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Erratum

Connecting Solutions of the Lorentz Force Equation do Exist

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Due to a processing error the last paragraph of Sect. 3.1 on p. 358 was printed with an error. In addition, in Sect. 5.2 on pp. 365–366 the presentation of the *Example* was processed incorrectly. The second paragraph of Sect. 5.2 (two lines) must be removed and the last sentence in the same section must be replaced with 'Nevertheless, even though σ_0 maximizes in $\overline{C}_1$, it does not maximize in $\overline{C}_2$ (nor in the causal homotopy class C), in agreement with our results.' The corrected paragraphs read as follows.

p. 358:

In particular, if $x_0 \ll x_1$ the two points can be connected by means of a timelike geodesic (in fact, by one for each time like homotopy class in $\mathcal{C}_{x_0,x_1}$, as will be apparent below). If $x_1 \in E^+(x_0) = J^+(x_0)\backslash I^+(x_0)$ then x_0 and x_1 can still be joined by a lightlike geodesic, but this case does not make sense for the LFE. One can also wonder for the connectedness of x_0, x_1 by means of a geodesic even if they are not causally related, as in variational frameworks described below. Although this question has a geometrical interest (see for instance the survey [37]), it does not have a direct physical interpretation, nor equivalence for LFE.

pp. 365–366:

5.2. A remarkable example. Lemma 5.1 does not forbid the existence of a lightlike geodesic σ which maximizes the functional on the closure of a timelike class $\overline{\mathcal{C}}_{x_0,x_1}$. However, in that case the maximizer on $\mathcal{C}_{x_0,x_1} \supset \overline{\mathcal{C}}_{x_0,x_1}$ does not coincide with σ, as the following example shows.

Example. Let Σ be a surface embedded in $\mathbb{R}^3$ obtained by gluing the spherical cap $x^2 + y^2 + z^2 = r^2, z > -\frac{\sqrt{3}}{2}r + \epsilon_z$ with a cylinder $x^2 + y^2 = r^2/4, z < -\frac{\sqrt{3}}{2}r - \epsilon_z$,

The online version of the original article can be found at http://dx.doi.org/10.1007/s00220-006-1547-2 by making a smooth transition in the points with coordinate $z \in [-\frac{\sqrt{3}}{2}r - \epsilon_z, -\frac{\sqrt{3}}{2}r + \epsilon_z]$, for some positive $\epsilon_z < \frac{\sqrt{3}}{2}r$. Notice that this transition can be made smooth and depending only on the azimuthal angle θ in a small interval $(\frac{5}{6}\pi - \epsilon_\theta, \frac{5}{6}\pi + \epsilon_\theta)$, $\epsilon_\theta < \pi/6$. Only the details of this surface included in the spherical cap with $\theta \leq \pi/2 + \epsilon$, for some small positive $\epsilon < \pi/6$, will be relevant.

Let dl^2 be the induced Riemannian metric on Σ, and fix $q = (r, 0, 0) \in \Sigma$. Considier the natural product (globally hyperbolic) spacetime $M = \mathbb{R} \times \Sigma$, $g = dt^2 - dl^2$, with natural projection $\pi : M \to \Sigma$, and the fixed events $x_0 = (0, q)$, $x_1 = (2\pi r, q)$. The timelike curve $\lambda \mapsto (2\pi r\lambda, q)$ fix a timelike homotopy class $C_1 (:= C^{(1)}_{x_0,x_1})$. The connecting lightlike geodesic

$$\sigma_0(\lambda) = (2\pi r\lambda, c_0(\lambda)), \quad c_0(\lambda) = (r\cos 2\pi\lambda, r\sin 2\pi\lambda, 0), \quad \lambda \in [0, 1],$$

lies in the boundary $\dot{C}_1$, In fact, σ_0 can be reached by approximating the part c_0 with a constant-speed parametrization c_α of $\Sigma \cap \Pi_\alpha$, where $\Pi_\alpha \subset \mathbb{R}^3$ is the plane through q, orthogonal to the plane $y = 0$, which makes an oriented positive angle $\alpha < \pi/2$ with the plane $z = 0$ (c_α is contained in the region $z > 0$ except in the tangent point q). However, by letting $\alpha < 0$ we can find a second timelike homotopy class C_2 such that $\sigma_0 \in \dot{C}_2$; of course, C_1 and C_2 are contained in the same causal homotopy class $\mathcal{C}$. Notice that c_0 passes through the antipodal point $-q = (-r, 0, 0)$, which is also a conjugate point of q; thus, σ_0 also contains a conjugate point.

Fix $q/m > 0$ (resp. $q/m < 0$), and let $F = \mathcal{B}\pi^*\Omega = d\omega$ be on M, where Ω is the volume 2-form of Σ (with the orientation induced by the outer normal in the spherical cap), and where $\mathcal{B} : \Sigma \to \mathbb{R}$ is a non-negative (resp. non-positive) function, with $\mathcal{B} \equiv B > 0$ (resp. < 0) constant for $\theta \leq \pi/2$, acid monotonically decreasing (resp. increasing) to 0 for $\theta \in (\pi/2, \pi/2 + \epsilon]$. The charged-particle action I_{x_0,x_1} is given by two contributions. The electromagnetic term reads

$$\frac{q}{m}\int_\sigma \omega = \frac{q}{m}\int_R \mathcal{B}\Omega \tag{17}$$

where, without loss of generality, $\sigma(\lambda) = (2\pi r\lambda, c(\lambda))$ and $\partial R = c$. For a given length $L \leq 2\pi r$ of c this integral is maximized in C_1 by the circle c_α with length L, namely c^L. Indeed, the maximizer must be a circle in order to maximize the area, and it is tangent to c_0 since, otherwise, its enclosed surface R would include regions where $\mathcal{B} < B$ (resp. $\mathcal{B} > B$). Thus

$$\left|\frac{q}{m}\int_\sigma \omega\right| \leq \frac{q}{m} B A[c^L], \tag{18}$$

where $A[c^L]$ is the area contained in c^L. And the equality holds iff $c = c^L$ (up to a reparametrization with the same winding number).

The contribution of the length of σ in I_{x_0,x_1} is:

$$\int_\sigma \mathrm{d}s = \int_0^{2\pi r} \sqrt{1 - \left(\frac{\mathrm{d}l}{\mathrm{d}t}\right)^2}\,\mathrm{d}t \leq 2\pi r\sqrt{1 - \left(\frac{l[c]}{2\pi r}\right)^2}, \tag{19}$$

where $l[c]$ is the length of $c = \pi \circ \sigma$, and the equality holds when the speed of c is constant. We have then

$$I_{x_0,x_1}[\sigma] \leq 2\pi r\sqrt{1 - \left(\frac{l[c]}{2\pi r}\right)^2} + \frac{q}{m} B A[c^{l[c]}], \tag{20}$$

where the equality holds iff $\pi \circ \sigma = c^{l[c]}$. But in terms of the angle $0 \leq \alpha \leq \pi/2$ with $c_\alpha = c^l$, we have $l = 2\pi r \cos\alpha$ and $A[c^l] = 2\pi r^2(1 - \sin\alpha)$. Hence if $\frac{q}{m} Br > 1$,

$$I_{x_0,x_1}[\sigma] \leq 2\pi r^2 \frac{q}{m} B + 2\pi r \left(1 - \frac{q}{m} Br\right) \sin\alpha \leq 2\pi r^2 \frac{q}{m} B = I_{x_0,x_1}[\sigma_0], \quad (21)$$

and the equality holds iff $\alpha = 0$ and the projection of σ is $c_0 (= c^{2\pi r})$, i.e. iff $\sigma = \sigma_0$.

Nevertheless, even though σ_0 maximizes in $\overline{C}_1$, it does not maximize in $\overline{C}_2$ (nor in the causal homotopy class $\mathcal{C}$), in agreement with our results.