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Linear adjoint restriction estimates for paraboloid

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Abstract
We prove a class of modified paraboloid restriction estimates with a loss of angular derivatives for the full set of paraboloid restriction conjecture indices. This result generalizes the paraboloid restriction estimate in radial case from [Shao, Rev. Mat. Iberoam. 25(2009), 1127–1168], as well as the result from [Miao et al. Proc. AMS 140(2012), 2091–2102]. As an application, we show a local smoothing estimate for a solution of the linear Schrödinger equation under the assumption that the initial datum has additional angular regularity.

Keywords Linear adjoint restriction estimate · Local restriction estimate · Bessel function · Spherical harmonics · Local smoothing

Mathematics Subject Classification 42B37 · 42B10 · 42B25 · 35Q55

1 Introduction

Let S be a non-empty smooth compact subset of the paraboloid,

$$\left\{ (\tau, \xi) \int \mathbb{R} \times \mathbb{R}^n : \tau = |\xi|^2 \right\},$$

where $n \geq 1$. We denote by $d\sigma$ the pull-back of the n-dimensional Lebesgue measure $d\xi$ under the projection map $(\tau, \xi) \mapsto \xi$. Let f be a Schwartz function and define the inverse space-time Fourier transform of the measure $f d\sigma$

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$$(f d\sigma)^\vee(t,x) = \int_S f(\tau,\xi) e^{2\pi i(x\cdot\xi + t\tau)} d\sigma(\xi) \qquad (1.1)$$
$$= \int_{\mathbb{R}^n} f(|\xi|^2,\xi) e^{2\pi i(x\cdot\xi + t|\xi|^2)} d\xi.$$

The classical linear adjoint restriction estimate for the paraboloid reads

$$\|(f d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \le C_{p,q,n,S}\|f\|_{L^p(S;d\sigma)}, \qquad (1.2)$$

where $1 \le p, q \le \infty$. The famous restriction problem is to find the optimal range of p and q such that the estimate (1.2) holds. It is known that the condition

$$q > \frac{2(n+1)}{n} \quad \text{and} \quad \frac{n+2}{q} \le \frac{n}{p'}, \qquad (1.3)$$

is necessary for (1.2), see [24,29]. Here p' denotes the conjugate exponent of p. The adjoint restriction estimate conjecture on paraboloid reads as follows.

Conjecture 1.1 *The inequality* (1.2) *holds true if and only if inequalities* (1.3) *are valid.*

There is a large amount of literature on this problem. For $n = 1$, Conjecture 1.1 was proved by Fefferman-Stein [11] for the non-endpoint case and by Zygmund [36] for the endpoint case. Conjecture 1.1 in high dimension case becomes much more difficult. For $n \ge 2$, Tomas [33] showed (1.2) for $q > 2(n+2)/n$, and Stein [25] fixed the limit case $q = 2(n+2)/n$. Bourgain [1] further proved estimate (1.2) for $q > 2(n+2)/n - \epsilon_n$ with some $\epsilon_n > 0$; in particular, $\epsilon_n = \frac{2}{15}$ when $n = 2$. Further improvements were made by Moyua-Vargas-Vega [16] and Wolff [34]. Tao [31] used the bilinear argument to show that estimate (1.2) holds true for $q > 2(n+3)/(n+1)$ with $n \ge 2$. This result was improved by Bourgain-Guth [2] when $n \ge 4$. This conjecture is so difficult that it remains open up to now. For more details, we refer the reader to [2,29–32,34].

On the other hand, the restriction conjecture becomes simpler (but not trivial) when a test function has some angular regularity. For example, Conjecture 1.1 is proved by Shao [22] when test functions are cylindrically symmetric and are supported on a dyadic subset of the paraboloid in the form of

$$\Big\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n : \ M \le |\xi| \le 2M, \quad \tau = |\xi|^2, \quad M \in 2^{\mathbb{Z}}\Big\}.$$

Indeed, many famous conjectures in harmonic analysis (such as Fourier restriction estimates, Bochner-Riesz estimate etc.) have easier counterparts when the corresponding operators act on radial functions. Let $\mathbb{S}^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ and $L^q_{\text{sph}} := L^q_\theta(\mathbb{S}^{n-1})$, the intermediate situation is to replace the $L^q(\mathbb{R}^n)$ by $L^q_{r^{n-1}dr}L^2_{\text{sph}}$ in (1.2). This intermediate case has been settled for adjoint restriction estimates for a cone by the authors of [17]. More precisely, if S is a non-empty smooth compact subset of the cone:

$$S = \big\{(\tau,\xi) \in \mathbb{R}\times\mathbb{R}^n : \ \ \tau = |\xi|\big\},$$

then for $q > 2n/(n-1)$ and $(n+1)/q \le (n-1)/p'$ we have

$$\|(f d\sigma)^\vee\|_{L^q_t(\mathbb{R}; L^q_{r^{n-1}dr}L^2_{\text{sph}})} \le C_{p,q,n,S}\|f\|_{L^p(S;d\sigma)}. \qquad (1.4)$$

The L^2_{sph}-norm allows us to use spherical harmonic expanding, so the problem is converted to $L^q(\ell^2)$-bounds for sequences of operators $\{H_k\}$ where each H_k is an operator acting on radial functions. The pioneering paper using such intermediate space is the Mockenhaupt Diploma in which he proved weighted L^p inequalities and then sharp $L^p_{\mathrm{rad}}(L^2_{\mathrm{sph}}) \to L^p_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ estimates for the disc multiplier operator, see either Mockenhaupt [14] or Córdoba [5]. Sharp endpoint bounds for the disk multiplier were obtained by Carbery-Romera-Soria [4]. Müller-Seeger [15] established some sharp mixed spacetime $L^p_{\mathrm{rad}}(L^2_{\mathrm{sph}})$ estimates in order to study a local smoothing of solutions for the linear wave equation. Córdoba-Latorre [9] revisited some classical conjecture including restriction estimate in harmonic analysis in this kind of mixed space-time. Gigante-Soria [12] studied a related mixed norm problem for Schrödinger maximal operators. Concerning the sphere restriction conjecture, Carli-Grafakos [7] also treated the same problem for spherically-symmetric functions and Cho-Guo-Lee [8] showed a restriction estimate for $q > 2(n+1)/n$ and $s \geq (n+2)/q - n/2$

$$\left\| \int_{\mathbb{S}^n} e^{2\pi i x\cdot\xi} f(\xi) d\sigma(\xi) \right\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{H^s(\mathbb{S}^n)}, \quad x \in \mathbb{R}^{n+1}, \tag{1.5}$$

where $d\sigma$ is the induced Lebesgue measure on $\mathbb{S}^n$ and $H^s(\mathbb{S}^n)$ denote the L^2-Sobolev space of order s on the sphere. An advantage of the proof consists in a fact that inequality (1.5) is based on L^2-spaces. The advantage of using the L^2-based Hilbert space also allows us to use effective the TT^* arguments to obtain Strichartz estimate with a wider range of admissible indexes by compensating with extra regularity in angular direction; see Sterbenz [21] for wave equation, Cho-Lee [9] for general dispersive equations and the authors [18] for wave equation with an inverse-square potential. Concerning other results in this direction, Cho-Hwang-Kwon-Lee [10] studied profile decompositions of fractional Schrödinger equations under the angular regularity assumption.

In this paper, we prove that estimate (1.2) holds for all p, q in (1.3) by compensating with some loss of angular derivatives. Our strategy is to use a spherical harmonic expanding as well as localized restriction estimates. In contrast to the radial case, e.g. [7,22], the main difficulty comes from the asymptotic behavior of the Bessel function $J_\nu(r)$ when $\nu \gg 1$. It is worth to point out that the method of treating cone restriction [17] is not valid since it can not be used to exploit the curvature property of paraboloid multiplier $e^{it|\xi|^2}$. We note that the bilinear argument used in [22], which is in spirit of Carleson-Sjölin argument or equivalently the TT^* argument, can be used to deal with the oscillation of the paraboloid multiplier. To use this argument, one needs to write the Bessel function $J_\nu(r) \sim c_\nu r^{-1/2} e^{ir}$ when $r \gg 1$. This expression works well for small ν (corresponding to the radial case) but it seems complicate to write the Bessel function in that form when $\nu \gg 1$. Indeed, as in [37], one can do this when $\nu^2 \ll r$, but it will cause more loss of derivative for the case $\nu \lesssim r \lesssim \nu^2$, since it is difficult to capture simultaneously the oscillation and decay behavior of $J_\nu(r)$. Our new idea here is to establish a $L^4_{t,x}$-localized restriction estimate by directly analyzing the kernel associated with the Bessel function. The key ingredient is to explore the decay and oscillation property of $J_\nu(r)$ for $r \gg \nu$, and resonant property of paraboloid multiplier. We also have to overcome low decay shortage of $J_\nu(r)$ (when $\nu \sim r \gg 1$) by compensating a loss of angular regularity.

Before stating the main theorem, we introduce some notation. Incorporating the angular regularity, we set the infinitesimal generators of the rotations on Euclidean space:

$$\Omega_{j,k} := x_j \partial_k - x_k \partial_j$$

and define for $s \in \mathbb{R}$

$$\Delta_\theta := \sum_{j<k} \Omega_{j,k}^2, \quad |\Omega|^s = (-\Delta_\theta)^{\frac{s}{2}}.$$

Hence Δ_θ is the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$. Define the Sobolev norm $\|\cdot\|_{H^{s,p}_{\text{sph}}(\mathbb{R}^n)}$ by setting

$$\|g\|^p_{H^{s,p}_{\text{sph}}(\mathbb{R}^n)} = \int_0^\infty \int_{\mathbb{S}^{n-1}} |(1-\Delta_\theta)^{s/2} g(r\theta)|^p d\theta\, r^{n-1} dr. \tag{1.6}$$

Given a constant A, we briefly write $A+\epsilon$ as A_+ or $A-\epsilon$ as A_- for $0<\epsilon \ll 1$.

Our main result is the following one.

Theorem 1.1 *Let $n \geq 2$. The following estimates hold for all Schwartz functions f*

- *if $q_0 = (2(n+1)/n)_+$ and $(n+2)/q_0 = n/p_0'$, then*

$$\|(f d\sigma)^\vee\|_{L^{q_0}_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \leq C_{p,q_0,n,S} \|f(|\xi|^2,\xi)\|_{H^{\sigma_0,p_0}_{\text{sph}}(\mathbb{R}^n_\xi)}, \tag{1.7}$$

  where $\sigma_0 = (n-2)\left(\frac{1}{2}-\frac{1}{q_0}\right)+\frac{2}{q_0}$;

- *if $1 \leq q, p \leq \infty$ satisfy (1.3), then*

$$\|(f d\sigma)^\vee\|_{L^{q}_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \leq C_{p,q,n,S} \|(1+|\Omega|)^s f\|_{L^p(S;d\sigma)}, \tag{1.8}$$

where $s = s(q,n) = \sigma_0\alpha$ and $0 \leq \alpha \leq 1$ satisfying $1/q = \alpha/q_0 + (1-\alpha)/q_1$. Here $q_1 = q(n)_+$ with $q(n) = 2 + 12/(4n+1-k)$ if $n+1 \equiv k (mod\ 3), k = -1, 0, 1$ as in Bourgain-Guth [2, Theorem 1].

Remark 1.1 Estimate (1.8) is an interpolation consequence of (1.7) and L^p-estimates in Bourgain-Guth [2]. Inequality (1.8) leads to the linear adjoint restriction estimate when $q \in (2(n+1)/n, q(n)]$ with some loss of angular derivatives.

Remark 1.2 Since the sphere $\mathbb{S}^n = \{(\tau,\xi) : |\tau|^2 + |\xi|^2 = 1\}$ is closely related to the paraboloid in sense of Taylor expansion $\sqrt{1-\rho^2} = 1 - \frac{1}{2}\rho^2 + O(\rho^4)$ near $\rho = 0$, it seems to be possible to show some modified version of (1.5) with $H^{s,p}(\mathbb{S}^n)$-norm on right hand side.

As an application of the modified restriction estimate, we show a result on the local smoothing estimate for the Schödinger equation for initial data with additional conditions angular regularity by Rogers's argument in [20]. Our result here extend [20, Theorem 1] from $q > 2(n+3)/(n+1)$ to $q > 2(n+1)/n$ under the assumption that initial data has additional angular regularity.

More precisely, we have the following local smoothing result.

Corollary 1.1 *Let $n \geq 2$, $q > 2(n+1)/n$ and s be as in Theorem 1.1. Then*

$$\|e^{it\Delta} u_0\|_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} \leq C \left\|(1+|\Omega|)^s u_0\right\|_{W^{\alpha,q}(\mathbb{R}^n)}, \tag{1.9}$$

where $\alpha > 2n(1/2-1/q) - 2/q$ and $W^{\alpha,q}(\mathbb{R}^n)$ is the Sobolev space.

This paper is organized as follows: In Sect. 2, we introduce notation and present some basic facts about spherical harmonics and Bessel functions. Furthermore, we use the stationary phase argument to prove some properties of Bessel functions. Section 3 is devoted to the proof of Theorem 1.1. In Sect. 4, we prove the key Proposition 3.1. We prove Corollary 1.1 in the final section.

2 Preliminaries

2.1 Notation

We use $A \lesssim B$ to denote the statement that $A \leq CB$ for some large constant C which may vary from line to line and depend on various parameters, and similarly employ $A \sim B$ to denote the statement that $A \lesssim B \lesssim A$. We also use $A \ll B$ to denote the statement $A \leq C^{-1}B$. If a constant C depends on a special parameter other than the above, we shall write it explicitly by subscripts. For instance, C_ϵ should be understood as a positive constant not only depending on p, q, n and S, but also on ϵ. Throughout this paper, pairs of conjugate indices are written as p, p', where $\frac{1}{p} + \frac{1}{p'} = 1$ with $1 \leq p \leq \infty$. Let $R > 0$ be a dyadic number, we define the dyadic annulus in $\mathbb{R}^n$ by

$$A_R := \left\{ x \in \mathbb{R}^n : \quad R/2 \leq |x| \leq R \right\}, \quad S_R := [R/2, R].$$

For each $M \in 2^{\mathbb{Z}}$, we define $\mathbb{L}_M$ to be the class of Schwartz functions supported on a dyadic subset of the paraboloid in the form of

$$\left\{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n : M \leq |\xi| \leq 2M, \tau = |\xi|^2\right\}. \tag{2.1}$$

2.2 Spherical harmonics expansions and Bessel function

We recall an expansion formula with respect to the spherical harmonics. Let

$$\xi = \rho\omega \quad \text{and} \quad x = r\theta \quad \text{with} \quad \omega, \theta \in \mathbb{S}^{n-1}. \tag{2.2}$$

For every $g \in L^2(\mathbb{R}^n)$, we have the expansion formula

$$g(\xi) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} a_{k,\ell}(\rho) Y_{k,\ell}(\omega),$$

where

$$\left\{Y_{k,1}, \ldots, Y_{k,d(k)}\right\}$$

is the orthogonal basis of the spherical harmonics space of degree k on $\mathbb{S}^{n-1}$. This space is recorded by $\mathcal{H}^k$ and it has the dimension

$$d(k) = \frac{2k + n - 2}{k} C_{n+k-3}^{k-1} \simeq \langle k \rangle^{n-2}.$$

It is clear that we have the orthogonal decomposition of $L^2(\mathbb{S}^{n-1})$

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}^k.$$

It follows that

$$\|g(\xi)\|_{L^2_\omega} = \|a_{k,\ell}(\rho)\|_{\ell^2_{k,\ell}}. \tag{2.3}$$

Using the spherical harmonic expansion, as well as [19,28], we define the action of $(1-\Delta_\omega)^{s/2}$ on g as follows

$$(1-\Delta_\omega)^{s/2} g = \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} (1+k(k+n-2))^{s/2} a_{k,\ell}(\rho) Y_{k,\ell}(\omega). \tag{2.4}$$

Given $s, s' \geq 0$ and $p, q \geq 1$, define

$$\|g\|_{H^{s,q}_\rho H^{s',p}_\omega} := \left\|(1-\Delta)^{\frac{s}{2}}\big((1-\Delta_\omega)^{\frac{s'}{2}} g\big)\right\|_{L^q_{\mu(\rho)}(\mathbb{R}^+; L^p_\omega(\mathbb{S}^{n-1}))},$$

where $\mu(\rho) = \rho^{n-1} d\rho$.

For our purpose, we need the inverse Fourier transform of $a_{k,\ell}(\rho) Y_{k,\ell}(\omega)$. We recall the Bochner-Hecke formula, see [13] and [26, Theorem 3.10]

$$\check{g}(r\theta) = \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} 2\pi i^k Y_{k,\ell}(\theta) r^{-\frac{n-2}{2}} \int_0^\infty J_{\nu(k)}(2\pi r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} d\rho. \tag{2.5}$$

Here $\nu(k) = k + \frac{n-2}{2}$ and the Bessel function $J_\nu(r)$ of order ν is defined by

$$J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu+\frac{1}{2})\Gamma(1/2)} \int_{-1}^{1} e^{isr}(1-s^2)^{(2\nu-1)/2} \mathrm{d}s,$$

where $\nu > -1/2$ and $r > 0$. It is easy to verify that there exists a constant C independent of ν such that

$$|J_\nu(r)| \leq \frac{Cr^\nu}{2^\nu \Gamma(\nu+\frac{1}{2})\Gamma(1/2)} \Big(1 + \frac{1}{\nu+1/2}\Big). \tag{2.6}$$

To investigate a behavior of asymptotic bound on ν and r, we recall the Schläfli integral representation [35] of the Bessel function: for $r \in \mathbb{R}^+$ and $\nu > -\frac{1}{2}$

$$\begin{aligned} J_\nu(r) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ir\sin\theta - i\nu\theta} d\theta - \frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-(r\sinh s+\nu s)} ds \\ &=: \tilde{J}_\nu(r) - E_\nu(r). \end{aligned} \tag{2.7}$$

Clearly, $E_\nu(r) = 0$ when $\nu \in \mathbb{Z}^+$. An easy computation shows that

$$|E_\nu(r)| = \left|\frac{\sin(\nu\pi)}{\pi}\int_0^\infty e^{-(r\sinh s+\nu s)} ds\right| \leq C(r+\nu)^{-1}. \tag{2.8}$$

There is a number of references for the asymptotic behavior of a Bessel function, see e.g. [9,23,25,35]. We recall some properties of a Bessel function for a convenience.

Lemma 2.1 (Asymptotics of Bessel functions) *Let $\nu \gg 1$ and let $J_\nu(r)$ be the Bessel function of order ν defined as above. Then there exists a large constant C and small constant c independent of ν and r such that:*

- *When $r \leq \frac{\nu}{2}$, we have*

$$|J_\nu(r)| \leq Ce^{-c(\nu+r)}; \tag{2.9}$$

- *When* $\frac{\nu}{2} \le r \le 2\nu$, *we have*

$$|J_\nu(r)| \le C\nu^{-\frac{1}{3}}(\nu^{-\frac{1}{3}}|r-\nu|+1)^{-\frac{1}{4}}; \tag{2.10}$$

- *When* $r \ge 2\nu$, *we have*

$$J_\nu(r) = r^{-\frac{1}{2}}\sum_{\pm} a_\pm(\nu,r)e^{\pm ir} + E(\nu,r), \tag{2.11}$$

where $|a_\pm(\nu,r)| \le C$ *and* $|E(\nu,r)| \le Cr^{-1}$.

3 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by using some localized linear estimates whose proof are postpone to the next section. Since inequality (1.7) is a special case of (1.8), we aim to prove (1.8). Since (1.8) is a direct consequence of the Stein-Tomas inequality [25] for the case $p \le 2$, it suffices to prove (1.8) for the case $p \ge 2$. More precisely, we will only establish the estimate for $q > 2(n+1)/n$, $(n+2)/q = n/p'$ with $p \ge 2$

$$\|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \le C_{p,q,n,S}\|(1+|\Omega|)^s f\|_{L^p(S;d\sigma)}. \tag{3.1}$$

Recall the notation $\mathbb{L}_M$ and A_R in the Sect. 2.1. We decompose f into a sum of dyadic supported functions

$$f = \sum_M f_M,$$

where $f_M = f\chi_{\{(\tau,\xi):\tau=|\xi|^2, M\le|\xi|\le 2M\}} \in \mathbb{L}_M$. It follows that

$$\begin{aligned}
\|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} &= \Big\|\sum_M (f_M d\sigma)^\vee\Big\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \\
&= \Big(\sum_R \Big\|\sum_M (f_M d\sigma)^\vee\Big\|^q_{L^q_{t,x}(\mathbb{R}\times A_R)}\Big)^{\frac{1}{q}} \\
&\lesssim \Big(\sum_R \Big(\sum_M \|(f_M d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\Big)^q\Big)^{\frac{1}{q}}.
\end{aligned} \tag{3.2}$$

To prove (3.1), we need localized linear restriction estimates.

Proposition 3.1 *Assume* $f \in \mathbb{L}_1$ *and* $R > 0$ *is a dyadic number. Then the following linear restriction estimates hold true.*

- *Let* $q = 2$, *then*

$$\|(fd\sigma)^\vee\|_{L^2_{t,x}(\mathbb{R}\times A_R)} \lesssim \min\left\{R^{\frac{1}{2}}, R^{\frac{n}{2}}\right\}\|f\|_{L^2(S;d\sigma)}. \tag{3.3}$$

- *Let* $q = 3p'$ *with* $2 \le p \le 4$ *and* $\sigma = (n-2)(\frac{1}{2}-\frac{1}{q}) + \frac{2}{q}$, $0 < \epsilon \ll 1$, *then*

$$\|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \lesssim \min\left\{R^{(n-1)(\frac{1}{q}-\frac{1}{2})+\epsilon}, R^{\frac{n}{q}}\right\}\left\|(1+|\Omega|)^\sigma f\right\|_{L^p(S;d\sigma)}. \tag{3.4}$$

We postpone the proof of Proposition 3.1 to the next section, and we complete the proof of Theorem 1.1 by this proposition. By a scaling argument, we conclude from (3.3) that

$$\|(f_M d\sigma)^\vee\|_{L^2_{t,x}(\mathbb{R}\times A_R)} \lesssim \min\left\{(RM)^{\frac{1}{2}}, (RM)^{\frac{n}{2}}\right\} M^{n-\frac{n+2}{2}-\frac{n}{2}} \|f_M\|_{L^2(S;d\sigma)}.$$

For any (q, p) satisfying

$$q > 2(n+1)/n, \quad (n+2)/q = n/p' \quad \text{with} \quad p \geq 2,$$

let $\alpha = 2 - \frac{3}{q} - \frac{1}{p}$, then we choose $\bar{q} = 3\bar{p}'$ such that

$$\frac{1}{q} = \frac{1-\alpha}{2} + \frac{\alpha}{\bar{q}}, \qquad \frac{1}{p} = \frac{1-\alpha}{2} + \frac{\alpha}{\bar{p}}.$$

From (3.4), we have that for $\bar{q} = 3\bar{p}'$ with $2 \leq \bar{p} \leq 4$ and $\bar{\sigma} = (n-2)(\frac{1}{2} - \frac{1}{\bar{q}}) + \frac{2}{\bar{q}}$

$$\begin{aligned}
&\|(f_M d\sigma)^\vee\|_{L^{\bar{q}}_{t,x}(\mathbb{R}\times A_R)} \\
&\lesssim \min\left\{(RM)^{(n-1)(\frac{1}{\bar{q}}-\frac{1}{2})+\bar{\epsilon}}, (RM)^{\frac{n}{\bar{q}}}\right\} M^{n-\frac{n+2}{\bar{q}}-\frac{n}{\bar{p}}} \left\|(1+|\Omega|)^{\bar{\sigma}} f_M\right\|_{L^{\bar{p}}(S;d\sigma)},
\end{aligned}$$

where $0 < \bar{\epsilon} \ll 1$. Therefore we obtain by an interpolation theorem

$$\begin{aligned}
&\|(f_M d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \\
&\lesssim \min\{(RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}[1-\frac{2(n+1)}{qn}]+\epsilon}\} \left\|(1+|\Omega|)^{\sigma} f_M\right\|_{L^p(S;d\sigma)}.
\end{aligned} \tag{3.5}$$

Here $0 < \epsilon := \bar{\epsilon}\alpha \ll 1$. According to (3.2), we obtain

$$\begin{aligned}
&\|(f d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \\
&\lesssim \left(\sum_R \left(\sum_M \min\left\{(RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}[1-\frac{2(n+1)}{qn}]+\epsilon}\right\} \|(1+|\Omega|)^{\sigma} f_M\|_{L^p(S;d\sigma)}\right)^q\right)^{\frac{1}{q}}.
\end{aligned}$$

Since $q > 2(n+1)/n$, $\epsilon \ll 1$, and R, M are both dyadic number, we have

$$\sup_{R>0}\left(\sum_M \min\left\{(RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}[1-\frac{2(n+1)}{qn}]+\epsilon}\right\}\right) < \infty,$$

$$\sup_{M>0}\left(\sum_R \min\left\{(RM)^{\frac{n}{q}}, (RM)^{-\frac{n-1}{2}[1-\frac{2(n+1)}{qn}]+\epsilon}\right\}\right) < \infty.$$

Note that for $q > 2(n+1)/n > p \geq 2$, we have by the Schur lemma and embedding inequality

$$\begin{aligned}
\|(f d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} &\lesssim \left(\sum_M \|(1+|\Omega|)^{\sigma} f_M\|^p_{L^p(S;d\sigma)}\right)^{\frac{1}{p}} \\
&= \left\|(1+|\Omega|)^{\sigma} f\right\|_{L^p(S;d\sigma)}.
\end{aligned}$$

Choosing $q = q_0 = (2(n+1)/n)_+$ and $(n+2)/q_0 = n/p_0'$, we have

$$\|(f d\sigma)^\vee\|_{L^{q_0}_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \left\|(1+|\Omega|)^{\sigma_0} f\right\|_{L^{p_0}(S;d\sigma)}.$$

This implies (1.7). Interpolating this inequality with the restriction estimate by Bourgain-Guth [2, Theorem 1], we prove (3.1). Hence, the proof of estimate (1.8) is completed.

4 Localized restriction estimate

In this section we prove Proposition 3.1. We start our proof by recalling

$$(f(\tau,\xi)d\sigma)^\vee(t,x)=\int_{\mathbb{R}^n} g(\xi)e^{2\pi i(x\cdot\xi+t|\xi|^2)}d\xi, \tag{4.1}$$

where $g(\xi)=f(|\xi|^2,\xi)\in\mathcal{S}(\mathbb{R}^n)$ with supp $g\subset\{\xi:|\xi|\in[1,2]\}$. We apply the spherical harmonic expansion to g to obtain

$$g(\xi)=\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}a_{k,\ell}(\rho)Y_{k,\ell}(\omega).$$

Recalling $\nu(k)=k+(n-2)/2$, we have by (2.5)

$$(fd\sigma)^\vee(t,x)=2\pi r^{-\frac{n-2}{2}}\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}i^kY_{k,\ell}(\theta)\int_0^\infty e^{-2\pi it\rho^2}J_{\nu(k)}(2\pi r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)d\rho. \tag{4.2}$$

Here we insert a harmless smooth bump function φ supported on the interval $(1/2,4)$ into the above integral, since $a_{k,\ell}(\rho)$ is supported on $[1,2]$. Now we estimate the quantity $\|(fd\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)}$. To this end, we first prove the following lemma.

Lemma 4.1 *Let $\mu(r)=r^{n-1}dr$ and $\omega(k)$ be a weight specified below. For $q\geq 2$, we have*

$$\begin{aligned}&\left\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\Big|\int_0^\infty e^{it\rho^2}J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n-2}{2}}\rho d\rho\Big|^2\Big)^{\frac12}\right\|_{L^q_t(\mathbb{R};L^q_{\mu(r)}(S_R))}\\&\quad\lesssim\left\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\big\|J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n-2}{2}+\frac{1}{q'}}\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}\right\|_{L^q_{\mu(r)}(S_R)}.\end{aligned} \tag{4.3}$$

Proof Since $q\geq 2$, the Minkowski inequality and the Fubini theorem show that the left hand side of (4.3) is bounded by

$$\left\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\Big\|\int_0^\infty e^{it\rho^2}J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n-2}{2}}\rho\,d\rho\Big\|^2_{L^q_t(\mathbb{R})}\Big)^{\frac12}\right\|_{L^q_{\mu(r)}(S_R)}.$$

We rewrite this by making the variable change $\rho^2\rightsquigarrow\rho$

$$\left\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\Big\|\int_0^\infty e^{it\rho}J_{\nu(k)}(r\sqrt{\rho})a_{k,\ell}(\sqrt{\rho})\varphi(\sqrt{\rho})\rho^{\frac{n-2}{4}}\,d\rho\Big\|^2_{L^q_t(\mathbb{R})}\Big)^{\frac12}\right\|_{L^q_{\mu(r)}(S_R)}. \tag{4.4}$$

We use the Hausdorff-Young inequality with respect to t and we change variables back to obtain

$$\text{LHS of (4.3)} \lesssim \Big\| r^{-\frac{n-2}{2}} \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \big\| J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\varphi(\rho)\rho^{(n-2)/2+1/q'} \big\|_{L^{q'}_\rho}^2 \Big)^{\frac{1}{2}} \Big\|_{L^q_{\mu(r)}(S_R)}.$$

□

Now we prove that the inequalities (3.3) and (3.4) with $R \lesssim 1$. For doing this, we need

Lemma 4.2 *Let $q \geq 2$ and $R \lesssim 1$, we have the following estimate*

$$\|(f\, d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \lesssim R^{\frac{n}{q}} \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \big\| a_{k,\ell}(\rho)\varphi(\rho) \big\|_{L^{q'}_\rho}^2 \Big)^{\frac{1}{2}}, \tag{4.5}$$

where $\omega(k) = (1+k)^{2(n-1)(1/2-1/q)}$.

We postpone the proof of this lemma for a moment. Note that for $q' \leq 2 \leq p$, we use (4.5), (2.4), the Minkowski inequality and the Hölder inequality to obtain

$$\begin{aligned} \|(f\, d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \lesssim\ & R^{\frac{n}{q}} \Big\| \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \big| a_{k,\ell}(\rho) \big|^2 \Big)^{\frac{1}{2}} \varphi(\rho) \Big\|_{L^{q'}_\rho} \\ & \lesssim R^{\frac{n}{q}}\ \|g\|_{L^{q'}_\rho H^m_\omega(\mathbb{S}^{n-1})} \lesssim R^{\frac{n}{q}}\ \|g\|_{L^p_\rho H^{m,p}_\omega(\mathbb{S}^{n-1})}, \end{aligned}$$

where $m = (n-1)(\frac{1}{2} - \frac{1}{q})$. In particular, for $q = 2$ and $4 \leq q \leq 6$, this proves (3.3) and (3.4) when $R \lesssim 1$. Hence it suffices to consider the case $R \gg 1$ once we prove Lemma 4.2.

Proof of Lemma 4.2 By scaling argument in variables t, x and (4.2), we obtain

$$\begin{aligned} & \|(f\, d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \\ & \lesssim \Big\| r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\, d\rho \Big\|_{L^q_{t,x}(\mathbb{R}\times A_R)}. \end{aligned} \tag{4.6}$$

By Sobolev's embedding, (2.3) and (2.4), we have

$$\begin{aligned} & \|(f\, d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \\ & \lesssim \Big\| r^{-\frac{n-2}{2}} \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \Big| \int_0^\infty e^{it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n-2}{2}} \rho\, d\rho \Big|^2 \Big)^{\frac{1}{2}} \Big\|_{L^q_t(\mathbb{R}; L^q_{\mu(r)}(S_R))}. \end{aligned}$$

By Lemma 4.1, it is enough to show

$$\begin{aligned} & \Big\| r^{-\frac{n-2}{2}} \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \big\| J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\varphi(\rho)\rho^{(n-2)/2+1/q'} \big\|_{L^{q'}_\rho}^2 \Big)^{\frac{1}{2}} \Big\|_{L^q_{\mu(r)}(S_R)} \\ & \qquad \lesssim R^{\frac{n}{q}} \Big(\sum_{k=0}^{\infty} \sum_{\ell=1}^{d(k)} \omega(k) \big\| a_{k,\ell}(\rho)\varphi(\rho) \big\|_{L^{q'}_\rho}^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Writing briefly $\nu=\nu(k)$, and noting that $R<r<2R$ and $1<\rho<2$, we have by (2.6)

$$\Big\|r^{-\frac{n-2}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\big\|J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{(n-2)/2+1/q'}\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}\Big\|_{L^q_{\mu(r)}([R,2R])}$$
$$\lesssim\Big(\int_R^{2R} r^{-\frac{(n-2)q}{2}}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\Big|\frac{(4r)^\nu}{2^\nu\Gamma(\nu+\frac12)\Gamma(\frac12)}\Big|^2\big\|a_{k,\ell}(\rho)\rho^\nu\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac q2}r^{n-1}\,dr\Big)^{\frac1q}$$
$$\lesssim R^{\frac nq}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\Big[\frac{(2R)^{\nu-\frac{n-2}{2}}}{\Gamma(\nu+\frac12)}\Big]^2\big\|a_{k,\ell}(\rho)\rho^\nu\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}$$
$$\lesssim R^{\frac nq}\Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}\omega(k)\big\|a_{k,\ell}(\rho)\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}.$$

In the last inequality, we use the Stirling formula $\Gamma(\nu+1)\sim\sqrt{\nu}(\nu/e)^\nu$ and the fact that $R\lesssim 1$ and $\nu\ge(n-2)/2$. □

Now we are in a position to prove Proposition 3.1 when $R\gg 1$. We first prove (3.3) by making use of (4.1). Since supp $g\subset\{\xi:|\xi|\in[1,2]\}$, we may assume $|\xi_n|\sim 1$. Then we freeze one spatial variable, say x_n, with $|x_n|\lesssim R$ and free other spatial variables $x'=(x_1,\dots,x_{n-1})$. After making the change of variables $\eta_j=\xi_j,\ \eta_n=|\xi|^2$ with $j=1,\dots n-1$, we use the Plancherel theorem on the spacetime Fourier transform in (t,x') to obtain (3.3).

When $R\gg 1$, inequality (3.4) is a consequence of the interpolation theorem and the following proposition.

Proposition 4.1 *Assume $f\in\mathbb{L}_1$ and $R\gg 1$ is a dyadic number. For every small constant $0<\epsilon\ll 1$, we have the following inequalities*

- *For $q=4$, we have*

$$\|(f\,d\sigma)^\vee\|_{L^4_{t,x}(\mathbb{R}\times A_R)}\lesssim R^{-\frac{n-1}{4}+\epsilon}\|(1+|\Omega|)^{\frac n4}f\|_{L^4(S;\,d\sigma)}. \tag{4.7}$$

- *For $q=6$, we have*

$$\|(f\,d\sigma)^\vee\|_{L^6_{t,x}(\mathbb{R}\times A_R)}\lesssim R^{-\frac{n-1}{3}+\epsilon}\|(1+|\Omega|)^{\frac{n-1}{3}}f\|_{L^2(S;\,d\sigma)}. \tag{4.8}$$

Remark 4.1 It seems to be possible to remove the ϵ-loss in (4.8), but we do not purchase this option here because we do not need it in this paper.

To prove this proposition, we firstly show

Lemma 4.3 *Assume $f\in\mathbb{L}_1$ and $R\gg 1$. We have the following estimate*

$$\|(f\,d\sigma)^\vee\|_{L^4_{t,x}(\mathbb{R}\times A_R)}\lesssim R^{-\frac{n-1}{4}+\epsilon}\|g\|_{L^4_\rho H^{\frac n4,4}_\omega(\mathbb{S}^{n-1})}, \tag{4.9}$$

where $0<\epsilon\ll 1$, and $g(\xi)=f(|\xi|^2,\xi)$.

Proof By the scaling argument and (4.2), it suffices to estimate the quantity

$$\Big\|r^{-\frac{n-2}{2}}\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)}i^kY_{k,\ell}(\theta)\int_0^\infty e^{-it\rho^2}J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac n2}\varphi(\rho)\,d\rho\Big\|_{L^4_{t,x}(\mathbb{R}\times A_R)}. \tag{4.10}$$

In the following, we consider the three cases. For the first two cases, we establish the estimates for general $q \geq 4$ so that we can use them directly for $q = 6$ later.

- Case 1: $k \in \Omega_1 := \{k : R \ll \nu(k)\}$. Let $\omega(k) = (1+k)^{2(n-1)(1/2-1/q)}$ again. We have by a similar argument as in the proof of Lemma 4.2:

$$\begin{aligned}&\Big\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta)\int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\,d\rho\Big\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\\&\lesssim \Big\| r^{-\frac{n-2}{2}}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)\Big|\int_0^\infty e^{it\rho^2}J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{\frac{n-2}{2}}\rho\,d\rho\Big|^2\Big)^{\frac12}\Big\|_{L^q_t(\mathbb{R};L^q_{\mu(r)}(S_R))}\\&\lesssim \Big\| r^{-\frac{n-2}{2}}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)\big\|J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{(n-2)/2+1/q'}\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}\Big\|_{L^q_{\mu(r)}(S_R)}.\end{aligned}$$

Recall that for $R \gg 1$ and $k \in \Omega_1$, we have $|J_{\nu(k)}(r)| \lesssim e^{-c(r+\nu)}$ by (2.9). Using $R < r < 2R$ and $1 < \rho < 2$, we obtain

$$\begin{aligned}&\Big\| r^{-\frac{n-2}{2}}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)\big\|J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\varphi(\rho)\rho^{(n-2)/2+1/q'}\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}\Big\|_{L^q_{\mu(r)}([R,2R])}\\&\lesssim \Big(\int_R^{2R} r^{-\frac{(n-2)q}{2}}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)e^{-(r+\nu)}\big\|a_{k,\ell}(\rho)\rho^\nu\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac q2} r^{n-1}\,dr\Big)^{\frac1q}\\&\lesssim e^{-cR}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)e^{-\nu(k)}\big\|a_{k,\ell}(\rho)\rho^\nu\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}\\&\lesssim e^{-cR}\Big(\sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)}\omega(k)\big\|a_{k,\ell}(\rho)\varphi(\rho)\big\|^2_{L^{q'}_\rho}\Big)^{\frac12}.\end{aligned}$$

By Minkowski's inequality and Hölder's inequality, we obtain

$$\begin{aligned}&\Big\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta)\int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\,d\rho\Big\|_{L^q_{t,x}(\mathbb{R}\times A_R)}\\&\qquad\lesssim e^{-cR}\Big\|\Big(\sum_{k=0}^\infty\sum_{\ell=1}^{d(k)}\omega(k)\big|a_{k,\ell}(\rho)\big|^2\Big)^{\frac12}\varphi(\rho)\Big\|_{L^p_\rho}.\end{aligned}\tag{4.11}$$

Applying this with $q = 4 = p$, we have

$$\begin{aligned}&\Big\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta)\int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\,d\rho\Big\|_{L^4_{t,x}(\mathbb{R}\times A_R)}\\&\lesssim e^{-cR}\Big\|\Big(\sum_{k=0}^\infty\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\big|a_{k,\ell}(\rho)\big|^2\Big)^{\frac12}\varphi(\rho)\Big\|_{L^4_\rho}\\&\lesssim R^{-\frac{n-1}{4}+\epsilon}\|g\|_{L^4_\rho H^{(n-1)/4,4}_\omega(\mathbb{S}^{n-1})}.\end{aligned}$$

- Case 2: $k \in \Omega_2 := \{k : \nu(k) \sim R\}$. Recalling $g(\xi) = f(|\xi|^2, \xi)$, and using the Sobolev embedding, the Strichartz estimate and the fact $\operatorname{supp} g \subset \{\xi \in \mathbb{R}^n : |\xi| \in [1, 2]\}$, we have for $q \geq 4$ and $\frac{2}{q} = n(\frac{1}{2} - \frac{1}{r})$

$$\|(f\,d\sigma)^\vee\|_{L^q_{t,x}(\mathbb{R}\times\mathbb{R}^n)} \lesssim \|(f\,d\sigma)^\vee\|_{L^q(\mathbb{R}; H^m_r(\mathbb{R}^n))} \lesssim \|\hat{g}\|_{H^m(\mathbb{R}^n)} \lesssim \|g\|_{L^2(\mathbb{R}^n)} \tag{4.12}$$

where $m = \frac{(q-2)n-4}{2q} \geq 0$ since $n \geq 2$. If $g = \bigoplus_{k\in\Omega_2} \mathcal{H}^k$, then

$$\begin{aligned} \|g\|^2_{L^2_\omega(\mathbb{S}^{n-1})} &= \sum_{k\in\Omega_2}\sum_{\ell=1}^{d(k)} |a_{k,\ell}|^2 \\ &\lesssim R^{-2(n-1)(1/2-1/q)} \sum_{k\in\Omega_2}\sum_{\ell=1}^{d(k)} (1+k)^{2(n-1)(1/2-1/q)} |a_{k,\ell}|^2 \\ &\lesssim R^{-2(n-1)(1/2-1/q)} \|g\|^2_{H_\omega^{(n-1)(\frac{1}{2}-\frac{1}{q}),2}(\mathbb{S}^{n-1})}. \end{aligned} \tag{4.13}$$

Since $\operatorname{supp} g \subset \{\xi \in \mathbb{R}^n : |\xi| \in [1, 2]\}$ and $p \geq 2$, we have by Hölder's inequality and (4.12)

$$\begin{aligned} &\left\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_2}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right\|_{L^q_{t,x}(\mathbb{R}\times A_R)} \\ &\lesssim R^{-(n-1)(1/2-1/q)} \|g\|_{L^p_\rho H_\omega^{(n-1)(\frac{1}{2}-\frac{1}{q}),p}(\mathbb{S}^{n-1})}. \end{aligned} \tag{4.14}$$

In particular, when $q = p = 4$, inequality (4.14) implies that

$$\begin{aligned} &\Bigg\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_2}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \\ &\int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \Bigg\|_{L^4_{t,x}(\mathbb{R}\times A_R)} \\ &\lesssim R^{-(n-1)/4} \|g\|_{L^4_\rho H_\omega^{(n-1)/4,4}(\mathbb{S}^{n-1})}. \end{aligned} \tag{4.15}$$

- Case 3: $k \in \Omega_3 := \{k : \nu(k) \ll R\}$. We need the following lemma about the oscillation and decay property of a Bessel function. This lemma was proved by Barcelo-Cordoba [3].

Lemma 4.4 (Oscillation and asymptotic property, [3]). *Let $\nu > 1/2$ and $r > \nu + \nu^{1/3}$. There exists a constant number C independent of r and ν such that*

$$J_\nu(r) = \sqrt{\frac{2}{\pi}} \frac{\cos\theta(r)}{(r^2-\nu^2)^{1/4}} + h_\nu(r), \tag{4.16}$$

where $\theta(r) = (r^2-\nu^2)^{1/2} - \nu \arccos\frac{\nu}{r} - \frac{\pi}{4}$ and

$$|h_\nu(r)| \leq C\left(\left(\frac{\nu^2}{(r^2-\nu^2)^{7/4}} + \frac{1}{r}\right) 1_{[\nu+\nu^{1/3},2\nu]}(r) + \frac{1}{r} 1_{[2\nu,\infty)}(r)\right). \tag{4.17}$$

Note that $\nu(k) = k + (n-2)/2$ and $k \in \Omega_3$, we can write

$$J_\nu(r) = I_\nu(r) + \bar{I}_\nu(r) + h_\nu(r), \quad \text{where } |h_\nu(r)| \lesssim r^{-1},$$

and

$$I_\nu(r) = \frac{\sqrt{2/\pi}\, e^{i\theta(r)}}{\left(r^2 - \nu^2\right)^{1/4}}.$$

A simple computation yields to

$$\begin{cases} \theta'(r) = (r^2 - \nu^2)^{1/2} r^{-1}, \\ \theta''(r) = (r^2 - \nu^2)^{-1/2} - (r^2 - \nu^2)^{1/2} r^{-2} = (r^2 - \nu^2)^{-1/2} \nu^2 r^{-2}, \\ \theta'''(r) = \frac{\nu^2}{r}(r^2 - \nu^2)^{-3/2} \nu^2 r^{-2} \left(-3 + \frac{2\nu^2}{r^2}\right). \end{cases} \tag{4.18}$$

Using Sobolev embedding on sphere and Minkowski's inequality, we estimate

$$\begin{aligned} &\left\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right\|_{L^4_{t,x}(\mathbb{R}\times A_R)} \\ &\lesssim R^{-\frac{n-2}{2}} \left\| \left(\sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} (1+k)^{(n-1)/2} \left| \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right|^2 \right)^{1/2} \right\|_{L^4_t(\mathbb{R}; L^4_{\mu(r)}(S_R))} \\ &\lesssim R^{-\frac{n-3}{4}} \left\| \left(\sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} (1+k)^{(n-1)/2} \left| \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right|^2 \right)^{1/2} \right\|_{L^4_t(\mathbb{R}; L^4_r(S_R))}. \end{aligned}$$

Since $J_\nu(r) = I_\nu(r) + \bar{I}_\nu(r) + h_\nu(r)$, it suffices to estimate two terms

$$\begin{aligned} &\left(\sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} (1+k)^{(n-1)/2} \left\| \int_0^\infty e^{-it\rho^2} h_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right\|^2_{L^4_t(\mathbb{R}; L^4_r(S_R))} \right)^{1/2} \\ &\quad \lesssim R^{-3/4} \|g\|_{L^4_\rho H^{\frac{n-1}{4},4}_\omega(\mathbb{S}^{n-1})} \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} &\left\| \left(\sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} (1+k)^{(n-1)/2} \left| \int_0^\infty e^{-it\rho^2} I_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \right|^2 \right)^{1/2} \right\|_{L^4_t(\mathbb{R}; L^4_r(S_R))} \\ &\quad \lesssim R^{-1/2+\epsilon} \|g\|_{L^4_\rho H^{\frac{n}{4},4}_\omega(\mathbb{S}^{n-1})}. \end{aligned} \tag{4.20}$$

For the first purpose, we consider the operator

$$T_\nu(a)(t,r) = \chi\left(\frac{r}{R}\right) \int_0^\infty e^{-it\rho^2} h_\nu(r\rho) a(\rho) \rho^{\frac{n}{2}} \varphi(\rho) d\rho$$

where $|h_\nu(r)| \leq C/r$. By a similar argument as in the proof of Lemma 4.1, it is easy to see

$$\|T_\nu(a)(t,r)\|_{L^q_{t,r}} \leq R^{-1/q'} \|a\varphi\|_{L^{q'}_\rho}. \tag{4.21}$$

Hence we have

$$\begin{aligned}
&\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\Big\|\int_0^\infty e^{-it\rho^2}h_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\,d\rho\Big\|^2_{L^4_t(\mathbb{R};L^4_r(S_R))}\Big)^{1/2}\\
&\lesssim R^{-3/4}\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\Big\|a_{k,\ell}(\rho)\varphi(\rho)\Big\|^2_{L^{4/3}}\Big)^{1/2}\\
&\lesssim R^{-3/4}\Big\|\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\big|a_{k,\ell}(\rho)\big|^2\Big)^{1/2}\varphi\Big\|_{L^{4/3}}\\
&\lesssim R^{-3/4}\|g\|_{L^4_\rho H_\omega^{\frac{n-1}{4},4}(\mathbb{S}^{n-1})}
\end{aligned}$$

which implies (4.19).

Next we prove (4.20). To this end, let $\beta(\rho)=\rho^{\frac{n}{2}}\varphi(\rho)$, we see that

$$\begin{aligned}
&\left\|\left(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\left|\int_0^\infty e^{-it\rho^2}I_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\mathrm{d}\rho\right|^2\right)^{1/2}\right\|^4_{L^4_t(\mathbb{R};L^4_r(S_R))}\\
&=\Big\|\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\int_{\mathbb{R}^2}e^{-it(\rho_1^2-\rho_2^2)}I_{\nu(k)}(r\rho_1)\overline{I_{\nu(k)}(r\rho_2)}\\
&\quad\times a_{k,\ell}(\rho_1)\overline{a_{k,\ell}(\rho_2)}\beta(\rho_1)\beta(\rho_2)\mathrm{d}\rho_1\mathrm{d}\rho_2\Big\|^2_{L^2_t(\mathbb{R};L^2_r(S_R))}\\
&\le\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big\|\int_{\mathbb{R}^2}e^{-it(\rho_1^2-\rho_2^2)}I_{\nu(k)}(r\rho_1)\overline{I_{\nu(k)}(r\rho_2)}\\
&\quad\times\sum_{\ell=1}^{d(k)}a_{k,\ell}(\rho_1)\overline{a_{k,\ell}(\rho_2)}\beta(\rho_1)\beta(\rho_2)\mathrm{d}\rho_1\mathrm{d}\rho_2\Big\|_{L^2_t(\mathbb{R};L^2_r(S_R))}\Big)^2\\
&=\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big(\int_{\mathbb{R}^4}\sum_{\ell=1}^{d(k)}a_{k,\ell}(\rho_1)\overline{a_{k,\ell}(\rho_2)}\sum_{\ell'=1}^{d(k)}\overline{a_{k,\ell'}(\rho_3)}a_{k,\ell'}(\rho_4)\beta(\rho_1)\beta(\rho_2)\beta(\rho_3)\beta(\rho_4)\\
&\quad\int_{\mathbb{R}}e^{-it(\rho_1^2-\rho_2^2+\rho_3^2-\rho_4^2)}dt\,K(R,\nu;\rho_1,\rho_2,\rho_3,\rho_4)d\rho_1d\rho_2d\rho_3d\rho_4\Big)^{1/2}\Big)^2
\end{aligned}\tag{4.22}$$

where the kernel

$$\begin{aligned}
&K(R,\nu;\rho_1,\rho_2,\rho_3,\rho_4)\\
&=\int_0^\infty\frac{\chi(\frac{r}{R})e^{i(\theta(\rho_1r)-\theta(\rho_2r)+\theta(\rho_3r)-\theta(\rho_4r))}}{\big((r\rho_1)^2-\nu^2\big)^{1/4}\big((r\rho_2)^2-\nu^2\big)^{1/4}\big((r\rho_3)^2-\nu^2\big)^{1/4}\big((r\rho_4)^2-\nu^2\big)^{1/4}}dr.
\end{aligned}\tag{4.23}$$

Now we analyze the kernel K. Let

$$\phi(r;\rho_1,\rho_2,\rho_3,\rho_4)=\theta(\rho_1r)-\theta(\rho_2r)+\theta(\rho_3r)-\theta(\rho_4r).$$

Hence if $\rho_1^2-\rho_2^2=\rho_4^2-\rho_3^2$, we have by (4.18)

$$\begin{aligned}\phi_r' &= (\rho_1^2-\rho_2^2)r\Big(\frac{1}{\sqrt{(r\rho_1)^2-\nu^2}+\sqrt{(r\rho_2)^2-\nu^2}}-\frac{1}{\sqrt{(r\rho_3)^2-\nu^2}+\sqrt{(r\rho_4)^2-\nu^2}}\Big)\\ &= \frac{(\rho_1^2-\rho_2^2)(\rho_3^2-\rho_2^2)r^3}{\Big(\sqrt{(r\rho_1)^2-\nu^2}+\sqrt{(r\rho_2)^2-\nu^2}\Big)\Big(\sqrt{(r\rho_3)^2-\nu^2}+\sqrt{(r\rho_4)^2-\nu^2}\Big)}\\ &\times\quad\Big(\frac{1}{\sqrt{(r\rho_3)^2-\nu^2}+\sqrt{(r\rho_2)^2-\nu^2}}+\frac{1}{\sqrt{(r\rho_1)^2-\nu^2}+\sqrt{(r\rho_4)^2-\nu^2}}\Big).\end{aligned}$$

Since $k\in\Omega_3$, one has $r\gg\nu(k)$. Therefore we have

$$|\phi_r'|\geq|\rho_1^2-\rho_2^2|\cdot|\rho_3^2-\rho_2^2|.$$

Applying integration by parts with respect to r to (4.23), we have for any $N\geq 0$

$$K(R,\nu;\rho_1,\rho_2,\rho_3,\rho_4)\lesssim R^{-1}\left(1+R|\rho_1^2-\rho_2^2|\cdot|\rho_3^2-\rho_2^2|\right)^{-N}, \tag{4.24}$$

when $\rho_1^2-\rho_2^2=\rho_4^2-\rho_3^2$. Let $b_{k,\ell}(\rho)=2a_{k,\ell}(\sqrt{\rho})\beta(\sqrt{\rho})/\sqrt{\rho}$, from (4.22) and (4.24), it suffices to estimate

$$\begin{aligned}&\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big(\int_{\mathbb{R}^4}\delta(\rho_1-\rho_2+\rho_3-\rho_4)K(R,\nu(k);\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_4})\\ &\quad\times\sum_{\ell=1}^{d(k)}b_{k,\ell}(\rho_1)\overline{b_{k,\ell}(\rho_2)}\sum_{\ell'=1}^{d(k)}\overline{b_{k,\ell'}(\rho_3)}b_{k,\ell'}(\rho_4)d\rho_1d\rho_2d\rho_3d\rho_4\Big)^{1/2}\Big)^2\\ &=\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big(\int_{\mathbb{R}^3}K(R,\nu(k);\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_1-\rho_2+\rho_3})\\ &\quad\times\sum_{\ell=1}^{d(k)}b_{k,\ell}(\rho_1)\overline{b_{k,\ell}(\rho_2)}\sum_{\ell'=1}^{d(k)}\overline{b_{k,\ell'}(\rho_3)}b_{k,\ell'}(\rho_1-\rho_2+\rho_3)d\rho_1d\rho_2d\rho_3\Big)^{1/2}\Big)^2\\ &\leq R^{-1}\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big(\int_{\mathbb{R}^3}(1+R|\rho_1-\rho_2||\rho_3-\rho_2|)^{-N}\\ &\quad\times\sum_{\ell=1}^{d(k)}\big|b_{k,\ell}(\rho_1)\overline{b_{k,\ell}(\rho_2)}\big|\sum_{\ell'=1}^{d(k)}\big|\overline{b_{k,\ell'}(\rho_3)}b_{k,\ell'}(\rho_1-\rho_2+\rho_3)\big|d\rho_1d\rho_2d\rho_3\Big)^{1/2}\Big)^2\\ &\lesssim R^{-1}\Big(\sum_{k\in\Omega_3}(1+k)^{(n-1)/2}\Big(\int_{\mathbb{R}^3}(1+R|\rho_1-\rho_2||\rho_3-\rho_2|)^{-N}\\ &\quad\times b_k(\rho_1)b(\rho_2)b_k(\rho_3)b_k(\rho_1-\rho_2+\rho_3)d\rho_1d\rho_2d\rho_3\Big)^{1/2}\Big)^2\end{aligned}$$

where $b_k(\rho)=\big(\sum_{\ell=1}^{d(k)}|b_{k,\ell}(\rho)|^2\big)^{1/2}$. Then we aim to estimate

$$\int_{\mathbb{R}^3}\frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1-\rho_2+\rho_3)}{(1+R|\rho_1-\rho_2||\rho_3-\rho_2|)^N}d\rho_1d\rho_2d\rho_3\lesssim R^{-1+\epsilon}\|b\|_{L^4}^4. \tag{4.25}$$

Indeed once we have proved (4.25), we show

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{(n-1)/2}\Big|\int_0^\infty e^{-it\rho^2}I_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\mathrm{d}\rho\Big|^2\Big)^{1/2}\Big\|^2_{L^4_t(\mathbb{R};L^4_r(S_R))}\\
&\lesssim R^{-1+\epsilon}\sum_{k\in\Omega_3}(1+k)^{(n-1)/2+\frac{1}{2}+\epsilon}(1+k)^{-\frac{1}{2}-\epsilon}\,\|b_k\|^2_{L^4}\\
&\lesssim R^{-1+2\epsilon}\Big(\sum_{k\in\Omega_3}(1+k)^n\Big\|(\sum_{\ell=1}^{d(k)}|b_{k,\ell}(\rho)|^2)^{1/2}\Big\|^4_{L^4}\Big)^{1/2}\\
&\lesssim R^{-1+2\epsilon}\Big\|\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{\frac{n}{2}}|a_{k,\ell}(\rho)|^2\Big)^{1/2}\Big\|^2_{L^4}\end{aligned}$$

which implies (4.20). Therefore, it remains to prove

$$\int_{\mathbb{R}^3}\frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1-\rho_2+\rho_3)}{(1+R|\rho_1-\rho_2|\cdot|\rho_3-\rho_2|)^N}d\rho_1d\rho_2d\rho_3\lesssim R^{-1+\epsilon}\|b\|^4_{L^4}. \tag{4.26}$$

For $R=2^{k_0}\gg 1$, we decompose the integral into

$$\begin{aligned}&\int_{\mathbb{R}^3}\frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1-\rho_2+\rho_3)}{(1+R|\rho_1-\rho_2||\rho_3-\rho_2|)^N}d\rho_1d\rho_2d\rho_3\\
&=\Big(\sum_{\{(i,j)\in\mathbb{N}^2;i+j\geq k_0\}}+\sum_{\{(i,j)\in\mathbb{N}^2;i+j\lesssim k_0\}}R^{-N}2^{N(i+j)}\Big)\\
&\quad\int b(\rho_2)d\rho_2\int_{|\rho_1-\rho_2|\sim 2^{-i}}b(\rho_1)d\rho_1\int_{|\rho_3-\rho_2|\sim 2^{-j}}b(\rho_3)b(\rho_1-\rho_2+\rho_3)d\rho_3.\end{aligned}\tag{4.27}$$

To estimate it, we need the following lemma.

Lemma 4.5 *We have the following estimate for the integral*

$$\int b(\rho_2)d\rho_2\int_{|\rho_1-\rho_2|\sim 2^{-i}}b(\rho_1)d\rho_1\int_{|\rho_3-\rho_2|\sim 2^{-j}}b(\rho_3)b(\rho_1-\rho_2+\rho_3)d\rho_3\lesssim 2^{-(i+j)}\|b\|^4_{L^4}. \tag{4.28}$$

Proof We first have by Hölder's inequality

$$
\begin{aligned}
&\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} b(\rho_3)b(\rho_1-\rho_2+\rho_3)d\rho_3 \\
&\lesssim \left(\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_1-\rho_2+\rho_3)|^2 d\rho_3\right)^{1/2} \\
&\lesssim \left(\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int\limits_{|\rho|\sim 2^{-j}} |b(\rho_1+\rho)|^2 d\rho\right)^{1/2} \\
&\lesssim \left(\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_3)|^2 d\rho_3 \int\limits_{|\rho-\rho_1|\sim 2^{-j}} |b(\rho)|^2 d\rho\right)^{1/2}. \qquad (4.29)
\end{aligned}
$$

Let I be the left hand side of (4.28). We estimate I by (4.29) and Hölder's inequality

$$
\begin{aligned}
&\int b(\rho_2) \int\limits_{|\rho_1-\rho_2|\sim 2^{-i}} \Big(\int\limits_{|\rho_1-\rho|\sim 2^{-j}} |b(\rho)|^2 d\rho\Big)^{1/2} b(\rho_1)d\rho_1 \Big(\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_3)|^2 d\rho_3\Big)^{1/2} d\rho_2 \\
&\lesssim \|b\|_{L^4} \Big\| \int\limits_{|\rho_1-\rho_2|\sim 2^{-i}} \Big(\int\limits_{|\rho_1-\rho|\sim 2^{-j}} |b(\rho)|^2 d\rho\Big)^{1/2} |b(\rho_1)| d\rho_1 \Big\|_{L^2_{\rho_2}} \Big\| \Big(\int\limits_{|\rho_3-\rho_2|\sim 2^{-j}} |b(\rho_3)|^2 d\rho_3\Big)^{1/2} \Big\|_{L^4} \\
&\lesssim \|b\|_{L^4} \left\| \chi_i * \big((\chi_j * |b|^2)^{\frac12} |b|\big) \right\|_{L^2} \left\|\chi_j * |b|^2\right\|_{L^2}^{1/2},
\end{aligned}
$$

where $\chi_j = \chi_j(\rho) = \chi(2^j\rho)$ and $\chi \in C_c^\infty([\frac14, 4])$. It is easy to see by the Young inequality

$$
\left\|\chi_j * |b|^2\right\|_{L^2}^{1/2} \lesssim \|\chi_j\|_{L^1}^{1/2} \|b\|_{L^4} \lesssim 2^{-j/2} \|b\|_{L^4},
$$

and

$$
\begin{aligned}
\left\| \chi_i * \Big((\chi_j * |b|^2)^{\frac12} |b|\Big) \right\|_{L^2} &\lesssim \|\chi_i\|_{L^1} \left\| (\chi_j * |b|^2)^{\frac12} |b| \right\|_{L^2} \\
&\lesssim \|\chi_i\|_{L^1} \left\|\chi_j * |b|^2\right\|_{L^2}^{\frac12} \|b\|_{L^4} \\
&\lesssim \quad 2^{-i} 2^{-j/2} \|b\|_{L^4}^2.
\end{aligned}
$$

Collecting the above estimates, we obtain

$$
I \lesssim 2^{-(i+j)} \|b\|_{L^4}^4.
$$

This completes the proof of Lemma 4.5. □

Now we return to prove (4.26). Applying Lemma 4.5 to (4.27), we have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^3} \frac{b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1-\rho_2+\rho_3)}{(1+R|\rho_1-\rho_2||\rho_3-\rho_2|)^N} d\rho_1 d\rho_2 d\rho_3 \\
&\lesssim \Bigg(\sum_{\{(i,j)\in\mathbb{N}^2; i+j\geq k_0\}} 2^{-(i+j)} + R^{-N} \sum_{\{(i,j)\in\mathbb{N}^2; i+j\lesssim k_0\}} 2^{(N-1)(i+j)} \Bigg) \|b\|_{L^4}^4 \\
&\lesssim R^{-1+\epsilon} \|b\|_{L^4}^4. \qquad (4.30)
\end{aligned}
$$

Hence we prove (4.26), and so, we finish the proof of (4.7). □

We next prove (4.8) in Proposition 4.1. We need to prove the following lemma.

Lemma 4.6 *Let $R \gg 1$ and $f \in \mathbb{L}_1$, we have the following estimate for every $0 < \epsilon \ll 1$*

$$\|(f\, d\sigma)^{\vee}\|_{L^6_{t,x}(\mathbb{R}\times A_R)} \lesssim R^{-\frac{n-1}{3}+\epsilon}\|g\|_{L^2_\rho H_\omega^{\frac{n-1}{3},2}(\mathbb{S}^{n-1})}, \tag{4.31}$$

where $g(\xi) = f(|\xi|^2, \xi)$.

Proof It suffices to estimate, by a scaling argument, the following quantity

$$\left\| r^{-\frac{n-2}{2}} \sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\, d\rho \right\|_{L^6_{t,x}(\mathbb{R}\times A_R)}. \tag{4.32}$$

We divide the above integral into three cases.

- Case 1: $k \in \Omega_1 := \{k : R \ll \nu(k)\}$. Using (4.11) with $q = 6$, we prove

$$\begin{aligned}&\left\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_1}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\, d\rho \right\|_{L^6_{t,x}(\mathbb{R}\times A_R)}\\ &\lesssim e^{-cR}\left\| \Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{d(k)} (1+k)^{2(n-1)/3}\left|a_{k,\ell}(\rho)\right|^2\Big)^{\frac12}\varphi(\rho)\right\|_{L^2_\rho} \lesssim e^{-cR}\|g\|_{L^2_\rho H_\omega^{\frac{n-1}{3},2}(\mathbb{S}^{n-1})}.\end{aligned}$$

- Case 2: $k \in \Omega_2 := \{k : \nu(k) \sim R\}$. Applying (4.14) with $q = 6$ and $p = 2$, we show

$$\begin{aligned}&\left\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_2}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\, d\rho \right\|_{L^6_{t,x}(\mathbb{R}\times A_R)}\\ &\lesssim R^{-(n-1)/3}\|g\|_{L^2_\rho H_\omega^{\frac{n-1}{3},2}(\mathbb{S}^{n-1})}.\end{aligned} \tag{4.33}$$

- Case 3: $k \in \Omega_3 := \{k : \nu(k) \ll R\}$. We introduce the operator

$$T_\nu(a)(t,r) = \chi\Big(\frac{r}{R}\Big)\int_0^\infty e^{-it\rho^2} h_\nu(r\rho)a(\rho)\rho^{\frac{n}{2}}\varphi(\rho)d\rho$$

where $|h_\nu(r)| \le C/r$ and the operator

$$H_\nu(a)(t,r) = \chi\Big(\frac{r}{R}\Big)\int_0^\infty e^{-it\rho^2} I_\nu(r\rho)a(\rho)\rho^{\frac{n}{2}}\varphi(\rho)d\rho,$$

where $\nu = \nu(k) = k + (n-2)/2$. Since

$$J_\nu(r) = I_\nu(r) + \bar{I}_\nu(r) + h_\nu(r),$$

our aim here is to estimate

$$\begin{aligned}&\Big\| r^{-\frac{n-2}{2}}\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta)\int_0^\infty e^{-it\rho^2}J_{\nu(k)}(r\rho)a_{k,\ell}(\rho)\rho^{\frac{n}{2}}\varphi(\rho)\,d\rho\Big\|_{L^6_{t,x}(\mathbb{R}\times A_R)}\\ &\lesssim R^{-\frac{n-1}{3}+\frac12}\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{2(n-1)/3}\Big(\big\|T_{\nu(k)}(a_{k,\ell})(t,r)\big\|^2_{L^6_t(\mathbb{R};L^6_r(S_R))}\\ &\qquad+\big\|H_{\nu(k)}(a_{k,\ell})(t,r)\big\|^2_{L^6_t(\mathbb{R};L^6_r(S_R))}\Big)\Big)^{1/2}.\end{aligned}$$

By making use of (4.21) with $q=6$, we have

$$\|T_\nu(a)(t,r)\|_{L^6_{t,r}}\le R^{-5/6}\|a\varphi\|_{L^{6/5}}.$$

This implies that

$$\begin{aligned}&\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{2(n-1)/3}\big\|T_{\nu(k)}(a_{k,\ell})(t,r)\big\|^2_{L^6_t(\mathbb{R};L^6_r(S_R))}\Big)^{1/2}\\ &\lesssim R^{-5/6}\Big\|\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{2(n-1)/3}\big|a_{k,\ell}(\rho)\big|^2\Big)^{1/2}\varphi\Big\|_{L^{6/5}}\\ &\lesssim R^{-5/6}\|g\|_{L^2_\rho H^{\frac{n-1}{3},2}_\omega(\mathbb{S}^{n-1})}.\end{aligned}\tag{4.34}$$

On the other hand, by (2.11), one has $|I_\nu(r)|\lesssim r^{-1/2}$ when $k\in\Omega_3$. Consider the operator

$$H_\nu(a)(t,r)=\chi\Big(\frac{r}{R}\Big)\int_0^\infty e^{-it\rho^2}I_\nu(r\rho)a(\rho)\rho^{\frac{n}{2}}\varphi(\rho)d\rho,$$

where $\nu=\nu(k)=k+(n-2)/2$ with $k\in\Omega_3$.

On the one hand, it is easy to see

$$\|H_\nu(a)(t,r)\|_{L^\infty_{t,r}(\mathbb{R}\times\mathbb{R}^n)}\lesssim R^{-1/2}\|a\varphi\|_{L^1}.$$

On the other hand, we have the claim that for any $\epsilon>0$

$$\|H_\nu(a)(t,r)\|_{L^4_{t,r}(\mathbb{R}\times\mathbb{R})}\lesssim R^{-1/2+\epsilon}\|a\varphi\|_{L^4_\rho}.\tag{4.35}$$

We postpone the proof of this claim to the end of this section. Hence, by the interpolation of the above two estimates, for any $\epsilon>0$, we obtain that

$$\|H_\nu(a)(t,r)\|_{L^6_{t,r}(\mathbb{R}\times\mathbb{R}^n)}\lesssim R^{-1/2+\epsilon}\|a\varphi\|_{L^2}.$$

This shows

$$\begin{aligned}&\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{2(n-1)/3}\big\|H_{\nu(k)}(a_{k,\ell})(t,r)\big\|^2_{L^6_t(\mathbb{R};L^6_r(S_R))}\Big)^{1/2}\\ &\lesssim R^{-1/2+\epsilon}\Big(\sum_{k\in\Omega_3}\sum_{\ell=1}^{d(k)}(1+k)^{2(n-1)/3}\big\|a_{k,\ell}(\rho)\varphi(\rho)\big\|^2_{L^2}\Big)^{1/2}\\ &\lesssim R^{-1/2+\epsilon}\|g\|_{L^2_\rho H^{\frac{n-1}{3},2}_\omega(\mathbb{S}^{n-1})}.\end{aligned}\tag{4.36}$$

Collecting (4.34) and (4.36) yields

$$\Big\| r^{-\frac{n-2}{2}} \sum_{k\in\Omega_3} \sum_{\ell=1}^{d(k)} i^k Y_{k,\ell}(\theta) \int_0^\infty e^{-it\rho^2} J_{\nu(k)}(r\rho) a_{k,\ell}(\rho) \rho^{\frac{n}{2}} \varphi(\rho)\, d\rho \Big\|_{L^6_{t,x}(\mathbb{R}\times A_R)} \lesssim R^{-\frac{n-1}{3}+\epsilon} \|g\|_{L^2_\rho H_\omega^{\frac{n-1}{3},2}(\mathbb{S}^{n-1})}.$$

This implies (4.31), which completes the proof of Lemma 4.6. □

The proof of claim (4.35) The same argument in the proof the (4.20) shows the claim (4.35). Recall the kernel (4.23), it is enough to estimate the integral

$$\begin{aligned} &\|H_\nu(a)(t,r)\|^4_{L^4_{t,r}(\mathbb{R}\times\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^4}\int_{\mathbb{R}} e^{-it(\rho_1^2-\rho_2^2+\rho_3^2-\rho_4^2)} K(R,\nu;\rho_1,\rho_2,\rho_3,\rho_4) a(\rho_1)a(\rho_2)a(\rho_3)a(\rho_4) \\ &\qquad \beta(\rho_1)\beta(\rho_2)\beta(\rho_3)\beta(\rho_4) dt d\rho_1 d\rho_2 d\rho_3 d\rho_4, \end{aligned}$$

where $\beta(\rho) = \rho^{\frac{n}{2}}\varphi(\rho)$. For $b(\rho) = 2a(\sqrt{\rho})\beta(\sqrt{\rho})/\sqrt{\rho}$, therefore we obtain

$$\begin{aligned} &\|H_\nu(a)(t,r)\|^4_{L^4_{t,r}(\mathbb{R}\times\mathbb{R}^n)} \\ &= \int_{\mathbb{R}^4} \delta(\rho_1-\rho_2+\rho_3-\rho_4) K(R,\nu;\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_4}) b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_4) d\rho_1 d\rho_2 d\rho_3 d\rho_4 \\ &= \int_{\mathbb{R}^3} K(R,\nu;\sqrt{\rho_1},\sqrt{\rho_2},\sqrt{\rho_3},\sqrt{\rho_1-\rho_2+\rho_3}) b(\rho_1)b(\rho_2)b(\rho_3)b(\rho_1-\rho_2+\rho_3) d\rho_1 d\rho_2 d\rho_3 \\ &\lesssim R^{-2+\epsilon}\|b\|^4_{L^4} \lesssim R^{-2+\epsilon}\|a\varphi\|^4_{L^4}. \end{aligned}$$

where we use the kernel estimate (4.24) and (4.26) in the first inequality. □

5 Local smoothing estimate

K. M. Rogers [20] developed an argument showing that a restriction estimate implies a local smoothing estimate under some suitable conditions. For the sake of convenience, we closely follow this argument to prove Corollary 1.1. In fact, by making use of the standard Littlewood-Paley argument, it can be reduced to prove the claim

$$\|e^{it\Delta}(1-\Delta_\theta)^{-s/2}u_0\|_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} \lesssim N^{(2n(1/2-1/q)-2/q)_+}\|u_0\|_{L^q_x}, \quad \forall\, N \gg 1 \tag{5.1}$$

where

$$\operatorname{supp} \mathcal{F}((1-\Delta_\theta)^{-s/2}u_0) \subset \{\xi : |\xi| \le N\}.$$

Here we denote by $\mathcal{F}$ the Fourier transform. We also use the notation $\hat{h}$ to express the Fourier transform of h. Let $h = (1-\Delta_\theta)^{-s/2}u_0$. Denote by P_N the Littlewood-Paley projector, i.e.

$$P_N h = \mathcal{F}^{-1}\Big(\chi\Big(\frac{|\xi|}{N}\Big)\hat{h}\Big), \quad \chi \in \mathbb{C}_c^\infty([1/2,1]).$$

By the Littlewood-Paley theory and the claim (5.1), one has for $\alpha > 2n(1/2 - 1/q) - 2/q$

$$\begin{aligned}
\|e^{it\Delta}h\|^2_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} &\lesssim \|e^{it\Delta}P_{\lesssim 1}h\|^2_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} + \sum_{N\gg 1}\left\|e^{it\Delta}P_N h\right\|^2_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} \\
&\lesssim \|u_0\|^2_{L^q_x(\mathbb{R}^n)} + \sum_{N\gg 1} N^{2[2n(1/2-1/q)-2/q]+}\left\|P_N u_0\right\|^2_{L^q_x} \\
&\lesssim \|u_0\|^2_{L^q_x(\mathbb{R}^n)} + \left\|\Big(\sum_{N\gg 1} N^{q\alpha}\,|P_N u_0|^q\Big)^{1/q}\right\|^2_{L^q_x} \\
&\lesssim \|u_0\|^2_{L^q_x(\mathbb{R}^n)} + \left\|\Big(\sum_{N\gg 1} N^{2\alpha}\,|P_N u_0|^2\Big)^{1/2}\right\|^2_{L^q_x} \\
&\simeq \|u_0\|^2_{W^{\alpha,q}_x(\mathbb{R}^n)}.
\end{aligned}$$

Here we use Hölder's inequality for the third inequality, Sobolev imbedding for the fourth one. Hence we have

$$\|e^{it\Delta}u_0\|_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} \lesssim \|(1-\Delta_\theta)^{s/2}u_0\|_{W^{\alpha,q}_x(\mathbb{R}^n)}.$$

Now we are left to prove claim (5.1). Assume supp $\hat{f} \subset [0,1]$. Note that

$$e^{it\Delta}f = \frac{1}{(it)^{n/2}}\int_{\mathbb{R}^n} e^{i|x-y|^2/t} f(y)dy, \quad \forall\, t \in \mathbb{R}\backslash\{0\}.$$

On the other hand, we have for $t \neq 0$

$$\begin{aligned}
e^{it\Delta}f &= \int_{\mathbb{R}^n} e^{i(t|\xi|^2+x\cdot\xi)}\hat{f}(\xi)d\xi = e^{-\frac{i|x|^2}{4t}}\int_{\mathbb{R}^n} e^{it|\xi+\frac{x}{2t}|^2}\hat{f}(\xi)d\xi \\
&= \frac{1}{(it)^{n/2}}e^{-\frac{i|x|^2}{4t}}\left(e^{i\frac{\Delta}{t}}\hat{f}\right)\left(-\frac{x}{2t}\right).
\end{aligned}$$

So we have for every dyadic number N

$$\begin{aligned}
\|e^{it\Delta}f\|_{L^q_{t,x}(|t|\sim N^2;|x|\lesssim N^2)} &\lesssim N^{-n}\left\|\left(e^{i\frac{\Delta}{t}}\hat{f}\right)\left(-\frac{\bullet}{2t}\right)\right\|_{L^q_{t,x}(|t|\sim N^2;|x|\lesssim N^2)} \\
&\lesssim N^{-n+\frac{2n+4}{q}}\left\|e^{it\Delta}\hat{f}\right\|_{L^q_{t,x}(|t|\sim N^{-2};|x|\lesssim 1)}.
\end{aligned}$$

By making use of Theorem 1.1, we obtain for $q > 2(n+1)/n$ and $\frac{n+2}{q} = \frac{n}{p'}$

$$\left\|e^{it\Delta}\hat{f}\right\|_{L^q_{t,x}(|t|\sim N^{-2};|x|\lesssim 1)} \lesssim \|f\|_{L^p_{\mu(r)}(\mathbb{R}^+;H^{s,p}_\theta(\mathbb{S}^{n-1}))}. \tag{5.2}$$

This yields

$$\|e^{it\Delta}f\|_{L^q_{t,x}(|t|\sim N^2;|x|\lesssim N^2)} \lesssim N^{-n+\frac{2n+4}{q}}\|f\|_{L^p_{\mu(r)}(\mathbb{R}^+;H^{s,p}_\theta(\mathbb{S}^{n-1}))}.$$

This implies that

$$\|e^{it\Delta}(1-\Delta_\theta)^{-s/2}f\|_{L^q_{t,x}(|t|\sim N^2;|x|\lesssim N^2)} \lesssim N^{-n+\frac{2n+4}{q}}\|f\|_{L^p_x}. \tag{5.3}$$

For the sake of convenience, we recall [20, Lemma 8]

Lemma 5.1 *Let $q \geq p_1 \geq p_0$, $r \geq 1$ and $I \subset [0, R^2]$. If one has*

$$\|e^{it\Delta} f\|_{L^q_x(B_{R^2}; L^r_t(I))} \leq C R^s \|f\|_{L^{p_0}(\mathbb{R}^n)}$$

where $R \gg 1$, and f is frequency supported in unite ball $\mathbb{B}^n$. Then for all $\epsilon > 0$

$$\|e^{it\Delta} f\|_{L^q_x(\mathbb{R}^n; L^r_t(I))} \leq C_\epsilon R^{s+2n(\frac{1}{p_0}-\frac{1}{p_1})+\epsilon} \|f\|_{L^{p_1}(\mathbb{R}^n)}.$$

Since $q > p$ when $q > 2(n+1)/n$, for any $0 < \epsilon \ll 1$, we have by this lemma

$$\begin{aligned}
&\|e^{it\Delta}(1-\Delta_\theta)^{-s/2} f\|_{L^q_{t,x}(|t|\sim N^2; x\in\mathbb{R}^n)} \\
&\lesssim N^{-n+\frac{2n+4}{q}+2n(\frac{1}{p}-\frac{1}{q})+\epsilon} \|f\|_{L^q_x} \\
&\lesssim N^{n(1-\frac{2}{q})+\epsilon} \|f\|_{L^q_x}.
\end{aligned}$$

Using the scaling argument, if

$$\operatorname{supp} \widehat{f_{k,N}} \subset B_{2^{k/2}N} := \left\{\xi : |\xi| \in [0, 2^{k/2}N]\right\}, \quad \forall\, k \geq 0,$$

then

$$\|e^{it\Delta}(1-\Delta_\theta)^{-\frac{s}{2}} f_{k,N}\|_{L^q_{t,x}([2^{-k},2^{-k+1}]\times\mathbb{R}^n)} \lesssim N^{n(1-\frac{2}{q})+\epsilon} (2^{\frac{k}{2}} N)^{-\frac{2}{q}} \left\|f_{k,N}\right\|_{L^q_x}. \tag{5.4}$$

Since

$$\operatorname{supp}\hat{h} \subset \{\xi : |\xi| \in [N/2, N]\} \subset B_{2^{k/2}N}, \quad \forall k \geq 2,$$

we replace $(1-\Delta_\theta)^{-s/2} f_{k,N}$ by h to obtain

$$\begin{aligned}
\|e^{it\Delta} h\|_{L^q_{t,x}([0,1]\times\mathbb{R}^n)} &= \left(\sum_{k\geq 0} \|e^{it\Delta}(1-\Delta_\theta)^{-s/2} u_0\|^q_{L^q_{t,x}([2^{-k},2^{-k+1}]\times\mathbb{R}^n)}\right)^{1/q} \\
&\lesssim \left(\sum_{k\geq 0} 2^{-k}\right)^{1/q} N^{(2n(1/2-1/q)-2/q)+} \|u_0\|_{L^q_x}.
\end{aligned} \tag{5.5}$$

This proves inequality (5.1).

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