q}(z^{-1}, w^{-1}; rz\overline{w}) = \frac{q_0^2 \theta(-rz\overline{w}(z\overline{w})^{-1})}{\theta(-rz\overline{w}, (z\overline{w})^{-1})} = \frac{q_0^2 \theta(-r)}{\theta(-rz\overline{w}, (z\overline{w})^{-1})} = -z\overline{w} \frac{q_0^2 \theta(-r)}{\theta(-z\overline{w}r, z\overline{w})},$$

where (2.13) was used. Hence, $S_{\mathbb{A}_q}(z^{-1}, w^{-1}; rz\overline{w}) = -z\overline{w}\{\theta(-r)/\theta(-r(z\overline{w})^2)\} S_{\mathbb{A}_q}(z, w; rz\overline{w})$ and (C.7) gives $S_{\mathbb{A}_q}(z, w; (rz\overline{w})^{-1}) = z\overline{w}\{\theta(-r)/\theta(-r(z\overline{w})^2)\} S_{\mathbb{A}_q}(z, w; rz\overline{w})$. Then (C.6) proves the proposition. □

C.3. Relation between $K_{\mathbb{A}_q}$ and $S_{\mathbb{A}_q}$. We prove the following relation between the Bergman kernel $K_{\mathbb{A}_q}$ and the Szegő kernel $S_{\mathbb{A}_q}$ of an annulus.

Proposition C.2. *The equality*

$$S_{\mathbb{A}_q}(z, w)^2 = K_{\mathbb{A}_q}(z, w) + \frac{a}{z\overline{w}}, \quad z, w \in \mathbb{A}_q, \quad \text{(C.8)}$$

holds, where

$$a = a(q) = e_2 + \frac{P}{12} + \frac{1}{2 \log q} = -2 \sum_{n \in \mathbb{N}} \frac{(-1)^n n q^n}{1 - q^{2n}} + \frac{1}{2 \log q}. \quad \text{(C.9)}$$

Proof. By Proposition 2.1, $S_{\mathbb{A}_q}(z, w)^2 = f_q^{\mathrm{JK}}(z\overline{w}, -q)^2$. Since $f^{\mathrm{JK}}(z, a) = f^{\mathrm{JK}}(z, a/q^2)/z$ is given by (2.29), we have $f^{\mathrm{JK}}(z\overline{w}, -q) = f^{\mathrm{JK}}(z\overline{w}, -q^{-1})/(z\overline{w})$, and hence

$$S_{\mathbb{A}_q}(z, w)^2 = \frac{1}{z\overline{w}} f^{\mathrm{JK}}(z\overline{w}, -q) f^{\mathrm{JK}}(z\overline{w}, -q^{-1}). \quad \text{(C.10)}$$

Here we use (2.51) in Lemma 2.5 given in Sect. 2.4 [19]. Then

$$f^{\mathrm{JK}}(z\overline{w}, -q) f^{\mathrm{JK}}(z\overline{w}, -q^{-1}) = \wp(\pi + \pi\tau_q) - \wp(\phi_{z\overline{w}}) = e_2 - \wp(\phi_{z\overline{w}}),$$

where we have used the setting (2.34), the notation (2.36) and the evenness of $\wp(z)$. The equality (C.10) is thus written as $S_{\mathbb{A}_q}(z, w)^2 = -\wp(\phi_{z\overline{w}})/(z\overline{w}) + e/(z\overline{w})$. Now we use (C.2). Then (C.8) is obtained with a given by the first expression in (C.9). If we set $z = -q/\overline{w}$ in (C.8), then Lemma 2.3 gives an equality, $0 = K_{\mathbb{A}_q}(-q/\overline{w}, w) - a/q$. By (C.1) with a short calculation, the second expression for a in (C.9) is obtained. □

Remark 14. The relationship (C.8) between $S_{\mathbb{A}_q}$ and $K_{\mathbb{A}_q}$ with an additional term a is concluded from a general theory (see, for instance, Exercise 3 in Section 6, Chapter VII of [60], and Chapter 25 of [7]). It was shown in [12] that a is readily determined by Lemma 2.3 as shown above, if the equality (C.8) is established. Here we showed direct proof of (C.8) using the equality (2.51) between f^{JK} and $\wp$ [19]. By the explicit formulas (C.9) for a, we see that $\lim_{q\to 0} a(q) = 0$. Therefore, the relation (C.8) is reduced in the limit $q \to 0$ to $S_{\mathbb{D}}(z, w)^2 = K_{\mathbb{D}}(z, w)$, $z, w \in \mathbb{D}$, which is a special case of (2.9), as expected.

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