



On the Parabolic and Hyperbolic Liouville Equations

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Received: 11 August 2019 / Accepted: 24 May 2021
Published online: 28 September 2021 – © The Author(s) 2021

Abstract: We study the two-dimensional stochastic nonlinear heat equation (SNLH) and stochastic damped nonlinear wave equation (SdNLW) with an exponential nonlinearity $\lambda\beta e^{\beta u}$, forced by an additive space-time white noise. (i) We first study SNLH for general $\lambda \in \mathbb{R}$. By establishing higher moment bounds of the relevant Gaussian multiplicative chaos and exploiting the positivity of the Gaussian multiplicative chaos, we prove local well-posedness of SNLH for the range $0 < \beta^2 < \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$. Our argument yields stability under the noise perturbation, thus improving Garban's local well-posedness result (2020). (ii) In the defocusing case $\lambda > 0$, we exploit a certain sign-definite structure in the equation and the positivity of the Gaussian multiplicative chaos. This allows us to prove global well-posedness of SNLH for the range: $0 < \beta^2 < 4\pi$. (iii) As for SdNLW in the defocusing case $\lambda > 0$, we go beyond the Da Prato-Debussche argument and introduce a decomposition of the nonlinear component, allowing us to recover a sign-definite structure for a rough part of the unknown, while the other part enjoys a stronger smoothing property. As a result, we reduce SdNLW into a system of equations (as in the paracontrolled approach for the dynamical Φ_3^4 -model) and prove local well-posedness of SdNLW for the range: $0 < \beta^2 < \frac{32-16\sqrt{3}}{5}\pi \simeq 0.86\pi$. This result (translated to the context of random data well-posedness for the deterministic nonlinear wave equation with an exponential nonlinearity) solves an open question posed by Sun and Tzvetkov (2020). (iv) When $\lambda > 0$, these models formally preserve the associated Gibbs measures with the exponential nonlinearity. Under the same assumption on β as in (ii) and (iii) above, we prove almost sure global well-posedness (in particular for SdNLW) and invariance of the Gibbs measures in both the parabolic and hyperbolic settings. (v) In Appendix, we present an argument for proving local well-posedness of SNLH for general $\lambda \in \mathbb{R}$ *without* using the positivity of the Gaussian multiplicative chaos. This proves local well-posedness of SNLH for the range $0 < \beta^2 < \frac{4}{3}\pi \simeq 1.33\pi$, slightly smaller than that in (i), but provides Lipschitz continuity of the solution map in initial data as well as the noise.

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1. Introduction

1.1. Parabolic and hyperbolic Liouville equations. We study the two-dimensional stochastic heat and wave equations with exponential nonlinearities, driven by an additive space-time white noise forcing. More precisely, we consider the following stochastic nonlinear heat equations (SNLH) on the two-dimensional torus $\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z})^2$:

$$\begin{cases} \partial_t u + \frac{1}{2}(1 - \Delta)u + \frac{1}{2}\lambda\beta e^{\beta u} = \xi & (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2 \\ u|_{t=0} = u_0, \end{cases} \quad (1.1)$$

and stochastic damped nonlinear wave equations (SdNLW) on \mathbb{T}^2 :

$$\begin{cases} \partial_t^2 u + \partial_t u + (1 - \Delta)u + \lambda\beta e^{\beta u} = \sqrt{2}\xi & (t, x) \in \mathbb{R}_+ \times \mathbb{T}^2, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1), \end{cases} \quad (1.2)$$

where $\beta, \lambda \in \mathbb{R} \setminus \{0\}$ and ξ denotes a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$. Our main goal is to establish local and global well-posedness of these equations for certain ranges of the parameter $\beta^2 > 0$ and also prove invariance of the associated Gibbs measures in the defocusing case $\lambda > 0$. As we see below, due to the exponential nonlinearity, the difficulty of these equations depends sensitively on the value of $\beta^2 > 0$ as well as the sign of λ .

Our study is motivated by a number of perspectives. From the viewpoint of analysis on singular stochastic PDEs, the equations (1.1) and (1.2) on \mathbb{T}^2 are very interesting

models. The main sources of the difficulty of these equations come from the roughness of the space-time white noise forcing and the non-polynomial nature of the nonlinearity. The first difficulty can already be seen at the level of the associated linear equations whose solutions (namely, stochastic convolutions) are known to be merely distributions for the spatial dimension $d \geq 2$. This requires us to introduce a proper *renormalization*, adapted to the exponential nonlinearity, to give a precise meaning to the equations. In recent years, we have seen a tremendous development in the study of singular stochastic PDEs, in particular in the parabolic setting [17, 19, 20, 30, 34, 38, 39, 42, 49, 55]. Over the last few years, we have also witnessed a rapid progress in the theoretical understanding of nonlinear wave equations with singular stochastic forcing and/or rough random initial data [15, 23, 24, 35–37, 58–65, 69, 77]. On the two-dimensional torus \mathbb{T}^2 , the stochastic heat and wave equations with a monomial nonlinearity u^k (see (1.3) and (1.4) below) have been studied in [20, 35, 37]. In particular, in the seminal work [20], Da Prato and Debussche introduced the so-called Da Prato-Debussche trick¹ (see Sect. 1.3) which set a new standard in the study of singular stochastic PDEs. We point out that many of the known results focus on polynomial nonlinearities and thus it is of great interest to extend the existing solution theory to the case of non-polynomial nonlinearities. We will come back and elaborate further this viewpoint later. Furthermore, in this paper, we study both SNLH (1.1) and SdNLW (1.2), which allows us to point out similarity and difference between the analysis of the stochastic heat and wave equations. See also [57] for a comparison of the stochastic heat and wave equations on \mathbb{T}^2 with a quadratic nonlinearity driven by fractional derivatives of a space-time white noise.

Another important point of view comes from mathematical physics. It is well known that many of singular stochastic PDEs studied in the references mentioned above correspond to parabolic and hyperbolic² stochastic quantization equations for various models arising in Euclidean quantum field theory; namely, the resulting dynamics preserves a certain Gibbs measure on an infinite-dimensional state space of distributions. See [70, 74]. For example, the well-posedness results in [20, 35, 37] show that, for an odd integer $k \geq 3$, the Φ_2^{k+1} -measure³ is invariant under the dynamics of the parabolic Φ_2^{k+1} -model on \mathbb{T}^2 :

$$\partial_t u + \frac{1}{2}(1 - \Delta)u + u^k = \xi \tag{1.3}$$

and the hyperbolic Φ_2^{k+1} -model on \mathbb{T}^2 :

$$\partial_t^2 u + \partial_t u + (1 - \Delta)u + u^k = \sqrt{2}\xi, \tag{1.4}$$

respectively. From this point of view, when $\lambda > 0$, the equations (1.1) and (1.2) correspond to the parabolic and hyperbolic stochastic quantization equations for the $\exp(\Phi)_2$ -measure constructed in [2] (see (1.15) and (1.23) below); namely, they formally preserve the associated Gibbs measures with the exponential nonlinear potential. This provides another motivation to study well-posedness of the equations (1.1) and (1.2). We also point out that the $\exp(\Phi)_2$ -measure and the resulting Gaussian multiplicative chaos play an important role in Liouville quantum gravity [21, 22, 25, 26, 46, 66]; see also a recent paper [30] for a nice exposition and further references therein. We also mention the works [1, 4] on the elliptic $\exp(\Phi)_2$ -model.

¹ See also the work by McKean [54] and Bourgain [10].

² This is the so-called “canonical” stochastic quantization equation. See [74].

³ In the hyperbolic case, it is coupled with the white noise measure μ_0 on the $\partial_t u$ -component. See (1.23).

Let us now come back to the viewpoint of analysis on singular stochastic PDEs and discuss the known results for the stochastic heat and wave equations with non-polynomial nonlinearities. In the one-dimensional case, the stochastic convolution (for the heat or wave equation) has positive regularity and thus there is no need for renormalization. In this case, the well-posedness theory for (1.1) and (1.2) on the one-dimensional torus \mathbb{T} and invariance of the associated Gibbs measures (when $\lambda > 0$) follow in a straightforward manner [3, 77]. In the two-dimensional case, the stochastic convolution is only a distribution, making the problem much more delicate. To illustrate this, we first discuss the case of the sine-Gordon models on \mathbb{T}^2 studied in [19, 42, 63, 64]. In the parabolic setting, Hairer-Shen [42] and Chandra-Hairer-Shen [19] studied the following parabolic sine-Gordon model on \mathbb{T}^2 :

$$\partial_t u + \frac{1}{2}(1 - \Delta)u + \sin(\beta u) = \xi. \tag{1.5}$$

In this series of work, they observed that the difficulty of the problem depends sensitively on the value of $\beta^2 > 0$. By comparing the regularities of the relevant singular stochastic terms,⁴ we can compare this sine-Gordon model (1.5) with the Φ_d^3 - and Φ_d^4 -models, at least at a heuristic level; for example, the Φ_d^3 -model (and the Φ_d^4 -model, respectively) formally corresponds to (1.5) with $d = 2 + \frac{\beta^2}{2\pi}$ (and $d = 2 + \frac{\beta^2}{4\pi}$, respectively). In terms of the actual well-posedness theory, the Da Prato-Debussche trick [20] along with a standard Wick renormalization yields local well-posedness of (1.5) for $0 < \beta^2 < 4\pi$. For the sine-Gordon model (1.5) on \mathbb{T}^2 , there is an infinite number of thresholds: $\beta^2 = \frac{j}{j+1}8\pi$, $j \in \mathbb{N}$, where one encounters new divergent stochastic objects, requiring further renormalizations. By using the theory of regularity structures [39], Chandra, Hairer, and Shen proved local well-posedness of (1.5) for the entire subcritical regime $0 < \beta^2 < 8\pi$. More recently, the authors with P. Sosoe studied the hyperbolic counterpart of the sine-Gordon problem [63, 64]. Due to a weaker smoothing property of the wave propagator, however, the resulting solution theory is much less satisfactory than that in the parabolic case; in the damped wave case, local well-posedness was established only for $0 < \beta^2 < 2\pi$. See also Remark 1.19(ii) below. It is this lack of strong smoothing in the wave case which makes the problems in the hyperbolic setting much more analytically challenging than those in the parabolic setting,⁵ and one of our main goals in this paper is to make a progress in the solution theory of the more challenging SdNLW (1.2) with the exponential nonlinearity. See also Remark 1.10.

In terms of regularity analysis, SNLH (1.1) and SdNLW (1.2) with the exponential nonlinearity can also be formally compared to the Φ_d^3 - and Φ_d^4 -models by the heuristic argument mentioned above, which yields the same correspondence as in the sine-Gordon case. While the sine-Gordon model enjoys a certain charge cancellation property [42, 63], there is no such cancellation property in the exponential model under consideration, which provides an additional difficulty in studying the regularity property of the relevant stochastic term (see Proposition 1.12 below). See also [30] for a discussion on intermittency of the problem with an exponential nonlinearity.

⁴ Namely, compare the regularities of the imaginary Gaussian multiplicative chaos with the stochastic convolution for the Φ_d^3 -model and with the renormalized square power of the stochastic convolution for the Φ_d^4 -model.

⁵ We mention the recent works [15, 36, 59, 60] on the paracontrolled approach to study the stochastic wave equations on the three-dimensional torus \mathbb{T}^3 , which are substantially more involved than the paracontrolled approach in the parabolic setting [17, 55]. Note that a standard application of the Da Prato-Debussche trick suffices to handle the quadratic nonlinearity on \mathbb{T}^3 in the parabolic setting [27], while it is not the case in the hyperbolic setting considered in [36].

In a recent paper [30], motivated from the viewpoint of Liouville quantum gravity, Garban studied the stochastic nonlinear heat equation (1.1) on \mathbb{T}^2 with an exponential nonlinearity $e^{\beta u}$:

$$\partial_t u - \frac{1}{2} \Delta u + \frac{1}{(2\pi)^{\frac{2}{3}}} e^{\beta u} = \xi. \tag{1.6}$$

See also (1.59) below. By studying the regularity property of the Gaussian multiplicative chaos (see (1.39) below) and applying Picard’s iteration argument, he proved local well-posedness of (1.6) for $0 < \beta^2 < \frac{8\pi}{7+4\sqrt{3}} \simeq 0.57\pi$.⁶ Furthermore, by exploiting the positivity of the Gaussian multiplicative chaos, he also proved local well-posedness for the range: $\frac{8\pi}{7+4\sqrt{3}} \leq \beta^2 < \frac{8\pi}{(1+\sqrt{2})^2} \simeq 1.37\pi$. This latter result is without stability under the perturbation of the noise and, in particular, the solution u was not shown to be a limit of the solutions with regularized noises.

Before we state our first main result on SNLH (1.1), let us introduce some notations. Given $N \in \mathbb{N}$, we denote by \mathbf{P}_N a smooth frequency projector onto the (spatial) frequencies $\{n \in \mathbb{Z}^2 : |n| \leq N\}$, associated with a Fourier multiplier

$$\chi_N(n) = \chi(N^{-1}n) \tag{1.7}$$

for some fixed non-negative even function $\chi \in C_c^\infty(\mathbb{R}^2)$ with $\text{supp } \chi \subset \{\xi \in \mathbb{R}^2 : |\xi| \leq 1\}$ and $\chi \equiv 1$ on $\{\xi \in \mathbb{R}^2 : |\xi| \leq \frac{1}{2}\}$. Let $\{g_n\}_{n \in \mathbb{Z}^2}$ and $\{h_n\}_{n \in \mathbb{Z}^2}$ be sequences of mutually independent standard complex-valued⁷ Gaussian random variables on a probability space (Ω, P) conditioned so that $g_{-n} = \overline{g_n}$ and $h_{-n} = \overline{h_n}$, $n \in \mathbb{Z}^2$. Moreover, we assume that $\{g_n\}_{n \in \mathbb{Z}^2}$ and $\{h_n\}_{n \in \mathbb{Z}^2}$ are independent from the space-time white noise ξ in the equations (1.1) and (1.2). Then, we define random functions w_0 and w_1 by setting

$$w_0^\omega = \sum_{n \in \mathbb{Z}^2} \frac{g_n(\omega)}{\langle n \rangle} e_n \quad \text{and} \quad w_1^\omega = \sum_{n \in \mathbb{Z}^2} h_n(\omega) e_n, \tag{1.8}$$

where $\langle n \rangle = \sqrt{1 + |n|^2}$ and $e_n(x) = \frac{1}{2\pi} e^{in \cdot x}$ as in (2.1). Lastly, given $s \in \mathbb{R}$, let μ_s denote the Gaussian measure on $\mathcal{D}'(\mathbb{T}^2)$ with the density:

$$d\mu_s = Z_s^{-1} e^{-\frac{1}{2} \|u\|_{H^s}^2} du. \tag{1.9}$$

On \mathbb{T}^2 , it is well known that μ_s is a Gaussian probability measure supported on $W^{s-1-\varepsilon, p}(\mathbb{T}^2)$ for any $\varepsilon > 0$ and $1 \leq p \leq \infty$. Note that the laws of w_0 and w_1 in (1.8) are given by the massive Gaussian free field μ_1 and the white noise measure μ_0 , respectively.

We study the following truncated SNLH:

$$\begin{cases} \partial_t u_N + \frac{1}{2}(1 - \Delta)u_N + \frac{1}{2}\lambda\beta C_N e^{\beta u_N} = \mathbf{P}_N \xi \\ u_N|_{t=0} = u_{0,N} \end{cases} \tag{1.10}$$

for a suitable renormalization constant $C_N > 0$, with initial data $u_{0,N}$ of the form:

$$u_{0,N} = v_0 + \mathbf{P}_N w_0, \tag{1.11}$$

where v_0 is a given deterministic function and w_0 is as in (1.8). We now state our first local well-posedness result for SNLH (1.1).

⁶ Here, the numerology is converted to our scaling convention. See Remark 1.16 below.

⁷ This means that $g_0, h_0 \sim \mathcal{N}_{\mathbb{R}}(0, 1)$ and $\text{Re}g_n, \text{Im}g_n, \text{Re}h_n, \text{Im}h_n \sim \mathcal{N}_{\mathbb{R}}(0, \frac{1}{2})$ for $n \neq 0$.

Theorem 1.1. (local well-posedness in the general case) *Let $\lambda \neq 0$ and $0 < \beta^2 < \beta_{\text{heat}}^2 := \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$. Then, there exists a sequence of positive constants $\{C_N\}_{N \in \mathbb{N}}$, tending to 0, (see (1.40) below) such that the stochastic nonlinear heat equation (1.1) is locally well-posed in the following sense; given $v_0 \in L^\infty(\mathbb{T}^2)$, there exist an almost surely positive stopping time $\tau = \tau(\|v_0\|_{L^\infty}, \beta, \lambda)$ and a non-trivial⁸ stochastic process $u \in C([0, \tau]; H^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any small $T > 0$, on the event $\{\tau \geq T\}$, the solution u_N to the truncated SNLH (1.10) with initial data $u_{0,N}$ of the form (1.11) converges in probability to u in $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$.*

Formally speaking, the limit u in Theorem 1.1 is a solution to the following ‘‘equation’’:

$$\begin{cases} \partial_t u + \frac{1}{2}(1 - \Delta)u + \frac{1}{2}\infty^{-1} \cdot \lambda \beta e^{\beta u} = \xi \\ u|_{t=0} = v_0 + w_0. \end{cases} \tag{1.12}$$

We will describe a precise meaning of this limiting equation in Sect. 1.3.

Note that the model (1.6) studied in [30] corresponds to (a massless version of) our model (1.1) with $\lambda = \frac{2\beta^{-1}}{(2\pi)^{\frac{1}{2}}}$. In view of the symmetry (in law) for (1.1): $(u, \xi, \beta, \lambda) \mapsto (-u, -\xi, -\beta, \lambda)$, Garban’s result covers both⁹ $\lambda > 0$ and $\lambda < 0$ as in Theorem 1.1. After rescaling, the upper bound $\frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$ on β^2 in Theorem 1.1 agrees with the ‘‘critical’’ value $\gamma_{\text{pos}} = 2\sqrt{2} - 2$ in [30]. See Remark 1.16 below. Namely, the ranges of the parameter β^2 in Theorem 1.1 and [30, Theorems 1.7 and 1.11] agree. The difference between the result in [30] and Theorem 1.1 for the range $\frac{8\pi}{7+4\sqrt{3}} \simeq 0.57\pi \leq \beta^2 < \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$ appears in the approximation property of the solution. In [30], Garban proved local well-posedness of the limiting equation (1.12) in the Da Prato-Debussche formulation but without continuity in the noise. In Sect. 4, we will prove convergence of the solution u_N of the truncated SNLH (1.10) to the limit u , thus establishing continuity in the noise.

In proving Theorem 1.1, we apply the Da Prato-Debussche trick as in [30]. By exploiting the positivity of the Gaussian multiplicative chaos, we construct a solution by a standard Picard’s iteration argument. For this purpose, we study higher moment bounds of the Gaussian multiplicative chaos. This is done with two different approaches: the first one using the Brascamp-Lieb inequality [13]¹⁰ (see Lemma 2.11 below), and the other one relying on Kahane’s classical approach.

This local well-posedness result by a contraction argument does not directly provide continuity in the noise since in studying the difference of Gaussian multiplicative chaoses, we can no longer exploit any positivity. In order to prove convergence of the solutions u_N to the truncated SNLH (1.10), we employ a more robust energy method

⁸ Here, non-triviality means that the limiting process u is not zero or a linear solution. As we see below, the limiting process u admits a decomposition $u = v + z + \Psi$, where $z = P(t)v_0$ denotes the (deterministic) linear solution defined in (1.44), Ψ denotes the stochastic convolution defined in (1.35), and the residual term v satisfies the nonlinear equation (1.46). See, for example, [41, 58, 63], where in contrast some triviality phenomena appear. A similar comment applies in the following statements.

⁹ What is important is the sign of λ , not its magnitude. Furthermore, as for the local well-posedness theory, there is no essential difference between the massive and massless case.

¹⁰ This is not to be confused with the Brascamp-Lieb concentration inequality [14, Theorem 5.1] in probability theory, which was used in the study of the Gibbs measure for the defocusing nonlinear Schrödinger equations on the real line [11].

(namely, an a priori bound and a compactness argument) and combine it with the uniqueness of a solution to the limiting equation (1.12) in the Da Prato-Debussche formulation. This in turn yields the continuity in the noise property. See also Remark 1.3 (ii) below.

In the defocusing case $\lambda > 0$, we can improve the local well-posedness result of Theorem 1.1 on two aspects. The first one is that the defocusing nonlinearity allows us to prove a global-in-time result in place of a local one. The second and less obvious one is that we can improve on the range of $\beta^2 > 0$. Namely, the defocusing nature of the nonlinearity also improves the local Cauchy theory.

Theorem 1.2. (global well-posedness in the defocusing case) *Let $\lambda > 0$ and $0 < \beta^2 < (\beta_{\text{heat}}^*)^2 := 4\pi$. Let $\{C_N\}_{N \in \mathbb{N}}$ be as in Theorem 1.1. Then, the stochastic nonlinear heat equation (1.1) is globally well-posed in the following sense; given $v_0 \in L^\infty(\mathbb{T}^2)$, there exists a non-trivial stochastic process $u \in C(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution u_N to the truncated SNLH (1.10) with initial data $u_{0,N}$ of the form (1.11) converges in probability to u in $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$.*

When $\lambda > 0$, Eq. (1.10) indeed has a sign-definite structure; see (1.47) for example. We exploit such a sign-definite structure at the level of the Da Prato-Debussche formulation to prove Theorem 1.2. For $\beta^2 \geq \frac{8\pi}{3+2\sqrt{2}}$, we need to employ an energy method even to prove existence of solutions. Both the sign-definite structure and the positivity of the Gaussian multiplicative chaos play an important role. We then prove uniqueness by establishing an energy estimate for the difference of two solutions. Continuity in the noise is shown by an analogous argument to that in the proof of Theorem 1.1. Theorem 1.2 thus shows that there is a significant improvement from [30] on the range of β^2 from $0 < \beta^2 < \frac{8\pi}{7+4\sqrt{3}} \simeq 0.57\pi$ in [30] to $0 < \beta^2 < 4\pi$ when $\lambda > 0$. This answers Question 7.1 in [30], showing that the value γ_{pos} in [30] does not correspond to a critical threshold, at least in the $\lambda > 0$ case. In view of the heuristic comparison to the Φ_d^4 -model mentioned above, the range: $0 < \beta^2 < 4\pi$ in Theorem 1.2 corresponds to the sub- Φ_3^4 case. Note that in this range, the Da Prato-Debussche trick and a contraction argument suffice for the parabolic sine-Gordon model [42, 64].

Remark 1.3. (i) For the sake of the argument, Theorems 1.1 and 1.2 are stated for the initial data $u_{0,N}$ of the form (1.11). By a slight modification of the argument, however, we can also treat general deterministic initial data $u_{0,N} = v_0 \in L^\infty(\mathbb{T}^2)$. See Remark 1.19 below. A similar comment applies to Theorem 1.8 for SdNLW (1.2).

(ii) In Appendix A, we present a local well-posedness argument in the sense of Theorem 1.1, in particular for any $\lambda \in \mathbb{R} \setminus \{0\}$, for the slightly smaller range $0 < \beta^2 < \frac{4}{3}\pi \simeq 1.33\pi$ than that in Theorem 1.1, but without using the positivity of the Gaussian multiplicative chaos or any sign-definite structure of the equation. This argument also provides stronger Lipschitz dependence on initial data and noise. See also Remark 4.3.

(iii) The well-posedness results in Theorem 1.1 and Theorem A.1 for general $\lambda \neq 0$ are directly applicable to the following parabolic sinh-Gordon equation on \mathbb{T}^2 :

$$\partial_t u + \frac{1}{2}(1 - \Delta)u + \frac{1}{2}\beta \sinh(\beta u) = \xi, \tag{1.13}$$

providing local well-posedness of (1.13) for the same range of β^2 , in particular, with continuity in the noise. The model (1.13) corresponds to the so-called cosh-interaction in quantum field theory. See Remark 1.18 below.

We now investigate an issue of invariant measures for (1.1) when $\lambda > 0$. Define the energy E_{heat} by

$$E_{\text{heat}}(u) = \frac{1}{2} \int_{\mathbb{T}^2} |\langle \nabla \rangle u|^2 dx + \lambda \int_{\mathbb{T}^2} e^{\beta u} dx, \tag{1.14}$$

where $\langle \nabla \rangle = \sqrt{1 - \Delta}$. The condition $\lambda > 0$ guarantees that the problem is defocusing. Note that the equation (1.1) formally preserves the Gibbs measure ρ_{heat} associated with the energy E_{heat} , whose density is formally given by

$$“d\rho_{\text{heat}} = Z^{-1} e^{-E_{\text{heat}}(u)} du = Z^{-1} \exp\left(-\lambda \int_{\mathbb{T}^2} e^{\beta u} dx\right) d\mu_1”, \tag{1.15}$$

where μ_1 is the massive Gaussian free field defined in (1.9). In view of the low regularity of the support of μ_1 , we need to apply a renormalization to the density in (1.15) so that ρ_{heat} can be realized as a weighted Gaussian measure on $\mathcal{D}'(\mathbb{T}^2)$.

In order to preserve the sign-definite structure of the equation for $\lambda > 0$, we can not use an arbitrary approximation to the identity for regularization but we need to use those with non-negative convolution kernels. Let ρ be a smooth, non-negative, even function compactly supported in $\mathbb{T}^2 \simeq [-\pi, \pi)^2$ and such that $\int_{\mathbb{R}^2} \rho(x) dx = 1$. Then, given $N \in \mathbb{N}$, we define a smoothing operator \mathbf{Q}_N by setting

$$\mathbf{Q}_N f = \rho_N * f = \sum_{n \in \mathbb{Z}^2} (2\pi \widehat{\rho}_N(n)) \widehat{f}(n) e_n, \tag{1.16}$$

where the mollifier ρ_N is defined by

$$\rho_N(x) = N^2 \rho(Nx). \tag{1.17}$$

We then define the truncated Gibbs measure $\rho_{\text{heat},N}$ by

$$d\rho_{\text{heat},N} = Z_N^{-1} \exp\left(-\lambda C_N \int_{\mathbb{T}^2} e^{\beta \mathbf{Q}_N u} dx\right) d\mu_1, \tag{1.18}$$

where C_N is the renormalization constant from Theorem 1.1 but with \mathbf{Q}_N instead of \mathbf{P}_N . As a corollary to the analysis on the Gaussian multiplicative chaos (see Proposition 1.12 below), we have the following convergence result.

Proposition 1.4. *Let $\lambda > 0$ and $0 < \beta^2 < (\beta_{\text{heat}}^*)^2 = 4\pi$. The sequence $\{\rho_{\text{heat},N}\}_{N \in \mathbb{N}}$ of the renormalized truncated Gibbs measures converges in total variation to some limiting probability measure. With a slight abuse of notation, we denote the limit by ρ_{heat} . Then, the limiting renormalized Gibbs measure ρ_{heat} and the massive Gaussian free field μ_1 are mutually absolutely continuous.*

Remark 1.5. We only discuss the construction and invariance of the Gibbs measure in the defocusing case $\lambda > 0$. Indeed, in the focusing case $\lambda < 0$, the Gibbs measure (1.18) is not normalizable. More precisely, in [66, Appendix A], N. Tzvetkov and the authors showed that the partition function satisfies

$$Z_N = \int \exp\left(-\lambda C_N \int_{\mathbb{T}^2} e^{\beta \mathbf{Q}_N u} dx\right) d\mu_1 \longrightarrow \infty,$$

as $N \rightarrow \infty$ in the case $\lambda < 0$. See Proposition A.1 in [66]. See also [12, 16, 51, 59, 60, 67, 68, 73] on non-normalizability results for focusing Gibbs measures.

The truncated Gibbs measure $\rho_{\text{heat},N}$ is invariant under the following truncated SNLH:

$$\begin{cases} \partial_t u_N + \frac{1}{2}(1 - \Delta)u_N + \frac{1}{2}\lambda\beta C_N \mathbf{Q}_N e^{\beta \mathbf{Q}_N u_N} = \xi \\ u_N|_{t=0} = u_{0,N}^{\text{Gibbs}} \sim \rho_{\text{heat},N}. \end{cases} \tag{1.19}$$

See Lemma 5.2 below. By taking $N \rightarrow \infty$, we then have the following almost sure global well-posedness and invariance of the renormalized Gibbs measure ρ_{heat} for SNLH (1.1).

Theorem 1.6. *Let $\lambda > 0$ and $0 < \beta^2 < (\beta_{\text{heat}}^*)^2 = 4\pi$. Then, the stochastic nonlinear heat equation (1.1) is almost surely globally well-posed with respect to the random initial data distributed by the renormalized Gibbs measure ρ_{heat} . Furthermore, the renormalized Gibbs measure ρ_{heat} is invariant under the resulting dynamics.*

More precisely, there exists a non-trivial stochastic process $u \in C(\mathbb{R}_+; H^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution u_N to the truncated SNLH (1.19) with the random initial data $u_{0,N}^{\text{Gibbs}}$ distributed by the truncated Gibbs measure $\rho_{\text{heat},N}$ in (1.18) converges in probability to u in $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$. Furthermore, the law of $u(t)$ for any $t \in \mathbb{R}_+$ is given by the renormalized Gibbs measure ρ_{heat} .

A variant of Theorem 1.2 implies global well-posedness of (1.19). Then, in view of the mutual absolute continuity of the renormalized Gibbs measure ρ_{heat} and the massive Gaussian free field μ_1 and the convergence in total variation of the truncated Gibbs measure $\rho_{\text{heat},N}$ in (1.18) to the limiting renormalized Gibbs measure ρ_{heat} (Proposition 1.4), the proof of Theorem 1.6 follows from a standard argument. See Sect. 5.2.

Remark 1.7. Note that the positivity of the operator \mathbf{Q}_N is needed only for proving local well-posedness of the truncated SNLH (1.19) and that Proposition 1.4 holds with \mathbf{P}_N (or any approximation to the identity) in place of \mathbf{Q}_N . Then, noting that the proof of Theorem 1.1 does not exploit any sign-definite structure of the equation, we conclude that even if we replace \mathbf{Q}_N with \mathbf{P}_N in (1.19), the conclusion of Theorem 1.6 holds true for the range $0 < \beta^2 < \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$. Since Theorem 1.1 only provides local well-posedness, we need to use Bourgain’s invariant measure argument [9,10] to construct almost sure global-in-time dynamics. We refer to [37,40,59,60,65] for the implementation of Bourgain’s invariant measure argument in the context of singular SPDEs.

Next, we turn our attention to the stochastic damped nonlinear wave equation (1.2). Due to a weaker smoothing property of the associated linear operator, the problem in this hyperbolic setting is harder than that in the parabolic setting discussed above. In the following, we restrict our attention to the defocusing case ($\lambda > 0$), where we can hope to exploit a (hidden) sign-definite structure of the equation. Given $N \in \mathbb{N}$, we study the following truncated SdNLW:

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + \lambda\beta C_N e^{\beta u_N} = \sqrt{2}\mathbf{P}_N \xi \\ (u_N, \partial_t u_N)|_{t=0} = (u_{0,N}, u_{1,N}) \end{cases} \tag{1.20}$$

with the renormalization constant C_N from Theorem 1.1 and initial data $(u_{0,N}, u_{1,N})$ of the form:

$$(u_{0,N}, u_{1,N}) = (v_0, v_1) + (\mathbf{P}_N w_0, \mathbf{P}_N w_1), \tag{1.21}$$

where (v_0, v_1) is a pair of given deterministic functions and (w_0, w_1) is as in (1.8).

Theorem 1.8. *Let $\lambda > 0$, $0 < \beta^2 < \beta_{\text{wave}}^2 := \frac{32-16\sqrt{3}}{5}\pi \simeq 0.86\pi$, and $s > 1$. Then, the stochastic damped nonlinear wave equation (1.2) is locally well-posed in the following sense; given $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2) = H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2)$, there exist an almost surely positive stopping time $\tau = \tau(\|(v_0, v_1)\|_{\mathcal{H}^s}, \beta, \lambda)$ and a non-trivial stochastic process $(u, \partial_t u) \in C([0, \tau]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any small $T > 0$, on the event $\{\tau \geq T\}$, the solution $(u_N, \partial_t u_N)$ to the truncated SdNLW (1.20) with initial data $(u_{0,N}, u_{1,N})$ of the form (1.21) converges in probability to $(u, \partial_t u)$ in $C([0, T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$.*

Due to a weaker smoothing property of the linear wave operator, the range of β^2 in Theorem 1.8 is much smaller than that in Theorem 1.2 and we can only prove local well-posedness for SdNLW (1.2). Furthermore, we do not know how to prove local well-posedness of SdNLW (1.2) in the focusing case ($\lambda < 0$). Namely, there is no analogue of Theorem 1.1 in this hyperbolic setting at this point.

As in the proof of Theorem 1.2, we proceed with the Da Prato-Debussche trick but the proof of Theorem 1.8 in the hyperbolic setting is more involved than that of Theorem 1.2 in the parabolic setting. Due to the oscillatory nature of the Duhamel integral operator (see (1.32) below) associated with the damped Klein-Gordon operator $\partial_t^2 + \partial_t + (1 - \Delta)$, we can not exploit any sign-definite structure as it is. We point out, however, that near the singularity, the kernel for the Duhamel integral operator is essentially non-negative. This observation motivates us to decompose the residual term v in the Da Prato-Debussche argument as $v = X + Y$, where the low regularity part X enjoys a sign-definite structure and the other part Y enjoys a stronger smoothing property. As a result, we reduce the equation (1.20) to a system of equations; see (1.54) below. This decomposition of the unknown into a less regular but structured part X and a smoother part Y is reminiscent of the paracontrolled approach to the dynamical Φ_3^4 -model in [17, 55]. See also [36]. We will describe an outline of the proof of Theorem 1.8 in Sect. 1.3.

Lastly, we study the Gibbs measure ρ_{wave} for SdNLW (1.2) associated with the energy:

$$E_{\text{wave}}(u, \partial_t u) = E_{\text{heat}}(u) + \frac{1}{2} \int_{\mathbb{T}^2} (\partial_t u)^2 dx,$$

where E_{heat} is as in (1.14). As in the parabolic case, we need to introduce a renormalization. Define the truncated Gibbs measure $\rho_{\text{wave},N}$ by

$$d\rho_{\text{wave},N}(u, \partial_t u) = Z_N^{-1} d(\rho_{\text{heat},N} \otimes \mu_0)(u, \partial_t u), \tag{1.22}$$

where μ_0 is the white noise measure defined in (1.9). Then, it follows from Proposition 1.4 that when $0 < \beta^2 < 4\pi$, the truncated Gibbs measure $\rho_{\text{wave},N}$ converges in total variation to the renormalized Gibbs measure ρ_{wave} given by

$$d\rho_{\text{wave}}(u, \partial_t u) = Z^{-1} d(\rho_{\text{heat}} \otimes \mu_0)(u, \partial_t u). \tag{1.23}$$

Now, consider the following truncated SdNLW:

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + \lambda \beta C_N \mathbf{Q}_N e^{\beta \mathbf{Q}_N u_N} = \sqrt{2} \xi \\ (u_N, \partial_t u_N)|_{t=0} = (u_{0,N}^{\text{Gibbs}}, u_{1,N}^{\text{Gibbs}}), \end{cases} \tag{1.24}$$

where \mathbf{Q}_N is the mollifier with a non-negative kernel defined in (1.16) and C_N is the renormalization constant from Theorem 1.1 but with \mathbf{Q}_N instead of \mathbf{P}_N . Decomposing the truncated SdNLW (1.24) into the deterministic nonlinear wave dynamics:

$$\partial_t^2 u_N + (1 - \Delta)u_N + \lambda\beta C_N \mathbf{Q}_N e^{\beta \mathbf{Q}_N u_N} = 0$$

and the Ornstein-Uhlenbeck process (for $\partial_t u_N$):

$$\partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N = \sqrt{2}\xi,$$

we see that the truncated Gibbs measure $\rho_{\text{wave},N}$ is invariant under the truncated SdNLW (1.24). See Section 4 in [37]. As a result, we obtain the following almost sure global well-posedness of (1.2) and invariance of the renormalized Gibbs measure ρ_{wave} .

Theorem 1.9. *Let $\lambda > 0$ and $0 < \beta^2 < \beta_{\text{wave}}^2 = \frac{32-16\sqrt{3}}{5}\pi \simeq 0.86\pi$. Then, the stochastic damped nonlinear wave equation (1.2) is almost surely globally well-posed with respect to the renormalized Gibbs measure ρ_{wave} . Furthermore, the renormalized Gibbs measure ρ_{wave} is invariant under the resulting dynamics.*

More precisely, there exists a non-trivial stochastic process $(u, \partial_t u) \in C(\mathbb{R}_+; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$ for any $\varepsilon > 0$ such that, given any $T > 0$, the solution $(u_N, \partial_t u_N)$ to the truncated SdNLW (1.24) with the random initial data $(u_{0,N}^{\text{Gibbs}}, u_{1,N}^{\text{Gibbs}})$ distributed by the truncated Gibbs measure $\rho_{\text{wave},N}$ in (1.22) converges in probability to $(u, \partial_t u)$ in $C([0, T]; \mathcal{H}^{-\varepsilon}(\mathbb{T}^2))$. Furthermore, the law of $(u(t), \partial_t u(t))$ for any $t \in \mathbb{R}_+$ is given by the renormalized Gibbs measure ρ_{wave} .

Unlike Theorem 1.2 in the parabolic setting, Theorem 1.8 does not yield global well-posedness of SdNLW (1.2). Therefore, in order to prove Theorem 1.9, we need to employ Bourgain’s invariant measure argument [9, 10] to first prove almost sure global well-posedness by exploiting invariance of the truncated Gibbs measure $\rho_{\text{heat},N}$ for the truncated dynamics (1.24). Since such an argument is by now standard, we omit details. See, for example, [15, 59, 60, 65, 77] in the context of the (stochastic) nonlinear wave equations.

Remark 1.10. In [77], Sun and Tzvetkov studied the following (deterministic) dispersion-generalized nonlinear wave equation (NLW) on \mathbb{T}^d with the exponential nonlinearity:

$$\partial_t^2 u + (1 - \Delta)^\alpha u + e^u = 0 \tag{1.25}$$

and the associated Gibbs measure ρ_α . When $\alpha > \frac{d}{2}$, they proved almost sure global well-posedness of (1.25) with respect to the Gibbs measure ρ_α and invariance of ρ_α . We point out that, when $\alpha > \frac{d}{2}$, a solution u is a function and no normalization is required. As such, the analysis in [77] also applies to¹¹

$$\partial_t^2 u + (1 - \Delta)^\alpha u + e^{\beta u} = 0 \tag{1.26}$$

for any $\beta \in \mathbb{R} \setminus \{0\}$ and a precise value of β is irrelevant in this non-singular setting.

When $d = 2$, their result barely misses the $\alpha = 1$ case, corresponding to the wave equation, and the authors in [77] posed the $\alpha = 1$ case on \mathbb{T}^2 as an interesting and challenging open problem. By adapting the proofs of Theorems 1.8 and 1.9 to the deterministic NLW setting, our argument yields almost sure global well-posedness of (1.26) for $\alpha = 1$ and $0 < \beta^2 < \beta_{\text{wave}}^2$ with respect to the (renormalized) Gibbs measure ρ_1 ($= \rho_{\text{wave}}$ in (1.23)) and invariance of ρ_{wave} , thus answering the open question in an affirmative manner in this regime of β^2 .

¹¹ In the massless case: $\partial_t^2 u + (-\Delta)^\alpha u + e^{\beta u} = 0$, by scaling analysis, we can reduce the problem to the $\beta = 1$ case (on a dilated torus, where the analysis in [77] still applies).

1.2. *On the Gaussian multiplicative chaos.* In this subsection, we go over a renormalization procedure for our problems. In the following, we present a discussion in terms of the frequency truncation operator \mathbf{P}_N but exactly the same results hold for the smoothing operator \mathbf{Q}_N defined in (1.16). We begin by studying the following linear stochastic heat equation with a regularized noise:

$$\begin{cases} \partial_t \Psi_N^{\text{heat}} + \frac{1}{2}(1 - \Delta)\Psi_N^{\text{heat}} = \mathbf{P}_N \xi \\ \Psi_N^{\text{heat}}|_{t=0} = \mathbf{P}_N w_0, \end{cases}$$

where w_0 is the random distribution defined in (1.8), distributed according to the massive Gaussian free field μ_1 . Then, the truncated stochastic convolution Ψ_N^{heat} is given by

$$\Psi_N^{\text{heat}}(t) = P(t)\mathbf{P}_N w_0 + \int_0^t P(t - t')\mathbf{P}_N dW(t'), \tag{1.27}$$

where $P(t) = e^{\frac{t}{2}(\Delta-1)}$ denotes the linear heat operator defined by

$$P(t)f = e^{\frac{t}{2}(\Delta-1)}f = \sum_{n \in \mathbb{Z}^2} e^{-\frac{t}{2}(1+|n|^2)} \widehat{f}(n)e_n \tag{1.28}$$

and W denotes the cylindrical Wiener process on $L^2(\mathbb{T}^2)$ defined by

$$W(t) = \sum_{n \in \mathbb{Z}^2} B_n(t)e_n. \tag{1.29}$$

Here, $\{B_n\}_{n \in \mathbb{Z}^2}$ is a family of mutually independent complex-valued Brownian motions conditioned so that $B_{-n} = \overline{B_n}$, $n \in \mathbb{Z}^2$. By convention, we normalize B_n such that $\text{Var}(B_n(t)) = t$ and assume that $\{B_n\}_{n \in \mathbb{Z}^2}$ is independent from w_0 and w_1 in (1.8).

Given $N \in \mathbb{N}$, we have $\Psi_N^{\text{heat}} \in C(\mathbb{R}_+ \times \mathbb{T}^2)$. For each fixed $t \geq 0$ and $x \in \mathbb{T}^2$, it is easy to see that $\Psi_N^{\text{heat}}(t, x)$ is a mean-zero real-valued Gaussian random variable with variance (independent of $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$):

$$\begin{aligned} \sigma_N^{\text{heat}} &= \mathbb{E}[\Psi_N^{\text{heat}}(t, x)^2] = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2} \chi_N^2(n) \left(\frac{e^{-t\langle n \rangle^2}}{\langle n \rangle^2} + \int_0^t \left[e^{-\frac{1}{2}(t-t')\langle n \rangle^2} \right]^2 dt' \right) \\ &= \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2} \chi_N^2(n) \frac{1}{\langle n \rangle^2} \sim \log N \longrightarrow \infty, \end{aligned} \tag{1.30}$$

as $N \rightarrow \infty$. This essentially shows that $\{\Psi_N(t)\}_{N \in \mathbb{N}}$ is almost surely unbounded in $W^{0,p}(\mathbb{T}^2)$ for any $1 \leq p \leq \infty$.

In the case of the wave equation, we consider the following linear stochastic damped wave equation with a regularized noise:

$$\begin{cases} \partial_t^2 \Psi_N^{\text{wave}} + \partial_t \Psi_N^{\text{wave}} + (1 - \Delta)\Psi_N^{\text{wave}} = \sqrt{2}\mathbf{P}_N \xi, \\ (\Psi_N^{\text{wave}}, \partial_t \Psi_N^{\text{wave}})|_{t=0} = (\mathbf{P}_N w_0, \mathbf{P}_N w_1), \end{cases}$$

where w_0 and w_1 are as in (1.8). Then, the stochastic convolution Ψ_N^{wave} in this case is given by

$$\Psi_N^{\text{wave}}(t) = \partial_t \mathcal{D}(t)\mathbf{P}_N w_0 + \mathcal{D}(t)\mathbf{P}_N (w_0 + w_1) + \sqrt{2} \int_0^t \mathcal{D}(t - t')\mathbf{P}_N dW(t'), \tag{1.31}$$

where the linear damped wave operator $\mathcal{D}(t)$ is given by

$$\mathcal{D}(t) = e^{-\frac{t}{2}} \frac{\sin\left(t\sqrt{\frac{3}{4} - \Delta}\right)}{\sqrt{\frac{3}{4} - \Delta}}, \tag{1.32}$$

viewed as a Fourier multiplier operator:

$$\mathcal{D}(t)f = e^{-\frac{t}{2}} \sum_{n \in \mathbb{Z}^2} \frac{\sin\left(t\sqrt{\frac{3}{4} + |n|^2}\right)}{\sqrt{\frac{3}{4} + |n|^2}} \widehat{f}(n)e_n. \tag{1.33}$$

One can easily derive the propagator $\mathcal{D}(t)$ in (1.32) by writing the linear damped wave equation $\partial_t^2 u + \partial_t u + (1 - \Delta)u = 0$ on the Fourier side:

$$\partial_t^2 \widehat{u}(t, n) + \partial_t \widehat{u}(t, n) + \langle n \rangle^2 \widehat{u}(t, n) = 0$$

and solving it directly for each spatial frequency $n \in \mathbb{Z}^2$. Then, a standard variation-of-parameter argument yields the expression (1.31). By a direct computation using (1.31) and (1.33), we obtain, for any $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$,

$$\sigma_N^{\text{wave}} = \mathbb{E}[\Psi_N^{\text{wave}}(t, x)^2] = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2} \chi_N^2(n) \frac{1}{\langle n \rangle^2} \sim \log N \longrightarrow \infty, \tag{1.34}$$

as $N \rightarrow \infty$.

In the following, we set

$$\Psi_N = \Psi_N^{\text{heat}} \text{ or } \Psi_N^{\text{wave}} \quad \text{and} \quad \sigma_N = \sigma_N^{\text{heat}} = \sigma_N^{\text{wave}}.$$

Since we do not study the stochastic heat and wave equations at the same time, their meaning will be clear from the context.

By a standard argument, we then have the following regularity and convergence result for the (truncated) stochastic convolution. See, for example, [35, Proposition 2.1] in the context of the wave equation.

Lemma 1.11. *Given any $T, \varepsilon > 0$ and finite $p \geq 1$, $\{\Psi_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2)))$, converging to some limit Ψ in $L^p(\Omega; C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2)))$. Moreover, Ψ_N converges almost surely to the same limit $\Psi \in C([0, T]; W^{-\varepsilon, \infty}(\mathbb{T}^2))$.*

Clearly, the limiting stochastic convolution is given by formally taking $N \rightarrow \infty$ in (1.27) or (1.31). Namely, in the heat case, we have

$$\Psi(t) = \Psi^{\text{heat}}(t) = P(t)w_0 + \int_0^t P(t - t')dW(t'), \tag{1.35}$$

while in the wave case, it is given by

$$\Psi(t) = \Psi^{\text{wave}}(t) = \partial_t \mathcal{D}(t)w_0 + \mathcal{D}(t)(w_0 + w_1) + \sqrt{2} \int_0^t \mathcal{D}(t - t')dW(t'). \tag{1.36}$$

Next, we study the Gaussian multiplicative chaos formally given by

$$e^{\beta\Psi_N} = \sum_{k=0}^{\infty} \frac{\beta^k}{k!} \Psi_N^k(t).$$

Since $\Psi_N^k, k \geq 2$, does not have any nice limiting behavior as $N \rightarrow \infty$, we now introduce the Wick renormalization:

$$:\Psi_N^k(t, x): \stackrel{\text{def}}{=} H_k(\Psi_N(t, x); \sigma_N), \tag{1.37}$$

where H_k denotes the k th Hermite polynomial, defined through the generating function:

$$e^{tx - \frac{\sigma^2}{2}t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma). \tag{1.38}$$

From (1.37) and (1.38), the (renormalized) Gaussian multiplicative chaos is then given by

$$\begin{aligned} \Theta_N(t, x) &= :e^{\beta\Psi_N(t,x)}: \stackrel{\text{def}}{=} \sum_{k=0}^{\infty} \frac{\beta^k}{k!} :\Psi_N^k(t, x): \\ &= e^{-\frac{\beta^2}{2}\sigma_N} e^{\beta\Psi_N(t,x)}. \end{aligned} \tag{1.39}$$

We also set $C_N = C_N(\beta)$ by

$$C_N = e^{-\frac{\beta^2}{2}\sigma_N} \longrightarrow 0, \tag{1.40}$$

as $N \rightarrow \infty$.

The following proposition provides the regularity and convergence properties of the Gaussian multiplicative chaos Θ_N .

Proposition 1.12. (i) Given $0 < \beta^2 < 8\pi$, let $1 \leq p < \frac{8\pi}{\beta^2}$ and define $\alpha = \alpha(p)$ by

$$\frac{(p-1)\beta^2}{4\pi} < \alpha(p) < 2. \tag{1.41}$$

Then, given any $T > 0$, the sequence of stochastic processes Θ_N is uniformly bounded in

$$L^p(\Omega; L^p([0, T]; W^{-\alpha,p}(\mathbb{T}^2))).$$

(ii) Given $0 < \beta^2 < 4\pi$, let $1 < p < \frac{8\pi}{\beta^2}$ and $\alpha(p)$ as in (1.41). Then, given any $T > 0$, $\{\Theta_N\}_{N \in \mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega; L^p([0, T]; W^{-\alpha,p}(\mathbb{T}^2)))$ and hence converges to some limit Θ in the same class. In particular, Θ_N converges in probability to Θ in $L^p([0, T]; W^{-\alpha,p}(\mathbb{T}^2))$.

In the following, we write the limit Θ as

$$\Theta = :e^{\beta\Psi} := \lim_{N \rightarrow \infty} \Theta_N = \lim_{N \rightarrow \infty} C_N e^{\beta\Psi_N}. \tag{1.42}$$

We point out that by applying Fubini’s theorem, a proof of Proposition 1.12 reduces to analysis for fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$. Therefore, the proof is identical for $\Psi_N = \Psi_N^{\text{heat}}$ and Ψ_N^{wave} .

In [30], Garban established an analogous result on the Gaussian multiplicative chaos but in the context of the space-time Hölder regularity; see [30, Theorem 3.10]. See also [1, Theorem 6] for an analogous approach in the elliptic setting, working in the L^p -based Besov spaces but only for $1 < p \leq 2$.

In the case of a polynomial nonlinearity [35,36], the p th moment bound follows directly from the second moment estimate combined with the Wiener chaos estimate (see, for example, Lemma 2.5 in [36]), since the stochastic objects in [35,36] all belong to Wiener chaoses of finite order. However, the Gaussian multiplicative chaos Θ_N in (1.39) does *not* belong to any Wiener chaos of finite order. Therefore, we need to estimate all the higher moments by hand. The approach in [30] is based on Kahane’s convexity inequality [46]; see Lemma 3.4. In Sect. 3, we first compute higher even moments, using the Brascamp-Lieb inequality [8,13,52]. See Lemma 2.11 and Corollary 2.12. We believe that our approach based on the Brascamp-Lieb inequality is of independent interest. In order to compare this approach with Kahane’s, we also provide a proof of Proposition 1.12 based on Kahane’s inequality. See Propositions 3.2 and 3.6 as well as Appendix B.

We conclude this subsection by briefly discussing a proof of Proposition 1.4.

Proof of Proposition 1.4. As mentioned above, the proof of Proposition 1.12 is based on reducing the problem for fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$. In particular, it follows from the proof of Proposition 1.12 presented in Sect. 3 that $\Theta_N(0)$ at time $t = 0$ converges to $\Theta(0)$ in $L^p(\Omega; W^{-\alpha,p}(\mathbb{T}^2))$. Then, by restricting to the (spatial) zeroth Fourier mode, we obtain convergence in probability (with respect to the Gaussian free field μ_1 in (1.9)) of the density

$$R_N = \exp\left(-\lambda C_N \int_{\mathbb{T}^2} e^{\beta \mathbf{Q}_N u} dx\right) = \exp\left(-2\pi\lambda \widehat{\Theta}_N(0, 0)\right) \tag{1.43}$$

to

$$R = \exp\left(-\lambda \int_{\mathbb{T}^2} :e^{\beta u} : dx\right) = \exp\left(-2\pi\lambda \widehat{\Theta}(0, 0)\right).$$

Moreover, by the positivity of Θ_N and λ , the density R_N in (1.43) is uniformly bounded by 1. Putting together, we conclude the $L^p(\mu_1)$ -convergence of the density R_N to R by a standard argument (see [79, Remark 3.8]). More precisely, the L^p -convergence of R_N follows from the uniform L^p -bound on R_N and the softer convergence in probability. \square

1.3. Outline of the proof. In the following, we briefly describe an outline of the proofs of Theorems 1.1, 1.2, 1.6, 1.8, and 1.9.

• **Parabolic case:** Given $v_0 \in L^\infty(\mathbb{T}^2)$, we consider the truncated SNLH (1.10). We proceed with the Da Prato-Debussche trick and write a solution u_N to (1.10) as

$$u_N = v_N + z + \Psi_N,$$

where $\Psi_N = \Psi_N^{\text{heat}}$ is the truncated stochastic convolution in (1.27) and z denotes the linear solution given by

$$z = P(t)v_0. \tag{1.44}$$

Then, the residual term v_N satisfies the following equation:

$$\begin{cases} \partial_t v_N + \frac{1}{2}(1 - \Delta)v_N + \frac{1}{2}\lambda\beta e^{\beta z} e^{\beta v_N} \Theta_N = 0 \\ v_N|_{t=0} = 0, \end{cases} \tag{1.45}$$

where $\Theta_N = :e^{\beta\Psi_N^{\text{heat}}}$ denotes the Gaussian multiplicative noise defined in (1.39).

When $0 < \beta^2 < \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$, we prove local well-posedness of (1.45) by a standard contraction argument. The key ingredients are Proposition 1.12 on the regularity of the Gaussian multiplicative chaos Θ_N and the positivity of the nonlinearity, in particular the positivity of Θ_N (see Lemma 2.14). In studying continuity in the noise, we can no longer exploit any positivity. For this part of the argument, we use a more robust energy method and combine it with the uniqueness of a solution to the limiting equation (see (1.46) below).

Theorem 1.1 follows once we prove the following local well-posedness result for (1.45).

Theorem 1.13. *Let $\lambda \neq 0$ and $0 < \beta^2 < \beta_{\text{heat}}^2 = \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$. Given any $v_0 \in L^\infty(\mathbb{T}^2)$, the Cauchy problem (1.45) is uniformly locally well-posed in the following sense; there exists $T_0 = T_0(\|v_0\|_{L^\infty}, \beta, \lambda) > 0$ such that given $0 < T \leq T_0$ and $N \in \mathbb{N}$, there exists a set $\Omega_N(T) \subset \Omega$ such that*

(i) *for any $\omega \in \Omega_N(T)$, there exists a unique solution v_N to (1.45) in the class:*

$$C([0, T]; W^{s,p}(\mathbb{T}^2)) \subset C([0, T]; L^\infty(\mathbb{T}^2))$$

for some appropriate $0 < s < 1$ and $p \geq 2$, satisfying $sp > 2$. (ii) *there exists a uniform estimate on the probability of the complement of $\Omega_N(T)$:*

$$P(\Omega_N(T)^c) \longrightarrow 0,$$

uniformly in $N \in \mathbb{N}$, as $T \rightarrow 0$,

Furthermore, there exist an almost surely positive stopping time $\tau = \tau(\|v_0\|_{L^\infty}, \beta)$ and a stochastic process $v \in C([0, T]; W^{s,p}(\mathbb{T}^2))$ such that, given any small $T > 0$, on the event $\{\tau \geq T\}$, the sequence $\{v_N\}_{N \in \mathbb{N}}$ converges in probability to v in $C([0, T]; W^{s,p}(\mathbb{T}^2))$.

The limit v satisfies the following equation:

$$\begin{cases} \partial_t v + \frac{1}{2}(1 - \Delta)v + \frac{1}{2}\lambda\beta e^{\beta z} e^{\beta v} \Theta = 0 \\ v|_{t=0} = 0, \end{cases} \tag{1.46}$$

where Θ is the limit of Θ_N constructed in Proposition 1.12. Then, $u = v + z + \Psi$ formally satisfies the equation (1.12).

Next, we discuss the $\lambda > 0$ case. In this case, the equation (1.45) enjoys a sign-definite structure. By writing (1.45) in the Duhamel formulation, we have

$$v_N(t) = -\frac{1}{2}\lambda\beta \int_0^t P(t-t')(e^{\beta z} e^{\beta v_N} \Theta_N)(t') dt'.$$

Since the kernel for $P(t) = e^{\frac{t}{2}(\Delta-1)}$ and the integrand $e^{\beta z} e^{\beta v_N} \Theta_N$ are both positive, we see that

$$\beta v_N \leq 0. \tag{1.47}$$

This observation shows that the nonlinearity $e^{\beta v_N}$ is in fact bounded, allowing us to rewrite (1.45) as

$$\begin{cases} \partial_t v_N + \frac{1}{2}(1 - \Delta)v_N + \frac{1}{2}\lambda\beta e^{\beta z} F(\beta v_N)\Theta_N = 0 \\ v_N|_{t=0} = 0, \end{cases} \tag{1.48}$$

where F is a smooth bounded function such that

$$F(x) = e^x \tag{1.49}$$

for $x \leq 0$ and $F|_{\mathbb{R}_+} \in C_c^\infty(\mathbb{R}_+; \mathbb{R}_+)$. In particular, F is Lipschitz. By making use of this particular structure and the positivity of the Gaussian multiplicative chaos Θ_N , we prove a stronger well-posedness result, from which Theorem 1.2 follows.

Theorem 1.14. *Let $\lambda > 0$ and $0 < \beta^2 < (\beta_{\text{heat}}^*)^2 = 4\pi$. Given any $v_0 \in L^\infty(\mathbb{T}^2)$, any $T > 0$, and any $N \in \mathbb{N}$, there exists a unique solution v_N to (1.45) in the energy space:*

$$\mathcal{Z}_T = C([0, T]; L^2(\mathbb{T}^2)) \cap L^2([0, T]; H^1(\mathbb{T}^2)) \tag{1.50}$$

almost surely such that v_N converges in probability to some limit v in the class \mathcal{Z}_T . Furthermore, v is the unique solution to the equation (1.46) in the class \mathcal{Z}_T .

For Theorem 1.14, a contraction argument does not suffice even for constructing solutions and thus we proceed with an energy method. Namely, we first establish a uniform (in N) a priori bound for a solution to (1.48). Then, by applying a compactness lemma (Lemma 2.16) and extracting a convergent subsequence, we prove existence of a solution. Uniqueness follows from an energy consideration for the difference of two solutions in the energy space \mathcal{Z}_T . As for continuity in the noise, in particular convergence of v_N to v , we lose the positivity of the stochastic term (i.e. $\Theta_N - \Theta$ is not positive). We thus first establish convergence in some weak norm and then combine this with strong convergence (up to a subsequence) via the compactness argument mentioned above and the uniqueness of the limit v as a solution to (1.46) in the energy space \mathcal{Z}_T .

• **Hyperbolic case:** Next, we discuss the stochastic damped nonlinear wave equation when $\lambda > 0$. Let $N \in \mathbb{N} \cup \{\infty\}$. Given $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2)$, let u_N be the solution to (1.20). Proceeding with the Da Prato-Debussche trick $u_N = v_N + z + \Psi_N^{\text{wave}}$, the residual term v_N satisfies the following equation:

$$\begin{cases} \partial_t^2 v_N + \partial_t v_N + (1 - \Delta)v_N + \lambda\beta e^{\beta z} e^{\beta v_N} \Theta_N = 0 \\ (v_N, \partial_t v_N)|_{t=0} = (0, 0), \end{cases} \tag{1.51}$$

where $\Theta_N =: e^{\beta\Psi_N^{\text{wave}}}$: for $N \in \mathbb{N}$, $\Theta_\infty = \Theta = \lim_{N \rightarrow \infty} \Theta_N$ constructed in Proposition 1.12, and z denotes the linear solution given by

$$z(t) = \partial_t \mathcal{D}(t)v_0 + \mathcal{D}(t)(v_0 + v_1), \tag{1.52}$$

satisfying the following linear equation:

$$\begin{cases} \partial_t^2 z + \partial_t z + (1 - \Delta)z = 0 \\ (z, \partial_t z)|_{t=0} = (v_0, v_1). \end{cases}$$

Since the smoothing property of the wave operator is weaker than that of the heat equation, there is no uniform (in N) L^∞ -control for v_N (which is crucial in bounding the nonlinearity $e^{\beta v_N}$) and thus we need to exploit a sign-definite structure as in SNLH (1.1) for $\lambda > 0$ discussed above. The main issue is the oscillatory nature of the kernel for $\mathcal{D}(t)$ defined in (1.32). In particular, unlike the case of the heat equation, there is no explicit sign-definite structure for (1.51).

In the following, we drop the subscript N for simplicity of notations. Write (1.51) in the Duhamel formulation:

$$v(t) = -\lambda\beta \int_0^t \mathcal{D}(t - t')(e^{\beta z} e^{\beta v} \Theta) dt',$$

where $\mathcal{D}(t)$ is as in (1.32). The main point is that while the kernel for $\mathcal{D}(t)$ is not sign-definite, it is essentially non-negative near the singularity. This motivates us to introduce a further decomposition of the unknown:

$$v = X + Y, \tag{1.53}$$

where (X, Y) solves the following system of equations:

$$\begin{aligned} X(t) &= -\lambda\beta \int_0^t e^{-\frac{(t-t')}{2}} S(t - t')(e^{\beta z} e^{\beta X} e^{\beta Y} \Theta)(t') dt', \\ Y(t) &= -\lambda\beta \int_0^t (\mathcal{D}(t - t') - e^{-\frac{(t-t')}{2}} S(t - t'))(e^{\beta z} e^{\beta X} e^{\beta Y} \Theta)(t') dt'. \end{aligned} \tag{1.54}$$

Here, $S(t)$ denotes the forward propagator for the standard wave equation: $\partial_t^2 u - \Delta u = 0$ with initial data $(u, \partial_t u)|_{t=0} = (0, u_1)$ given by

$$S(t) = \frac{\sin(t|\nabla|)}{|\nabla|}. \tag{1.55}$$

The key point is that, in view of the positivity of the kernel for $S(t)$ (see Lemma 2.5 below), there is a sign-definite structure for the X -equation when $\lambda > 0$ and we have

$$\beta X \leq 0.$$

With F as in (1.49), we can then write (1.54) as

$$\begin{aligned} X(t) &= -\lambda\beta \int_0^t e^{-\frac{(t-t')}{2}} S(t - t')(e^{\beta z} F(\beta X) e^{\beta Y} \Theta)(t') dt', \\ Y(t) &= -\lambda\beta \int_0^t (\mathcal{D}(t - t') - e^{-\frac{(t-t')}{2}} S(t - t'))(e^{\beta z} F(\beta X) e^{\beta Y} \Theta)(t') dt'. \end{aligned} \tag{1.56}$$

Thus, the nonlinear contribution $F(\beta X)$ from X is bounded thanks to the sign-definite structure. This is crucial since, as we see below, X does not have sufficient regularity to be in $L^\infty(\mathbb{T}^2)$. While X and Y both enjoy the Strichartz estimates, the difference of the propagators in the Y -equation provides an extra smoothing, gaining two derivatives (see Lemma 2.6 below). This smoothing of two degrees allows us to place Y in $C([0, T]; H^s(\mathbb{T}^2))$ for some $s > 1$ and to make sense of $e^{\beta Y}$. In Sect. 6, we prove the following theorem.

Theorem 1.15. *Let $\lambda > 0$, $0 < \beta^2 < \beta_{\text{wave}}^2 = \frac{32-16\sqrt{3}}{5}\pi \simeq 0.86\pi$, and $s > 1$. Suppose that a deterministic positive distribution Θ satisfies the regularity property stated in Proposition 1.12. Namely, $\Theta \in L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2))$ for $1 \leq p < \frac{8\pi}{\beta^2}$, where $\alpha = \alpha(p)$ is as in (1.41). Then, given $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2)$, there exist $T = T(\|(v_0, v_1)\|_{\mathcal{H}^s}, \|\Theta\|_{L^p_T W_x^{-\alpha,p}}) > 0$ and a unique solution (X, Y) to (1.56) in the class:*

$$\mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2} \subset C([0, T]; H^{s_1}(\mathbb{T}^2)) \times C([0, T]; H^{s_2}(\mathbb{T}^2))$$

for some $0 < s_1 < 1 < s_2$ and some (α, p) satisfying (1.41). Moreover, the solution (X, Y) depends continuously on

$$(v_0, v_1, \Theta) \in \mathcal{H}^s(\mathbb{T}^2) \times L^p([0, T]; W^{-\alpha+\varepsilon,p}(\mathbb{T}^2))$$

for sufficiently small $\varepsilon > 0$ (such that the pair $(\alpha + \varepsilon, p)$ satisfies the condition (1.41)).

Here, the spaces $\mathcal{X}_T^{s_1}$ and $\mathcal{Y}_T^{s_2}$ are defined by

$$\mathcal{X}_T^{s_1} = C([0, T]; H^{s_1}(\mathbb{T}^2)) \cap C^1([0, T]; H^{s_1-1}(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)), \tag{1.57}$$

$$\mathcal{Y}_T^{s_2} = C([0, T]; H^{s_2}(\mathbb{T}^2)) \cap C^1([0, T]; H^{s_2-1}(\mathbb{T}^2)), \tag{1.58}$$

for some suitable s_1 -admissible pair (q, r) . See Sect. 2.4. Note that Theorem 1.8 directly follows from Theorem 1.15. As for Theorem 1.9, a small modification of the proof of Theorem 1.15 yields the result. See Sect. 6 for details.

We point out that this reduction of (1.51) to the system (1.56), involving the decomposition of the unknown (in the Da Prato-Debussche argument) into a less regular but structured part and a smoother part, has some similarity to the paracontrolled approach to the dynamical Φ_3^4 -model.¹² Once we arrive at the system (1.56), we can apply the Strichartz estimates for the X -equation (Lemma 2.8) and the extra smoothing for the Y -equation (Lemma 2.6) along with the positivity of Θ (Lemma 2.14) to construct a solution (X, Y) by a standard contraction argument.

We conclude this introduction by stating some remarks and comments.

Remark 1.16. In [30], Garban studied the closely related massless stochastic nonlinear heat equation with an exponential nonlinearity on $(\mathbb{R}/\mathbb{Z})^2$:

$$\partial_t X - \frac{1}{4\pi} \Delta X + e^{\gamma X} = \tilde{\xi}, \tag{1.59}$$

where $\tilde{\xi}$ is a space-time white noise on $\mathbb{R}_+ \times (\mathbb{R}/\mathbb{Z})^2$. By setting

$$u(t, x) = \frac{1}{\sqrt{2\pi}} X\left(\frac{t}{2\pi}, \frac{x}{2\pi}\right) \quad \text{and} \quad \xi(t, x) = \frac{1}{(2\pi)^{\frac{3}{2}}} \tilde{\xi}\left(\frac{t}{2\pi}, \frac{x}{2\pi}\right),$$

we see that ξ is a space-time white noise on $\mathbb{R}_+ \times \mathbb{T}^2$ and that u satisfies the massless equation (1.6) with coupling constant

$$\beta = \sqrt{2\pi} \gamma.$$

This provides the conversion of the parameters γ in [30] and β in this paper.

¹² This is not to be confused with the Da Prato-Debussche trick or its higher order variants, where we decompose an unknown into a sum of a less regular but *explicitly known* (random) distribution and a smoother remainder. The point of the decomposition (1.53) is that both X and Y are unknown.

Remark 1.17. As mentioned before, the massive equation (1.1) (with $\lambda > 0$) arises as the stochastic quantization of the so-called Høegh-Krohn model [2, 43] in Euclidean quantum field theory, while the massless model (1.59) treated in [30] comes from the stochastic quantization of Liouville Conformal Field Theory (LCFT). In [66], with N. Tzvetkov, we extended the results of this paper on the stochastic nonlinear heat equation (1.6) on the torus \mathbb{T}^2 to the case of a massless stochastic nonlinear heat equation with “punctures” on any closed Riemannian surface, thus addressing properly the stochastic quantization of LCFT. See Theorem 1.4 in [66]. We point out that the corresponding problem in the hyperbolic case, i.e. the massless analogue of Theorem 1.15 for the “canonical” stochastic quantization of LCFT, was not treated in [66] and remains open. See also Remark 4.4 in [66].

Remark 1.18. (stochastic quantization of the $\cosh(\Phi)_2$ -model) The parabolic sinh-Gordon equation (1.13) formally preserves (a renormalized version of) the Gibbs measure of the form:

$$“d\rho_{\sinh} = Z^{-1} e^{-E_{\sinh}(u)}(u)du”,$$

associated with the energy:

$$E_{\sinh}(u) = \frac{1}{2} \int_{\mathbb{T}^2} |(\nabla)u|^2 dx + \int_{\mathbb{T}^2} \cosh(\beta u) dx.$$

In view of Proposition 1.12, we can proceed as in the proof of Proposition 1.4 and construct the renormalized Gibbs measure ρ_{\sinh} as a limit of the truncated Gibbs measure:

$$d\rho_{\sinh,N} = Z_N^{-1} \exp\left(-C_N \int_{\mathbb{T}^2} \cosh(\beta \mathbf{Q}_N u)\right) d\mu_1 \tag{1.60}$$

for $0 < \beta^2 < 4\pi$, where μ_1 is the massive Gaussian free field defined in (1.9) and C_N is the renormalization constant defined in (1.40) but with \mathbf{Q}_N instead of \mathbf{P}_N .

As in the case of the truncated SNLH (1.19), it is easy to see that the truncated Gibbs measure $\rho_{\sinh,N}$ in (1.60) is invariant under the following truncated sinh-Gordon equation:

$$\partial_t u_N + \frac{1}{2}(1 - \Delta)u_N + \frac{1}{2}\beta C_N \mathbf{Q}_N \sinh(\beta \mathbf{Q}_N u_N) = \xi. \tag{1.61}$$

Since the equation (1.61) does not enjoy any sign-definite structure, we can not apply (the proof of) Theorem 1.2. On the other hand, our proof of Theorem 1.1 is applicable to study (1.61), yielding local well-posedness of (1.61) for the range $0 < \beta^2 < \frac{8\pi}{3+2\sqrt{2}} \simeq 1.37\pi$. The key point is that, unlike [30, Theorem 1.11], this local well-posedness result yields convergence of the solution u_N of the truncated sinh-Gordon equation (1.61) to some limit u . Combining this local well-posedness result with Bourgain’s invariant measure argument [9, 10], we then obtain almost sure global well-posedness for the parabolic sinh-Gordon equation (1.13) and invariance of the renormalized Gibbs measure ρ_{\sinh} in the sense of Theorem 1.9.

Note that these results for the sinh-Gordon equation hold only in the parabolic setting since, when $\lambda < 0$, we do not know how to handle SdNLW (1.2) for any $\beta^2 > 0$.

Remark 1.19. (i) In Theorem 1.1, we treat initial data $u_{0,N}$ of the form (1.11). Due to the presence of the random part $\mathbf{P}_N w_0$ of the initial data, the variance σ_N^{heat} in (1.30) is time-independent, which results in the time-independent renormalization constant C_N in Theorem 1.1. It is, however, possible to treat deterministic initial data $u_{0,N} = v_0 \in L^\infty(\mathbb{T}^2)$. In this case, the associated truncated stochastic convolution $\tilde{\Psi}_N^{\text{heat}}$ is given by

$$\tilde{\Psi}_N^{\text{heat}}(t) = \int_0^t P(t-t') \mathbf{P}_N dW(t')$$

whose variance $\tilde{\sigma}_N^{\text{heat}}$ is now time-dependent and given by

$$\begin{aligned} \tilde{\sigma}_N^{\text{heat}}(t) &= \mathbb{E}[\tilde{\Psi}_N^{\text{heat}}(t, x)^2] = \frac{1}{4\pi^2} \sum_{n \in \mathbb{Z}^2} \chi_N^2(n) \frac{1 - e^{-t\langle n \rangle^2}}{\langle n \rangle^2} \\ &\approx -\frac{1}{2\pi} \log N^{-1} + \frac{1}{2\pi} \log(\sqrt{t} \vee N^{-1}) = \frac{1}{2\pi} \log(1 \vee \sqrt{t}N), \end{aligned} \tag{1.62}$$

where $A \vee B = \max(A, B)$. Here, the third step of (1.62) follows from Lemmas 2.2 and 2.3 below, by viewing $e^{t(\Delta-1)}$ as a regularization operator \mathbf{Q}_N with a regularizing parameter $t^{-\frac{1}{2}}$. By comparing (1.30) and (1.62), we see that $\tilde{\sigma}_N^{\text{heat}}(t) < \sigma_N^{\text{heat}}$, which allows us to establish an analogue of Proposition 1.12 in this case. As a result, we obtain an analogue of Theorem 1.1 but with a time-dependent renormalization constant. A similar comment applies to Theorem 1.8 in the wave case.

(ii) In [63], the authors (with P.Sosoe) studied the (undamped) stochastic hyperbolic sine-Gordon equation on \mathbb{T}^2 :

$$\partial_t^2 u + (1 - \Delta)u + \lambda \sin(\beta u) = \xi. \tag{1.63}$$

Due to the undamped structure, the variance of the truncated stochastic convolution $\Psi_N(t, x)$ behaves like $\sim t \log N$; compare this with (1.34) and (1.62). This time dependence allows us to make the variance as small as we like for any $\beta^2 > 0$ by taking $t > 0$ sufficiently small. As a result, we proved local well-posedness of the renormalized version of (1.63) for any $\beta^2 > 0$, with a (random) time of existence $T \lesssim \beta^{-2}$.

Similarly, if we consider the undamped stochastic nonlinear wave equation (SNLW) with an exponential nonlinearity:

$$\partial_t^2 u + (1 - \Delta)u + \lambda \beta e^{\beta u} = \sqrt{2}\xi, \tag{1.64}$$

then we see that Proposition 1.12 holds with the regularity α given by (1.41) with β^2 replaced by $\beta^2 T$. Thus, given any $\beta^2 > 0$, we can make $\alpha > 0$ arbitrarily small by taking $T \sim \beta^{-2} > 0$ small. See also Proposition 1.1 in [63]. This allows us to prove local well-posedness of SNLW (1.64) for any $\beta^2 > 0$.

Note that in view of (1.62), due to the exponential convergence to equilibrium for the linear stochastic heat equation, we have $\tilde{\sigma}_N^{\text{heat}}(t) \sim \sigma_N$ as soon as $t \gtrsim N^{-2+\theta}$ for some (small) $\theta > 0$, and thus the regularization effect as in the wave case can only be captured at time scales $t \ll N^{-2+\theta}$, which prevents us from building a local solution with deterministic initial data for arbitrary $\beta^2 > 0$ in the parabolic case.

Remark 1.20. As we mentioned above, in the recent work [1], Albeverio, De Vecchi and Gubinelli investigated the elliptic analogue of (1.1) and (1.2), namely the authors studied the following singular elliptic SPDE:

$$(1 - \Delta_{x,z})\phi + \alpha : e^{\alpha\phi} : = \xi \tag{1.65}$$

for $\phi : (x, z) \in \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \phi(x, z) \in \mathbb{R}$, where ξ is a space-time white noise on \mathbb{R}^4 . Here, due to scaling considerations, the coupling constant corresponds to $\alpha = 2\sqrt{\pi}\beta$. The authors of [1] then proved that (1.65) is well-posed in the regime $0 < \alpha^2 < \alpha_{\max}^2 = 4(8 - 4\sqrt{3})\pi \cdot (4\pi)$; see [1, Theorem 25 and Proposition 36]. In particular, note that $\beta_{\text{heat}}^* = \sqrt{4\pi} < \frac{\alpha_{\max}}{2\sqrt{\pi}} < \sqrt{8\pi}$. Their proof also relies on the Da Prato-Debussche trick, writing ϕ as $\phi = (1 - \Delta)^{-1}\xi + \bar{\phi}$ and solving the corresponding elliptic PDE for the nonlinear component $\bar{\phi}$. One of the benefits of the elliptic setting is that, due to the dimension being $d = 4$, the L^2 -regime corresponds to $0 < \alpha^2 < 8\pi \cdot (4\pi)$, namely to the full sub-critical regime $0 < \beta^2 < 8\pi$ for the reduced coupling constant $\beta = \frac{\alpha}{2\sqrt{\pi}}$. This in particular yields an analogue of Proposition 1.12 for the (elliptic) Gaussian multiplicative chaos $: e^{\alpha(1-\Delta)^{-1}\xi} :$ in the entire range $0 < \alpha^2 < 8\pi \cdot (4\pi)$ for which the construction of the $\exp(\Phi)_2$ -measure holds, by just working in L^2 -based Sobolev spaces. See [1, Lemma 22]. Note that the same approach here only gives the convergence of Θ_N for $0 < \beta^2 < 4\pi$. The well-posedness of the elliptic SPDE (1.65) then follows from an argument similar as that in Sect. 5 adapted to the elliptic setting. Heuristically speaking, this should provide well-posedness in the whole range $0 < \alpha^2 < 8\pi \cdot (4\pi)$. However, there seems to be an issue similar to that discussed after (6.9). Namely, $\bar{\phi}$ does not have sufficient regularity to use an analogue of the condition (i) in Lemma 2.14 for bounding the product of a distribution and a measure, which instead forces the use of an analogue of condition (ii) in Lemma 2.14. This in turn restricts the range of admissible $\alpha^2 > 0$.

Remark 1.21. (i) In [44], Hoshino, Kawabi, and Kusuoka studied SNLH (1.1) with $\lambda = 1$ and independently established Theorem 1.2 and Theorem 1.6. While the analytical part of the argument is analogous,¹³ the approaches for studying the Gaussian multiplicative chaos Θ_N ([44, Theorem 2.4] and Proposition 1.12 above) are quite different. The proof in [44] is based on the Fourier side approach as in [35,56], establishing only the second moment bound. On the other hand, our argument is based on the physical side approach as in our previous work [63,64] on the hyperbolic sine-Gordon model. By employing the Brascamp-Lieb inequality (and Kahane’s convexity inequality), we also obtain higher moment bounds on the Gaussian multiplicative chaos, which is a crucial ingredient to prove Theorem 1.1 for SNLH (1.1) with general $\lambda \in \mathbb{R} \setminus \{0\}$ and Theorem 1.8 on SdNLW (1.2).

After the submission of this paper, the same authors proved well-posedness and invariance of the Gibbs measure for the parabolic SPDE (1.1) in the full “ L^1 ” regime $0 < \beta^2 < 8\pi$; see [45]. This relies on arguments similar to those presented in Sect. 5 but working in L^p -based spaces with $1 < p < 2$ instead of the L^2 -based Sobolev spaces used in the proof of Theorem 1.6. In particular, this requires extending the convergence part of Proposition 1.12 to the case $1 < p < 2$.

¹³ The sign-definite structure of the equation in the defocusing case also plays an important role in [44]. See, for example, the proof of Lemma 3.10 in [44].

(ii) In a recent preprint [73], the second author studied the fractional nonlinear Schrödinger equation with an exponential nonlinearity on a d -dimensional compact Riemannian manifold:

$$i \partial_t u + (-\Delta)^{\frac{\alpha}{2}} + \lambda \beta e^{\beta |u|^2} u = 0$$

with the dispersion parameter $\alpha > d$. In the defocusing case ($\lambda > 0$), under some assumption, the author proved almost sure global well-posedness and invariance of the associated Gibbs measure. See [73] for precise statements. In the focusing case ($\lambda < 0$), it was shown that the Gibbs measure is not normalizable for any $\beta > 0$. See also Remark 1.5. Our understanding of the Schrödinger problem, however, is far from being satisfactory at this point and it is of interest to investigate further issues in this direction.

This paper is organized as follows. In Sect. 2, we introduce notations and state various tools from deterministic analysis. In Sect. 3, we study the regularity and convergence properties of the Gaussian multiplicative chaos (Proposition 1.12). In Sect. 4, we prove local well-posedness of SNLH (1.1) for general $\lambda \in \mathbb{R} \setminus \{0\}$ (Theorem 1.1). In Sect 5, we discuss the $\lambda > 0$ case for SNLH (1.1) and present proofs of Theorems 1.2 and 1.6. Section 6 is devoted to the study of SdNLW (1.2). In Appendix A, we present a simple contraction argument to prove local well-posedness of SNLH (1.46) for any $\lambda \in \mathbb{R} \setminus \{0\}$, in the range $0 < \beta^2 < \frac{4}{3}\pi \simeq 1.33\pi$ without using the positivity of the Gaussian multiplicative chaos. Lastly, in Appendix B, we present a proof of Lemma 3.5, which is crucial in establishing moment bounds for the Gaussian multiplicative chaos.

2. Deterministic Toolbox

In this section, we introduce some notations and go over preliminaries from deterministic analysis. In Sects. 2.2, 2.3, and 2.4, we recall key properties of the kernels of elliptic, heat, and wave equations. We also state the Schauder estimate (Lemma 2.4) and the Strichartz estimates (Lemma 2.8). In Sect. 2.5, we state other useful lemmas from harmonic and functional analysis.

2.1. *Notations.* We first introduce some notations. We set

$$e_n(x) \stackrel{\text{def}}{=} \frac{1}{2\pi} e^{in \cdot x}, \quad n \in \mathbb{Z}^2, \tag{2.1}$$

for the orthonormal Fourier basis in $L^2(\mathbb{T}^2)$. Given $s \in \mathbb{R}$, we define the Sobolev space $H^s(\mathbb{T}^2)$ by the norm:

$$\|f\|_{H^s(\mathbb{T}^2)} = \|\langle n \rangle^s \widehat{f}(n)\|_{\ell^2(\mathbb{Z}^2)},$$

where $\widehat{f}(n)$ is the Fourier coefficient of f and $\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}$. We also set

$$\mathcal{H}^s(\mathbb{T}^2) \stackrel{\text{def}}{=} H^s(\mathbb{T}^2) \times H^{s-1}(\mathbb{T}^2).$$

Given $s \in \mathbb{R}$ and $p \geq 1$, we define the L^p -based Sobolev space (Bessel potential space) $W^{s,p}(\mathbb{T}^2)$ by the norm:

$$\|f\|_{W^{s,p}} = \|\langle \nabla \rangle^s f\|_{L^p} = \|\mathcal{F}^{-1}(\langle n \rangle^s \widehat{f}(n))\|_{L^p}.$$

When $p = 2$, we have $H^s(\mathbb{T}^2) = W^{s,2}(\mathbb{T}^2)$. When we work with space-time function spaces, we use short-hand notations such as $C_T H_x^s = C([0, T]; H^s(\mathbb{T}^2))$.

For $A, B > 0$, we use $A \lesssim B$ to mean that there exists $C > 0$ such that $A \leq CB$. By $A \sim B$, we mean that $A \lesssim B$ and $B \lesssim A$. We also use a subscript to denote dependence on an external parameter; for example, $A \lesssim_\alpha B$ means $A \leq C(\alpha)B$, where the constant $C(\alpha) > 0$ depends on a parameter α . Given two functions f and g on \mathbb{T}^2 , we write

$$f \approx g$$

if there exist some constants $c_1, c_2 \in \mathbb{R}$ such that $f(x) + c_1 \leq g(x) \leq f(x) + c_2$ for any $x \in \mathbb{T}^2 \setminus \{0\} \cong [-\pi, \pi)^2 \setminus \{0\}$. Given $A, B \geq 0$, we also set $A \vee B = \max(A, B)$ and $A \wedge B = \min(A, B)$.

Given a random variable X , we use $\text{Law}(X)$ to denote its distribution.

2.2. Bessel potential and Green’s function. In this subsection, we recall several facts about the Bessel potentials and the Green’s function for $(1 - \Delta)$ on \mathbb{T}^2 . See also Section 2 in [63].

For $\alpha > 0$, the Bessel potential of order α on \mathbb{T}^d is given by $(\nabla)^{-\alpha} = (1 - \Delta)^{-\frac{\alpha}{2}}$ viewed as a Fourier multiplier operator. Its convolution kernel is given by

$$J_\alpha(x) \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^d} \frac{\chi_N(n)}{\langle n \rangle^\alpha} e_n(x), \tag{2.2}$$

where the limit is interpreted in the sense of distributions on \mathbb{T}^d . We recall from [63, Lemma 2.2] the following local description of these kernels.

Lemma 2.1. *For any $0 < \alpha < d$, the distribution J_α agrees with an integrable function, which is smooth away from the origin. Furthermore, there exist a constant $c_{\alpha,d} > 0$ and a smooth function R on \mathbb{T}^d such that*

$$J_\alpha(x) = c_{\alpha,d} |x|^{\alpha-d} + R(x)$$

for all $x \in \mathbb{T}^d \setminus \{0\} \cong [-\pi, \pi)^d \setminus \{0\}$.

An important remark is that the coefficient $c_{\alpha,d}$ is positive; see (4.2) in [5]. This in particular means that the singular part of the Bessel potential J_α is positive. We will use this remark in Lemma 2.14 below to establish a refined product estimate involving positive distributions.

In the following, we focus on $d = 2$. The borderline case $\alpha = d = 2$ corresponds to the Green’s function G for $1 - \Delta$. On \mathbb{T}^2 , G is given by

$$G \stackrel{\text{def}}{=} (1 - \Delta)^{-1} \delta_0 = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{1}{\langle n \rangle^2} e_n. \tag{2.3}$$

It is well known that G is an integrable function, smooth away from the origin, and that it satisfies the asymptotics

$$G(x) = -\frac{1}{2\pi} \log |x| + R(x), \quad x \in \mathbb{T}^2 \setminus \{0\}, \tag{2.4}$$

for some smooth function R on \mathbb{T}^2 . See (2.5) in [63].

We also recall the following description of the truncated Green’s function $\mathbf{P}_N G$, where \mathbf{P}_N is the smooth frequency projector with the symbol χ_N in (1.7). See Lemma 2.3 and Remark 2.4 in [63].

Lemma 2.2. *Let $N_2 \geq N_1 \geq 1$. Then, we have*

$$\mathbf{P}_{N_1} \mathbf{P}_{N_2} G(x) \approx -\frac{1}{2\pi} \log(|x| + N_1^{-1})$$

for any $x \in \mathbb{T}^2 \setminus \{0\}$. Similarly, we have

$$|\mathbf{P}_{N_j}^2 G(x) - \mathbf{P}_{N_1} \mathbf{P}_{N_2} G(x)| \lesssim (1 \vee -\log(|x| + N_j^{-1})) \wedge (N_1^{-1} |x|^{-1})$$

for $j = 1, 2$ and any $x \in \mathbb{T}^2 \setminus \{0\}$.

In establishing invariance of the Gibbs measures (Theorems 1.6 and 1.9), we need to consider the truncated dynamics (1.19) and (1.24) with the truncated nonlinearity. In order to preserve the sign-definite structure, it is crucial that we use the smoothing operator \mathbf{Q}_N defined in (1.16) with a non-negative kernel. In particular, we need to construct the Gaussian multiplicative chaos Θ_N with the smoothing operator \mathbf{Q}_N in place of \mathbf{P}_N . For this purpose, we state an analogue of Lemma 2.2 for the truncation of the Green’s function by \mathbf{Q}_N .

Lemma 2.3. *Let $N_2 \geq N_1 \geq 1$. Then we have*

$$\mathbf{Q}_{N_1} \mathbf{Q}_{N_2} G(x) \approx -\frac{1}{2\pi} \log(|x| + N_1^{-1}) \tag{2.5}$$

for any $x \in \mathbb{T}^2 \setminus \{0\}$. Similarly, we have

$$|\mathbf{Q}_{N_j}^2 G(x) - \mathbf{Q}_{N_1} \mathbf{Q}_{N_2} G(x)| \lesssim (1 \vee -\log(|x| + N_j^{-1})) \wedge (N_1^{-1} |x|^{-1})$$

for $j = 1, 2$ and any $x \in \mathbb{T}^2 \setminus \{0\}$.

Proof. We mainly follow the proof of Lemma 2.3 in [63]. We only show (2.5) for $N_1 = N_2 = N$, since the other claims follow from a straightforward modification. Fix $x \in \mathbb{T}^2 \setminus \{0\} \cong [-\pi, \pi)^2 \setminus \{0\}$.

• **Case 1:** We first treat the case $|x| \lesssim N^{-1}$. Since $\rho \in C_c^\infty(\mathbb{R}^2)$, we have

$$|\partial_\xi^k \widehat{\rho}_N(\xi)| \lesssim N^{-|k|} \langle N^{-1} \xi \rangle^{-\ell} \tag{2.6}$$

for any $k \in (\mathbb{Z}_{\geq 0})^2$, $\ell \in \mathbb{N}$, and $\xi \in \mathbb{R}^2$. Then, by (2.3), the mean value theorem, and (2.6) with $|k| = 0$ and $\ell = 2$, we have

$$\begin{aligned} |\mathbf{Q}_N^2 G(x) - \mathbf{Q}_N^2 G(0)| &= 2\pi \left| \sum_{n \in \mathbb{Z}^2} \frac{\widehat{\rho}_N(n)^2}{\langle n \rangle^2} (e_n(x) - e_n(0)) \right| \\ &\lesssim \sum_{n \in \mathbb{Z}^2} \frac{\widehat{\rho}_N(n)^2}{\langle n \rangle} |x| \lesssim \sum_{|n| \leq N} \langle n \rangle^{-1} |x| + \sum_{|n| \geq N} N^2 |n|^{-3} |x| \\ &\lesssim N|x| \lesssim 1. \end{aligned} \tag{2.7}$$

Similarly, by (2.3), the mean value theorem with $\widehat{\rho}_N(0) = \frac{1}{2\pi}$, and (2.6) with $\ell = 1$, we have

$$\begin{aligned} \left| \mathbf{Q}_N^2 G(0) - \frac{1}{4\pi^2} \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \right| &\lesssim \left| \sum_{|n| \leq N} \frac{4\pi^2 \widehat{\rho}_N(n)^2 - 1}{\langle n \rangle^2} \right| + C \sum_{|n| \geq N} \frac{N}{\langle n \rangle^2 |n|} \\ &\lesssim \sum_{|n| \leq N} \frac{|n|}{N \langle n \rangle^2} + 1 \\ &\lesssim 1. \end{aligned} \tag{2.8}$$

Hence, from (2.7) and (2.8), we conclude that

$$\mathbf{Q}_N^2 G(x) \approx \frac{1}{4\pi^2} \sum_{|n| \leq N} \frac{1}{\langle n \rangle^2} \approx \frac{1}{2\pi} \log N \approx -\frac{1}{2\pi} \log(|x| + N^{-1}),$$

where we used Lemma 3.2 in [41] at the second step.

• **Case 2:** Next, we consider the case $|x| \gg N^{-1}$. Since G is integrable and ρ_N is non-negative and integrates to 1, we have

$$\begin{aligned} |\mathbf{Q}_N^2 G(x) - G(x)| &= \left| \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \rho_N(x-y) \rho_N(y-z) (G(z) - G(x)) dz dy \right| \\ &\lesssim \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \rho_N(x-y) \rho_N(y-z) \left| \log \left(\frac{|z|}{|x|} \right) \right| dz dy + 1, \end{aligned} \tag{2.9}$$

where, at the second step, we used (2.4) and the fact that R in (2.4) is smooth. Since ρ_N is supported in a ball of radius $O(N^{-1})$ centered at 0, we have $|x - z| \lesssim |x - y| + |y - z| \lesssim N^{-1}$ in the above integrals, which implies that $|x| \sim |z|$ under the assumption $|x| \gg N^{-1}$. Hence, the log term in (2.9) is bounded and we obtain

$$|\mathbf{Q}_N^2 G(x) - G(x)| \lesssim \int_{\mathbb{T}^2} \int_{\mathbb{T}^2} \rho_N(x-y) \rho_N(y-z) dz dy + 1 \sim 1. \tag{2.10}$$

Therefore, from (2.4) and (2.10), we have

$$\mathbf{Q}_N^2 G(x) \approx G(x) \approx -\frac{1}{2\pi} \log |x| \approx -\frac{1}{2\pi} \log(|x| + N^{-1}).$$

This concludes the proof of Lemma 2.3. □

2.3. On the heat kernel and the Schauder estimate. In this subsection, we summarize the properties of the linear heat propagator $P(t)$ defined in (1.28). We denote the kernel of $P(t)$ by

$$P_t \stackrel{\text{def}}{=} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} e^{-\frac{t}{2} \langle n \rangle^2} e_n.$$

Then, we have the following lemma by passing the corresponding result on \mathbb{R}^2 to the periodic torus \mathbb{T}^2 via the Poisson summation formula (see [33, Theorem 3.2.8]). See also (2.1) in [63].

Lemma 2.4. *Let $t > 0$. (i) P_t is a positive smooth function. (ii) Let $\alpha \geq 0$ and $1 \leq p \leq q \leq \infty$. Then, we have*

$$\|P(t)f\|_{L^q(\mathbb{T}^2)} \lesssim t^{-\frac{\alpha}{2} - (\frac{1}{p} - \frac{1}{q})} \|(\nabla)^{-\alpha} f\|_{L^p(\mathbb{T}^2)} \tag{2.11}$$

for any $f \in L^p(\mathbb{T}^2)$.

Proof. By the Poisson summation formula, and the positivity of the heat kernel on \mathbb{R}^2 , we have

$$P_t = \frac{1}{2\pi} e^{-\frac{t}{2}} \sum_{n \in \mathbb{Z}^2} e^{-\frac{t}{2}|n|^2} e_n = \frac{1}{2\pi} e^{-\frac{t}{2}} \sum_{n \in \mathbb{Z}^2} \mathcal{F}_{\mathbb{R}^2}^{-1}(e^{-\frac{t}{2}|\cdot|^2})(x + 2\pi n) > 0,$$

where $\mathcal{F}_{\mathbb{R}^2}^{-1}$ denotes the inverse Fourier transform on \mathbb{R}^2 . This proves (i).

The Schauder estimate on \mathbb{R}^2 follows from Young’s inequality and estimating the kernel on \mathbb{R}^2 in some Sobolev norm. As for the Schauder estimate (2.11) on \mathbb{T}^2 , we apply Young’s inequality and then use the Poisson summation formula to pass an estimate on (fractional derivatives of) the heat kernel on \mathbb{T}^2 to that in a weighted Lebesgue space on \mathbb{R}^2 . This proves (ii). \square

2.4. On the kernel of the wave operator and the Strichartz estimates. Next, we turn our attention to the linear operators for the (damped) wave equations. Let $S(t)$ be the forward propagator for the standard wave equation defined in (1.55). We denote its kernel by S_t , which can be written as the following distribution:

$$S_t \stackrel{\text{def}}{=} \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{\sin(t|n|)}{|n|} e_n,$$

where we set $\frac{\sin(t|0|)}{|0|} = t$ by convention.

We say that a distribution T is positive if its evaluation $T(\varphi)$ at any non-negative test function φ is non-negative. We have the following positivity result for S_t .

Lemma 2.5. *For any $t \geq 0$, the distributional kernel S_t on the two-dimensional torus \mathbb{T}^2 is positive.*

Proof. As a distribution, we have

$$S_t = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^2} \frac{\sin(t|n|)}{|n|} e_n = \lim_{N \rightarrow \infty} \sum_{n \in \mathbb{Z}^2} \widehat{\rho}_N(n) \frac{\sin(t|n|)}{|n|} e_n,$$

where ρ_N is as in (1.16). In particular, we can use the Poisson summation formula to write

$$\begin{aligned} S_t(x) &= \lim_{N \rightarrow \infty} \frac{1}{2\pi} \sum_{m \in \mathbb{Z}^2} \int_{\mathbb{R}^2} \widehat{\rho}_N(\xi) \frac{\sin(t|\xi|)}{|\xi|} e^{i(x+2\pi m)\cdot\xi} d\xi \\ &= \lim_{N \rightarrow \infty} \sum_{m \in \mathbb{Z}^2} \frac{\sin(t|\nabla|)}{|\nabla|} \rho_N(x + 2\pi m). \end{aligned} \tag{2.12}$$

Let u_N be the solution to the following linear wave equation on \mathbb{R}^2 :

$$\begin{cases} \partial_t^2 u_N - \Delta u_N = 0, \\ (u_N, \partial_t u_N)|_{t=0} = (0, \rho_N). \end{cases} \tag{2.13}$$

It is well known (see, for example, (27) on p.74 in [28]) that in the two-dimensional case, the solution u_N to (2.13) is given by the following Poisson’s formula:

$$u_N(t, x) = \frac{1}{2\pi} \int_{B(x,t)} \frac{\rho_N(y)}{\sqrt{t^2 - |x - y|^2}} dy \geq 0$$

for any $x \in \mathbb{R}^2$ and $t \geq 0$, where $B(x, t) \subset \mathbb{R}^2$ is the ball of radius t centered at x in \mathbb{R}^2 . Hence, from (2.12), we conclude that

$$S_t(x) = \lim_{N \rightarrow \infty} \sum_{m \in \mathbb{Z}^2} u_N(x + 2\pi m) \geq 0. \tag{2.14}$$

We point out that the sum in (2.14) (for fixed $N \in \mathbb{N}$) is convergent thanks to the compact support of ρ_N and the finite speed of propagation for the wave equation. \square

The next lemma shows that the operators $\mathcal{D}(t)$ in (1.32) and $e^{-\frac{t}{2}} S(t)$ in (1.55) are close in the sense that their difference provides an extra smoothing property. This extra smoothing plays a crucial role for estimating Y in (1.56).

Lemma 2.6. *Let $t \geq 0$ and $s \in \mathbb{R}$. (i) The operator $\mathcal{D}(t) - e^{-\frac{t}{2}} S(t)$ is bounded from $H^s(\mathbb{T}^2)$ to $H^{s+2}(\mathbb{T}^2)$. (ii) The operator $\partial_t(\mathcal{D}(t) - e^{-\frac{t}{2}} S(t))$ is bounded from $H^s(\mathbb{T}^2)$ to $H^{s+1}(\mathbb{T}^2)$.*

Proof. (i) It suffices to show that the symbol of $\langle \nabla \rangle^2 (e^{\frac{t}{2}} \mathcal{D}(t) - S(t))$ is bounded. Since $\langle n \rangle \sim \sqrt{\frac{3}{4} + |n|^2}$ for any $n \in \mathbb{Z}^2$, it suffices to bound, for $n \neq 0$,

$$\begin{aligned} & \left(\frac{3}{4} + |n|^2\right) \left(\frac{\sin\left(t\sqrt{\frac{3}{4} + |n|^2}\right)}{\sqrt{\frac{3}{4} + |n|^2}} - \frac{\sin(t|n|)}{|n|} \right) \\ &= \sqrt{\frac{3}{4} + |n|^2} \left(\sin\left(t\sqrt{\frac{3}{4} + |n|^2}\right) - \sin(t|n|) \right) \\ & \quad + \left(\frac{3}{4} + |n|^2\right) \sin(t|n|) \left(\frac{1}{\sqrt{\frac{3}{4} + |n|^2}} - \frac{1}{|n|} \right) \\ &=: \text{I} + \text{II}. \end{aligned}$$

By the mean value theorem, we have

$$|\text{I}| \lesssim \langle n \rangle \left| \sqrt{\frac{3}{4} + |n|^2} - |n| \right| \lesssim \langle n \rangle \frac{1}{\sqrt{\frac{3}{4} + |n|^2} + |n|} \lesssim 1.$$

Similarly, we can bound the second term by

$$|\text{II}| \lesssim \langle n \rangle^2 \frac{1}{|n| \sqrt{\frac{3}{4} + |n|^2} (|n| + \sqrt{\frac{3}{4} + |n|^2})} \lesssim 1.$$

This proves (i).

(ii) In this case, we show the boundedness of the symbol for

$$\begin{aligned} & \langle \nabla \rangle \partial_t (\mathcal{D}(t) - e^{-\frac{t}{2}} S(t)) \\ &= -\frac{1}{2} \langle \nabla \rangle (\mathcal{D}(t) - e^{-\frac{t}{2}} S(t)) + e^{-\frac{t}{2}} \langle \nabla \rangle \left(\cos \left(t \sqrt{\frac{3}{4} - \Delta} \right) - \cos(t|\nabla|) \right) \\ &=: \text{III} + \text{IV}. \end{aligned}$$

The symbol of III is clearly bounded by the argument above. As for the symbol of IV, it follows from the mean value theorem that

$$\langle n \rangle \left[\cos \left(t \sqrt{\frac{3}{4} + |n|^2} \right) - \cos(t|n|) \right] \lesssim \langle n \rangle \left(\sqrt{\frac{3}{4} + |n|^2} - |n| \right) \lesssim 1.$$

This completes the proof of Lemma 2.6. □

Next, we state the Strichartz estimates for the linear wave equation.

Definition 2.7. Given $0 < s < 1$, we say that a pair (q, r) of exponents (and a pair (\tilde{q}, \tilde{r}) , respectively) is *s-admissible* (and *dual s-admissible*, respectively), if $1 \leq \tilde{q} \leq 2 \leq q \leq \infty$ and $1 < \tilde{r} \leq 2 \leq r < \infty$ and if they satisfy the following scaling and admissibility conditions:

$$\frac{1}{q} + \frac{2}{r} = 1 - s = \frac{1}{\tilde{q}} + \frac{2}{\tilde{r}} - 2, \quad \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, \quad \text{and} \quad \frac{2}{\tilde{q}} + \frac{1}{\tilde{r}} \geq \frac{5}{2}.$$

Given $\frac{1}{4} < s < \frac{3}{4}$, we fix the following *s-admissible* and *dual s-admissible* pairs:

$$(q, r) = \left(\frac{3}{s}, \frac{6}{3 - 4s} \right) \quad \text{and} \quad (\tilde{q}, \tilde{r}) = \left(\frac{3}{2 + s}, \frac{6}{7 - 4s} \right). \tag{2.15}$$

In Sect. 6, we will only use these pairs.

Let $0 < T \leq 1$, $\frac{1}{4} < s < \frac{3}{4}$ and fix the *s-admissible* pair (q, r) and the *dual s-admissible* pair (\tilde{q}, \tilde{r}) given in (2.15). We then define the Strichartz space:

$$\mathcal{X}_T^s = C([0, T]; H^s(\mathbb{T}^2)) \cap C^1([0, T]; H^{s-1}(\mathbb{T}^2)) \cap L^q([0, T]; L^r(\mathbb{T}^2)) \tag{2.16}$$

and its “dual” space:

$$\mathcal{N}_T^s = L^1([0, T]; H^{s-1}(\mathbb{T}^2)) + L^{\tilde{q}}([0, T]; L^{\tilde{r}}(\mathbb{T}^2)). \tag{2.17}$$

We now state the Strichartz estimates. The Strichartz estimates on \mathbb{R}^d are well-known; see [32, 48, 53]. Thanks to the finite speed of propagation, the same estimates also hold on \mathbb{T}^d locally in time.

Lemma 2.8. *The solution u to the linear wave equation:*

$$\begin{cases} \partial_t^2 u - \Delta u = F \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \end{cases}$$

satisfies the following Strichartz estimate:

$$\|u\|_{\mathcal{X}_T^s} \lesssim \|(u_0, u_1)\|_{\mathcal{H}^s} + \|F\|_{\mathcal{N}_T^s},$$

uniformly in $0 < T \leq 1$.

We also recall from [35] the following interpolation result for \mathcal{X}_T^s and \mathcal{N}_T^s . See (3.22) and (3.23) in [35] for the proof.

Lemma 2.9. *The following continuous embeddings hold:*

(i) *Let $0 \leq \alpha \leq s$ and $2 \leq q_1, r_1 \leq \infty$ satisfy the scaling condition:*

$$\frac{1}{q_1} = \frac{1 - \alpha/s}{q} + \frac{\alpha/s}{\infty} \quad \text{and} \quad \frac{1}{r_1} = \frac{1 - \alpha/s}{r} + \frac{\alpha/s}{2}.$$

Then, we have

$$\|u\|_{L_T^{q_1} W_x^{\alpha, r_1}} \lesssim \|u\|_{\mathcal{X}_T^s}.$$

(ii) *Let $0 \leq \alpha \leq 1 - s$ and $1 \leq \tilde{q}_1, \tilde{r}_1 \leq 2$ satisfy the scaling condition:*

$$\frac{1}{\tilde{q}_1} = \frac{1 - \alpha/(1 - s)}{\tilde{q}} + \frac{\alpha/(1 - s)}{1} \quad \text{and} \quad \frac{1}{\tilde{r}_1} = \frac{1 - \alpha/(1 - s)}{\tilde{r}} + \frac{\alpha/(1 - s)}{2}.$$

Then, we have

$$\|u\|_{\mathcal{N}_T^s} \lesssim \|u\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{r}_1}}.$$

2.5. *Some useful results from nonlinear analysis.* We conclude this section by presenting some further results from harmonic and functional analysis.

We first state the Brascamp-Lieb inequality [13]. This inequality plays an important role in the proof of Proposition 1.12. In particular, it allows us to establish a good bound on the p th moment of the Gaussian multiplicative chaos Θ_N when $p > 2$. The version we present here is due to [8].

Definition 2.10. We say that a pair (\mathbf{B}, \mathbf{q}) is a *Brascamp-Lieb datum*, if, for some $m \in \mathbb{N} \cup \{0\}$ and $d, d_1, \dots, d_m \in \mathbb{N}$, $\mathbf{B} = (B_1, \dots, B_m)$ is a collection of linear maps from \mathbb{R}^d to \mathbb{R}^{d_j} , $j = 1, \dots, m$, and $\mathbf{q} = (q_1, \dots, q_m) \in \mathbb{R}_+^m$.

We now state the m -linear Brascamp-Lieb inequality.

Lemma 2.11. (Theorem 1.15 in [8]) *Let (\mathbf{B}, \mathbf{q}) be a Brascamp-Lieb datum. Suppose that the following conditions hold:*

- *Scaling condition:*

$$\sum_{j=1}^m q_j d_j = d. \tag{2.18}$$

- *Dimension condition: for all subspace $V \subset \mathbb{R}^d$, there holds*

$$\dim(V) \leq \sum_{j=1}^m q_j \dim(B_j V). \tag{2.19}$$

Then, there exists a positive constant $BL(\mathbf{B}, \mathbf{q}) < \infty$ such that

$$\int_{\mathbb{R}^d} \prod_{j=1}^m f_j(B_j x)^{q_j} dx \leq BL(\mathbf{B}, \mathbf{q}) \prod_{j=1}^m \left(\int_{\mathbb{R}^{d_j}} f_j(y) dy \right)^{q_j}$$

for any non-negative functions $f_j \in L^1(\mathbb{R}^{d_j})$, $j = 1, \dots, m$.

We point out that the conditions (2.18) and (2.19) guarantee that the Brascamp-Lieb data is non-degenerate, i.e. the maps B_j , $j = 1, \dots, m$, are surjective and their common kernel is trivial. See [8, Remarks 1.16].

For our purpose, we only need the following special version of Lemma 2.11.

Corollary 2.12. *Let $p \in \mathbb{N}$. Then, we have*

$$\begin{aligned} \int_{(\mathbb{T}^2)^{2p}} \prod_{1 \leq j < k \leq 2p} |f_{j,k}(\pi_{j,k}(x))|^{\frac{1}{2p-1}} dx \\ \lesssim \prod_{1 \leq j < k \leq 2p} \left(\int_{(\mathbb{T}^2)^2} |f_{j,k}(x_j, x_k)| dx_j dx_k \right)^{\frac{1}{2p-1}} \end{aligned} \tag{2.20}$$

for any $f_{j,k} \in L^1(\mathbb{T}^2 \times \mathbb{T}^2)$. Here, $\pi_{j,k}$ denotes the projection defined by $\pi_{j,k}(x) = \pi_{j,k}(x_1, \dots, x_{2p}) = (x_j, x_k)$ for $x = (x_1, \dots, x_{2p}) \in (\mathbb{T}^2)^{2p}$.

This is precisely the geometric Brascamp-Lieb inequality stated in [8, Example 1.6]. For readers' convenience, we include its reduction to Lemma 2.11.

Proof. Write $(\mathbb{R}^2)^{2p} = \prod_{\ell=1}^{2p} \mathbb{R}_\ell^2$ and define projections $\pi_\ell : (\mathbb{R}^2)^{2p} \rightarrow \mathbb{R}_\ell^2$ and $\pi_{j,k} : (\mathbb{R}^2)^{2p} \rightarrow \mathbb{R}_j^2 \times \mathbb{R}_k^2$ for $j \neq k$ in the usual way. Now, we set $\mathbf{B} = (\pi_{j,k} : 1 \leq j < k \leq 2p)$ and

$$\mathbf{q} = \left(\frac{1}{2p-1}, \dots, \frac{1}{2p-1} \right) \in \mathbb{R}_+^{p(2p-1)}.$$

It is also easy to check that the scaling condition (2.18) holds since $d_{j,k} = 4$, $1 \leq j < k \leq 2p$ and $m = p(2p-1)$ and $q_{j,k} = \frac{1}{2p-1}$, while the total dimension is $d = 4p$.

As for the dimension condition (2.19), first note that

$$\dim(\pi_{j,k} V) = \dim(\pi_j V) + \dim(\pi_k V)$$

for $j \neq k$. Then, we have

$$\dim(V) \leq \sum_{j=1}^{2p} \dim(\pi_j V) = \frac{1}{2p-1} \sum_{1 \leq j < k \leq 2p} \dim(\pi_{j,k} V),$$

verifying (2.19).

The desired estimate (2.20) follows from extending $f_{j,k}$ on $(\mathbb{T}^2)^2$ as a compactly supported measurable function on \mathbb{R}^4 by extending it by 0 outside of $(\mathbb{T}^2)^2 \simeq [-\pi, \pi]^4$ and applying Lemma 2.11. \square

We now recall several product estimates. See Lemma 3.4 in [35] for the proofs.

Lemma 2.13. *Let $0 \leq s \leq 1$. (i) Suppose that $1 < p_j, q_j, r < \infty$, $\frac{1}{p_j} + \frac{1}{q_j} = \frac{1}{r}$, $j = 1, 2$. Then, we have*

$$\|\langle \nabla \rangle^s (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^s f\|_{L^{p_1}(\mathbb{T}^d)} \|g\|_{L^{q_1}(\mathbb{T}^d)} + \|f\|_{L^{p_2}(\mathbb{T}^d)} \|\langle \nabla \rangle^s g\|_{L^{q_2}(\mathbb{T}^d)}.$$

(ii) Suppose that $1 < p, q, r < \infty$ satisfy $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$. Then, we have

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^r(\mathbb{T}^d)} \lesssim \|\langle \nabla \rangle^s f\|_{L^q(\mathbb{T}^d)} \|\langle \nabla \rangle^{-s} g\|_{L^p(\mathbb{T}^d)}. \tag{2.21}$$

Note that while Lemma 2.13 (ii) was shown only for $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + \frac{s}{d}$ in [35], the general case $\frac{1}{p} + \frac{1}{q} \leq \frac{1}{r} + \frac{s}{d}$ follows from the inclusion $L^{r_1}(\mathbb{T}^d) \subset L^{r_2}(\mathbb{T}^d)$ for $r_1 \geq r_2$.

The next lemma shows that an improvement over (2.21) in Lemma 2.13 (ii) is possible if g happens to be a positive distribution.

Lemma 2.14. *Let $0 \leq s \leq 1$ and $1 < p < \infty$. Then, we have*

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^p(\mathbb{T}^d)} \lesssim \|f\|_{L^\infty(\mathbb{T}^d)} \|\langle \nabla \rangle^{-s} g\|_{L^p(\mathbb{T}^d)} \tag{2.22}$$

for any $f \in L^\infty(\mathbb{T}^d)$ and any positive distribution $g \in W^{-s,p}(\mathbb{T}^d)$, satisfying one of the following two conditions: (i) $f \in C(\mathbb{T}^d)$ or (ii) $f \in W^{s,q}(\mathbb{T}^d)$ for some $1 < q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} < 1 + \frac{s}{d}$.

This lemma plays an important role in estimating a product involving the non-negative Gaussian multiplicative chaos Θ_N . In studying continuity in the noise, we need to estimate the difference of the Gaussian multiplicative chaoses. In this case, there is no positivity to exploit and hence we instead apply Lemma 2.13 (ii).

Proof. We consider $0 < s \leq 1$ since the $s = 0$ case corresponds to Hölder’s inequality. Since g is a positive distribution, it can be identified with a positive Radon measure on \mathbb{T}^2 ; see for example [29, Theorem 7.2]. If $f \in C(\mathbb{T}^d)$, then the product fg is a well-defined function in $L^1(\mathbb{T}^d)$. With ρ_N as in (1.17), we have $f_N g \stackrel{\text{def}}{=} (\rho_N * f)g \rightarrow fg$ in $L^1(\mathbb{T}^d)$, in particular in the distributional sense. Hence, from Fatou’s lemma, we have

$$\|\langle \nabla \rangle^{-s} (fg)\|_{L^p} \leq \liminf_{N \rightarrow \infty} \|\langle \nabla \rangle^{-s} (f_N g)\|_{L^p}. \tag{2.23}$$

Since ρ_M is non-negative, we see that $g_M = \rho_M * g$ is a well-defined smooth, positive distribution which converges to g in $W^{-s,p}(\mathbb{T}^d)$. Then, it follows from Lemma 2.13 (ii) that, for each fixed $N \in \mathbb{N}$, $f_N g_M$ converges to $f_N g$ in $W^{-s,p}(\mathbb{T}^d)$ as $M \rightarrow \infty$. Hence, it suffices to prove (2.22) for $f_N g_M$, $N, M \in \mathbb{N}$. Indeed, if (2.22) holds for $f_N g_M$, $N, M \in \mathbb{N}$, then by (2.23), (2.22) for $f_N g_M$, the convergence of $\langle \nabla \rangle^{-s} g_M = \rho_M * (\langle \nabla \rangle^{-s} g)$ to $\langle \nabla \rangle^{-s} g$ in $L^p(\mathbb{T}^d)$, and Young’s inequality with $\|\rho_N\|_{L^1} = 1$, we obtain

$$\begin{aligned} \|\langle \nabla \rangle^{-s} (fg)\|_{L^p} &\leq \liminf_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \|\langle \nabla \rangle^{-s} (f_N g_M)\|_{L^p} \\ &\lesssim \liminf_{N \rightarrow \infty} \lim_{M \rightarrow \infty} \|f_N\|_{L^\infty} \|\langle \nabla \rangle^{-s} g_M\|_{L^p} \\ &\leq \liminf_{N \rightarrow \infty} \|f_N\|_{L^\infty} \|\langle \nabla \rangle^{-s} g\|_{L^p} \\ &\leq \|f\|_{L^\infty} \|\langle \nabla \rangle^{-s} g\|_{L^p}. \end{aligned}$$

It remains to prove (2.22) for $f_N g_M$. By Lemma 2.1, we have

$$\begin{aligned} \| \langle \nabla \rangle^{-s} (f_N g_M) \|_{L^p} &= \| J_s * (f_N g_M) \|_{L^p} \\ &\lesssim \left\| \int_{\mathbb{T}^d} |x - y|^{s-d} |f_N(y) g_M(y)| dy \right\|_{L^p} \end{aligned}$$

Since g_m is non-negative,

$$\lesssim \| f_N \|_{L^\infty} \left\| | \cdot |^{s-d} * g_M \right\|_{L^p}$$

Using Lemma 2.1 again,

$$\begin{aligned} &\sim \| f_N \|_{L^\infty} \left\| (J_s - R) * g_M \right\|_{L^p} \\ &\leq \| f_N \|_{L^\infty} \left(\| \langle \nabla \rangle^{-s} g_M \|_{L^p} + \| (\langle \nabla \rangle^s R) * (\langle \nabla \rangle^{-s} g_M) \|_{L^p} \right) \\ &\lesssim \| f_N \|_{L^\infty} \| \langle \nabla \rangle^{-s} g_M \|_{L^p}, \end{aligned}$$

where in the last step we used the fact that R is smooth. This shows (2.22) for $f_M g_M$ and hence for $f \in C(\mathbb{T}^d)$ and a positive distribution $g \in W^{-s,p}(\mathbb{T}^d)$.

In view of Lemma 2.13 (ii), the condition (ii) guarantees that the product operation $(f, g) \in W^{s,q}(\mathbb{T}^d) \times W^{-s,p}(\mathbb{T}^d) \mapsto fg \in W^{-s,1+\varepsilon}(\mathbb{T}^d)$ for some small $\varepsilon > 0$ is a continuous bilinear map. Namely, it suffices to prove (2.22) for $f_N g_M = (\rho_N * f)(\rho_M * g)$, which we already did above. This completes the proof of Lemma 2.14. \square

Next, we recall the following fractional chain rule from [31]. The fractional chain rule on \mathbb{R}^d was essentially proved in [18].¹⁴ As for the estimates on \mathbb{T}^d , see [31].

Lemma 2.15. *Let $0 < s < 1$. (i) Suppose that F is a Lipschitz function with Lipschitz constant $K > 0$. Then, for any $1 < p < \infty$, we have*

$$\| |\nabla|^s F(u) \|_{L^p(\mathbb{T}^d)} \lesssim K \| |\nabla|^s u \|_{L^p(\mathbb{T}^d)}.$$

(ii) Suppose that $F \in C^1(\mathbb{R})$ satisfies

$$|F'(\tau x + (1 - \tau)y)| \leq c(\tau)(|F'(x)| + |F'(y)|)$$

for every $\tau \in [0, 1]$ and $x, y \in \mathbb{R}$, where $c \in L^1([0, 1])$. Then for $1 < p, q, r < \infty$ with $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, we have

$$\| |\nabla|^s F(u) \|_{L^r(\mathbb{T}^d)} \lesssim \| F'(u) \|_{L^p(\mathbb{T}^d)} \| |\nabla|^s u \|_{L^q(\mathbb{T}^d)}.$$

Lastly, we state a tool from functional analysis. The following classical Aubin-Lions lemma [6] provides a criterion for compactness. See also [75, Corollary 4 on p. 85].

Lemma 2.16. *Let $\mathcal{X}_{-1}, \mathcal{X}_0, \mathcal{X}_1$ be Banach spaces satisfying the continuous embeddings $\mathcal{X}_1 \subset \mathcal{X}_0 \subset \mathcal{X}_{-1}$ such that the embedding $\mathcal{X}_1 \subset \mathcal{X}_0$ is compact. Suppose that B is bounded in $L^p([0, T]; \mathcal{X}_1)$ such that $\{\partial_t u : u \in B\}$ is bounded in $L^q([0, T]; \mathcal{X}_{-1})$ for some $T > 0$ and finite $p, q \geq 1$. Then, B is relatively compact in $L^p([0, T]; \mathcal{X}_0)$. Moreover, if B is bounded in $L^\infty([0, T]; \mathcal{X}_1)$ and $\{\partial_t u : u \in B\}$ is bounded in $L^q([0, T]; \mathcal{X}_{-1})$ for some $q > 1$, then B is relatively compact in $C([0, T]; \mathcal{X}_0)$.*

¹⁴ As pointed out in [76], the proof in [18] needs a small correction, which yields the fractional chain rule in a less general context. See [47, 76, 78].

3. Gaussian Multiplicative Chaos

In this section, we establish the regularity and convergence properties of the Gaussian multiplicative chaos $\Theta_N = :e^{\beta\Psi_N}$: claimed in Proposition 1.12, where Ψ_N denotes the truncated stochastic convolution for either the heat equation or the wave equation. These properties are of central importance for the study of the truncated SNLH (1.10) and the truncated SdNLW (1.20). As in the case of the sine-Gordon model studied in [42,63], the main difficulty comes from the fact that the processes Θ_N do not belong to any Wiener chaos of finite order. There is, however, a major difference from the analysis on the imaginary Gaussian multiplicative chaos $:e^{i\beta\Psi_N}$: studied for the sine-Gordon model in [42,63]. As for the imaginary Gaussian multiplicative chaos, the regularity depends only on the values of β^2 . On the other hand, the regularity of Θ_N depends not only on the values of β^2 but also on the integrability index (either for moments or space-time integrability). In particular, for higher moments, the regularity gets worse. This phenomenon is referred to as intermittency in [30]. See Remark 3.3 below.

3.1. Preliminaries. Since the definition (1.39) of Θ_N involves polynomials of arbitrarily high degrees, it seems more convenient to study Θ_N on the physical space, as in the case of the sine-Gordon equation [63], rather than in the frequency space as in [35]. For this purpose, we first recall the main property of the covariance function:

$$\Gamma_{N_1, N_2}(t, x - y) \stackrel{\text{def}}{=} \mathbb{E}[\Psi_{N_1}(t, x)\Psi_{N_2}(t, y)]$$

for the truncated stochastic convolution $\Psi_{N_j} = \Psi_{N_j}^{\text{heat}}$ or $\Psi_{N_j}^{\text{wave}}$, where the truncation may be given by the smooth frequency projector \mathbf{P}_N or the smoothing operator \mathbf{Q}_N with a positive kernel defined in (1.16). When $N = N_1 = N_2$, we set

$$\Gamma_N = \Gamma_{N, N}.$$

As stated in Sect. 1.2, the results in this section hold for both \mathbf{P}_N and \mathbf{Q}_N .

The next lemma follows as a corollary to Lemmas 2.2 and 2.3. See Lemma 2.7 in [63] for the proof.

Lemma 3.1. *Let $N_2 \geq N_1 \geq 1$. Then we have*

$$\Gamma_{N_1, N_2}(t, x - y) \approx -\frac{1}{2\pi} \log(|x - y| + N_1^{-1})$$

for any $t \geq 0$. Similarly, we have

$$\begin{aligned} &|\Gamma_{N_j}(t, x - y) - \Gamma_{N_1, N_2}(t, x - y)| \\ &\lesssim (1 \vee -\log(|x - y| + N_j^{-1})) \wedge (N_1^{-1}|x - y|^{-1}) \end{aligned} \tag{3.1}$$

for $j = 1, 2$ and $t \geq 0$.

3.2. *Estimates on the even moments.* In this subsection, we prove the following proposition for the uniform control on the even moments of the random variables $\Theta_N(t, x)$ for any fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$ and $N \in \mathbb{N}$.

Proposition 3.2. *Let $0 < \beta^2 < 8\pi$. Then, the following statements hold. (i) For any $t \geq 0, x \in \mathbb{T}^2$, and $N \in \mathbb{N}$, we have $\mathbb{E}[|\Theta_N(t, x)|] = 1$, (ii) Let $p \geq 2$ be even. Let $0 < \alpha < 2$ and $(p - 1)\frac{\beta^2}{4\pi} < \min(1, \alpha)$. Then, for any $T > 0$, we have*

$$\sup_{t \in [0, T], x \in \mathbb{T}^2, N \in \mathbb{N}} \mathbb{E} \left[|\langle \nabla \rangle^{-\alpha} \Theta_N(t, x)|^p \right] \leq C(T).$$

(iii) *Let $0 < \alpha < 2$ and $\frac{\beta^2}{4\pi} < \min(1, \alpha)$. Then, there exists small $\varepsilon > 0$ such that*

$$\sup_{t \in [0, T], x \in \mathbb{T}^2} \mathbb{E} \left[|\langle \nabla \rangle^{-\alpha} (\Theta_{N_1}(t, x) - \Theta_{N_2}(t, x))|^2 \right] \leq C(T) N_1^{-\varepsilon}$$

for any $T > 0$ and any $N_2 \geq N_1 \geq 1$.

Proof. For fixed $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^2$, $\Psi_N(t, x)$ is a mean-zero Gaussian random variable with variance σ_N . Hence, from the positivity of Θ_N and (1.39), we have

$$\mathbb{E}[|\Theta_N(t, x)|] = e^{-\frac{\beta^2}{2}\sigma_N} \mathbb{E}[e^{\beta\Psi_N(t, x)}] = 1.$$

This proves (i).

Next, we consider (ii). Let $p = 2m, m \in \mathbb{N}$. Fix $(t, x) \in [0, T] \times \mathbb{T}^2$. Recalling $\langle \nabla \rangle^{-\alpha} f = J_\alpha * f$, where J_α is as in (2.2), we have

$$\begin{aligned} & \mathbb{E} \left[|\langle \nabla \rangle^{-\alpha} \Theta_N(t, x)|^{2m} \right] \\ &= e^{-m\beta^2\sigma_N} \mathbb{E} \left[\left| \int_{\mathbb{T}^2} J_\alpha(x - y) e^{\beta\Psi_N(t, y)} dy \right|^{2m} \right] \\ &= e^{-m\beta^2\sigma_N} \int_{(\mathbb{T}^2)^{2m}} \mathbb{E} \left[e^{\beta \sum_{j=1}^{2m} \Psi_N(t, y_j)} \right] \left(\prod_{j=1}^{2m} J_\alpha(x - y_j) \right) d\vec{y} \tag{3.2} \\ &= e^{-m\beta^2\sigma_N} \int_{(\mathbb{T}^2)^{2m}} \exp \left(\frac{\beta^2}{2} \mathbb{E} \left[\left| \sum_{j=1}^{2m} \Psi_N(t, y_j) \right|^2 \right] \right) \left(\prod_{j=1}^{2m} J_\alpha(x - y_j) \right) d\vec{y}. \end{aligned}$$

where $d\vec{y} = dy_1 \cdots dy_{2m}$ and we used the fact that $\sum_{j=1}^{2m} \Psi_N(t, y_j)$ is a Gaussian random variable at the last step. From the definition (1.30) of σ_N and Lemma 3.1, we have

$$\begin{aligned} & \exp \left(\frac{\beta^2}{2} \mathbb{E} \left[\left| \sum_{j=1}^{2m} \Psi_N(t, y_j) \right|^2 \right] \right) \\ &= e^{m\beta^2\sigma_N} \exp \left(\beta^2 \sum_{1 \leq j < k \leq 2m} \mathbb{E} [\Psi_N(t, y_j) \Psi_N(t, y_k)] \right) \tag{3.3} \\ &\lesssim e^{m\beta^2\sigma_N} \prod_{1 \leq j < k \leq 2m} (|y_j - y_k| + N^{-1})^{-\frac{\beta^2}{2\pi}}. \end{aligned}$$

Hence, from (3.2) and (3.3), we obtain

$$\begin{aligned}
 & \mathbb{E} \left[|(\nabla)^{-\alpha} \Theta_N(t, x)|^{2m} \right] \\
 & \lesssim \int_{(\mathbb{T}^2)^{2m}} \left(\prod_{1 \leq j < k \leq 2m} (|y_j - y_k| + N^{-1})^{-\frac{\beta^2}{2\pi}} \right) \left(\prod_{j=1}^{2m} |J_\alpha(x - y_j)| \right) d\vec{y} \\
 & = \int_{(\mathbb{T}^2)^{2m}} \prod_{1 \leq j < k \leq 2m} \frac{|J_\alpha(x - y_j) J_\alpha(x - y_k)|^{\frac{1}{2m-1}}}{(|y_j - y_k| + N^{-1})^{\frac{\beta^2}{2\pi}}} d\vec{y}.
 \end{aligned} \tag{3.4}$$

By applying the geometric Brascamp-Lieb inequality (Corollary 2.12) and proceeding as in the proof of Proposition 1.1 in [63] to bound the resulting integral, we then obtain

$$\begin{aligned}
 \text{RHS of (3.4)} & \lesssim \prod_{1 \leq j < k \leq 2m} \left(\int_{(\mathbb{T}^2)^2} \frac{|J_\alpha(x - y) J_\alpha(x - z)|}{(|y - z| + N^{-1})^{(2m-1)\frac{\beta^2}{2\pi}}} dy dz \right)^{\frac{1}{2m-1}} \\
 & = \left(\int_{(\mathbb{T}^2)^2} \frac{|J_\alpha(x - y) J_\alpha(x - z)|}{(|y - z| + N^{-1})^{(2m-1)\frac{\beta^2}{2\pi}}} dy dz \right)^m \\
 & \lesssim 1,
 \end{aligned}$$

uniformly in $t \in [0, T]$, $x \in \mathbb{T}^2$, and $N \in \mathbb{N}$, provided $(2m - 1)\beta^2 < 4\pi \min(1, \alpha)$ and $0 < \alpha < 2$.

Lastly, Part (iii) for the case $p = 2$ follows from the last part of the proof of Proposition 1.1 in [63] (with $t = 2$), provided that $\beta^2 < 4\pi \min(1, \alpha)$ and $0 < \alpha < 2$. The second estimate (3.1) in Lemma 3.1 is needed here. This completes the proof of Proposition 3.2. \square

Remark 3.3. When $p = 2$, the proof of Proposition 3.2 is identical to that in [63, Proposition 1.1]. For $p > 2$, however, the bounds are quite different. In computing higher moments for the imaginary Gaussian multiplicative chaos $: e^{i\beta\Psi_N} :$, it was crucial to exploit certain cancellation property [42, 63]. Namely, in the ‘‘multipole picture’’ for the imaginary Gaussian multiplicative chaos (and more generally log-correlated Gaussian fields [50]), there is a ‘‘charge cancellation’’ in estimating higher moments of $: e^{i\beta\Psi_N} :$ due to its complex nature.

In the current setting, i.e. without the ‘‘ i ’’ in the exponent, there is no such cancellation taking place; the charges accumulate and contribute to worse estimates in the sense that the higher moment estimates require more smoothing. This is the source of the so-called intermittency phenomenon [30], which is quantified by the dependence on p for the choice of α in Proposition 3.2 (ii) above.

3.3. Kahane’s approach. Proposition 3.2 in the previous subsection allows to get part of the result claimed in Proposition 1.12. Indeed, using Fubini’s theorem and arguing as in the proof of Proposition 1.1 in [63], interpolating between (ii) and (iii) in Proposition 3.2 above implies the convergence of $\{\Theta_N\}_{N \in \mathbb{N}}$ in $L^p(\Omega; L^p([0, T]; W^{-\alpha, p}(\mathbb{T}^2)))$ in the case of even $p \geq 2$, for all $\alpha = \alpha(p)$ as in (1.41).

Note, however, that when $p \in (1, 2)$ or $p > 2$ is not even, we only get a weaker result than Proposition 1.12. Indeed, when $p > 2$ is not even, if $2m < p < 2m + 2$ for some $m \in \mathbb{N}$

\mathbb{N} , Proposition 3.2 provides convergence in both $L^{2m}(\Omega; L^{2m}([0, T]; W^{-\alpha, 2m}(\mathbb{T}^2)))$ and $L^{2m+2}(\Omega; L^{2m+2}([0, T]; W^{-\alpha, 2m+2}(\mathbb{T}^2)))$, which by interpolation provides convergence in $L^p(\Omega; L^p([0, T]; W^{-\alpha, p}(\mathbb{T}^2)))$ for $\alpha = \alpha(p)$ as in (1.41). Such an argument then imposes the condition

$$0 < \beta^2 < \frac{4\pi}{(2m + 2) - 1}, \tag{3.5}$$

which gives a smaller range than the natural one $0 < \beta^2 < \frac{4\pi}{p-1}$. The condition (3.5) comes from the requirement that $\{\Theta_N\}_{N \in \mathbb{N}}$ be uniformly bounded in $L^{2m+2}(\Omega; L^{2m+2}([0, T]; W^{-\alpha, 2m+2}(\mathbb{T}^2)))$, On the other hand, in the case $p \in (1, 2)$, interpolating between (i) and (iii) of Proposition 3.2 provides the convergence of $\{\Theta_N\}_{N \in \mathbb{N}}$ in $L^p(\Omega; L^p([0, T]; W^{-\alpha, p}(\mathbb{T}^2)))$ only for in a more restricted range $\alpha > \frac{(p-1)\beta^2}{2\pi p} (> \alpha(p))$.

The argument presented in the previous subsection still has the advantage of being applicable to a large class of processes. Namely, whenever the k -points correlation functions can be expressed as a product, the use of the Brascamp-Lieb inequality (Corollary 2.12) allows to decouple them into a product of 2-points correlation functions. As pointed out above, however, this only works for even $p \geq 2$, which restricts the range of admissible $\beta^2 > 0$ in studying (1.1) or (1.2).

In this subsection, we instead follow the classical approach of Kahane [46] which relies on the following comparison inequality for the renormalized exponential of Gaussian random variables. See, for example, [71, Theorem 2.1] and [72, Corollary A.2].

Lemma 3.4. (Kahane’s convexity inequality) *Given $n \in \mathbb{N}$, let $\{X_j\}_{j=1}^n$ and $\{Y_j\}_{j=1}^n$ be two centered Gaussian vectors satisfying*

$$\mathbb{E}[X_j X_k] \leq \mathbb{E}[Y_j Y_k]$$

for all $j, k = 1, \dots, n$. Then, for any sequence $\{p_j\}_{j=1}^n$ of non-negative numbers and any convex function $F : [0, \infty) \rightarrow \mathbb{R}$ with at most polynomial growth at infinity, it holds

$$\mathbb{E} \left[F \left(\sum_{j=1}^n p_j e^{X_j - \frac{1}{2} \mathbb{E}[X_j^2]} \right) \right] \leq \mathbb{E} \left[F \left(\sum_{j=1}^n p_j e^{Y_j - \frac{1}{2} \mathbb{E}[Y_j^2]} \right) \right].$$

As an application of Lemma 3.4, one has the following bound on the moments of the random measure¹⁵ $\mathcal{M}_N(t, \cdot)$, $t \geq 0$ defined by

$$\mathcal{M}_N(t, A) = \int_A \Theta_N(t, x) dx \tag{3.6}$$

for $A \in \mathcal{B}(\mathbb{T}^2)$, where $\mathcal{B}(\mathbb{T}^2)$ is the Borel σ -algebra of \mathbb{T}^2 .

Lemma 3.5. *For any $0 < \beta^2 < 8\pi$ and $1 \leq p < \frac{8\pi}{\beta^2}$, we have*

$$\sup_{t \in \mathbb{R}_+, A \in \mathcal{B}(\mathbb{T}^2), N \in \mathbb{N}} \mathbb{E} \left[\mathcal{M}_N(t, A)^p \right] < \infty.$$

¹⁵ In the literature, this random measure is also referred to as a multiplicative chaos. See [72].

Lemma 3.5 is a classical result in the theory of Gaussian multiplicative chaoses. See for example Proposition 3.5 in [72]. We present a self-contained proof in Appendix B below.

With the bounds of Lemmas 3.4 and 3.5, we can prove the following uniform estimate on $\{\Theta_N\}_{N \in \mathbb{N}}$.

Proposition 3.6. *Let $0 < \beta^2 < 8\pi$, $1 \leq p < \frac{8\pi}{\beta^2}$, and $0 < \alpha < 2$ such that $\alpha > (p - 1)\frac{\beta^2}{4\pi}$. Then, we have for any $T > 0$*

$$\sup_{t \in [0, T], x \in \mathbb{T}^2, N \in \mathbb{N}} \mathbb{E} \left[|\langle \nabla \rangle^{-\alpha} \Theta_N(t, x)|^p \right] \leq C(T). \tag{3.7}$$

Note that in Proposition 3.6, we do not need to assume that p is even. The uniform bound in Proposition 1.12(i) follows from (3.7), while the convergence part of Proposition 1.12 follows from interpolating (3.7) in Proposition 3.6 and Proposition 3.2(iii) and using the same argument as in the proof of Proposition 1.1 in [63]. When $1 < p < 2$, the use of Proposition 3.2(iii) imposes the condition $0 < \beta^2 < 4\pi$, which yields the restriction on the range of β^2 in Proposition 1.12(ii).

Proof of Proposition 3.6. We split the proof into two steps.

• **Step 1: multifractal spectrum.** We first establish the following bound on the moments of the random measure $\mathcal{M}_N(t)$ over small balls:

$$\sup_{t \in [0, T], x_0 \in \mathbb{T}^2, N \in \mathbb{N}} \mathbb{E} \left[\mathcal{M}_N(t, B(x_0, r))^p \right] \lesssim r^{(2 + \frac{\beta^2}{4\pi})p - \frac{\beta^2}{4\pi} p^2} \tag{3.8}$$

for any $r \in (0, 1)$.

By a change of variables, the positivity of Θ_N , and a Riemann sum approximation, we have

$$\begin{aligned} \mathbb{E} \left[\mathcal{M}_N(t, B(x_0, r))^p \right] &= r^{2p} \mathbb{E} \left[\left(\int_{B(0,1)} \Theta_N(t, x_0 + ry) dy \right)^p \right] \\ &\leq r^{2p} \mathbb{E} \left[\left(\int_{\mathbb{T}^2} \Theta_N(t, x_0 + ry) dy \right)^p \right] \\ &= r^{2p} \lim_{J \rightarrow \infty} \mathbb{E} \left[\left(\sum_{j,k=1}^J \frac{4\pi^2}{J^2} e^{\beta \Psi_N(t, x_0 + ry_{j,k}) - \frac{\beta^2}{2} \sigma_N} \right)^p \right], \end{aligned} \tag{3.9}$$

where $y_{j,k}$, $j, k = 1, \dots, J$, is given by $y_{j,k} = (-\pi + \frac{2\pi}{J}(j - 1), -\pi + \frac{2\pi}{J}(k - 1)) \in \mathbb{T}^2 \simeq [-\pi, \pi)^2$. From Lemma 3.1, we can bound the covariance function as

$$\begin{aligned} &\mathbb{E} \left[\Psi_N(t, x_0 + ry_{j_1, k_1}) \Psi_N(t, x_0 + ry_{j_2, k_2}) \right] \\ &\leq -\frac{1}{2\pi} \log (r |y_{j_1, k_1} - y_{j_2, k_2}| + N^{-1}) + C \\ &\leq -\frac{1}{2\pi} \log (|y_{j_1, k_1} - y_{j_2, k_2}| + (rN)^{-1}) - \frac{1}{2\pi} \log r + C \\ &\leq -\frac{1}{2\pi} \log (|y_{j_1, k_1} - y_{j_2, k_2}| + N^{-1}) - \frac{1}{2\pi} \log r + C \\ &\leq \mathbb{E} \left[(\Psi_N(t, y_{j_1, k_1}) + h_r) (\Psi_N(t, y_{j_2, k_2}) + h_r) \right] \end{aligned} \tag{3.10}$$

for any $0 < r < 1$ and $j_1, j_2, k_1, k_2 = 1, \dots, J$, where h_r is a mean-zero Gaussian random variable with variance $-\frac{1}{2\pi} \log r + C$, independent from Ψ_N . Then, by applying Kahane’s convexity inequality (Lemma 3.4) with the convex function $x \mapsto x^p$, a Riemann sum approximation, and the independence of h_r from Ψ_N , it follows from (3.9) that

$$\begin{aligned} \mathbb{E}\left[\mathcal{M}_N(t, B(x_0, r))^p\right] &\leq r^{2p} \lim_{J \rightarrow \infty} \mathbb{E}\left[\left(\sum_{j,k=1}^J \frac{4\pi^2}{J^2} e^{\beta(\Psi_N(t,y_{j,k})+h_r)-\frac{\beta^2}{2}\mathbb{E}[(\Psi_N(t,y_{j,k})+h_r)^2]}\right)^p\right] \\ &= r^{2p} \mathbb{E}\left[\left(\int_{\mathbb{T}^2} e^{\beta\Psi_N(t,y)-\frac{\beta^2}{2}\sigma_N} e^{\beta h_r-\frac{\beta^2}{2}\left(-\frac{1}{2\pi}\log r+C\right)} dy\right)^p\right] \\ &= r^{2p} \mathbb{E}\left[e^{p\beta h_r-p\frac{\beta^2}{2}\left(\frac{1}{2\pi}\log\frac{1}{r}+C\right)}\right] \mathbb{E}\left[\left(\int_{\mathbb{T}^2} e^{\beta\Psi_N(t,y)-\frac{\beta^2}{2}\sigma_N} dy\right)^p\right] \\ &= r^{2p} e^{(p^2-p)\frac{\beta^2}{2}\left(-\frac{1}{2\pi}\log r+C\right)} \mathbb{E}\left[\left(\int_{\mathbb{T}^2} e^{\beta\Psi_N(t,y)-\frac{\beta^2}{2}\sigma_N} dy\right)^p\right] \\ &\lesssim r^{(2+\frac{\beta^2}{4\pi})p-\frac{\beta^2}{4\pi}p^2} \mathbb{E}\left[\mathcal{M}_N(t, \mathbb{T}^2)^p\right]. \end{aligned}$$

Hence, the bound (3.8) follows Lemma 3.5.

• **Step 2:** From (2.2), Lemma 2.1, where the remainder R is bounded on \mathbb{T}^2 , and Minkowski’s integral inequality, we have

$$\begin{aligned} \mathbb{E}\left[|\langle \nabla \rangle^{-\alpha} \Theta_N(t, x)|^p\right] &= \mathbb{E}\left[\left|\int_{\mathbb{T}^2} J_\alpha(x-y)\Theta_N(t, y)dy\right|^p\right] \\ &\lesssim \mathbb{E}\left[\left(\int_{\mathbb{T}^2} |x-y|^{\alpha-2}\Theta_N(t, y)dy\right)^p\right] \\ &\lesssim \left\{\sum_{\ell \geq 0} 2^{-(\alpha-2)\ell} \mathbb{E}\left[\left(\int_{|x-y|\sim 2^{-\ell}} \Theta_N(t, y)dy\right)^p\right]^{\frac{1}{p}}\right\}^p \\ &\lesssim \sup_{\ell \geq 0} 2^{-(\alpha-2-\varepsilon)p\ell} \mathbb{E}\left[\mathcal{M}_N(t, B(x, 2^{-\ell}))^p\right] \end{aligned}$$

for $\varepsilon > 0$, uniformly in $t \in \mathbb{R}_+$, $x \in \mathbb{T}^2$, and $N \in \mathbb{N}$. Then, using (3.8), we obtain

$$\mathbb{E}\left[|\langle \nabla \rangle^{-\alpha} \Theta_N(t, x)|^p\right] \lesssim \sup_{\ell \geq 0} 2^{-(\alpha-2-\varepsilon)p\ell} 2^{\frac{\beta^2}{4\pi}p^2\ell-(2+\frac{\beta^2}{4\pi})p\ell} \lesssim 1$$

by choosing $\varepsilon > 0$ sufficiently small, provided that $\alpha > (p-1)\frac{\beta^2}{4\pi}$. This proves (3.7). □

4. Parabolic Liouville Equation I: General Case

In this section, we present a proof of Theorem 1.13. Namely, we prove local well-posedness of the truncated SNLH (1.45) for $v_N = u_N - z - \Psi_N$ in the Da Prato-Debussche formulation in the range:

$$0 < \beta^2 < \beta_{\text{heat}}^2 \stackrel{\text{def}}{=} \frac{8\pi}{3+2\sqrt{2}}$$

without assuming the positivity of λ . Here, z denotes the deterministic linear solution defined in (1.44) and Ψ_N denotes the truncated stochastic convolution defined in (1.27).

Writing (1.45) in the Duhamel formulation, we have

$$v_N = -\frac{1}{2}\lambda\beta \int_0^t P(t-t')(e^{\beta z} e^{\beta v_N} \Theta_N)(t') dt'. \tag{4.1}$$

Given $v_0 \in L^\infty(\mathbb{T}^2)$ and a space-time distribution Θ , we define a map Φ by

$$\Phi(v) = \Phi_{v_0, \Theta}(v) \stackrel{\text{def}}{=} -\frac{1}{2}\lambda\beta \int_0^t P(t-t')(e^{\beta P(t)v_0} e^{\beta v} \Theta)(t') dt'. \tag{4.2}$$

Then, (4.1) can be written as the following fixed point problem:

$$v_N = \Phi_{v_0, \Theta_N}(v_N).$$

In the following, we fix $0 < \alpha, s < 1$ and $p \geq 2$ such that

$$p' \frac{\alpha + s}{2} < 1 \quad \text{and} \quad sp > 2. \tag{4.3}$$

See (4.9) below for a concrete choice of these parameters. Then, we have the following deterministic well-posedness result for the fixed point problem:

$$v = \Phi_{v_0, \Theta}(v). \tag{4.4}$$

Proposition 4.1. *Let α, s, p be as above. Then, given any $v_0 \in L^\infty(\mathbb{T}^2)$ and $R > 0$, there exists $T = T(\|v_0\|_{L^\infty}, R) > 0$ such that given any positive distribution $\Theta \in L^p([0, T]; W^{-\alpha, p}(\mathbb{T}^2))$ satisfying*

$$\|\Theta\|_{L_T^p W_x^{-\alpha, p}} \leq R, \tag{4.5}$$

there exists a unique solution $v \in C([0, T]; W^{s, p}(\mathbb{T}^2))$ to (4.4), depending continuously on the initial data v_0 .

Note that we do not claim any continuity of the solution v in Θ for Proposition 4.1.

Proof. Fix $R > 0$. We prove that there exists $T = T(\|v_0\|_{L^\infty}, R) > 0$ such that $\Phi_{v_0, \Theta}$ is a contraction on the ball $B \subset C([0, T]; W^{s, p}(\mathbb{T}^2))$ of radius $O(1)$ centered at the origin.

Let $v \in B$. Then, by Sobolev’s embedding theorem (with $sp > 2$), we have $v \in C([0, T]; C(\mathbb{T}^2))$. For $v_0 \in L^\infty(\mathbb{T}^2)$, we also have $z \in C([0, T]; C(\mathbb{T}^2))$. In particular, $e^{\beta z} e^{\beta v}(t)$ is continuous in $x \in \mathbb{T}^2$ for any $t \in (0, T]$. Then, by the Schauder estimate (Lemma 2.4 (ii)), Lemma 2.14, and Young’s inequality with (4.3), we have

$$\begin{aligned} \|\Phi(v)\|_{C_T W_x^{s, p}} &\lesssim \left\| \int_0^t (t-t')^{-\frac{(s+\alpha)}{2}} \|\langle \nabla \rangle^{-\alpha} (e^{\beta z} e^{\beta v} \Theta)(t')\|_{L_x^p} dt' \right\|_{L_T^\infty([0, T])} \\ &\lesssim \|e^{\beta z} e^{\beta v}\|_{L_{T, x}^\infty} \left\| (\mathbf{1}_{[0, T]} \cdot |\cdot|^{-\frac{(s+\alpha)}{2}}) * \mathbf{1}_{[0, T]} \Theta \right\|_{W_x^{-\alpha, p}} \Big\|_{L_T^\infty} \\ &\lesssim T^\theta e^C \|v_0\|_{L^\infty} e^{C\|v\|_{L_T^\infty W_x^{s, p}}} \|\Theta\|_{L_T^p W_x^{-\alpha, p}} \\ &\lesssim T^\theta R e^{C\|v_0\|_{L^\infty}} \lesssim 1 \end{aligned} \tag{4.6}$$

for $v \in B$ and a positive distribution Θ satisfying (4.5), by choosing $T = T(\|v_0\|_{L^\infty}, R) > 0$ sufficiently small.

By the fundamental theorem of calculus, we have

$$e^{\beta v_1} - e^{\beta v_2} = \beta(v_1 - v_2) \int_0^1 e^{\beta\tau v_1 + \beta(1-\tau)v_2} d\tau. \tag{4.7}$$

Then, proceeding as in (4.6) with (4.7), we have

$$\begin{aligned} \|\Phi(v_1) - \Phi(v_2)\|_{C_T W_x^{s,p}} &\lesssim T^\theta \left\| e^{\beta z} (e^{\beta v_1} - e^{\beta v_2}) \right\|_{L_{T,x}^\infty} \|\Theta\|_{L_T^p W_x^{-\alpha,p}} \\ &\lesssim T^\theta Re^C \|v_0\|_{L^\infty} e^{C(\|v_1\|_{L_{T,x}^\infty} + \|v_2\|_{L_{T,x}^\infty})} \|v_1 - v_2\|_{L_{T,x}^\infty} \\ &\lesssim T^\theta Re^C \|v_0\|_{L^\infty} \|v_1 - v_2\|_{C_T W_x^{s,p}} \end{aligned} \tag{4.8}$$

for $v_1, v_2 \in B$ and a positive distribution Θ satisfying (4.5).

Hence, from (4.6) and (4.8), we see that Φ is a contraction on B by taking $T = T(\|v_0\|_{L^\infty}, R) > 0$ sufficiently small. The continuity of the solution v in initial data follows from a standard argument and hence we omit details. \square

Remark 4.2. In the proof of Proposition 4.1, a contraction argument shows the uniqueness of the solution v only in the ball $B \subset C([0, T]; W^{s,p}(\mathbb{T}^2))$. By a standard continuity argument, we can upgrade the uniqueness statement to hold in the entire $C([0, T]; W^{s,p}(\mathbb{T}^2))$. Since such an argument is standard, we omit details.

Now, let Θ_N be the Gaussian multiplicative chaos in (1.39). In view of Proposition 1.12, in order to determine the largest admissible range for β^2 , we aim to maximize

$$\beta^2 < \frac{4\pi\alpha}{p-1} < \frac{p-2}{p(p-1)} 8\pi =: h(p),$$

where we used both of the inequalities in (4.3). A direct computation shows that h has a unique maximum in $[2, \infty)$ reached at $p = p_* = 2 + \sqrt{2}$, for which we have

$$h(p_*) = \max_{p \geq 2} h(p) = \frac{8\pi}{3 + 2\sqrt{2}} = \beta_{\text{heat}}^2.$$

Therefore, for $\beta^2 < \beta_{\text{heat}}^2$, we see that the constraints (4.3) are satisfied by taking

$$\begin{aligned} p &= 2 + \sqrt{2}, & s &= 2 - \sqrt{2} + \varepsilon, & \text{and} \\ \alpha &= (p-1) \frac{\beta_{\text{heat}}^2}{4\pi} - 2\varepsilon = 2(\sqrt{2}-1) - 2\varepsilon \end{aligned} \tag{4.9}$$

for sufficiently small $\varepsilon > 0$ such that $\alpha > (p-1) \frac{\beta^2}{4\pi}$. With this choice of the parameters, Proposition 4.1 with Proposition 1.12 establishes local well-posedness of (4.1).

In the remaining part of this section, we fix the parameters α, s , and p as in (4.9) and proceed with a proof of Theorem 1.13.

Proof of Theorem 1.13. Given $v_0 \in L^\infty(\mathbb{T}^2)$ and Θ_N in (1.39), let $v_N = \Phi_{v_0, \Theta_N}(v_N)$ be the solution to (4.1) given by Proposition 4.1. Proceeding as in the proof of Theorem 1.2 in [63], it suffices to prove the continuity of the solution map $\Phi = \Phi_{v_0, \Theta}$ constructed in Proposition 4.1 with respect to Θ .

In the proof of Proposition 4.1, the positivity of the distribution Θ played an important role, allowing us to apply Lemma 2.14. In studying the difference $\Theta_N - \Theta$, we lose such positivity and can no longer apply Lemma 2.14. This prevents us from showing convergence of v_N in $C([0, T]; W^{s,p}(\mathbb{T}^2))$ directly. We instead use a compactness argument.

Let us take a sequence of positive distributions Θ_N converging to some limit Θ in $L^p([0, T]; W^{-\alpha,p}(\mathbb{T}^2)) \cap L^r([0, T]; W^{-s+\varepsilon,r}(\mathbb{T}^2))$, where r is defined by

$$r = \frac{4\pi(s - \varepsilon)}{\beta_{\text{heat}}^2} + 1 = 2 + \frac{\sqrt{2}}{2} \tag{4.10}$$

with s as in (4.9). Note that the pair $(s - \varepsilon, r)$ (in place of (α, p)) satisfies (1.41) for any $\beta^2 < \beta_{\text{heat}}^2$.

Let us then denote by v_N and v the corresponding solutions to (1.45) and (1.46), respectively, constructed in Proposition 4.1. We first show an extra regularity for these solutions:

$$\partial_t v_N \in L^p([0, T]; W^{s-2,p}(\mathbb{T}^2)).$$

Indeed, using Eq. (1.45) with $p < \infty$ and $s - 2 < -\alpha$, we have

$$\begin{aligned} \|\partial_t v_N\|_{L_T^p W_x^{s-2,p}} &= \left\| \frac{1}{2}(\Delta - 1)v - \frac{1}{2}\lambda\beta e^{\beta z} e^{\beta v_N} \Theta_N \right\|_{L_T^p W_x^{s-2,p}} \\ &\lesssim \|v_N\|_{L_T^\infty W_x^{s,p}} + \|e^{\beta z} e^{\beta v} \Theta_N\|_{L_T^p W_x^{-\alpha,p}}. \end{aligned}$$

Note that both of the terms on the right-hand side are already bounded in the proof of Proposition 4.1 (by switching the order of Lemma 2.14 and Young’s inequality in (4.6)).

Next, observe that by taking $\tilde{s} > s$, sufficiently close to s , we can repeat the proof of Proposition 4.1 without changing the range of $\beta^2 < \beta_{\text{heat}}^2$. This shows that $\{v_N\}_{N \in \mathbb{N}}$ is bounded in $C([0, T]; W^{\tilde{s},p}(\mathbb{T}^2))$. Then, by Rellich’s lemma and the Aubin-Lions lemma (Lemma 2.16), we see that the embedding:

$$A_T \stackrel{\text{def}}{=} C([0, T]; W^{\tilde{s},p}(\mathbb{T}^2)) \cap \{\partial_t v \in L^p([0, T]; W^{s-2,p}(\mathbb{T}^2))\} \subset C([0, T]; W^{s,p}(\mathbb{T}^2))$$

is compact. Since $\{v_N\}_{N \in \mathbb{N}}$ is bounded in A_T , given any subsequence of $\{v_N\}_{N \in \mathbb{N}}$, we can extract a further subsequence $\{v_{N_k}\}_{k \in \mathbb{N}}$ such that v_{N_k} converges to some limit \tilde{v} in $C([0, T]; W^{s,p}(\mathbb{T}^2))$. In the following, we show that $\tilde{v} = v$. This implies that the limit is independent of the choice of subsequences and hence the entire sequence $\{v_N\}_{N \in \mathbb{N}}$ converges to v in $C([0, T]; W^{s,p}(\mathbb{T}^2))$.

It remains to prove $\tilde{v} = v$. In the following, we first show that $v_{N_k} = \Phi_{v_0, \Theta_{N_k}}(v_{N_k})$ converges to $\Phi_{v_0, \Theta}(\tilde{v})$ in $L^1([0, T]; W^{s', p}(\mathbb{T}^2))$ for some $s' \leq -s$. From (4.2), we have

$$\begin{aligned} & \|\Phi_{v_0, \Theta_{N_k}}(v_{N_k}) - \Phi_{v_0, \Theta}(\tilde{v})\|_{L_T^1 W_x^{s', p}} \\ & \lesssim \left\| \int_0^t P(t-t')(e^{\beta z} e^{\beta \tilde{v}}(\Theta_{N_k} - \Theta))(t') dt' \right\|_{L_T^1 W_x^{s', p}} \\ & \quad + \left\| \int_0^t P(t-t')(e^{\beta z}(e^{\beta v_{N_k}} - e^{\beta \tilde{v}})\Theta_{N_k})(t') dt' \right\|_{L_T^1 W_x^{s', p}} \\ & =: \text{I} + \text{II}. \end{aligned} \tag{4.11}$$

By the Schauder estimate (Lemma 2.4), Young's inequality, Lemma 2.13(ii) with $\frac{1}{r} + \frac{1}{p} < \frac{1}{r} + \frac{s}{2}$ (which is guaranteed by $sp > 2$), we have

$$\begin{aligned} \text{I} & \lesssim \left\| |\cdot|^{-\left(\frac{1}{r} - \frac{1}{p}\right)} * \|e^{\beta z} e^{\beta \tilde{v}}(\Theta_{N_k} - \Theta)\|_{W_x^{-s+\varepsilon, r}} \right\|_{L_T^1} \\ & \lesssim \|e^{\beta z} e^{\beta \tilde{v}}(\Theta_{N_k} - \Theta)\|_{L_T^1 W_x^{-s+\varepsilon, r}} \\ & \lesssim \|e^{\beta(z+\tilde{v})}\|_{L_T^{r'} W_x^{s-\varepsilon, p}} \|\Theta_{N_k} - \Theta\|_{L_T^r W_x^{-s+\varepsilon, r}}. \end{aligned} \tag{4.12}$$

By Sobolev's inequality and the fractional chain rule (Lemma 2.15(ii)), we have

$$\|\nabla|^{s-\varepsilon} e^{\beta(z+\tilde{v})}(t)\|_{L_x^p} \lesssim \|\nabla|^{s+\tilde{v}} e^{\beta(z+\tilde{v})}(t)\|_{L_x^{\frac{p}{1+\varepsilon p/2}}} \lesssim \|e^{\beta(z+\tilde{v})}(t)\|_{L_x^{\frac{2}{\varepsilon}}} \|\nabla|^{s+\tilde{v}}(z+\tilde{v})(t)\|_{L_x^p}.$$

This yields

$$\begin{aligned} \|e^{\beta(z+\tilde{v})}\|_{L_T^{r'} W_x^{s-\varepsilon, p}} & \lesssim \|e^{\beta(z+\tilde{v})}\|_{L_{T,x}^\infty} \left(1 + \|z + \tilde{v}\|_{L_T^{r'} W_x^{s,p}}\right) \\ & \lesssim e^{C\|v_0\|_{L^\infty}} e^{C\|\tilde{v}\|_{L_T^\infty W_x^{s,p}}} \left(1 + \|v_0\|_{L^\infty} + \|\tilde{v}\|_{L_T^\infty W_x^{s,p}}\right). \end{aligned} \tag{4.13}$$

In the last step, we used the following bound which follows from the Schauder estimate (Lemma 2.4):

$$\|z\|_{L_T^{r'} W_x^{s,p}} \lesssim \|t^{-\frac{s}{2}}\|_{L^p} \|v_0\|_{L^p} \|L_T^{r'}\|_{L^\infty} \lesssim \|v_0\|_{L^\infty}$$

since $\frac{s}{2}r' < 1$ in view of (4.9) and (4.10). Therefore, from (4.12) and (4.13), we obtain

$$\text{I} \lesssim e^{C\|v_0\|_{L^\infty}} e^{C\|\tilde{v}\|_{L_T^\infty W_x^{s,p}}} \left(1 + \|v_0\|_{L^\infty} + \|\tilde{v}\|_{L_T^\infty W_x^{s,p}}\right) \|\Theta_{N_k} - \Theta\|_{L_T^r W_x^{-s+\varepsilon, r}}. \tag{4.14}$$

As for the second term II on the right-hand side of (4.11), we can use the positivity of Θ_{N_k} and proceed as in (4.8):

$$\text{II} \lesssim T^\theta e^{C(\|v_0\|_{L^\infty} + \|v_{N_k}\|_{L_T^\infty W_x^{s,p}} + \|\tilde{v}\|_{L_T^\infty W_x^{s,p}})} \|v_{N_k} - \tilde{v}\|_{L_T^\infty W_x^{s,p}} \|\Theta_{N_k}\|_{L_T^p W_x^{-\alpha, p}}. \tag{4.15}$$

Since $v_{N_k} \rightarrow \tilde{v}$ in $C([0, T]; W^{s,p}(\mathbb{T}^2))$ and $\Theta_{N_k} \rightarrow \Theta$ in $L^p([0, T]; W^{-\alpha, p}(\mathbb{T}^2)) \cap L^r([0, T]; W^{-s+\varepsilon, r}(\mathbb{T}^2))$, it follows from (4.11), (4.14), and (4.15) that $v_{N_k} = \Phi_{v_0, \Theta_{N_k}}$

(v_{N_k}) converges to $\Phi_{v_0, \Theta}(\tilde{v})$ in $L^1([0, T]; W^{s', p}(\mathbb{T}^2))$. By the uniqueness of the distributional limit, we conclude that

$$\tilde{v} = \Phi_{v_0, \Theta}(\tilde{v}). \tag{4.16}$$

Since \tilde{v} belongs to $C([0, T]; W^{s, p}(\mathbb{T}^2))$, we conclude from the uniqueness of the solution to (4.16) that $v = \tilde{v}$, where v denotes the unique fixed point to (4.16) in the class $C([0, T]; W^{s, p}(\mathbb{T}^2))$ constructed in Proposition 4.1. See also Remark 4.2. \square

Remark 4.3. While the argument above shows the continuity of the solution map in Θ , its dependence is rather weak. For the range $0 < \beta^2 < \frac{4}{3}\pi$, we can strengthen this result by proving local well-posedness and convergence *without* the positivity of Θ . This argument shows that, for the range $0 < \beta^2 < \frac{4}{3}\pi$, the solution map is also Lipschitz with respect to Θ , as in the hyperbolic case presented in Sect. 6 below. See Appendix A.

5. Parabolic Liouville Equation II: Using the Sign-definite Structure

In this section, we study SNLH (1.1) in the defocusing case ($\lambda > 0$) and present a proof of Theorem 1.2 and Theorem 1.6. As we will see below, the particular structure of the equation makes the exponential nonlinearity behave as a smooth bounded function. This allows us to treat the full range $0 < \beta^2 < 4\pi$ in this case.

5.1. *Global well-posedness.* In this subsection, we focus on the equation:

$$\begin{cases} \partial_t v + \frac{1}{2}(1 - \Delta)v + \frac{1}{2}\lambda\beta e^{\beta z} e^{\beta v} \Theta = 0 \\ v|_{t=0} = 0, \end{cases} \tag{5.1}$$

where $z = P(t)v_0$ for some $v_0 \in L^\infty(\mathbb{T}^2)$, Θ is a given *deterministic* positive space-time distribution, and $\lambda > 0$. In this case, as explained in Sect. 1.3, Eq. (5.1) can be written as

$$\begin{cases} \partial_t v + \frac{1}{2}(1 - \Delta)v + \frac{1}{2}\lambda\beta e^{\beta z} F(\beta v)\Theta = 0 \\ v|_{t=0} = 0, \end{cases} \tag{5.2}$$

where F is a smooth bounded and Lipschitz function defined in (1.49). Indeed, by writing (5.2) in the Duhamel formulation:

$$v(t) = -\frac{1}{2}\lambda\beta \int_0^t P(t-t')(e^{\beta z} F(\beta v)\Theta)(t')dt', \tag{5.3}$$

it follows from the non-negativity of λ , Θ , and F along with Lemma 2.4(i) that $\beta v \leq 0$. This means that the Cauchy problems (5.1) and (5.2) are equivalent.

Given $N \in \mathbb{N}$, consider the following equation:

$$\begin{cases} \partial_t v_N + \frac{1}{2}(1 - \Delta)v_N + \frac{1}{2}\lambda\beta e^{\beta z} F(\beta v_N)\Theta_N = 0 \\ v_N|_{t=0} = 0 \end{cases} \tag{5.4}$$

for some given smooth space-time non-negative function Θ_N . Then, since Θ_N is smooth and F is bounded and Lipschitz, we can apply a standard contraction argument to prove

local well-posedness of (5.4) in the class $C([0, \tau]; L^2(\mathbb{T}^2))$ for some small $\tau = \tau_N > 0$. Thanks to the boundedness of F , we can also establish an a priori bound on the L^2 -norm of the solution v_N on any time interval $[0, T]$; see (5.7) below. This shows global existence of v_N .

Our main goal in this subsection is to prove global well-posedness of (5.2).

Proposition 5.1. *Let $v_0 \in L^\infty(\mathbb{T}^2)$ and $\Theta \in L^2([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$ be a positive distribution for some $\varepsilon > 0$. Given $T > 0$, suppose that a sequence $\{\Theta_N\}_{N \in \mathbb{N}}$ of smooth non-negative functions converges to Θ in $L^2([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$. Then, the corresponding solution v_N to (5.4) converges to a limit v in the energy space \mathcal{Z}_T defined in (1.50). Furthermore, the limit v is the unique solution to (5.2) in the energy class \mathcal{Z}_T .*

In view of Proposition 1.12 with $p = 2$, given $0 < \beta^2 < 4\pi$, we can choose $\varepsilon > 0$ sufficiently small such that $\frac{\beta^2}{4\pi} < 1 - \varepsilon$, which guarantees that the Gaussian multiplicative chaos Θ_N in (1.39) belongs to $L^2([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$ for any $T > 0$, almost surely. Moreover, Θ_N converges in probability to Θ in (1.42) in the same class. Then, Theorem 1.14 follows from Proposition 5.1 above.

Proof of Proposition 5.1. With a slight abuse of notation, we set

$$\Phi = \Phi_{v_0, \Theta} \quad \text{and} \quad \Phi_N = \Phi_{v_0, \Theta_N},$$

where $\Phi_{v_0, \Theta}$ is defined in (4.2). In particular, we have

$$\begin{aligned} v_N &= \Phi_N(v_N) = \Phi_{v_0, \Theta_N}(v_N) \\ &= -\frac{1}{2}\lambda\beta \int_0^t P(t-t')(e^{\beta z} F(\beta v_N)\Theta_N)(t')dt'. \end{aligned} \tag{5.5}$$

Fix $T > 0$. Given $v_0 \in L^\infty(\mathbb{T}^2)$, we see that $z = P(t)v_0$ and v_N belong to $C((0, T]; C(\mathbb{T}^2))$ in view of the Schauder estimate (Lemma 2.4) and (5.5) with smooth Θ_N . Hence, we can apply Lemma 2.14 to estimate the product $e^{\beta z} F(\beta v_N)\Theta_N$ thanks to the positivity of Θ_N .

Fix small $\delta > 0$. Then, by the Schauder estimate (Lemma 2.4), Lemma 2.14, and Young’s inequality, we have

$$\begin{aligned} \|v_N\|_{L_T^2 H_x^{1+2\delta}} &\lesssim \left\| \int_0^t (t-t')^{-\frac{2+2\delta-\varepsilon}{2}} \|\langle \nabla \rangle^{-1+\varepsilon} (e^{\beta z} F(\beta v_N)\Theta_N)(t')\|_{L_x^2} dt' \right\|_{L_T^2} \\ &\lesssim \|e^{\beta z} F(\beta v_N)\|_{L_{T,x}^\infty} \left\| \int_0^t (t-t')^{-\frac{2+2\delta-\varepsilon}{2}} \|\Theta_N(t')\|_{H_x^{-1+\varepsilon}} dt' \right\|_{L_T^2} \\ &\lesssim e^C \|v_0\|_{L^\infty} \|\Theta_N\|_{L_T^2 H_x^{-1+\varepsilon}}, \end{aligned} \tag{5.6}$$

uniformly in $N \in \mathbb{N}$, provided that $2\delta < \varepsilon$. Here, we crucially used the boundedness of F . Similarly, we have

$$\begin{aligned} \|v_N\|_{L_T^\infty H_x^{2\delta}} &\lesssim e^C \|v_0\|_{L^\infty} \left\| \int_0^t (t-t')^{-\frac{1+2\delta-\varepsilon}{2}} \|\Theta_N(t')\|_{H_x^{-1+\varepsilon}} dt' \right\|_{L_T^\infty} \\ &\lesssim e^C \|v_0\|_{L^\infty} \|\Theta_N\|_{L_T^2 H_x^{-1+\varepsilon}} \end{aligned} \tag{5.7}$$

and

$$\begin{aligned} \|\partial_t v_N\|_{L_T^2 H_x^{-1+2\delta}} &= \left\| \frac{1}{2}(\Delta - 1)v_N - \frac{1}{2}\lambda\beta e^{\beta z} F(\beta v_N)\Theta_N \right\|_{L_T^2 H_x^{-1+2\delta}} \\ &\lesssim \|v_N\|_{L_T^2 H_x^{1+2\delta}} + \|e^{\beta z} F(\beta v_N)\Theta_N\|_{L_T^2 H_x^{-1+\varepsilon}} \\ &\lesssim e^C \|v_0\|_{L^\infty} \|\Theta_N\|_{L_T^2 H_x^{-1+\varepsilon}}, \end{aligned} \tag{5.8}$$

uniformly in $N \in \mathbb{N}$.

Given $s \in \mathbb{R}$, define \mathcal{Z}_T^s and $\tilde{\mathcal{Z}}_T^s$ by

$$\begin{aligned} \mathcal{Z}_T^s &= C([0, T]; H^s(\mathbb{T}^2)) \cap L^2([0, T]; H^{1+s}(\mathbb{T}^2)), \\ \tilde{\mathcal{Z}}_T^s &= \{v \in \mathcal{Z}_T^s : \partial_t v \in L^2([0, T]; H^{-1+s}(\mathbb{T}^2))\}. \end{aligned}$$

Then, it follows from Rellich’s lemma and the Aubin-Lions lemma (Lemma 2.16) that the embedding of $\tilde{\mathcal{Z}}_T^{2\delta} \subset \mathcal{Z}_T^{2\delta}$ is compact. Then, from (5.6), (5.7), and (5.8) along with the convergence of Θ_N to Θ in $L^2([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$, we see that $\{v_N\}_{N \in \mathbb{N}}$ is bounded in $\tilde{\mathcal{Z}}_T^{2\delta}$ and thus is precompact in \mathcal{Z}_T^δ . Hence, there exists a subsequence $\{v_{N_k}\}_{k \in \mathbb{N}}$ converging to some limit v in \mathcal{Z}_T^δ .

Next, we show that the limit v satisfies the Duhamel formulation (5.3). In particular, we prove that $\Phi_{N_k}(v_{N_k})$ converges to $\Phi(v)$ in $L^1([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$. Write

$$\begin{aligned} \|\Phi_{N_k}(v_{N_k}) - \Phi(v)\|_{L_T^1 H_x^{-1+\varepsilon}} &\lesssim \left\| \int_0^t P(t-t')(e^{\beta z} F(\beta v_{N_k})(\Theta_{N_k} - \Theta))(t') dt' \right\|_{L_T^1 H_x^{-1+\varepsilon}} \\ &\quad + \left\| \int_0^t P(t-t')(e^{\beta z} (F(\beta v_{N_k}) - F(\beta v))\Theta)(t') dt' \right\|_{L_T^1 H_x^{-1+\varepsilon}} \\ &=: \text{I} + \text{II}. \end{aligned} \tag{5.9}$$

By the Schauder estimate (Lemma 2.4), Young’s inequality, and Lemma 2.13 (ii), we have

$$\begin{aligned} \text{I} &\lesssim \|e^{\beta z} F(\beta v_{N_k})(\Theta_{N_k} - \Theta)\|_{L_T^1 W_x^{-1+\varepsilon, 1}} \\ &\lesssim \|e^{\beta z} F(\beta v_{N_k})\|_{L_T^2 W_x^{1-\varepsilon, \frac{1}{1-\varepsilon}}} \|\Theta_{N_k} - \Theta\|_{L_T^2 H_x^{-1+\varepsilon}} \end{aligned} \tag{5.10}$$

for sufficiently small $\varepsilon > 0$.

By the fractional Leibniz rule (Lemma 2.13 (i)), we have

$$\begin{aligned} &\|e^{\beta z} F(\beta v_{N_k})\|_{L_T^2 W_x^{1-\varepsilon, \frac{1}{1-\varepsilon}}} \\ &\lesssim \|e^{\beta z}\|_{L_T^2 H_x^{1-\varepsilon}} \|F(\beta v_{N_k})\|_{L_{T,x}^\infty} + \|e^{\beta z}\|_{L_{T,x}^\infty} \|F(\beta v_{N_k})\|_{L_T^2 H_x^{1-\varepsilon}}. \end{aligned} \tag{5.11}$$

By the fractional chain rule (Lemma 2.15 (ii)), we have

$$\begin{aligned} \|e^{\beta z}\|_{L_T^2 H_x^{1-\varepsilon}} &\sim \|e^{\beta z}\|_{L_{T,x}^2} + \|\nabla\|^{1-\varepsilon} e^{\beta z}\|_{L_{T,x}^2} \\ &\lesssim T^{\frac{1}{2}} e^{C\|z\|_{L_{T,x}^\infty}} + \|e^{\beta z}\|_{L_T^\infty L_x^4} \|\nabla\|^{1-\varepsilon} z\|_{L_T^2 L_x^4} \\ &\leq C(T) e^{C\|v_0\|_{L^\infty}} (1 + \|z\|_{L_T^2 W_x^{1-\varepsilon, 4}}) \\ &\leq C(T) e^{C\|v_0\|_{L^\infty}} (1 + \|v_0\|_{L^\infty}), \end{aligned} \tag{5.12}$$

where we used the Schauder estimate (Lemma 2.4) in the last step. Similarly, by the fractional chain rule (Lemma 2.15 (i)) along with the boundedness of F , we have

$$\begin{aligned} \|F(\beta v_{N_k})\|_{L_T^2 H_x^{1-\varepsilon}} &\sim \|F(\beta v_{N_k})\|_{L_T^2 L_x^\infty} + \||\nabla|^{1-\varepsilon} F(\beta v_{N_k})\|_{L_T^2 L_x^\infty} \\ &\lesssim T^{\frac{1}{2}} + \||\nabla|^{1-\varepsilon} v_{N_k}\|_{L_T^2 L_x^\infty} \\ &\leq C(T)(1 + \|v_{N_k}\|_{\mathcal{Z}_T^\delta}). \end{aligned} \tag{5.13}$$

Hence, putting (5.10), (5.11), (5.12), and (5.13) together, we obtain

$$I \lesssim e^{C\|v_0\|_{L^\infty}} \left(1 + \|v_0\|_{L^\infty} + \|v_{N_k}\|_{\mathcal{Z}_T^\delta}\right) \|\Theta_{N_k} - \Theta\|_{L_T^2 H_x^{-1+\varepsilon}}. \tag{5.14}$$

As for the second term Π in (5.9), we use the fundamental theorem of calculus and write

$$F(\beta v_{N_k}) - F(\beta v) = \beta(v_{N_k} - v)G(v_{N_k}, v), \tag{5.15}$$

where

$$G(v_1, v_2) = \int_0^1 F'(\tau\beta v_1 + (1 - \tau)\beta v_2) d\tau. \tag{5.16}$$

Since F is Lipschitz, we see that G is bounded. Since $v_{N_k}, v \in \mathcal{Z}_T^\delta$, we have $v_{N_k}(t), v(t) \in C(\mathbb{T}^2)$ for almost every $t \in [0, T]$. Then, by the Schauder estimate (Lemma 2.4), Lemma 2.14, and Hölder’s inequality, we have

$$\begin{aligned} \Pi &\lesssim \|e^{\beta z}(v_{N_k} - v)G(v_{N_k}, v)\Theta\|_{L_T^1 H_x^{-1+\varepsilon}} \\ &\lesssim \|e^{\beta z}(v_{N_k} - v)G(v_{N_k}, v)\|_{L_T^2 L_x^\infty} \|\Theta\|_{L_T^2 H_x^{-1+\varepsilon}} \\ &\lesssim e^{C\|z\|_{L_T^\infty}} \|v_{N_k} - v\|_{L_T^2 L_x^\infty} \|G(v_{N_k}, v)\|_{L_T^\infty L_x^\infty} \|\Theta\|_{L_T^2 H_x^{-1+\varepsilon}} \\ &\lesssim e^{C\|v_0\|_{L^\infty}} \|v_{N_k} - v\|_{\mathcal{Z}_T^\delta} \|\Theta\|_{L_T^2 H_x^{-1+\varepsilon}}. \end{aligned} \tag{5.17}$$

From (5.9), (5.14), and (5.17) along with the convergence of v_{N_k} to v in \mathcal{Z}_T^δ and Θ_{N_k} to Θ in $L^2([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$, we conclude that $\Phi_{N_k}(v_{N_k})$ converges to $\Phi(v)$ in $L^1([0, T]; H^{-1+\varepsilon}(\mathbb{T}^2))$. Since $v_{N_k} = \Phi_N(v_{N_k})$, this shows that

$$v = \lim_{k \rightarrow \infty} v_{N_k} = \lim_{k \rightarrow \infty} \Phi_{N_k}(v_{N_k}) = \Phi(v)$$

as distributions and hence as elements in \mathcal{Z}_T^δ since $v \in \mathcal{Z}_T^\delta$. This proves existence of a solution to (5.3) in $\mathcal{Z}_T^\delta \subset \mathcal{Z}_T$.

Lastly, we prove uniqueness of solutions to (5.3) in the energy space \mathcal{Z}_T . Let $v_1, v_2 \in \mathcal{Z}_T$ be two solutions to (5.3). Then, by setting $w = v_1 - v_2$, the difference w satisfies

$$\partial_t w + \frac{1}{2}(1 - \Delta)w + \frac{1}{2}\lambda\beta e^{\beta z}(F(\beta v_1) - F(\beta v_2))\Theta = 0. \tag{5.18}$$

Since $\beta v_j \leq 0, j = 1, 2$, it follows from (1.49) and (5.16) that

$$G(v_1, v_2) = \int_0^1 \exp(\tau\beta v_1 + (1 - \tau)\beta v_2) d\tau \geq 0.$$

Now, define an energy functional:

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \|w(t)\|_{L_x^2}^2 + \frac{1}{2} \int_0^t \|w(t')\|_{H_x^1}^2 dt' \geq 0.$$

Since $w \in \mathcal{Z}_T$, the energy functional $\mathcal{E}(t)$ is a well-defined differentiable function. Moreover, with (5.18) and (5.15), we have

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(t) &= \int_{\mathbb{T}^2} w(t)(2\partial_t w(t) + (1 - \Delta)w(t)) dx \\ &= -\lambda\beta^2 \int_{\mathbb{T}^2} w^2(t)e^{\beta z} G(v_1, v_2)\Theta(t) dx \\ &\leq 0 \end{aligned}$$

thanks to the positivity of G and Θ and the assumption that $\lambda > 0$. Since $w(0) = 0$, we conclude that $\mathcal{E}(t) = 0$ for any $t \geq 0$ and $v_1 \equiv v_2$. This proves uniqueness in the energy space \mathcal{Z}_T .

The solution $v \in \mathcal{Z}_T^\delta$ constructed in the existence part depends a priori on a choice of a subsequence v_{N_k} . The uniqueness in $\mathcal{Z}_T \supset \mathcal{Z}_T^\delta$, however, shows that the limit v is independent of the choice of a subsequence and hence the entire sequence $\{v_N\}_{N \in \mathbb{N}}$ converges to v in $\mathcal{Z}_T^\delta \subset \mathcal{Z}_T$. This completes the proof of Proposition 5.1. \square

5.2. On invariance of the Gibbs measure. In this subsection, we briefly go over the proof of Theorem 1.6. Given $N \in \mathbb{N}$, we consider the truncated SNLH (1.19) with initial data given by $u_N|_{t=0} = w_0$, where w_0 is as in (1.8) distributed by the massive Gaussian free field μ_1 . For this problem, there is no deterministic linear solution z and hence write u_N as $u_N = v_N + \Psi^{\text{heat}}$. Then, the residual term v_N satisfies

$$v_N(t) = -\frac{1}{2} \lambda\beta \int_0^t P(t - t') \mathbf{Q}_N(e^{\beta \mathbf{Q}_N v_N} \Theta_N)(t') dt', \tag{5.19}$$

where Θ_N is the Gaussian multiplicative chaos defined in terms of \mathbf{Q}_N . Since the smoothing operator \mathbf{Q}_N in (1.16) is equipped with a non-negative kernel, Eq. (5.19) enjoys the sign-definite structure:

$$\beta \mathbf{Q}_N v_N(t) = -\frac{1}{2} \lambda\beta^2 \int_0^t P(t - t') \mathbf{Q}_N^2(e^{\beta \mathbf{Q}_N v_N} \Theta_N)(t') dt' \leq 0.$$

Namely, we can rewrite (5.19) as

$$v_N(t) = -\frac{1}{2} \lambda\beta \int_0^t P(t - t') \mathbf{Q}_N(F(\beta \mathbf{Q}_N v_N) \Theta_N)(t') dt', \tag{5.20}$$

where F is as in (1.49).

In view of the uniform (in N) boundedness of \mathbf{Q}_N on $L^p(\mathbb{T}^2)$, $1 \leq p \leq \infty$, we can argue as in Sect. 5.1 to prove local well-posedness of (5.20) and establish an a priori bound on $\{v_N\}_{N \in \mathbb{N}}$ in $\mathcal{Z}_T^{2\delta} \subset \mathcal{Z}_T^\delta$. Then, by the Aubin-Lions lemma (Lemma 2.16), we see that there exists a subsequence $\{v_{N_k}\}_{k \in \mathbb{N}}$ converging to some limit v in \mathcal{Z}_T^δ . Moreover, the uniqueness argument for solutions to the limiting equation (5.3) remains

true. Therefore, in view of the argument in Sect. 5.1, it suffices to show that the limit v satisfies the equation (5.3).

With a slight abuse of notation, let Φ_{N_k} denotes the right-hand side of (5.20):

$$\Phi_{N_k}(v_{N_k})(t) \stackrel{\text{def}}{=} -\frac{1}{2}\lambda\beta \int_0^t P(t-t')\mathbf{Q}_{N_k}(F(\beta\mathbf{Q}_{N_k}v_{N_k})\Theta_{N_k})(t')dt'. \tag{5.21}$$

Then, it suffices to show that $\Phi_{N_k}(v_{N_k})$ converges to $\Phi(v)$ in $L^1([0, T]; H^{-1}(\mathbb{T}^2))$, where $\Phi = \Phi_{v_0, \Theta}$ is as in (4.2) (with $v_0 = 0$). From (4.2) and (5.21), we have

$$\begin{aligned} & \|\Phi_{N_k}(v_{N_k}) - \Phi(v)\|_{L^1_T H_x^{-1}} \\ & \lesssim \left\| \int_0^t P(t-t')(F(\beta\mathbf{Q}_{N_k}v_{N_k})(\Theta_{N_k} - \Theta))(t')dt' \right\|_{L^1_T H_x^{-1}} \\ & \quad + \left\| \int_0^t P(t-t')((F(\beta\mathbf{Q}_{N_k}v_{N_k}) - F(\beta v))\Theta)(t')dt' \right\|_{L^1_T H_x^{-1}} \\ & \quad + \left\| \int_0^t P(t-t')(\mathbf{Q}_{N_k} - \text{Id})(F(\beta\mathbf{Q}_{N_k}v_{N_k})\Theta_{N_k})(t')dt' \right\|_{L^1_T H_x^{-1}} \\ & =: \text{I} + \text{II} + \text{III}. \end{aligned} \tag{5.22}$$

The terms I and II can be handled exactly as in Sect. 5.1 and, hence, it remains to treat the extra term III.

When viewed as a Fourier multiplier operator, the symbol for \mathbf{Q}_N is given by $2\pi\widehat{\rho}_N$; see (1.16). Note that, for $0 < s_1 - s < 1$, the symbol

$$m_N(\xi) \stackrel{\text{def}}{=} N^{s_1-s} \langle \xi \rangle^{s-s_1} (2\pi\widehat{\rho}_N(\xi) - 1) \tag{5.23}$$

satisfies the bound

$$|\partial_\xi^k m_N(\xi)| \lesssim \langle \xi \rangle^{-|k|} \tag{5.24}$$

for any $k \in (\mathbb{Z}_{\geq 0})^2$. Indeed, when no derivatives hits $2\pi\widehat{\rho}_N - 1$, we can use the mean value theorem (as $2\pi\widehat{\rho}(0) = 1$) to get the bound

$$\begin{aligned} |N^{s_1-s} \partial_\xi^k (\langle \xi \rangle^{s-s_1}) \cdot (2\pi\widehat{\rho}_N(\xi) - 1)| & \lesssim N^{s_1-s} \langle \xi \rangle^{s-s_1-|k|} (1 \wedge N^{-1}|\xi|) \\ & \leq \langle \xi \rangle^{-|k|}, \end{aligned}$$

whereas when at least one derivative hits $2\pi\widehat{\rho}_N - 1$, we gain a negative power of N from $\widehat{\rho}_N(\xi) = \widehat{\rho}(N^{-1}\xi)$ and we use the fast decay of $\widehat{\rho}$ and its derivatives; with $|\alpha| + |\beta| = |k|$, we have

$$\begin{aligned} |N^{s_1-s} \partial_\xi^\alpha (\langle \xi \rangle^{s-s_1}) \cdot \partial_\xi^\beta (2\pi\widehat{\rho}_N(\xi) - 1)| & \lesssim N^{s_1-s-|\beta|} \langle \xi \rangle^{s-s_1-|\alpha|} \cdot (N|\xi|^{-1})^{s-s_1+|\beta|} \\ & \lesssim \langle \xi \rangle^{-|k|}, \end{aligned}$$

verifying (5.24).

Hence, by the transference principle ([33, Theorem 4.3.7]) and the Mihlin-Hörmander multiplier theorem ([33, Theorem 6.2.7]), the Fourier multiplier operator $N^{s_1-s}(\nabla)^{s-s_1}(\mathbf{Q}_N - \text{Id})$ with the symbol m_N in (5.23) is bounded from $L^p(\mathbb{T}^2)$ to $L^p(\mathbb{T}^2)$ for any

$1 < p < \infty$ with norm independent of N . This implies that the following estimate holds:

$$\|(\mathbf{Q}_N - \text{Id})f\|_{W^{s,p}(\mathbb{T}^2)} \lesssim N^{s-s_1} \|f\|_{W^{s_1,p}(\mathbb{T}^2)} \tag{5.25}$$

for any $0 < s_1 - s < 1$ and $1 < p < \infty$. Then, applying (5.25) and Lemma 2.14 again, we can bound III in (5.22) by

$$\begin{aligned} & \|(\mathbf{Q}_{N_k} - \text{Id})(F(\beta \mathbf{Q}_{N_k} v_{N_k}))^{\Theta_{N_k}}\|_{L_T^1 H_x^{-1}} \\ & \lesssim N_k^{-\varepsilon} \|F(\beta \mathbf{Q}_{N_k} v_{N_k})\|_{L_T^2 L_x^\infty} \|\Theta_{N_k}\|_{L_T^2 H_x^{-1+\varepsilon}} \\ & \lesssim N_k^{-\varepsilon} \|\Theta_{N_k}\|_{L_T^2 H_x^{-1+\varepsilon}}. \end{aligned} \tag{5.26}$$

Hence, from (5.22), the convergence of I and II to 0 as shown in Sect. 5.1, and (5.26), we conclude that $\Phi_{N_k}(v_{N_k})$ in (5.21) converges to $\Phi(v)$ in $L^1([0, T]; H^{-1}(\mathbb{T}^2))$. Combined with the uniqueness of the solution to (5.3) in \mathcal{Z}_T , this shows that the solution v_N to the truncated SNLH (5.20) converges to the solution v to SNLH (5.3) (with $z = 0$).

Lastly, we establish invariance of the Gibbs measure ρ_{heat} constructed in Proposition 1.4 under the dynamics of SNLH (1.1). In the following, we write $\Phi_N(t)$ and $\Phi(t)$ for the flow maps of the truncated SNLH (1.19) and SNLH (1.1), respectively, constructed above. Note that $\Phi(t)(u_0)$ is interpreted as $\Phi(t)(u_0) = \Psi + v$, where Ψ is the stochastic convolution defined in (1.27) (with $w_0 = u_0$) and v is the solution to (5.1) (with $z \equiv 0$). In the remaining part of this section, we take the space-time white noise $\xi = \xi^\omega$ in the equation to be on a probability space (Ω_1, \mathbb{P}) and use ω to denote the randomness coming from the space-time white noise. Moreover, we use \mathbb{E}_ω to denote an expectation with respect to the noise, namely, integration with respect to the probability measure \mathbb{P} . In the following, we write $\Phi^\omega(t)(u_0)$, when we emphasize the dependence of the solution on the noise. A similar comment applies to $\Phi_N(t)$. Given $N \in \mathbb{N}$, we use \mathcal{P}_t^N to denote the Markov semigroup associated with the truncated dynamics $\Phi_N^\omega(t)$:

$$\mathcal{P}_t^N(F)(u_0) = \mathbb{E}_\omega[F(\Phi_N^\omega(t)(u_0))] = \int_{\Omega_1} F(\Phi_N^\omega(t)(u_0)) d\mathbb{P}(\omega).$$

We first show invariance of the truncated Gibbs measure $\rho_{\text{heat},N}$ in (1.18) under the truncated dynamics (1.19).

Lemma 5.2. *Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Then, for any continuous and bounded function $F : H^{-\varepsilon}(\mathbb{T}^2) \rightarrow \mathbb{R}$, we have*

$$\int \mathcal{P}_t^N(F)(u_0) d\rho_{\text{heat},N}(u_0) = \int F(u_0) d\rho_{\text{heat},N}(u_0).$$

Proof. Since the truncated Gibbs measure $\rho_{\text{heat},N}$ in (1.18) truncated by \mathbf{Q}_N does not have a finite Fourier support, we first approximate it by

$$d\rho_{\text{heat},N,M} = Z_{N,M}^{-1} \exp\left(-\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta \mathbf{P}_M \mathbf{Q}_N u} dx\right) d\mu_1, \tag{5.27}$$

where \mathbf{P}_M is the Fourier multiplier with a compactly supported symbol χ_N in (1.7) and

$$C_{N,M} = e^{-\frac{\beta^2}{2} \sigma_{N,M}} = e^{-\frac{\beta^2}{2} \mathbb{E}[(\mathbf{P}_M \mathbf{Q}_N \Psi^{\text{heat}}(t,x))^2]} \longrightarrow C_N,$$

as $M \rightarrow \infty$. Here, Ψ^{heat} is as in (1.35).

Let

$$\Theta_N = e^{-\frac{\beta^2}{2}\sigma_N} e^{\beta\mathbf{Q}_N\Psi^{\text{heat}}} \quad \text{and} \quad \Theta_{N,M} = e^{-\frac{\beta^2}{2}\sigma_N} e^{\beta\mathbf{P}_N\mathbf{Q}_N\Psi^{\text{heat}}}.$$

Then, a slight modification of the proof of Proposition 1.12 shows that $\widehat{\Theta}_{N,M}(0, 0)$ converges to $\widehat{\Theta}_N(0, 0)$ in $L^p(\Omega)$ for $1 \leq p < \frac{8\pi}{\beta^2}$. Namely, we have

$$-\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta\mathbf{Q}_N\mathbf{P}_M u} dx \longrightarrow -\lambda C_N \int_{\mathbb{T}^2} e^{\beta\mathbf{Q}_N u} dx \tag{5.28}$$

in $L^p(\mu_1)$ for $1 \leq p < \frac{8\pi}{\beta^2}$ and also in probability. Let R_N be as in (1.43) and define $R_{N,M}$ by

$$R_{N,M} = \exp\left(-\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta\mathbf{P}_N\mathbf{Q}_N u} dx\right). \tag{5.29}$$

Then, it follows from (5.28) that $R_{N,M}$ converges to R_N in probability as $M \rightarrow \infty$. Moreover, by the positivity of Θ_N , $\Theta_{N,M}$, and λ , the densities R_N and $R_{N,M}$ are uniformly bounded by 1. As in the proof of Proposition 1.4, this implies the $L^p(\mu_1)$ -convergence of the density $R_{N,M}$ to R_N as $M \rightarrow \infty$, which in turn shows convergence in total variation $\rho_{\text{heat},N,M} \rightarrow \rho_{\text{heat},N}$ as $M \rightarrow \infty$.

Next, consider the truncated dynamics (1.19) with the Gaussian initial data $\text{Law}(u_N(0)) = \mu_1$. Then, proceeding as in the proof of Theorem 1.2, we see that the flow Φ_N of (1.19) is a limit in probability (with respect to $\mathbb{P} \otimes \mu_1(d\omega, du_0)$) in $C([0, T]; H^{-\varepsilon}(\mathbb{T}^2))$, $\varepsilon > 0$, of the flow $\Phi_{N,M}$ for the following truncated dynamics:

$$\begin{cases} \partial_t u_{N,M} + \frac{1}{2}(1 - \Delta)u_{N,M} + \frac{1}{2}\lambda\beta C_{N,M}\mathbf{P}_M\mathbf{Q}_N e^{\beta\mathbf{P}_M\mathbf{Q}_N u_{N,M}} = \xi \\ u_{N,M}|_{t=0} = u_0 \quad \text{with } \text{Law}(u_0) = \mu_1. \end{cases} \tag{5.30}$$

Let us now discuss invariance of $\rho_{\text{heat},N,M}$ under (5.30). Let Π_M be the sharp Fourier truncation on frequencies $\{|n| \leq M\}$. Then, from the definition (1.7) of \mathbf{P}_M , we have $\Pi_{2M}\mathbf{P}_M = \mathbf{P}_M$ for any $M \in \mathbb{N}$. In particular, with $\Pi_{2M}^\perp = \text{Id} - \Pi_{2M}$, we have $\Pi_{2M}^\perp\mathbf{P}_M = 0$. Then, it follows from (5.27) that the pushforward measure $(\Pi_{2M}^\perp)_\# \rho_{\text{heat},N,M}$ is Gaussian:

$$(\Pi_{2M}^\perp)_\# \rho_{\text{heat},N,M} = (\Pi_{2M}^\perp)_\# \mu_1.$$

Hence, we have the following decomposition:

$$\rho_{\text{heat},N,M} = (\Pi_{2M})_\# \rho_{\text{heat},N,M} \otimes (\Pi_{2M}^\perp)_\# \mu_1.$$

By writing

$$u_{N,M} = \Pi_{2M} u_{N,M} + \Pi_{2M}^\perp u_{N,M} =: u^{(1)} + u^{(2)},$$

where, for simplicity, we dropped the subscripts on the right-hand side, we see that the high frequency part $u^{(2)}$ satisfies the linear stochastic heat equation:

$$\partial_t u^{(2)} + \frac{1}{2}(1 - \Delta)u^{(2)} = \Pi_{2M}^\perp \xi. \tag{5.31}$$

Since this is a linear equation where spatial frequencies are decoupled,¹⁶ it is easy to check that the Gaussian measure $(\Pi_{2M}^{\frac{1}{2}})_{\#\mu_1}$ is invariant under (5.31).

The low frequency part $u^{(1)}$ satisfies the following equation:

$$\partial_t u^{(1)} + \frac{1}{2}(1 - \Delta)u^{(1)} + \mathcal{N}(u^{(1)}) = \Pi_{2M}\xi, \tag{5.32}$$

where the nonlinearity $\mathcal{N} = \mathcal{N}_{N,M}$ is given by

$$\mathcal{N}(u) = \mathcal{N}_{N,M}(u) = \frac{1}{2}\lambda\beta C_{N,M} \mathbf{P}_M \mathbf{Q}_N e^{\beta \mathbf{P}_M \mathbf{Q}_N u}. \tag{5.33}$$

On the Fourier side, (5.32) is a finite-dimensional system of SDEs. As such, one can easily check by hand that $(\Pi_{2M})_{\#\rho_{\text{heat},N,M}}$ is invariant under (5.32). In the following, we review this argument.

In the current real-valued setting, we have $\widehat{u^{(1)}}(-n) = \overline{\widehat{u^{(1)}}(n)}$. Then, by writing $\widehat{u^{(1)}}(n) = a_n + ib_n$ for $a_n, b_n \in \mathbb{R}$, we have

$$a_{-n} = a_n \quad \text{and} \quad b_{-n} = -b_n. \tag{5.34}$$

Defining the index sets $\Lambda = \Lambda(2M) \subset \mathbb{Z}^2$ and $\Lambda_0 = \Lambda_0(2M) \subset \mathbb{Z}^2$, $M \in \mathbb{N}$:

$$\Lambda = \{(\mathbb{N} \times \{0\}) \cup (\mathbb{Z} \times \mathbb{N})\} \cap \{n \in \mathbb{Z}^2 : |n| \leq 2M\} \quad \text{and} \quad \Lambda_0 = \Lambda \cup \{(0, 0)\},$$

we can write (5.32) as

$$\begin{aligned} da_n &= \left(-\frac{1}{2}\langle n \rangle^2 a_n - \text{Re} \widehat{\mathcal{N}(u^{(1)})}(n) \right) dt + d(\text{Re} B_n) \\ db_n &= \left(-\frac{1}{2}\langle n \rangle^2 b_n - \text{Im} \widehat{\mathcal{N}(u^{(1)})}(n) \right) dt + d(\text{Im} B_n) \end{aligned} \tag{5.35}$$

for $n \in \Lambda$ and

$$da_0 = \left(-\frac{1}{2}a_0 - \widehat{\mathcal{N}(u^{(1)})}(0) \right) dt + dB_0. \tag{5.36}$$

Here, $\{B_n\}_{n \in \Lambda_0}$ is a family of mutually independent complex-valued Brownian motions as in (1.29). Note that $\text{Var}(\text{Re} B_n(t)) = \text{Var}(\text{Im} B_n(t)) = \frac{1}{2}$ for $n \in \Lambda$, while $\text{Var}(B_0(t)) = t$.

Let F be a continuous and bounded function on $(\bar{a}, \bar{b}) = (a_m, b_n)_{m \in \Lambda_0, n \in \Lambda} \in \mathbb{R}^{2|\Lambda|+1}$. Then, by Ito’s lemma, the generator $\mathcal{L} = \mathcal{L}_{N,M}$ of the Markov semigroup associated with (5.35) and (5.36) is given by

$$\begin{aligned} \mathcal{L}F(\bar{a}, \bar{b}) &= \sum_{n \in \Lambda_0} \left[\left(-\frac{1}{2}\langle n \rangle^2 a_n - \text{Re} \widehat{\mathcal{N}(u^{(1)})}(n) \right) \partial_{a_n} F(\bar{a}, \bar{b}) + \frac{1}{4} \partial_{a_n}^2 F(\bar{a}, \bar{b}) \right] \\ &\quad + \sum_{n \in \Lambda} \left[\left(-\frac{1}{2}\langle n \rangle^2 b_n - \text{Im} \widehat{\mathcal{N}(u^{(1)})}(n) \right) \partial_{b_n} F(\bar{a}, \bar{b}) + \frac{1}{4} \partial_{b_n}^2 F(\bar{a}, \bar{b}) \right] \\ &\quad + \frac{1}{4} \partial_{a_0}^2 F(\bar{a}, \bar{b}). \end{aligned} \tag{5.37}$$

¹⁶ In particular, by writing (5.31) on the Fourier side, we see that $\widehat{u^{(2)}}(n)$ is the (independent) Ornstein-Uhlenbeck process for each frequency whose invariant measure is Gaussian.

The last term takes into account the different forcing in (5.36). In order to prove invariance of $\rho_{\text{heat},N,M}^{\text{low}} \stackrel{\text{def}}{=} (\Pi_{2M})\#\rho_{\text{heat},N,M}$ under the low-frequency dynamics (5.32), it suffices to prove

$$(\mathcal{L})^* \rho_{\text{heat},N,M}^{\text{low}} = 0.$$

By viewing $\rho_{\text{heat},N,M}^{\text{low}}$ as a measure on (\bar{a}, \bar{b}) with a slight abuse of notation, this is equivalent to proving

$$\int \mathcal{L}F(\bar{a}, \bar{b}) d\rho_{\text{heat},N,M}^{\text{low}}(\bar{a}, \bar{b}) = \int \mathcal{L}F(\bar{a}, \bar{b}) e^{-\mathcal{M}(u^{(1)})} d\bar{a}d\bar{b} = 0, \tag{5.38}$$

where $\mathcal{M}(u^{(1)})$ is given by

$$\mathcal{M}(u^{(1)}) = \lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta \mathbf{P}_M \mathbf{Q}_N u^{(1)}} dx + \sum_{n \in \Lambda} \langle n \rangle^2 (a_n^2 + b_n^2) + \frac{1}{2} a_0^2. \tag{5.39}$$

A direct computation with (5.34) shows

$$\begin{aligned} 2\pi \partial_{a_n} \mathcal{F}_x [(\mathbf{P}_M \mathbf{Q}_N u^{(1)})^k](0) &= 2k \text{Re} \mathcal{F}_x \left[\mathbf{P}_M \mathbf{Q}_N ((\mathbf{P}_M \mathbf{Q}_N u^{(1)})^{k-1}) \right](n), \\ 2\pi \partial_{b_n} \mathcal{F}_x [(\mathbf{P}_M \mathbf{Q}_N u^{(1)})^k](0) &= 2k \text{Im} \mathcal{F}_x \left[\mathbf{P}_M \mathbf{Q}_N ((\mathbf{P}_M \mathbf{Q}_N u^{(1)})^{k-1}) \right](n) \end{aligned} \tag{5.40}$$

for $n \in \Lambda$ and

$$2\pi \partial_{a_0} \mathcal{F}_x [(\mathbf{P}_M \mathbf{Q}_N u^{(1)})^k](0) = k \mathcal{F}_x \left[\mathbf{P}_M \mathbf{Q}_N ((\mathbf{P}_M \mathbf{Q}_N u^{(1)})^{k-1}) \right](0).$$

By the Taylor expansion with (5.40) and (5.33), we have

$$\begin{aligned} \partial_{a_n} \left(\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta \mathbf{P}_M \mathbf{Q}_N u^{(1)}} dx \right) &= \lambda C_{N,M} \cdot 2\pi \partial_{a_n} \mathcal{F}_x [e^{\beta \mathbf{P}_M \mathbf{Q}_N u^{(1)}}](0) \\ &= \lambda C_{N,M} \cdot 2\pi \sum_{k=0}^{\infty} \frac{\beta^k \partial_{a_n} \mathcal{F}_x [(\mathbf{P}_M \mathbf{Q}_N u^{(1)})^k](0)}{k!} \\ &= 4 \text{Re} \widehat{\mathcal{N}}(u^{(1)})(n) \end{aligned} \tag{5.41}$$

for $n \in \Lambda$. By a similar computation, we have

$$\partial_{b_n} \left(\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta \mathbf{P}_M \mathbf{Q}_N u^{(1)}} dx \right) = 4 \text{Im} \widehat{\mathcal{N}}(u^{(1)})(n) \tag{5.42}$$

for $n \in \Lambda$, and

$$\partial_{a_0} \left(\lambda C_{N,M} \int_{\mathbb{T}^2} e^{\beta \mathbf{P}_M \mathbf{Q}_N u^{(1)}} dx \right) = 2 \widehat{\mathcal{N}}(u^{(1)})(0). \tag{5.43}$$

Then, using (5.37), (5.39), (5.41), (5.42), and (5.43), we can rewrite the generator \mathcal{L} as

$$\begin{aligned} \mathcal{L}F(\bar{a}, \bar{b}) &= \sum_{n \in \Lambda} \left[-\frac{1}{4} \partial_{a_n} \mathcal{M}(u^{(1)}) \partial_{a_n} F(\bar{a}, \bar{b}) + \frac{1}{4} \partial_{a_n}^2 F(\bar{a}, \bar{b}) \right. \\ &\quad \left. - \frac{1}{4} \partial_{b_n} \mathcal{M}(u^{(1)}) \partial_{b_n} F(\bar{a}, \bar{b}) + \frac{1}{4} \partial_{b_n}^2 F(\bar{a}, \bar{b}) \right] \\ &\quad - \frac{1}{2} \partial_{a_0} \mathcal{M}(u^{(1)}) \partial_{a_0} F(\bar{a}, \bar{b}) + \frac{1}{2} \partial_{a_0}^2 F(\bar{a}, \bar{b}). \end{aligned} \tag{5.44}$$

Then, with (5.39) and (5.44), integration by parts yields

$$\begin{aligned} &\int \mathcal{L}F(\bar{a}, \bar{b}) e^{-\mathcal{M}(u^{(1)})} d\bar{a} d\bar{b} \\ &= \frac{1}{4} \sum_{n \in \Lambda} \int \left(\partial_{a_n} e^{-\mathcal{M}(u^{(1)})} \cdot \partial_{a_n} F(\bar{a}, \bar{b}) + \partial_{a_n}^2 F(\bar{a}, \bar{b}) \cdot e^{-\mathcal{M}(u^{(1)})} \right) d\bar{a} d\bar{b} \\ &\quad + \frac{1}{4} \sum_{n \in \Lambda} \int \left(\partial_{b_n} e^{-\mathcal{M}(u^{(1)})} \cdot \partial_{b_n} F(\bar{a}, \bar{b}) + \partial_{b_n}^2 F(\bar{a}, \bar{b}) \cdot e^{-\mathcal{M}(u^{(1)})} \right) d\bar{a} d\bar{b} \\ &\quad + \frac{1}{2} \int \left(\partial_{a_0} e^{-\mathcal{M}(u^{(1)})} \cdot \partial_{a_0} F(\bar{a}, \bar{b}) + \partial_{a_0}^2 F(\bar{a}, \bar{b}) \cdot e^{-\mathcal{M}(u^{(1)})} \right) d\bar{a} d\bar{b} \\ &= 0. \end{aligned}$$

This proves (5.38) and hence invariance of $\rho_{\text{heat},N,M}^{\text{low}} = (\Pi_{2M})_{\#} \rho_{\text{heat},N,M}$ under the low-frequency dynamics (5.32).

We are now ready to prove invariance of $\rho_{\text{heat},N}$ under $\Phi_N(t)$. This follows from (i) the convergence of $\rho_{\text{heat},N,M}$ to $\rho_{\text{heat},N}$ in total variation, (ii) the convergence of $\Phi_{N,M}^\omega(t)(u_0)$ to $\Phi_N^\omega(t)(u_0)$ in probability with respect to $\mathbb{P} \otimes \mu_1(\omega, u_0)$, and (iii) the invariance of $\rho_{\text{heat},N,M}$ under $\Phi_{N,M}(t)$.

Indeed, for any $F : H^{-\varepsilon}(\mathbb{T}^2) \rightarrow \mathbb{R}$, continuous and bounded, and any $t \geq 0$, we have

$$\begin{aligned} &\left| \int \mathbb{E}_\omega [F(\Phi_N^\omega(t)(u_0))] d\rho_{\text{heat},N}(u_0) - \int \mathbb{E}_\omega [F(\Phi_{N,M}^\omega(t)(u_0))] d\rho_{\text{heat},N,M}(u_0) \right| \\ &\leq Z_{N,M}^{-1} \int \left| \mathbb{E}_\omega [F(\Phi_N^\omega(t)(u_0))] - \mathbb{E}_\omega [F(\Phi_{N,M}^\omega(t)(u_0))] \right| R_{N,M}(u_0) d\mu_1(u_0) \\ &\quad + \left| \int \mathbb{E}_\omega [F(\Phi_N^\omega(t)(u_0))] d\rho_{\text{heat},N}(u_0) - \int \mathbb{E}_\omega [F(\Phi_N^\omega(t)(u_0))] d\rho_{\text{heat},N,M}(u_0) \right|, \end{aligned} \tag{5.45}$$

where $R_{N,M}$ is as in (5.29). The second term on the right-hand side tends to 0 as $M \rightarrow \infty$ since $\rho_{\text{heat},N,M}$ converges to $\rho_{\text{heat},N}$ in total variation. As for the first term, by the uniform bound $R_{N,M} \leq 1$, we have

$$\begin{aligned} &\int \left| \mathbb{E}_\omega [F(\Phi_N^\omega(t)(u_0))] - \mathbb{E}_\omega [F(\Phi_{N,M}^\omega(t)(u_0))] \right| R_{N,M}(u_0) d\mu_1(u_0) \\ &\leq \int |F(\Phi_N^\omega(t)(u_0)) - F(\Phi_{N,M}^\omega(t)(u_0))| d(\mathbb{P} \otimes \mu_1)(\omega, u_0) \\ &\leq \delta + 2\|F\|_{L^\infty} \cdot \mathbb{P} \otimes \mu_1 \left(|F(\Phi_N^\omega(t)(u_0)) - F(\Phi_{N,M}^\omega(t)(u_0))| > \delta \right) \end{aligned}$$

for any $\delta > 0$. In view of the convergence of $\Phi_{N,M}^\omega(t)(u_0)$ to $\Phi_N^\omega(t)(u_0)$ in probability with respect to $\mathbb{P} \otimes \mu_1(\omega, u_0)$ as $M \rightarrow \infty$, we then obtain

$$\lim_{M \rightarrow \infty} \int \left| \mathbb{E}_\omega[F(\Phi_N^\omega(t)(u_0))] - \mathbb{E}_\omega[F(\Phi_{N,M}^\omega(t)(u_0))] \right| R_{N,M}(u_0) d\mu_1(u_0) \leq \delta$$

Since the choice of $\delta > 0$ was arbitrary, we conclude that

$$\lim_{M \rightarrow \infty} \int \left| \mathbb{E}_\omega[F(\Phi_N^\omega(t)(u_0))] - \mathbb{E}_\omega[F(\Phi_{N,M}^\omega(t)(u_0))] \right| R_{N,M}(u_0) d\mu_1(u_0) = 0. \tag{5.46}$$

Hence, from (5.45), (5.46), and $Z_{N,M} \rightarrow Z_N$ together with the invariance of $\rho_{\text{heat},N,M}$ under $\Phi_{N,M}(t)$, we obtain

$$\begin{aligned} \int \mathbb{E}_\omega[F(\Phi_N^\omega(t)(u_0))] d\rho_{\text{heat},N}(u_0) &= \lim_{M \rightarrow \infty} \int \mathbb{E}_\omega[F(\Phi_{N,M}^\omega(t)(u_0))] d\rho_{\text{heat},N,M}(u_0) \\ &= \lim_{M \rightarrow \infty} \int F(u_0) d\rho_{\text{heat},N,M}(u_0) = \int F(u_0) d\rho_{\text{heat},N}(u_0). \end{aligned}$$

This concludes the proof of Lemma 5.2. □

With Lemma 5.2, we can finally prove invariance of the Gibbs measure ρ_{heat} in Theorem 1.6. Indeed, proceeding as in the proof of Lemma 5.2 above, we can easily deduce invariance of the Gibbs measure ρ_{heat} from (i) the convergence of the truncated Gibbs measures $\rho_{\text{heat},N}$ to the Gibbs measure ρ_{heat} in total variation (Proposition 1.4), (ii) the convergence in probability (with respect to $\mathbb{P} \otimes \mu_1$) of the truncated dynamics (5.20) to the full dynamics (5.3) (with $z = 0$) (Theorem 1.2), and (iii) the invariance of the truncated Gibbs measure $\rho_{\text{heat},N}$ in (1.18) under the truncated SNLH (1.19) (Lemma 5.2). (We also use the absolute continuity of the truncated Gibbs measure $\rho_{\text{heat},N}$ with respect to the massive Gaussian free field μ_1 , with the uniformly (in N) bounded density $R_N \leq 1$.) This concludes the proof of Theorem 1.6.

6. Hyperbolic Liouville Equation

In this section, we study the stochastic damped nonlinear wave equation (1.2) with the exponential nonlinearity. We restrict our attention to the defocusing case ($\lambda > 0$).

6.1. *Local well-posedness of SdNLW.* In this subsection, we present a proof of Theorem 1.15 on local well-posedness of the system (1.56):

$$\begin{aligned} X(t) &= \Phi_1(X, Y) \\ &\stackrel{\text{def}}{=} -\lambda\beta \int_0^t e^{-\frac{(t-t')}{2}} S(t-t')(e^{\beta z} F(\beta X)e^{\beta Y} \Theta)(t') dt', \\ Y(t) &= \Phi_2(X, Y) \\ &\stackrel{\text{def}}{=} -\lambda\beta \int_0^t (\mathcal{D}(t-t') - e^{-\frac{(t-t')}{2}} S(t-t'))(e^{\beta z} F(\beta X)e^{\beta Y} \Theta)(t') dt', \end{aligned} \tag{6.1}$$

where F is as in (1.49) and Θ is a positive distribution in $L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2))$ with α and $1 < p < \frac{8\pi}{\beta^2}$ satisfying (1.41). Here, $\mathcal{D}(t)$ and $S(t)$ are the linear propagators

defined in (1.32) and (1.55) and z denotes the linear solution in (1.52) with initial data $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for some $s > 1$.

We prove local well-posedness of (6.1) by a contraction argument for $(X, Y) \in \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$, where the Strichartz type spaces $\mathcal{X}_T^{s_1}$ and $\mathcal{Y}_T^{s_2}$ are defined in (1.57) and (1.58) for some $\frac{1}{4} < s_1 < \frac{3}{4}$ and $1 < s_2 < 2$ (to be chosen later). See also (2.16). In the following, we fix the following s_1 -admissible pair (q, r) and dual s_1 -admissible pair (\tilde{q}, \tilde{r}) (see Definition 2.7 for (dual) admissible pairs):

$$(q, r) = \left(\frac{3}{s_1}, \frac{6}{3 - 4s_1} \right) \quad \text{and} \quad (\tilde{q}, \tilde{r}) = \left(\frac{3}{2 + s_1}, \frac{6}{7 - 4s_1} \right). \tag{6.2}$$

We also fix $p \geq 2, 0 < \alpha \leq \min(s_1, 1 - s_1) < 1, 1 \leq \tilde{q} \leq \tilde{q}_1 \leq 2 \leq q \leq q_1 \leq \infty$, and $1 \leq \tilde{r} \leq \tilde{r}_1 \leq 2 \leq r_1 \leq r < \infty$, satisfying the following constraints: (i) For the interpolation lemma (Lemma 2.9):

$$\begin{aligned} \frac{1}{q_1} &= \frac{1 - \alpha/s_1}{q} + \frac{\alpha/s_1}{\infty}, & \frac{1}{r_1} &= \frac{1 - \alpha/s_1}{r} + \frac{\alpha/s_1}{2}, \\ \frac{1}{\tilde{q}_1} &= \frac{1 - \alpha/(1 - s_1)}{\tilde{q}} + \frac{\alpha/(1 - s_1)}{1}, & \frac{1}{\tilde{r}_1} &= \frac{1 - \alpha/(1 - s_1)}{\tilde{r}} + \frac{\alpha/(1 - s_1)}{2}, \end{aligned} \tag{6.3}$$

(ii) For Lemmas 2.13 (ii) and 2.14:

$$\frac{1}{r_1} + \frac{1}{p} \leq \frac{1}{\tilde{r}_1} + \frac{\alpha}{2}, \tag{6.4}$$

(iii) For Hölder’s inequality in time $\|fg\|_{L_T^{\tilde{q}_1}} \lesssim T^\theta \|f\|_{L_T^{q_1}} \|g\|_{L_T^p}$ for some $\theta > 0$:

$$\frac{1}{q_1} + \frac{1}{p} < \frac{1}{\tilde{q}_1}, \tag{6.5}$$

(iv) For Sobolev’s inequality $W^{-\alpha, \tilde{r}_1}(\mathbb{T}^2) \subset H^{s_2 - 2}(\mathbb{T}^2)$:

$$\frac{2 - s_2 - \alpha}{2} \geq \frac{1}{\tilde{r}_1} - \frac{1}{2}. \tag{6.6}$$

The constraints (i)–(iv) allow us to prove local well-posedness of the system (6.1).

We aim to obtain the best possible range $0 < \beta^2 < \beta_{\text{wave}}^2$ under the constraint (1.41) from Proposition 1.12:

$$\alpha \geq (p - 1) \frac{\beta_{\text{wave}}^2}{4\pi}. \tag{6.7}$$

First, note that from (6.3) with (6.2), $\frac{1}{r_1} - \frac{1}{\tilde{r}_1}$ depends only on α , not on s_1 . Then, by saturating (6.4) in the constraint (ii) above and substituting $\frac{1}{r_1} - \frac{1}{\tilde{r}_1} = \frac{4}{3}\alpha - \frac{2}{3}$, we obtain α in terms of p , which reduces (6.7) to

$$\beta_{\text{wave}}^2 \leq \frac{2p - 3}{5p(p - 1)} 8\pi.$$

The right-hand side is maximized when $p = \frac{3 + \sqrt{3}}{2} \simeq 2.37$, giving

$$\beta_{\text{wave}}^2 = \frac{32 - 16\sqrt{3}}{5} \pi \simeq 0.86\pi.$$

This in turn implies $\alpha = (p-1) \frac{\beta_{\text{wave}}^2}{4\pi} = \frac{2\sqrt{3}-2}{5}$. As for the other parameters, we have freedom to take any $s_1 \in [\alpha, 1 - \alpha]$ which determines the values of $q, r, q_1, r_1, \tilde{q}, \tilde{r}, \tilde{q}_1, \tilde{r}_1$. In the following, we set $s_1 = 1 - \alpha$ (which gives the best regularity for X). For the sake of concreteness, we choose the following parameters:

$$\begin{aligned} \beta_{\text{wave}}^2 &= \frac{32 - 16\sqrt{3}}{5}\pi, & p &= \frac{3 + \sqrt{3}}{2}, & \alpha &= \frac{2\sqrt{3} - 2}{5}, \\ s_1 &= 1 - \alpha, & s_2 &= s_1 + 1, & & \\ q &= \frac{15}{7 - 2\sqrt{3}}, & q_1 &= \frac{15}{9 - 4\sqrt{3}}, & \tilde{q}_1 &= 1, \\ r &= \frac{30}{8\sqrt{3} - 13}, & r_1 &= \frac{30}{16\sqrt{3} - 21}, & \tilde{r}_1 &= 2. \end{aligned} \tag{6.8}$$

We point out that the constraints (6.5) and (6.6) are satisfied with this choice of parameters.

Proof of Theorem 1.15. Let $0 < T < 1$ and $B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$ denotes the ball of radius $O(1)$ centered at the origin. We set

$$K = \|(v_0, v_1)\|_{\mathcal{H}^s} \quad \text{and} \quad R = \|\Theta\|_{L^p([0,1]; W_x^{-\alpha,p})}$$

for $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^2)$ for some $s > 1$ and a positive distribution $\Theta \in L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2))$.

• **Step 1:** Let $(X, Y) \in B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$. By the Strichartz estimate (Lemma 2.8) with the definitions (2.16) and (2.17) of the Strichartz space $\mathcal{X}_T^{s_1}$ and the dual space $\mathcal{N}_T^{s_1}$, Lemma 2.9, and Hölder’s inequality (with $\tilde{r}_1, \tilde{q}_1 \leq 2 < p$ in view of (6.8)), we have

$$\begin{aligned} \|\Phi_1(X, Y)\|_{\mathcal{X}_T^{s_1}} &\lesssim \|e^{\beta z} F(\beta X) e^{\beta Y} \Theta\|_{\mathcal{N}_T^{s_1}} \\ &\lesssim \|e^{\beta z} F(\beta X) e^{\beta Y} \Theta\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{r}_1}} \\ &\lesssim T^\theta \|e^{\beta z} F(\beta X) e^{\beta Y} \Theta\|_{L_T^p W_x^{-\alpha,p}} \end{aligned} \tag{6.9}$$

for some $\theta > 0$.

As in the parabolic case, we would like to exploit the positivity of Θ and apply Lemma 2.14 at this point. Unlike the parabolic case, however, the function X does not have sufficient regularity in order to apply Lemma 2.14 (i). Namely, we do not know if $X(t)$ is continuous (in x) for almost every $t \in [0, T]$. We instead rely on the hypothesis (ii) in Lemma 2.14.

In the following discussion, we only discuss spatial regularities holding for almost every $t \in [0, T]$. For simplicity, we suppress the time dependence. If we have

$$e^{\beta z} F(\beta X) e^{\beta Y} \in W^{\alpha, r_0}(\mathbb{T}^2) \tag{6.10}$$

for some $r_0 < r_1$ sufficiently close to r_1 , then the condition (6.4) guarantees the hypothesis (ii) in Lemma 2.14:

$$\frac{1}{r_0} + \frac{1}{p} \leq \left(\frac{1}{\tilde{r}_1} + \frac{\alpha}{2} \right) + \varepsilon < 1 + \frac{\alpha}{2} \tag{6.11}$$

for some small $\varepsilon > 0$, since $\tilde{r}_1 > 1$. We now verify (6.10). The fractional Leibniz rule (Lemma 2.13 (i)) with $\frac{1}{r_0} = \frac{1}{r_1} + \frac{1}{r_2}$ for some large but finite r_2 yields

$$\begin{aligned} \|e^{\beta z} F(\beta X)e^{\beta Y}\|_{W_x^{\alpha, r_0}} &\lesssim \|F(\beta X)\|_{W_x^{\alpha, r_1}} \|e^{\beta(z+Y)}\|_{L_x^{r_2}} \\ &\quad + \|F(\beta X)\|_{L_x^{r_2}} \|e^{\beta(z+Y)}\|_{W_x^{\alpha, r_1}}. \end{aligned} \tag{6.12}$$

Recall that F is Lipschitz. Hence, by the fractional chain rules (Lemma 2.15 (i)), we have

$$\begin{aligned} \|F(\beta X)\|_{W_x^{\alpha, r_1}} &\sim \|F(\beta X)\|_{L_x^{r_1}} + \|\lvert\nabla\rvert^\alpha F(\beta X)\|_{L_x^{r_1}} \\ &\lesssim 1 + \|X\|_{W_x^{\alpha, r_1}} < \infty, \end{aligned} \tag{6.13}$$

since Lemma 2.9 (i) ensures that $X \in W^{\alpha, r_1}(\mathbb{T}^2)$. Similarly, by the fractional chain rule (Lemma 2.15 (ii)), we have

$$\begin{aligned} \|e^{\beta(z+Y)}\|_{W_x^{\alpha, r_1}} &\sim \|e^{\beta(z+Y)}\|_{L_x^{r_1}} + \|\lvert\nabla\rvert^\alpha e^{\beta(z+Y)}\|_{L_x^{r_1}} \\ &\lesssim e^{C\|z+Y\|_{L_x^\infty}} + \|e^{\beta(z+Y)}\|_{L_x^{r_3}} \|\lvert\nabla\rvert^\alpha(z+Y)\|_{L_x^{r_1+\varepsilon}} \\ &\lesssim e^{C\|z+Y\|_{H_x^{1+\varepsilon}}} (1 + \|z+Y\|_{H_x^1}) < \infty \end{aligned} \tag{6.14}$$

for some large but finite r_3 and small $\varepsilon > 0$, since $z \in H^s(\mathbb{T}^2)$ and $Y \in H^{s_2}(\mathbb{T}^2)$ with $s, s_2 > 1$. In the last step, we used Sobolev’s inequality $\frac{1-\alpha}{2} \geq \frac{1}{2} - \frac{1}{r_1+\varepsilon}$, which is guaranteed from (6.8):

$$\frac{\alpha}{2} < \frac{1}{r_1} \tag{6.15}$$

and choosing $\varepsilon > 0$ sufficiently small. Putting (6.12), (6.13), and (6.14), we see that (6.10) is satisfied for almost every $t \in [0, T]$.

By applying Lemma 2.14 to (6.9), we have

$$\begin{aligned} \|\Phi_1(X, Y)\|_{\mathcal{X}_T^{s_1}} &\lesssim T^\theta \|e^{\beta z} F(\beta X)e^{\beta Y}\|_{L_{T,x}^\infty} \|\Theta\|_{L_T^p W_x^{-\alpha, p}} \\ &\lesssim T^\theta e^{C\|z+Y\|_{L_{T,x}^\infty}} \|\Theta\|_{L_T^p W_x^{-\alpha, p}} \\ &\lesssim T^\theta e^{CK} R. \end{aligned} \tag{6.16}$$

Next, by applying Lemma 2.6, Sobolev’s inequality with (6.6) and $p > \tilde{r}_1$, and proceeding as in (6.16), we have

$$\begin{aligned} \|\Phi_2(X, Y)\|_{\mathcal{Y}_T^{s_2}} &\lesssim \|e^{\beta z} F(\beta X)e^{\beta Y} \Theta\|_{L_T^1 H_x^{s_2-2}} \\ &\lesssim T^\theta \|e^{\beta z} F(\beta X)e^{\beta Y} \Theta\|_{L_T^p W_x^{-\alpha, p}} \\ &\lesssim T^\theta e^{CK} R. \end{aligned} \tag{6.17}$$

By choosing $T = T(K, R) > 0$ sufficiently small, the estimates (6.16) and (6.17) show boundedness of $\Phi = (\Phi_1, \Phi_2)$ on the ball $B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$.

• **Step 2:** Next, we establish difference estimates. Let $(X_1, Y_1), (X_2, Y_2) \in B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$. Write

$$\begin{aligned} & \|\Phi(X_1, Y_1) - \Phi(X_2, Y_2)\|_{\mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}} \\ & \leq \|\Phi_1(X_1, Y_1) - \Phi_1(X_2, Y_1)\|_{\mathcal{X}_T^{s_1}} + \|\Phi_2(X_1, Y_1) - \Phi_2(X_2, Y_1)\|_{\mathcal{Y}_T^{s_2}} \\ & \quad + \|\Phi_1(X_2, Y_1) - \Phi_1(X_2, Y_2)\|_{\mathcal{X}_T^{s_1}} + \|\Phi_2(X_2, Y_1) - \Phi_2(X_2, Y_2)\|_{\mathcal{Y}_T^{s_2}} \\ & =: I_1 + I_2 + II_1 + II_2. \end{aligned} \tag{6.18}$$

Recall from (5.15) and (5.16) that

$$F(\beta X_1) - F(\beta X_2) = \beta(X_1 - X_2)G(X_1, X_2).$$

Then, by the Strichartz estimate (Lemma 2.8), Lemma 2.9 (ii), and Lemma 2.13 (ii), we have

$$\begin{aligned} I_1 & \lesssim \|e^{\beta z}(F(\beta X_1) - F(\beta X_2))e^{\beta Y_1}\Theta\|_{\mathcal{N}_T^{s_1}} \\ & \lesssim \|e^{\beta z}(X_1 - X_2)G(X_1, X_2)e^{\beta Y_1}\Theta\|_{L_T^{\tilde{q}_1}W_x^{-\alpha, \tilde{r}_1}} \\ & \lesssim T^\theta \|X_1 - X_2\|_{L_T^{q_1}W_x^{\alpha, r_1}} \|e^{\beta z}G(X_1, X_2)e^{\beta Y_1}\Theta\|_{L_T^pW_x^{-\alpha, p}}, \end{aligned}$$

provided that

$$\theta = \frac{1}{\tilde{q}_1} - \frac{1}{q_1} - \frac{1}{p} > 0 \quad \text{and} \quad \frac{1}{r_1} + \frac{1}{p} \leq \frac{1}{\tilde{r}_1} + \frac{\alpha}{2}, \tag{6.19}$$

which are precisely the constraints (6.5) and (6.4). Then, applying Lemma 2.14 as in (6.16) along with the boundedness of G , we obtain

$$I_1 \lesssim T^\theta e^{CK} R \|X_1 - X_2\|_{\mathcal{X}_T^{s_1}}, \tag{6.20}$$

where we also used Lemma 2.9 (i) to estimate the norm of $X_1 - X_2$. As for I_2 , Lemma 2.6 and Sobolev’s inequality with (6.6) yield

$$\begin{aligned} I_2 & \lesssim \|e^{\beta z}(X_1 - X_2)G(X_1, X_2)e^{\beta Y_1}\Theta\|_{L_T^1H_x^{s_2-2}} \\ & \lesssim \|e^{\beta z}(X_1 - X_2)G(X_1, X_2)e^{\beta Y_1}\Theta\|_{L_T^1W_x^{-\alpha, \tilde{r}_1}}. \end{aligned}$$

Then, proceeding as above, we obtain

$$I_2 \lesssim T^\theta e^{CK} R \|X_1 - X_2\|_{\mathcal{X}_T^{s_1}}. \tag{6.21}$$

As for II_1 , by Lemmas 2.8 and 2.9 (ii), the fundamental theorem of calculus (as in (5.15) and (5.16)), Lemma 2.13 (ii) with (6.19), and then proceeding as in (6.16) with Lemma 2.14, we have

$$\begin{aligned}
 \text{II}_1 &\lesssim \|e^{\beta z} F(\beta X_2)(e^{\beta Y_1} - e^{\beta Y_2})\Theta\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{r}_1}} \\
 &\lesssim \|e^{\beta z} F(\beta X_2)(Y_1 - Y_2) \int_0^1 \exp(\tau\beta Y_1 + (1-\tau)\beta Y_2) d\tau \cdot \Theta\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{r}_1}} \\
 &\lesssim T\theta \|Y_1 - Y_2\|_{L_T^{q_1} W_x^{\alpha, r_1}} \|e^{\beta z} F(\beta X_2) \int_0^1 \exp(\tau\beta Y_1 + (1-\tau)\beta Y_2) d\tau \cdot \Theta\|_{L_T^p W_x^{-\alpha, p}} \\
 &\lesssim T^\theta e^{CK} R \|Y_1 - Y_2\|_{\mathcal{Y}_T^{s_2}}. \tag{6.22}
 \end{aligned}$$

In the last step, we use the embedding $H^{s_2}(\mathbb{T}^2) \subset W^{\alpha, r_1}(\mathbb{T}^2)$, which is guaranteed by (6.15) and $s_2 > 1$. Similarly, by applying Lemma 2.6 and Sobolev’s inequality with (6.6) and proceeding as in (6.22), we have

$$\begin{aligned}
 \text{II}_2 &\lesssim \|e^{\beta z} F(\beta X_2)(e^{\beta Y_1} - e^{\beta Y_2})\Theta\|_{L_T^1 W_x^{-\alpha, \tilde{r}_1}} \\
 &\lesssim T^\theta e^{CK} R \|Y_1 - Y_2\|_{\mathcal{Y}_T^{s_2}}. \tag{6.23}
 \end{aligned}$$

From Step 1, (6.18), (6.20), (6.21), (6.22), and (6.23), we conclude that $\Phi = (\Phi_1, \Phi_2)$ is a contraction on the ball $B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$, thus establishing local well-posedness of (6.1).

• **Step 3:** Continuous dependence of the solution (X, Y) on initial data (v_0, v_1) easily follows from the argument in Step 2. Hence, it remains to prove continuous dependence of the solution (X, Y) on the “noise” term Θ .

Let $(X_j, Y_j) \in B \subset \mathcal{X}_T^{s_1} \times \mathcal{Y}_T^{s_2}$ be solutions to (6.1) with a noise term $\Theta_j, j = 1, 2$. In estimating the difference, we can apply the argument in Step 2 to handle all the terms except for the following two terms:

$$\begin{aligned}
 &\left\| \int_0^t e^{-\frac{(t-t')}{2}} S(t-t')(e^{\beta z} F(\beta X_1)e^{\beta Y_1}(\Theta_1 - \Theta_2))(t') dt' \right\|_{\mathcal{X}_T^{s_1}} \\
 &+ \left\| \int_0^t (\mathcal{D}(t-t') - e^{-\frac{(t-t')}{2}} S(t-t'))(e^{\beta z} F(\beta X_1)e^{\beta Y_1}(\Theta_1 - \Theta_2))(t') dt' \right\|_{\mathcal{Y}_T^{s_2}} \\
 &=: \text{III}_1 + \text{III}_2.
 \end{aligned}$$

The main point is that the difference $\Theta_1 - \Theta_2$ does not enjoy positivity and hence we can not apply Lemma 2.14.

Let $r_0 < r_1$ sufficiently close to r_1 , satisfying (6.11):

$$\frac{1}{r_0} + \frac{1 - \varepsilon p}{p} \leq \frac{1}{\tilde{r}_1} + \frac{\alpha}{2}. \tag{6.24}$$

By the Strichartz estimate (Lemma 2.8), Lemma 2.9 (ii), and Lemma 2.13 (ii) with (6.24), we have

$$\begin{aligned}
 \text{III}_1 &\lesssim \|e^{\beta z} F(\beta X_1)e^{\beta Y_1}(\Theta_1 - \Theta_2)\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{r}_1}} \\
 &\lesssim T^\theta \|e^{\beta z} F(\beta X_1)e^{\beta Y_1}\|_{L_T^{q_1} W_x^{\alpha, r_0}} \|\Theta_1 - \Theta_2\|_{L_T^p W_x^{-\alpha, \frac{p}{1-\varepsilon p}}}.
 \end{aligned}$$

Then, applying (6.12), (6.13), and (6.14) along with Hölder’s inequality in time and Sobolev’s inequality, we obtain

$$\text{III}_1 \lesssim T^\theta e^{CK} (1 + K) \|\Theta_1 - \Theta_2\|_{L_T^p W_x^{-\alpha+2\varepsilon, p}}.$$

Thanks to Lemma 2.6 and the embedding $L^{\tilde{q}_1}([0, T]; W^{-\alpha, \tilde{r}_1}(\mathbb{T}^2)) \subset L^1([0, T]; H^{s_2-2}(\mathbb{T}^2))$ (see (6.6)), the second term III_2 can be handled in an analogous manner.

Let $0 < \beta^2 < \beta_{\text{wave}}^2$. Then, the pair (α, p) in (6.8) satisfies the condition (1.41). Then, by taking $\varepsilon > 0$ sufficiently small, we see that the pair $(\alpha - 2\varepsilon, p)$ also satisfies the condition (1.41). Hence, as Θ_2 tends to Θ_1 in $L^p([0, 1]; W^{-\alpha+2\varepsilon, p}(\mathbb{T}^2))$, we conclude that $\text{III}_1 + \text{III}_2 \rightarrow 0$, establishing the continuity of the solution map $(v_0, \Theta) \mapsto (X, Y)$. This completes the proof of Theorem 1.15. \square

6.2. *Almost sure global well-posedness and invariance of the Gibbs measure.* In this subsection, we briefly discuss a proof of Theorem 1.9. As mentioned in Sect. 1, the well-posedness result of Theorem 1.15 proved in the previous subsection is only local in time and hence we need to apply Bourgain’s invariant measure argument [9, 10] to extend the dynamics globally in time almost surely with respect to the Gibbs measure ρ_{wave} and then show invariance of the Gibbs measure ρ_{wave} .

Given $N \in \mathbb{N}$, we consider the following truncated SdNLW:

$$\begin{cases} \partial_t^2 u_N + \partial_t u_N + (1 - \Delta)u_N + \lambda\beta C_N \mathbf{Q}_N e^{\beta \mathbf{Q}_N u_N} = \sqrt{2}\xi \\ (u_N, \partial_t u_N)|_{t=0} = (\mathbf{Q}_N w_0, \mathbf{Q}_N w_1), \end{cases} \tag{6.25}$$

where \mathbf{Q}_N is as in (1.16) and (w_0, w_1) is as in (1.8). Namely, $\text{Law}(w_0, w_1) = \mu_1 \otimes \mu_0$.¹⁷

By writing $u_N = X_N + Y_N + \Psi$, where $\Psi = \Psi^{\text{wave}}$ is as in (1.36), we have

$$\begin{aligned} X_N(t) &= -\lambda\beta \int_0^t e^{-\frac{(t-t')}{2}} S(t-t') \mathbf{Q}_N (e^{\beta \mathbf{Q}_N X_N} e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt', \\ Y_N(t) &= -\lambda\beta \int_0^t (\mathcal{D}(t-t') - e^{-\frac{(t-t')}{2}} S(t-t')) \mathbf{Q}_N (e^{\beta \mathbf{Q}_N X_N} e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt'. \end{aligned}$$

By the positivity of the smoothing operator \mathbf{Q}_N , X_N enjoys the sign-definite structure:

$$\beta \mathbf{Q}_N X_N = -\lambda\beta^2 \int_0^t e^{-\frac{(t-t')}{2}} S(t-t') \mathbf{Q}_N^2 (e^{\beta \mathbf{Q}_N X_N} e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt' \leq 0,$$

thanks to $\lambda > 0$ and the positivity of the linear wave propagator $S(t)$ (Lemma 2.5). Hence, it is enough to consider

$$\begin{aligned} X_N(t) &= -\lambda\beta \int_0^t e^{-\frac{(t-t')}{2}} S(t-t') \mathbf{Q}_N (F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt', \\ Y_N(t) &= -\lambda\beta \int_0^t (\mathcal{D}(t-t') - e^{-\frac{(t-t')}{2}} S(t-t')) \mathbf{Q}_N (F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt', \end{aligned} \tag{6.26}$$

¹⁷ In view of the equivalence of $\mu_1 \otimes \mu_0$ and the Gibbs measure ρ_{wave} in (1.23), it suffices to study (6.25) with the initial data distributed by $\mu_1 \otimes \mu_0$.

where F is as in (1.49).

In view of the uniform (in N) boundedness of \mathbf{Q}_N on $L^p(\mathbb{T}^2)$, $1 \leq p \leq \infty$, we can argue as in Sect. 6.1 to prove local well-posedness of the system (6.26) in a uniform manner for any $N \in \mathbb{N}$. In order to prove convergence of the solution $((X_N, \partial_t X_N), (Y_N, \partial_t Y_N))$ to (6.26) towards the solution $((X, \partial_t X), (Y, \partial_t Y))$ of the untruncated dynamics (6.1), we can repeat the argument in Step 3 of the previous subsection to estimate the difference between $((X_N, \partial_t X_N), (Y_N, \partial_t Y_N))$ and $((X, \partial_t X), (Y, \partial_t Y))$. As in Sect. 5.2, we need to estimate the terms with $\mathbf{Q}_N - \text{Id}$:

$$\begin{aligned} & \left\| \int_0^t e^{-\frac{(t-t')}{2}} S(t-t') (\mathbf{Q}_N - \text{Id}) (F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt' \right\|_{\mathcal{X}_T^{s_1}} \\ & + \left\| \int_0^t (\mathcal{D}(t-t') - e^{-\frac{(t-t')}{2}} S(t-t')) (\mathbf{Q}_N - \text{Id}) (F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N)(t') dt' \right\|_{\mathcal{Y}_T^{s_2}} \\ & =: \text{IV}_1 + \text{IV}_2. \end{aligned}$$

The property (5.25) of \mathbf{Q}_N allows us to gain a negative power of N at a slight expense of regularity. By a slight modification of the argument from the previous subsection (see (6.16)), we have

$$\begin{aligned} \text{IV}_1 & \lesssim \left\| (\mathbf{Q}_N - \text{Id}) (F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N) \right\|_{L_T^{\tilde{q}_1} W_x^{-\alpha, \tilde{\gamma}_1}} \\ & \lesssim N^{-\varepsilon} \left\| F(\beta \mathbf{Q}_N X_N) e^{\beta \mathbf{Q}_N Y_N} \Theta_N \right\|_{L_T^{\tilde{q}_1} W_x^{-\alpha+\varepsilon, \tilde{\gamma}_1}} \\ & \lesssim T^\theta N^{-\varepsilon} \exp(C \|Y_N\|_{L_T^\infty H_x^{s_2}}) \left\| \Theta_N \right\|_{L_T^p W_x^{-\alpha+\varepsilon, p}}. \end{aligned} \tag{6.27}$$

Note that by choosing $\varepsilon > 0$ sufficiently small, the range $0 < \beta^2 < \beta_{\text{wave}}^2$ does not change even when we replace $-\alpha$ in (6.16) by $-\alpha + \varepsilon$ in (6.27). Similarly, we have

$$\text{IV}_2 \lesssim T^\theta N^{-\varepsilon} \exp(C \|Y_N\|_{L_T^\infty H_x^{s_2}}) \left\| \Theta \right\|_{L_T^p W_x^{-\alpha+\varepsilon, p}}. \tag{6.28}$$

The estimates (6.27) and (6.28) combined with the argument in the previous subsection allows us to prove the desired convergence of $((X_N, \partial_t X_N), (Y_N, \partial_t Y_N))$ to $((X, \partial_t X), (Y, \partial_t Y))$. The rest of the argument follows from applying Bourgain’s invariant measure argument [9, 10]. Since it is standard, we omit details. See, for example, [15, 37, 59, 60, 65, 77] for details.

Acknowledgement. The authors would like to thank Massimiliano Gubinelli for the references [1, 30] and also for helpful discussions. They are also grateful to Jonathan Bennett for a helpful conversation on the Brascamp-Lieb inequality. The authors are also thankful to Vincent Vargas, Rémi Rhodes, and Christophe Garban for helpful comments on a previous version of this article, which motivated them to write the follow-up paper [66]. Lastly, the authors wish to thank the anonymous referees for their helpful remarks which improved the quality of the paper.

T.O. was supported by the European Research Council (grant no. 637995 “ProbDynDispEq”) and grant no. 864138 “SingStochDispDyn”). T.R. was supported by the European Research Council (grant no. 637995 “ProbDynDispEq”). Y.W. was supported by supported by the EPSRC New Investigator Award (grant no. EP/V003178/1).

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Appendix A. On Local Well-Posedness of SNLH Without Using the Positivity

In this appendix, we revisit the fixed point problem (4.4) for SNLH:

$$v = \Phi_{v_0, \Theta}(v), \tag{A.1}$$

where the map $\Phi = \Phi_{v_0, \Theta}$ is defined in (4.2). In Sects. 4 and 5, we studied this problem by exploiting the positivity of Θ and furthermore the sign-definite structure of the equation when $\lambda > 0$. In the following, we study (4.4) for general $\lambda \in \mathbb{R} \setminus \{0\}$ and present a simple contraction argument *without* using any positivity of Θ for the range $0 < \beta^2 < \frac{4}{3}\pi \simeq 1.33\pi$. This simple argument provides Lipschitz dependence of a solution on initial data v_0 and noise Θ .

Let $0 < \alpha < 1$ and $p \geq 2$ such that

$$p' \left(\frac{\alpha}{2} + \frac{1}{p} \right) < 1 \quad \text{and} \quad 0 < \alpha \leq \frac{2}{p}. \tag{A.2}$$

Theorem A.1. *Let α, p be as above. Then, given any $v_0 \in H^{1+\varepsilon}(\mathbb{T}^2)$ and $\Theta \in L^p([0, 1]; W^{-\alpha, p}(\mathbb{T}^2))$ for some small $\varepsilon > 0$, there exist $T = T(\|v_0\|_{L^\infty}, \|\Theta\|_{L^p([0, 1]; W_x^{-\alpha, p})}) > 0$ and a unique solution $v \in C([0, T]; W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2))$ to (A.1), depending continuously on the initial data v_0 and the noise Θ .*

In view of Proposition 1.12 on the regularity of the Gaussian multiplicative chaos Θ_N , we see that Theorem A.1 provides local well-posedness of SNLH (1.45) for the range:

$$0 < \beta^2 < \frac{4\pi\alpha}{p-1} < 8\pi \frac{\min\left(\frac{1}{p}, 1 - \frac{2}{p}\right)}{p-1},$$

where we used both of the inequalities in (A.2). Hence, optimizing

$$\min \left(\max_{p \geq 3} \frac{1}{p(p-1)}, \max_{2 \leq p \leq 3} \frac{p-2}{p(p-1)} \right),$$

we find that the maximum is attained at $p = 3$, which gives the range $0 < \beta^2 < \frac{4}{3}\pi$. With $p = 3$, we can take $\alpha = \frac{2}{3} - \varepsilon$ for some small $\varepsilon > 0$ such that (A.2) is satisfied. We point out that our argument requires the initial data v_0 to belong to a smaller space $H^{1+\varepsilon}(\mathbb{T}^2) \subset L^\infty(\mathbb{T}^2)$.

Proof of Theorem A.1. Fix small $\varepsilon > 0$ such that

$$p' \left(\frac{\alpha + \varepsilon}{2} + \frac{1}{p} \right) < 1. \tag{A.3}$$

Given $v_0 \in H^{1+\varepsilon}(\mathbb{T}^2)$ and $\Theta \in L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2))$, we consider the map $\Phi = \Phi_{v_0, \Theta}$ defined in (4.2) and set $z = P(t)v_0$ as in (1.44). Let $B \subset C([0, T]; W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2))$ be the ball of radius $O(1)$ centered at the origin and set

$$K = \|v_0\|_{H^{1+\varepsilon}} \quad \text{and} \quad R = \|\Theta\|_{L^p([0,1]; W^{-\alpha,p})}.$$

Let $0 < T \leq 1$. By the Schauder estimate (Lemma 2.4) with $\frac{2}{\alpha} \geq p$ (as guaranteed in (A.2)), Lemma 2.13 (ii) with $\frac{1}{p} + \frac{1}{2/\alpha} = \frac{1}{p} + \frac{\alpha}{2}$, and Hölder’s inequality in time with (A.3), we have

$$\begin{aligned} \|\Phi(v)\|_{C_T W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} &\lesssim \left\| \int_0^t (t-t')^{-\frac{2\alpha+\varepsilon}{2} - (\frac{1}{p} - \frac{\alpha}{2})} e^{\beta z} e^{\beta v} \Theta(t') \|_{W_x^{-\alpha,p}} dt' \right\|_{L_T^\infty} \\ &\lesssim \|e^{\beta(z+v)}\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha}}} \int_0^T (t-t')^{-\frac{\alpha+\varepsilon}{2} - \frac{1}{p}} \|\Theta(t')\|_{W_x^{-\alpha,p}} dt' \quad (\text{A.4}) \\ &\lesssim T^\theta \|e^{\beta(z+v)}\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha}}} \|\Theta\|_{L_T^p W_x^{-\alpha,p}} \end{aligned}$$

for some $\theta > 0$. By the fractional chain rule (Lemma 2.15 (ii)) and the Sobolev embeddings:

$$\begin{aligned} H^{1+\varepsilon}(\mathbb{T}^2) &\subset W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2), \\ W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2) &\subset W^{\alpha, \frac{2}{\alpha-\varepsilon}}(\mathbb{T}^2) \cap L^\infty(\mathbb{T}^2), \end{aligned} \quad (\text{A.5})$$

we have

$$\begin{aligned} \|e^{\beta(z+v)}\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha}}} &\sim \|e^{\beta(z+v)}\|_{L_T^\infty L_x^{\frac{2}{\alpha}}} + \|\nabla|^\alpha e^{\beta(z+v)}\|_{L_T^\infty L_x^{\frac{2}{\alpha}}} \\ &\lesssim e^{C\|z+v\|_{L_T^\infty}} + \|e^{\beta(z+v)}\|_{L_T^\infty L_x^{\frac{2}{\alpha}}} \|\nabla|^\alpha(z+v)\|_{L_T^\infty L_x^{\frac{2}{\alpha-\varepsilon}}} \\ &\lesssim \exp\left(C(\|v_0\|_{H^{1+\varepsilon}} + \|v\|_{L_T^\infty W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}})\right) \\ &\quad \times \left(1 + \|v_0\|_{H^{1+\varepsilon}} + \|v\|_{L_T^\infty W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}}\right). \end{aligned} \quad (\text{A.6})$$

Hence, from (A.4) and (A.6), we have

$$\|\Phi(v)\|_{C_T W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} \lesssim T^\theta e^{CK} (1 + K) R \quad (\text{A.7})$$

for any $v \in B$.

Proceeding as in (A.4), we have

$$\|\Phi(v_1) - \Phi(v_2)\|_{C_T W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} \lesssim T^\theta \|e^{\beta z} (e^{\beta v_1} - e^{\beta v_2})\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha}}} \|\Theta\|_{L_T^p W_x^{-\alpha,p}}. \quad (\text{A.8})$$

By (4.7), the fractional Leibniz rule (Lemma 2.13 (i)), followed by the fractional chain rule as in (A.6), we have

$$\begin{aligned}
 & \|e^{\beta z}(e^{\beta v_1} - e^{\beta v_2})\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha}}} \\
 & \lesssim \|e^{\beta z}\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha-\varepsilon/2}}} \|v_1 - v_2\|_{L_{T,x}^\infty} e^{C(\|v_1\|_{L_{T,x}^\infty} + \|v_2\|_{L_{T,x}^\infty})} \\
 & \quad + e^{C\|z\|_{L_{T,x}^\infty}} \left\{ \|v_1 - v_2\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha-\varepsilon}}} e^{C(\|v_1\|_{L_{T,x}^\infty} + \|v_2\|_{L_{T,x}^\infty})} \right. \\
 & \quad \left. + \|v_1 - v_2\|_{L_{T,x}^\infty} \left\| \int_0^1 \exp(\tau\beta v_1 + (1-\tau)\beta v_2) d\tau \right\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha-\varepsilon/2}}} \right\} \\
 & \lesssim e^{CK} (1 + K) \left(\|v_1 - v_2\|_{L_{T,x}^\infty} + \|v_1 - v_2\|_{L_T^\infty W_x^{\alpha, \frac{2}{\alpha-\varepsilon}}} \right)
 \end{aligned} \tag{A.9}$$

for any $v_1, v_2 \in B$. Hence, from (A.8) and (A.9) with (A.5), we have

$$\|\Phi(v_1) - \Phi(v_2)\|_{C_T W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} \lesssim T^\theta e^{CK} (1 + K) R \|v_1 - v_2\|_{L_T^\infty W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} \tag{A.10}$$

for any $v_1, v_2 \in B$.

From (A.7) and (A.10), a contraction argument yields a solution map:

$$(v_0, \Theta) \in H^{1+\varepsilon}(\mathbb{T}^2) \times L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2)) \mapsto v \in C([0, T]; W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2))$$

for some $T = T(\|v_0\|_{H^{1+\varepsilon}}, \|\Theta\|_{L^p([0,1]; W_x^{-\alpha,p})}) \in (0, 1]$, where v is the unique fixed point of $\Phi_{v_0, \Theta}$ in the ball $B \subset C([0, T]; W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2))$. As for the Lipschitz dependence of the solution map on Θ , if we take $\Theta_1, \Theta_2 \in L^p([0, 1]; W^{-\alpha,p}(\mathbb{T}^2))$, then in estimating the difference $\Phi_{v_0, \Theta_1}(v_1) - \Phi_{v_0, \Theta_2}(v_2)$ for $v_1, v_2 \in B \subset C([0, T]; W^{\alpha+\varepsilon, \frac{2}{\alpha}}(\mathbb{T}^2))$, there is one additional term of the form:

$$\int_0^t P(t-t')(e^{\beta z} e^{\beta v_1}(\Theta_1 - \Theta_2))(t') dt'.$$

By proceeding as in (A.4) and (A.6), we can bound this additional term as

$$\begin{aligned}
 & \left\| \int_0^t P(t-t')(e^{\beta z} e^{\beta v_1}(\Theta_1 - \Theta_2))(t') dt' \right\|_{C_T W_x^{\alpha+\varepsilon, \frac{2}{\alpha}}} \\
 & \lesssim T^\theta e^{CK} (1 + K) \|\Theta_1 - \Theta_2\|_{L_T^p W_x^{-\alpha,p}}.
 \end{aligned}$$

This completes the proof of Theorem A.1. □

Appendix B. Moment Bounds for the Gaussian Multiplicative Chaos

In this last section, we give a proof of Lemma 3.5 on the uniform boundedness of the moments of the random measure $\mathcal{M}_N(t)$ in (3.6). We mainly follow the arguments in [7, 72].

First of all, in view of the positivity of $\Theta_N(t)$, it suffices to prove Lemma 3.5 with $A = \mathbb{T}^2$. Moreover, the bound for $p = 1$ being a consequence of Proposition 3.2 (i), we

may assume $p > 1$. We start by fixing some large number $K \gg 1$, independent of $N \in \mathbb{N}$, and we partition $\mathbb{T}^2 \simeq [-\pi, \pi]^2$ into cubes $C_{k,\ell} = x_{k,\ell}^K + [-\frac{\pi}{K}, \frac{\pi}{K}]^2$, $k, \ell = 1, \dots, K$ of side length $2\pi K^{-1}$ centered at $x_{k,\ell}^K = (-\pi + \frac{2\pi}{K}(k-1), -\pi + \frac{2\pi}{K}(\ell-1)) \in \mathbb{T}^2$. We then group these into four families of cubes:

$$\begin{aligned} \mathcal{M}_N(t, \mathbb{T}^2) &= \sum_{\substack{k,\ell=1 \\ k \text{ even}, \ell \text{ even}}}^K \int_{C_{k,\ell}} \Theta_N(t, x) dx + \sum_{\substack{k,\ell=1 \\ k \text{ even}, \ell \text{ odd}}}^K \int_{C_{k,\ell}} \Theta_N(t, x) dx \\ &+ \sum_{\substack{k,\ell=1 \\ k \text{ odd}, \ell \text{ even}}}^K \int_{C_{k,\ell}} \Theta_N(t, x) dx + \sum_{\substack{k,\ell=1 \\ k \text{ odd}, \ell \text{ odd}}}^K \int_{C_{k,\ell}} \Theta_N(t, x) dx \\ &=: \mathcal{M}_N^{(1)}(t) + \mathcal{M}_N^{(2)}(t) + \mathcal{M}_N^{(3)}(t) + \mathcal{M}_N^{(4)}(t). \end{aligned}$$

It follows from the (spatial) translation invariance of the law of $\Psi_N(t, \cdot)$ that $\mathcal{M}_N^{(j)}(t)$, $j = 1, \dots, 4$, have the same law. Hence, by Minkowski's inequality, we have

$$\mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^p] \leq C_p \mathbb{E}[\mathcal{M}_N^{(1)}(t, \mathbb{T})^p].$$

In order to estimate the last expectation, we proceed as in Step 1 of the proof of Proposition 3.6. Namely, by a change of variables and a Riemann sum approximation, we have

$$\begin{aligned} \mathbb{E}[\mathcal{M}_N^{(1)}(t, \mathbb{T}^2)^p] &= \mathbb{E}\left[\left(\sum_{\substack{k,\ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} \int_{\mathbb{T}^2} \Theta_N(t, x_{k,\ell}^K + K^{-1}y) dy\right)^p\right] \\ &= \lim_{J \rightarrow \infty} \mathbb{E}\left[\left(\sum_{i,j=1}^J \frac{4\pi^2}{J^2} \sum_{\substack{k,\ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} e^{\beta \Psi_N(t, x_{k,\ell}^K + K^{-1}x_{i,j}^J) - \frac{\beta^2}{2} \sigma_N}\right)^p\right]. \end{aligned}$$

Using Lemma 3.1, we can bound the covariance function by

$$\begin{aligned} &\mathbb{E}\left[\Psi_N(t, x_{k_1,\ell_1}^K + K^{-1}x_{i_1,j_1}^J) \Psi_N(t, x_{k_2,\ell_2}^K + K^{-1}x_{i_2,j_2}^J)\right] \\ &= \Gamma_N(t, x_{k_1,\ell_1}^K - x_{k_2,\ell_2}^K + K^{-1}(x_{i_1,j_1}^J - x_{i_2,j_2}^J)) \\ &\leq -\frac{1}{2\pi} \log\left(|x_{k_1,\ell_1}^K - x_{k_2,\ell_2}^K + K^{-1}(x_{i_1,j_1}^J - x_{i_2,j_2}^J)| + N^{-1}\right) + C \end{aligned} \tag{B.1}$$

for some constant $C > 0$ independent of J, K , and N . When $(k_1, \ell_1) = (k_2, \ell_2)$, we thus have the bound

$$\begin{aligned} &\mathbb{E}\left[\Psi_N(t, x_{k_1,\ell_1}^K + K^{-1}x_{i_1,j_1}^J) \Psi_N(t, x_{k_2,\ell_2}^K + K^{-1}x_{i_2,j_2}^J)\right] \\ &\leq -\frac{1}{2\pi} \log(|x_{i_1,j_1}^J - x_{i_2,j_2}^J| + (K^{-1}N)^{-1}) + \frac{1}{2\pi} \log K + C \\ &\leq -\frac{1}{2\pi} \log(|x_{i_1,j_1}^J - x_{i_2,j_2}^J| + N^{-1}) + \frac{1}{2\pi} \log K + C. \end{aligned} \tag{B.2}$$

See also (3.10). In the case $(k_1, \ell_1) \neq (k_2, \ell_2)$, we first note that $|x_{k_1, \ell_1}^K - x_{k_2, \ell_2}^K| \geq 2 \cdot \frac{2\pi}{K}$ since k_1, k_2, ℓ_1, ℓ_2 are all even. Then, with the trivial bound $|x_{i_1, j_1}^J - x_{i_2, j_2}^J| \leq \sqrt{2} \cdot 2\pi$, we have

$$|x_{k_1, \ell_1}^K - x_{k_2, \ell_2}^K + K^{-1}(x_{i_1, j_1}^J - x_{i_2, j_2}^J)| \geq (2 - \sqrt{2}) \frac{2\pi}{K}. \tag{B.3}$$

Thus, from (B.1) and (B.3), we have

$$\mathbb{E} \left[\Psi_N(t, x_{k_1, \ell_1}^K + K^{-1}x_{i_1, j_1}^J) \Psi_N(t, x_{k_2, \ell_2}^K + K^{-1}x_{i_2, j_2}^J) \right] \leq \frac{1}{2\pi} \log K + C. \tag{B.4}$$

Hence, from (B.2) and (B.4), we obtain

$$\begin{aligned} & \mathbb{E} \left[\Psi_N(t, x_{k_1, \ell_1}^K + K^{-1}x_{i_1, j_1}^J) \Psi_N(t, x_{k_2, \ell_2}^K + K^{-1}x_{i_2, j_2}^J) \right] \\ & \leq \mathbb{E} \left[(\psi_{N, k_1, \ell_1}(t, x_{i_1, j_1}^J) + h_K)(\psi_{N, k_2, \ell_2}(t, x_{i_2, j_2}^J) + h_K) \right], \end{aligned}$$

where $\psi_{N, k, \ell}$ are some independent¹⁸ copies of Ψ_N and h_K is a mean-zero Gaussian random variable with variance $\frac{1}{2\pi} \log K + C$ independent from $\psi_{N, k, \ell}$.

By applying Kahane’s convexity inequality (Lemma 3.4) and using the independence of h_K from $\psi_{N, k, \ell}$ with $\mathbb{E}[h_K^2] = \frac{1}{2\pi} \log K + C$, we have

$$\begin{aligned} & \mathbb{E} \left[\mathcal{M}_N^{(1)}(t, \mathbb{T}^2)^p \right] \\ & \leq \lim_{J \rightarrow \infty} \mathbb{E} \left[\left(\sum_{i, j=1}^J 4\pi^2 J^{-2} \sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} e^{\beta(\psi_{N, k, \ell}(t, x_{i, j}^J) + h_K) - \frac{\beta^2}{2}(\sigma_N + \mathbb{E}[h_K^2])} \right)^p \right] \\ & = \mathbb{E} \left[\left(\sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} e^{\beta h_K - \frac{\beta^2}{2} \mathbb{E}[h_K^2]} \int_{\mathbb{T}^2} e^{\beta \psi_{N, k, \ell}(t, y) - \frac{\beta^2}{2} \sigma_N} dy \right)^p \right] \\ & \leq CK^{(p^2-p)\frac{\beta^2}{4\pi}} \mathbb{E} \left[\left(\sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} \int_{\mathbb{T}^2} e^{\beta \psi_{N, k, \ell}(t, y) - \frac{\beta^2}{2} \sigma_N} dy \right)^p \right] \tag{B.5} \end{aligned}$$

for some constant $C > 0$ independent of K and N .

It then remains to bound the expectation in (B.5). Let $m \geq 2$ be an integer such that $m - 1 < p \leq m$. Then, by the embedding $\ell_m^{\frac{p}{m}} \subset \ell^1$, we have

$$\begin{aligned} & \mathbb{E} \left[\left(\sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2} \int_{\mathbb{T}^2} e^{\beta \psi_{N, k, \ell}(t, y) - \frac{\beta^2}{2} \sigma_N} dy \right)^p \right] \\ & \leq \mathbb{E} \left[\left\{ \sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2\frac{p}{m}} \left(\int_{\mathbb{T}^2} e^{\beta \psi_{N, k, \ell}(t, y) - \frac{\beta^2}{2} \sigma_N} dy \right)^{\frac{p}{m}} \right\}^m \right] \tag{B.6} \\ & =: \mathbb{E}[A_K]. \end{aligned}$$

¹⁸ In particular, we have $\mathbb{E}[\psi_{N, k_1, \ell_1}(t, x_{i_1, j_1}^J) \psi_{N, k_2, \ell_2}(t, x_{i_2, j_2}^J)] = 0$ when $(k_1, \ell_1) \neq (k_2, \ell_2)$.

We divide A_K into two pieces:

$$\begin{aligned}
 A_K &= \sum_{\substack{k, \ell=1 \\ k \text{ even}, \ell \text{ even}}}^K K^{-2p} \mathbb{E} \left[\left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k,\ell}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^p \right] \\
 &+ \sum_{(\mathbf{k}, \boldsymbol{\ell}) \in \Lambda_m} K^{-2p} \mathbb{E} \left[\prod_{j=1}^m \left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k_j, \ell_j}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^{\frac{p}{m}} \right] \\
 &=: A_K^{(1)} + A_K^{(2)},
 \end{aligned} \tag{B.7}$$

where the index set Λ_m is given by

$$\Lambda_m = \{(\mathbf{k}, \boldsymbol{\ell}) = (k_1, \dots, k_m, \ell_1, \dots, \ell_m) \in \{1, \dots, K\}^{2m} : k_j, \ell_j \text{ even}, (k_j, \ell_j) \text{ not all equal}\}.$$

Since $\psi_{N,k,\ell}$ are identically distributed, we can bound the diagonal term by

$$\mathbb{E}[A_K^{(1)}] \leq K^{2-2p} \mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^p]. \tag{B.8}$$

As for the second sum $A_K^{(2)}$ in (B.7), grouping the terms with the same values of (k, ℓ) together, each term within the sum can be written in the form

$$\mathbb{E} \left[\prod_{j=1}^n \left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k_j, \ell_j}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^{a_j \frac{p}{m}} \right] \tag{B.9}$$

for some $n \leq m$ and some $a_j \in \{0, \dots, m-1\}$ such that $\sum_{j=1}^n a_j = m$ and (k_j, ℓ_j) , $j = 1, \dots, n$, are all distinct. Noting that $\psi_{N,k,\ell}$ are independent and identically distributed, it follows from Hölder’s inequality with $a_j \frac{p}{m} \leq m-1$ that

$$\begin{aligned}
 \text{(B.9)} &\leq \prod_{j=1}^n \mathbb{E} \left[\left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k_j, \ell_j}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^{a_j \frac{p}{m}} \right] \\
 &\leq \prod_{j=1}^n \mathbb{E} \left[\left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k_j, \ell_j}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^{m-1} \right]^{a_j \frac{p}{m(m-1)}} \\
 &= \mathbb{E} \left[\left(\int_{\mathbb{T}^2} e^{\beta \psi_{N,k_1, \ell_1}(t,y) - \frac{\beta^2}{2} \sigma_N dy} \right)^{m-1} \right]^{\sum_{j=1}^n a_j \frac{p}{m(m-1)}} \\
 &= \mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^{m-1}]^{\frac{p}{m-1}}.
 \end{aligned} \tag{B.10}$$

Putting (B.5), (B.6), (B.7), (B.8), and (B.10) together, we obtain

$$\mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^p] \leq CK^{(p^2-p)\frac{\beta^2}{4\pi}} \cdot K^{2-2p} \mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^p] + C_{K,m} \mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^{m-1}]^{\frac{p}{m-1}}.$$

Under the assumption¹⁹ that $1 < p < \frac{8\pi}{\beta^2}$, the exponent $(p^2 - p)\frac{\beta^2}{4\pi} + 2 - 2p = (\frac{\beta^2}{4\pi}p - 2)(p - 1)$ of K in the first term on the right-hand side above is negative. Hence, by taking $K \gg 1$ (independent of N), we arrive at the bound:

$$\mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^p] \leq C_m \mathbb{E}[\mathcal{M}_N(t, \mathbb{T}^2)^{m-1}]^{\frac{p}{m-1}}, \tag{B.11}$$

¹⁹ Recall that we assume $p > 1$ in view of Proposition 3.2(i).

uniformly in $N \in \mathbb{N}$.

We now conclude the proof of Lemma 3.5 by induction on $m \geq 2$ with $m - 1 < p \leq m$. When $m = 2$, i.e. $p \in (1, 2]$, the conclusion of Lemma 3.5 follows from (B.11) and Proposition 3.2 (i). Now, given an integer $m \geq 3$, assume that Lemma 3.5 holds for all $1 < p \leq m - 1$. Fix $1 < p < \frac{8\pi}{\beta^2}$ such that $m - 1 < p \leq m$. Then, from (B.11) and the inductive hypothesis, we have

$$\sup_{t \in \mathbb{R}, N \in \mathbb{N}} \mathbb{E} \left[\mathcal{M}_N(t, \mathbb{T}^2)^p \right] \leq \sup_{t \in \mathbb{R}, N \in \mathbb{N}} C_m \mathbb{E} \left[\mathcal{M}_N(t, \mathbb{T}^2)^{m-1} \right]^{\frac{p}{m-1}} < \infty.$$

Therefore, by induction, we conclude the proof of Lemma 3.5.

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Communicated by M.Hairer