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A 2-adic automorphy lifting theorem for unitary groups over CM fields

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Abstract We prove a 'minimal' type automorphy lifting theorem for 2-adic Galois representations of unitary type, over imaginary CM fields. We use this to improve an automorphy lifting theorem of Kisin for GL_2.

Contents

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1 Introduction

In this paper, we study the deformation theory of 2-adic Galois representations of unitary type, over CM number fields. More precisely, let F be an imaginary CM number field with maximal totally real subfield F^+, and let $c \in \mathrm{Gal}(F/F^+)$ denote the non-trivial element. In this introduction, we say that a continuous representation

$$\rho\colon \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$$

is of unitary type if there is an isomorphism $\rho^c \cong \rho^\vee \otimes \epsilon^{1-n}$, where ϵ denotes the p-adic cyclotomic character. The deformation theory of such representations in the case that p is odd has been studied in a series of papers beginning with [8], and culminating in the paper [5], where the authors prove very general automorphy and potential automorphy theorems for such Galois representations.

In contrast, there has been no study to date of the deformation theory of representations with 2-adic coefficients, except when $n = 2$ (in which case, it is essentially equivalent to consider representations $\rho\colon G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ with no additional self-duality condition). In this connection, we mention the papers of Dickinson [10] (who proves a modularity lifting theorem for 2-adic representations under a supplementary local hypothesis at infinity) and Khare and Wintenberger [21] and Kisin [18] (who prove modularity lifting theorems for 2-adic representations without such a hypothesis). The additional difficulties that arise in the 2-adic case are of two main types: first, to get a good handle on the local lifting rings, especially at places dividing 2, and second, to get a good control of the relevant Galois cohomology groups.

In this paper, we prove an automorphy lifting theorem for 2-adic Galois representations of unitary type in arbitrary dimension:

Theorem 1.1 (Theorem 5.1) *Let $n \geq 2$. Let F be an imaginary CM number field with totally real subfield F^+. Fix a prime p and an isomorphism $\iota\colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and consider a continuous representation*

$$\rho\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p).$$

Suppose that ρ satisfies the following conditions:

(i) *There is an isomorphism $\rho^c \cong \rho^\vee \epsilon^{1-n}$.*
(ii) *The group $\overline{\rho}(G_{F(\zeta_p)}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is adequate, in the sense of Definition 2.20.*
(iii) *The representation ρ is almost everywhere unramified.*
(iv) *There exists a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that:*
    (a) *There is an isomorphism $\overline{r_\iota(\pi)} \cong \overline{\rho}$.*
    (b) *For each finite place v of F, we have $r_\iota(\pi)|_{G_{F_v}} \sim \rho|_{G_{F_v}}$ (this condition is automatic if π_v and $\rho|_{G_{F_v}}$ are both unramified). In particular, if $v|p$, then $\rho|_{G_{F_v}}$ and $r_\iota(\pi)|_{G_{F_v}}$ are potentially crystalline.*
(v) *If $p = 2$ and n is even, then there exists a place $v|\infty$ of F^+ at which the pair $(\overline{\rho}, \epsilon^{1-n})$ is strongly residually odd, in the sense of Definition 3.3.*

Then ρ is automorphic: there exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\Pi)$.

This is a theorem of 'minimal type'. We can phrase its conclusion more colloquially as follows: given two n-dimensional Galois representations ρ_1, ρ_2 of unitary type, which have 'the same' local behaviour at each place v of F and the same residual representation, the automorphy of one implies the automorphy of the other. We state the theorem with no restriction on p since even in the case where p is odd, we are able to make a slight improvement on existing results (cf. [32, Theorem 7.1]).

The main ingredients in the proof of Theorem 1.1 which are new to this paper are the definition of a representable deformation functor in the case $p = 2$, and the observation that one can carry out the procedure of killing the dual Selmer group, even in the case where the base field F contains pth roots of unity (see Proposition 2.21). It is here that we require the local hypothesis (v) at infinity. In the case $n = 2$, this condition is more-or-less equivalent to asking that for a given residual representation $\overline{\rho}\colon G_{F^+} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, there exist a place $v|\infty$ such that the image $\overline{\rho}(c_v)$ of complex conjugation at v is non-trivial. This is exactly the condition imposed by Dickinson [10].

The definition of 'adequate subgroup' that we use was first written down in the case $p = 2$ by Guralnick et al. [13]; it is pleasant to see that this turns out to be the right definition to prove automorphy lifting theorems in this case.

Having proved Theorem 1.1, we apply it to improve the aforementioned modularity lifting theorem of Kisin for GL_2. We are able to prove the following result.

Theorem 1.2 (Theorem 6.1) *Let F be a totally real number field, and let $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be a continuous representation satisfying the following conditions:*

- (i) *ρ is almost everywhere unramified.*
- (ii) *For each place $v|2$ of F, $\rho|_{G_{F_v}}$ is potentially crystalline. For each embedding $\tau\colon F_v \hookrightarrow \overline{\mathbb{Q}}_2$, $\mathrm{HT}_\tau(\rho) = \{0, 1\}$.*
- (iii) *$\overline{\rho}$ is absolutely irreducible, and has non-soluble image. There exists a place $v|\infty$ of F such that $\overline{\rho}(c_v)$ is non-trivial.*
- (iv) *There exists a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_F)$ and an isomorphism $\iota\colon \overline{\mathbb{Q}}_2 \cong \mathbb{C}$ such that $\overline{r_\iota(\pi)} \cong \overline{\rho}$.*

Then ρ is automorphic: there exists a RAESDC automorphic representation (σ, ψ) of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\sigma)$.

An analogous improvement in the case $p > 2$ has been made by Barnet-Lamb et al. [3]. As in that work, the main problem is to construct ordinary automorphic lifts of a given (automorphic) residual representation, and we accomplish this by using Theorem 1.1 in the case of $n = 4$, by tensoring together various 2-dimensional representations with given local properties. Our techniques are broadly similar to those of [3], the main wrinkle being that induction from index 2 subgroups does not preserve adequacy. Where the authors of [3] use automorphic induction from quadratic extensions, we must therefore use tensor product functoriality for $\mathrm{GL}_2 \times \mathrm{GL}_2$, and tensor together residual representations which have insoluble image.

We observe that Theorem 1.2 is not strictly stronger than the theorems of [18]; rather, we have exchanged a condition at p for an apparently milder condition at infinity. This condition at infinity could be removed if the analogous hypothesis (v) of Theorem 1.1 could be removed. It is possible that this could be done using the techniques of [18,21], although we have not tried to do this here. Given the important role played by 2-adic automorphy lifting theorems in the proof of Serre's conjecture [20,21], this seems like an interesting problem.

We now describe the organization of this paper. In Sect. 2, we define our deformation problem for Galois representations of unitary type, and study it using Galois cohomology. Where possible, we have given arguments that are independent of the residual characteristic p, although at some points it is impossible to avoid splitting up into cases (according as to whether p is even or odd). In Sect. 4, we define spaces of algebraic modular forms on definite unitary groups, and prove an automorphy lifting result under some additional local hypotheses. Given the foundations built up in Sect. 2, this section contains little that is new compared to [32, § 6], and we only sketch parts of the argument that remain unchanged. In Sect. 5, we prove Theorem 1.1, by reduction to the main result of Sect. 2. In Sect. 6, we apply this result to the proof of Theorem 1.2.

1.1 Notation

If E/F is a quadratic field extension, then we write $\delta_{E/F}\colon \mathrm{Gal}(E/F) \to \{\pm 1\}$ for the unique non-trivial character. A base number field F having been fixed, we will also choose algebraic closures $\overline{F}$ of F and $\overline{F}_v$ of F_v for every finite place v of F. If p is a prime, then we will write $\overline{\mathbb{Q}}_p$ for a fixed choice of algebraic closure of $\mathbb{Q}_p$, and val_p for the p-adic valuation on $\overline{\mathbb{Q}}_p$ normalized so that $\mathrm{val}_p(p) = 1$. These choices define the absolute Galois groups $G_F = \mathrm{Gal}(\overline{F}/F)$ and $G_{F_v} = \mathrm{Gal}(\overline{F}_v/F_v)$. We write $I_{F_v} \subset G_{F_v}$ for the inertia subgroup. We also fix embeddings $\overline{F} \hookrightarrow \overline{F}_v$, extending the canonical embeddings $F \hookrightarrow F_v$. This determines for each place v of F an embedding $G_{F_v} \to G_F$. We write $\mathbb{A}_F$ for the adele ring of F, and $\mathbb{A}_F^\infty = \prod'_{v \nmid \infty} F_v$ for its finite part. If v is a finite place of F, then we write $k(v)$ for the residue field at v and $q_v = \# k(v)$.

We write $\mathrm{Frob}_v \in G_{F_v}/I_{F_v}$ for the *geometric* Frobenius element. We write $\epsilon\colon G_F \to \mathbb{Z}_p^\times$ for the p-adic cyclotomic character; if v is a finite place of F, not dividing p, then $\epsilon(\mathrm{Frob}_v) = q_v^{-1}$. If $\rho\colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation, we say that ρ is de Rham if for each place $v|p$ of F, $\rho|_{G_{F_v}}$ is de Rham. In this case, we can associate to each embedding $\tau\colon F \hookrightarrow \overline{\mathbb{Q}}_p$ a multiset $\mathrm{HT}_\tau(\rho)$ of Hodge–Tate weights, which depends only on $\rho|_{G_{F_v}}$, where v is the place of F induced by τ. This multiset has n elements, counted with multiplicity. There are two natural normalizations for $\mathrm{HT}_\tau(\rho)$ which differ by a sign, and we choose the one with $\mathrm{HT}_\tau(\epsilon) = \{-1\}$ for every choice of τ.

We use geometric conventions for the Galois representations associated to automorphic forms. First, we use the normalizations of the local and global Artin maps $\mathrm{Art}_{F_v} : F_v^\times \to W_{F_v}^{\mathrm{ab}}$ and $\mathrm{Art}_F : \mathbb{A}_F^\times \to G_F^{\mathrm{ab}}$ which send uniformizers to geometric Frobenius elements. If $n \geq 1$ and v is a place of F, then we write rec_{F_v} for the local Langlands correspondence for $\mathrm{GL}_n(F_v)$, normalized as in [9, § 2.1]. If v is a finite place of F, then we define $\mathrm{rec}^T_{F_v}(\pi) = \mathrm{rec}_{F_v}(\pi \otimes |\cdot|^{(1-n)/2})$. Then $\mathrm{rec}^T_{F_v}$ commutes with automorphisms of $\mathbb{C}$, and so makes sense over any field Ω which is abstractly isomorphic to $\mathbb{C}$ (e.g. $\overline{\mathbb{Q}}_p$). We write $\|\cdot\|\colon \mathbb{A}_F^\times \to \mathbb{R}_{>0}$ for the standard norm character, which corresponds under global class field theory to the cyclotomic character ϵ.

If (r, N) is any Weil–Deligne representation, we write $(r, N)^{\text{F-ss}}$ for its Frobenius–semi-simplification. If v is a finite place of F and $\rho\colon G_{F_v} \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation, which is de Rham if $v|p$, then we write $\mathrm{WD}(\rho)$ for the associated Weil–Deligne representation, which is uniquely determined by ρ, up to isomorphism.

We will call a finite extension $E/\mathbb{Q}_p$ inside $\overline{\mathbb{Q}}_p$ a coefficient field. A coefficient field E having been fixed, we will write $\mathcal{O}$ or $\mathcal{O}_E$ for its ring of integers, k or k_E for its residue field, and λ or λ_E for its maximal ideal. If A is a complete Noetherian local $\mathcal{O}$-algebra with residue field k, then we write $\mathfrak{m}_A \subset A$ for its maximal ideal, and CNL_A for the category of complete Noetherian local A-algebras with residue field k. We endow each object $R \in \mathrm{CNL}_A$ with its profinite ($\mathfrak{m}_R$-adic) topology.

If Γ is a profinite group and $\rho\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation, then we can assume (after a change of basis) that ρ takes values in $\mathrm{GL}_n(\mathcal{O})$, for some choice of coefficient field E. The semi-simplification of the composite representation $\Gamma \to \mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(k)$

is independent of choices, up to isomorphism, and we will write $\overline{\rho}\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ for this semi-simplification.

If E is a coefficient field and $\overline{\rho}\colon \Gamma \to \mathrm{GL}_n(k)$ is a continuous representation, then we write $\operatorname{ad}\overline{\rho}$ for $\operatorname{End}_k(\overline{\rho})$, endowed with its structure of $k[\Gamma]$-module. We write $\operatorname{ad}^0\overline{\rho} \subset \operatorname{ad}\overline{\rho}$ for the submodule of trace 0 endomorphisms, and (if $\Gamma = G_F$ for a number field F) $\operatorname{ad}^0\overline{\rho}(1)$ for its twist by the cyclotomic character. We write $\operatorname{ad}_0\overline{\rho}$ for the quotient of $\operatorname{ad}\overline{\rho}$ by the Γ-invariant subspace of scalar endomorphisms. If M is a discrete $\mathbb{Z}[G_F]$-module, then we write $H^1(F, M)$ for the continuous Galois cohomology group with coefficients in M. Similarly, if M is a discrete $\mathbb{Z}[G_{F_v}]$-module, then we write $H^1(F_v, M)$ for the continuous Galois cohomology group with coefficients in M. If M is a discrete $k[G_F]$-module (resp. $k[G_{F_v}]$-module), then $H^1(F, M)$ (resp. $H^1(F_v, M)$) is a k-vector space, and we write $h^1(F, M)$ (resp. $h^1(F_v, M)$) for the dimension of this k-vector space, provided that it is finite.

2 Deformation theory

Let $n \geq 1$ be an integer, and define $\mathcal{G}_n = (\mathrm{GL}_n \times \mathrm{GL}_1) \rtimes \{1, \jmath\}$, an algebraic group over $\mathbb{Z}$. The semi-direct product is defined by the relation

$$\jmath(g, \mu) = (\mu\, {}^t g^{-1}, \mu)\jmath.$$

We let $\mathfrak{g}_n = \operatorname{Lie}\mathcal{G}_n$. There is a character $\nu\colon \mathcal{G}_n \to \mathbb{G}_m$ given by the formula $(g, \mu) \mapsto \mu,\ \jmath \mapsto -1$. Let Γ be a group, and let Δ be an index 2 subgroup. The reason for introducing the group $\mathcal{G}_n$ is the following lemma ([8, Lemma 2.1.1]):

Lemma 2.1 *Suppose that R is a ring, and let $c_0 \in \Gamma - \Delta$. Then the following two sets are in natural bijection:*

(i) *The set of homomorphisms $r\colon \Gamma \to \mathcal{G}_n(R)$ such that $r^{-1}(\mathcal{G}_n^0(R)) = \Delta$.*
(ii) *The set of triples $(\rho, \mu, \langle\cdot,\cdot\rangle)$, where $\rho\colon \Delta \to \mathrm{GL}_n(R)$ and $\mu\colon \Gamma \to R^\times$ are homomorphisms and $\langle\cdot,\cdot\rangle$ is a perfect R-linear pairing on R^n such that for all $x, y \in R^n$, $\delta \in \Delta$, we have*

$$\langle x, \rho(c_0^2)y\rangle = -\mu(c_0)\langle y, x\rangle \text{ and } \big\langle \rho(\delta)x, \rho(\delta^{c_0})y\big\rangle = \mu(\delta)\langle x, y\rangle.$$

Under this correspondence we have $\mu(\gamma) = (\nu \circ r)(\gamma)$ for all $\gamma \in \Gamma$, and $\langle x, y\rangle = {}^t x A^{-1} y$, where $r(c_0) = (A, -\mu(c_0))\jmath$. If Γ and R are topological groups and $\Delta \subset \Gamma$ is a closed subgroup, then continuous homomorphisms r correspond to pairs of continuous homomorphisms (ρ, μ).

The group $\mathcal{G}_n^0(R) = \mathrm{GL}_n(R) \times \mathrm{GL}_1(R)$ acts by conjugation on the set of homomorphisms $r\colon \Gamma \to \mathcal{G}_n(R)$ such that $r^{-1}(\mathcal{G}_n^0(R)) = \Delta$. We observe that, in general, the $\mathrm{GL}_1(R)$ factor does not act trivially; rather, under the dictionary of Lemma 2.1, it acts by rescaling the perfect pairing $\langle\cdot,\cdot\rangle$.

If $r\colon \Gamma \to \mathcal{G}_n(R)$ is a homomorphism such that $\Delta = r^{-1}(\mathcal{G}_n^0(R))$, then we will write $r|_\Delta\colon \Delta \to \mathrm{GL}_n(R)$ for the representation that arises by restricting r to Δ and then projecting to the GL_n factor of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$. Thus it makes sense to speak, for example, of the characteristic polynomial of an element $r|_\Delta(\delta)$, $\delta \in \Delta$. We write $\mathfrak{g}_n r$ for the $R[\Gamma]$-module, free as R-module, induced by the adjoint representation of $\mathcal{G}_n$.

Lemma 2.2 *Let k be a field, let $\rho\colon \Delta \to \mathrm{GL}_n(k)$ be an absolutely irreducible homomorphism, and let $\mu\colon \Gamma \to k^\times$ be a character such that $\rho^{c_0} \cong \rho^\vee \otimes \mu$. Then there exists a* *homomorphism* $r\colon \Gamma \to \mathcal{G}_n(k)$ *such that* $r|_\Delta = \rho$, $\nu \circ r|_\Delta = \mu$, *and* $r(c_0) \in \mathcal{G}_n(k) - \mathcal{G}_n^0(k)$. *This extension is unique up to* $\mathcal{G}_n^0(k)$*-conjugacy.*

Proof By hypothesis, there exists a perfect pairing $\langle \cdot, \cdot \rangle\colon k^n \times k^n \to k$ such that for all $\delta \in \Delta$, $x, y \in k^n$, $\langle \rho(\delta)x, \rho(\delta^{c_0})y \rangle = \mu(\delta)\langle x, y \rangle$. This pairing is unique up to $k^\times$-multiple (because ρ is absolutely irreducible). Define a new pairing by the formula $\langle x, y \rangle' = \langle x, \rho(c_0^2)y \rangle$. Then a calculation shows $\langle \rho(\delta)x, \rho(\delta^{c_0})y \rangle' = \mu(\delta^{c_0})\langle x, y \rangle' = \mu(\delta)\langle x, y \rangle'$, hence $\langle x, y \rangle' = \alpha \langle x, y \rangle$ for some $\alpha \in k^\times$.

Define another new pairing by $\langle x, y \rangle'' = \langle y, \rho(c_0^2)x \rangle'$. Then another calculation shows that

$$\langle x, y \rangle'' = \mu(c_0)^2 \langle x, y \rangle = \alpha^2 \langle x, y \rangle,$$

hence $\alpha = \pm\mu(c_0)$. After possibly replacing μ by its multiple by the non-trivial character of Γ/Δ, we see that the triple $(\rho, \mu, \langle \cdot, \cdot \rangle)$ is of the type appearing in the statement of Lemma 2.1, and the existence of r follows from this.

If r' is another such extension, then it corresponds under the dictionary of Lemma 2.1 to a triple $(\rho, \mu, \lambda\langle \cdot, \cdot \rangle)$ for some $\lambda \in k^\times$. We thus have $r' = (1, \lambda)r(1, \lambda)^{-1}$, showing that r and r' are indeed $\mathcal{G}_n^0(k)$-conjugate. $\square$

The following lemma is [8, Lemma 2.1.5].

Lemma 2.3 *Suppose that* Γ *is a profinite group and* $r\colon \Gamma \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ *is a continuous representation with* $\Delta = r^{-1}(\mathcal{G}_n^0(\overline{\mathbb{Q}}_p))$. *Then there exists a finite extension* $E/\mathbb{Q}_p$ *inside* $\overline{\mathbb{Q}}_p$ *and another continuous representation* $r'\colon \Gamma \to \mathcal{G}_n(\mathcal{O}_E)$ *which is* $\mathcal{G}_n^0(\overline{\mathbb{Q}}_p)$*-conjugate to* r.

Lemma 2.4 *Suppose that* $R \supset S$ *are complete Noetherian local rings with common residue field* k *and* $\mathfrak{m}_R \cap S = \mathfrak{m}_S$. *Suppose that* Γ *is a profinite group and* $r\colon \Gamma \to \mathcal{G}_n(R)$ *is a continuous representation with* $\Delta = r^{-1}(\mathcal{G}_n^0(R))$. *Suppose finally that* $r|_\Delta$ *mod* $\mathfrak{m}_R$ *is absolutely irreducible and that* $\mathrm{tr}\, r|_\Delta(\Delta) \subset S$ *and* $\nu \circ r$ *is valued in* $S^\times \subset R^\times$. *Then there exists a continuous representation* $r'\colon \Gamma \to \mathcal{G}_n(S)$ *and* $g \in \ker(\mathcal{G}_n(R) \to \mathcal{G}_n(k))$ *such that* $r' = grg^{-1}$.

Proof By the analogous result for GL_n (i.e. [8, Lemma 2.1.10]) and our assumption that $\nu \circ r$ is valued in $S^\times$, we can assume that $r(\Delta) \subset \mathcal{G}_n^0(S) = \mathrm{GL}_n(S) \times \mathrm{GL}_1(S)$. Choose $c_0 \in \Gamma - \Delta$, and write $r(c_0) = (A, -\mu)\jmath$. Let $\rho = r|_\Delta$. Then we have $\rho^{c_0} = A\rho^\vee A^{-1} \otimes (\nu \circ r)|_\Delta$, and hence (by [8, Lemma 2.1.9]) we can find $B \in \mathrm{GL}_n(S)$ such that $\rho^{c_0} = B\rho^\vee B^{-1} \otimes (\nu \circ r)|_\Delta$.

By Schur's lemma ([8, Lemma 2.1.8]), we have $A = \alpha B$ for some $\alpha \in R^\times$. Since R and S have the same residue field, we can assume (after multiplying B by an element of $S^\times$) that $\alpha \in 1 + \mathfrak{m}_R$. We then have

$$(1, \alpha)r(c_0)(1, \alpha)^{-1} = (1, \alpha)(A, -\mu)\jmath(1, \alpha)^{-1} = (\alpha^{-1}A, -\mu)\jmath \in \mathcal{G}_n(S).$$

It follows that $r' = (1, \alpha)r(1, \alpha^{-1})$ is valued in $\mathcal{G}_n(S)$, as desired. $\square$

A similar argument proves:

Lemma 2.5 *Suppose that* R *is a complete Noetherian local ring, that* Γ *is a profinite group and that* $r_1, r_2\colon \Gamma \to \mathcal{G}_n(R)$ *are continuous representations with* $\Delta = r_1^{-1}(\mathcal{G}_n^0(R)) = r_2^{-1}(\mathcal{G}_n^0(R))$, r_1 *mod* $\mathfrak{m}_R = r_2$ *mod* $\mathfrak{m}_R$, *and* $\nu \circ r_1 = \nu \circ r_2$. *Suppose moreover that* $r_1|_\Delta$ *mod* $\mathfrak{m}_R$ *is absolutely irreducible, and that for all* $\delta \in \Delta$, $\mathrm{tr}\, r_1|_\Delta(\delta) = \mathrm{tr}\, r_2|_\Delta(\delta)$. *Then there exists* $g \in \ker(\mathcal{G}_n(R) \to \mathcal{G}_n(k))$ *such that* $gr_1g^{-1} = r_2$.

2.1 Galois deformation theory

Let p be a prime, and let $E/\mathbb{Q}_p$ be a coefficient field with ring of integers $\mathcal{O}$, maximal ideal λ, and residue field $k = \mathcal{O}/\lambda$. If A is a group functor on the category of $\mathcal{O}$-algebras, then we define for $R \in \mathrm{CNL}_{\mathcal{O}} \widehat{A}(R) = \ker(A(R) \to A(k))$. Fix an imaginary CM number field F with maximal totally real subfield F^+. We assume that the following conditions are in effect:

- The extension F/F^+ is everywhere unramified.
- Each place of F^+ dividing p is split in F.

Let $c \in G_{F^+}$ be a fixed choice of complex conjugation. Let S_p denote the set of places of F^+ dividing p, S_∞ the set of places dividing ∞, and let S be a finite set of places of F^+ which contains $S_p \cup S_\infty$. We write $F(S)$ for the maximal Galois extension of F unramified outside S; then $F(S)/F^+$ is Galois. We define $G_{F^+,S} = \mathrm{Gal}(F(S)/F^+)$, $G_{F,S} = \mathrm{Gal}(F(S)/F)$. Then $G_{F,S} \subset G_{F^+,S}$ is an index 2 normal subgroup, and the quotient $G_{F^+,S}/G_{F,S}$ is generated by the image of c.

We fix a continuous representation $\overline{r}\colon G_{F^+,S} \to \mathcal{G}_n(k)$ such that $\overline{r}^{-1}(\mathcal{G}_n^0(k)) = G_{F,S}$ and $\overline{r}|_{G_{F,S}}$ is absolutely irreducible. We also fix a continuous character $\chi\colon G_{F^+,S} \to \mathcal{O}^\times$ such that $\overline{\chi} = \nu \circ \overline{r}$.

We can now give the definitions relating to the local and global deformation theory of the representation $\overline{r}$.

Definition 2.6 Let $v \in S$. We define the functor $\mathrm{Lift}_v^\square\colon \mathrm{CNL}_{\mathcal{O}} \to$ Sets of unrestricted liftings of $\overline{r}|_{G_{F_v^+}}$ as follows: if $R \in \mathrm{CNL}_{\mathcal{O}}$, then $\mathrm{Lift}_v^\square(R)$ is the set of homomorphisms $r_v\colon G_{F_v^+} \to \mathcal{G}_n(R)$ such that $\nu \circ r_v = \chi|_{G_{F_v^+}}$ and $r_v \bmod \mathfrak{m}_R = \overline{r}|_{G_{F_v^+}}$.

By definition, a local deformation problem is a representable subfunctor $\mathcal{D}_v \subset \mathrm{Lift}_v^\square$ such that for all $R \in \mathrm{CNL}_{\mathcal{O}}$, the subset $\mathcal{D}_v(R) \subset \mathrm{Lift}_v^\square$ is invariant under the conjugation action of the group $\widehat{\mathcal{G}}_n(R)$.

We remark that the above definition depends on the choice of character χ, although we do not include this in the notation. It is easy to see that for $v \in S$, the functor $\mathrm{Lift}_v^\square$ is a local deformation problem (i.e. that $\mathrm{Lift}_v^\square$ is represented by an object $R_v^\square \in \mathrm{CNL}_{\mathcal{O}}$).

Definition 2.7 A global deformation problem is a tuple

$$\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\mathcal{D}_v\}_{v \in S}),$$

where F, $\overline{r}$, $\mathcal{O}$, χ and S are as above and for each $v \in S$, $\mathcal{D}_v$ is a local deformation problem.

Let $R \in \mathrm{CNL}_{\mathcal{O}}$. A lifting of $\overline{r}$ to R of type $\mathcal{S}$ is, by definition, a homomorphism $r\colon G_{F^+,S} \to \mathcal{G}_n(R)$ satisfying the following conditions:

(i) We have $r \bmod \mathfrak{m}_R = \overline{r}$ and $\nu \circ r = \chi$.
(ii) For each $v \in S$, we have $r|_{G_{F_v^+}} \in \mathcal{D}_v(R)$.

Two liftings r_1, r_2 of type $\mathcal{S}$ are said to be strictly equivalent if there exists $g \in \widehat{\mathcal{G}}_n(R)$ such that $g r_1 g^{-1} = r_2$. A strict equivalence class of liftings of type $\mathcal{S}$ is called a deformation.

Lemma 2.8 *Let $\mathcal{S}$ be a global deformation problem, and let* $\mathrm{Def}_{\mathcal{S}}\colon \mathrm{CNL}_{\mathcal{O}} \to$ Sets *be the functor which assigns to each $R \in \mathrm{CNL}_{\mathcal{O}}$ the set of deformations of $\overline{r}$ to R of type $\mathcal{S}$. Then* $\mathrm{Def}_{\mathcal{S}}$ *is representable.*

Proof Let $\mathrm{Def}_{\mathcal{S}}^\square\colon \mathrm{CNL}_{\mathcal{O}} \to$ Sets be the functor of liftings of $\overline{r}$ of type $\mathcal{S}$. It is easy to see that $\mathrm{Def}_{\mathcal{S}}^\square$ is representable. Let $i\colon \widehat{\mathrm{GL}}_1 \to \widehat{\mathcal{G}}_n = \widehat{\mathrm{GL}}_n \times \widehat{\mathrm{GL}}_1$ be the map $\lambda \mapsto (\lambda \cdot 1_n, \lambda^2)$, and let $\widehat{\mathcal{H}}_n$ be the functor in groups on $\mathrm{CNL}_{\mathcal{O}}$ given by the formula $\widehat{\mathcal{H}}_n(R) = \widehat{\mathcal{G}}_n(R)/i(\widehat{\mathrm{GL}}_1(R))$.

It follows from [21, Proposition 2.5] that the functor $\widehat{\mathcal{H}}_n$ is representable by an object of $\mathrm{CNL}_{\mathcal{O}}$, formally smooth over $\mathcal{O}$. Moreover, $\widehat{\mathcal{H}}_n$ acts freely on $\mathrm{Def}_{\mathcal{S}}^{\square}$. Indeed, let $R \in \mathrm{CNL}_{\mathcal{O}}$, $r \in \mathrm{Def}_{\mathcal{S}}^{\square}(R)$, and suppose that $g \in \mathrm{Stab}_{\widehat{\mathcal{G}}_n(R)}(r)$. It follows from [8, Lemma 2.1.8] and our assumption that $\overline{r}|_{G_{F,S}}$ is absolutely irreducible that we can write $g = (\lambda \cdot 1_n, \mu)$, with $\lambda, \mu \in R^\times$. It follows from the equality $gr(c)g^{-1} = r(c)$ that in fact $\mu = \lambda^2$, hence $g \in i(\widehat{\mathrm{GL}}_1(R))$, hence the image of g in $\widehat{\mathcal{H}}_n(R)$ is trivial. The lemma now follows from another application of [21, Proposition 2.5]. □

We write $R_{\mathcal{S}}$ for the representing object of $\mathrm{Def}_{\mathcal{S}}$.

Definition 2.9 Let $\mathcal{S}$ be a global deformation problem, and let $T \subset S$. We define a T-framed lifting of $\overline{r}$ of type $\mathcal{S}$ to be a tuple $(r, \{\alpha_v\}_{v \in T})$, where r is a lifting of type $\mathcal{S}$ and for each $v \in T$, $\alpha_v \in \widehat{\mathcal{G}}_n(R)$. Two T-framed liftings $(r_1, \{\alpha_v\}_{v\in T})$ and $(r_2, \{\beta_v\}_{v\in T})$ are said to be strictly equivalent if there exists $g \in \widehat{\mathcal{G}}_n(R)$ such that $gr_1g^{-1} = r_2$ and $g\alpha_v = \beta_v$ for each $v \in T$. A strict equivalence class of T-framed liftings is called a T-framed deformation.

Lemma 2.10 *Let $\mathcal{S}$ be a global deformation problem, and let* $\mathrm{Def}_{\mathcal{S}}^T \colon \mathrm{CNL}_{\mathcal{O}} \to \mathrm{Sets}$ *be the functor which associates to each $R \in \mathrm{CNL}_{\mathcal{O}}$ the set of T-framed deformations of $\overline{r}$ to R of type $\mathcal{S}$. Then $\mathrm{Def}_{\mathcal{S}}^T$ is representable.*

Proof If $T = \emptyset$, then this is just Lemma 2.8. The general case follows easily from [21, Proposition 2.5]. □

We write $R_{\mathcal{S}}^T$ for the representing object of $\mathrm{Def}_{\mathcal{S}}^T$.

Lemma 2.11 *Let $\mathcal{S}$ be a global deformation problem and let $T \subset S$ be non-empty. Then $R_{\mathcal{S}}^T$ is a formally smooth $R_{\mathcal{S}}$-algebra of relative dimension $(n^2+1)\#T - 1$.*

Proof The map $R_{\mathcal{S}} \to R_{\mathcal{S}}^T$ arises from universality (forget the framing). The rest of the lemma is easy. □

Let $\mathcal{S}$ be a global deformation problem, and let $T \subset S$. Then there is a natural transformation

$$\mathrm{Def}_{\mathcal{S}}^T \to \prod_{v \in T} \mathcal{D}_v,$$

given by the formula

$$(r_1, \{\alpha_v\}_{v\in T}) \mapsto \left(\alpha_v r_1|_{G_{F_v^+}} \alpha_v^{-1}\right)_{v \in T}.$$

(it is easy to check that this is independent of the chosen representatives). If we write $R_v \in \mathrm{CNL}_{\mathcal{O}}$ for the representing object of $\mathcal{D}_v$ ($v \in T$), then there is a corresponding morphism in $\mathrm{CNL}_{\mathcal{O}}$:

$$R_{\mathcal{S},T}^{\mathrm{loc}} = \widehat{\otimes}_{v \in T} R_v \to R_{\mathcal{S}}^T.$$

2.2 Galois cohomology calculations

We continue with the notation of the previous section, and fix furthermore a choice of global deformation problem

$$\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\mathcal{D}_v\}_{v \in S}).$$

If $v \in S$, then the set $\mathrm{Lift}_v^{\square}(k[\epsilon])$ of liftings of $\overline{r}|_{G_{F_v^+}}$ to $k[\epsilon]$ can be identified with the group of cocycles $Z^1(F_v^+, \mathrm{ad}\,\overline{r})$, via the formula

$$r(\sigma) = (1 + \epsilon\phi(\sigma))\,\overline{r}(\sigma) \quad \left(\phi \in Z^1\left(F_v^+, \mathrm{ad}\,\overline{r}\right),\ \sigma \in G_{F_v^+}\right).$$

Here we identify $\mathrm{ad}\,\overline{r} = \mathrm{End}_k(\overline{r}) = \widehat{\mathrm{GL}}_n(k[\epsilon])$. Two such liftings r_1, r_2 are conjugate under the action of $\widehat{\mathcal{G}}_n(k[\epsilon])$ if and only if the associated cocycles ϕ_1, ϕ_2 have the same image in the group $H^1(F_v^+, \mathfrak{g}_n\overline{r})$. If v is split in F, then the exact sequence of $k[G_{F^+,S}]$-modules

$$0 \to \mathrm{ad}\,\overline{r} \to \mathfrak{g}_n\overline{r} \to k \to 0$$

splits over $k[G_{F_v^+}]$, and this is equivalent to asking that ϕ_1, ϕ_2 have the same image in $H^1(F_v^+, \mathrm{ad}\,\overline{r})$. We write $\mathcal{L}_v^1 \subset Z^1(F_v^+, \mathrm{ad}\,\overline{r})$ for the k-vector space of cocycles corresponding to liftings in $\mathcal{D}_v(k[\epsilon]) \subset \mathrm{Lift}_v^{\square}(k[\epsilon])$. We write $\mathcal{L}_v$ for the image of $\mathcal{L}_v^1$ in $H^1(F_v^+, \mathrm{ad}\,\overline{r})$, and $\ell_v^1 = \dim_k \mathcal{L}_v^1,\ \ell_v = \dim_k \mathcal{L}_v$.

Let $T \subset S$. We now want to define some global cohomology groups $H^i_{\mathcal{S},T}$ which can be used to analyze the relative tangent space of the morphism

$$R^{\mathrm{loc}}_{\mathcal{S},T} = \widehat{\otimes}_{v\in T} R_v \to R^T_{\mathcal{S}}.$$

To this end, we define a chain complex $C^i_{\mathcal{S},T}$ by the following formulae:

$$C^i_{\mathcal{S},T} = \begin{cases} 0 & i < 0; \\ C^0(F(S)/F^+, \mathfrak{g}_n\overline{r}) & i = 0; \\ C^1(F(S)/F^+, \mathrm{ad}\,\overline{r}) \oplus_{v\in T} C^0(F_v^+, \mathfrak{g}_n\overline{r}) & i = 1; \\ C^2(F(S)/F^+, \mathrm{ad}\,\overline{r}) \oplus_{v\in T} C^1(F_v^+, \mathrm{ad}\,\overline{r}) \oplus_{v\in S-T} C^1(F_v^+, \mathrm{ad}\,\overline{r})/\mathcal{L}_v^1 & i = 2; \\ C^i(F(S)/F^+, \mathrm{ad}\,\overline{r}) \oplus_{v\in S} C^{i-1}(F(S)/F^+, \mathrm{ad}\,\overline{r}) & i \geq 3. \end{cases}$$

The differentials are given by the formula

$$\partial(\phi, (\psi_v)_{v\in S}) = (\partial\phi, (\phi|_v - \partial\psi_v)_{v\in S}).$$

It is easy to check that this is indeed a complex, and that the cohomology groups $H^i_{\mathcal{S},T}$ of this complex fit into a long exact sequence

$$\begin{aligned} 0 &\longrightarrow H^0_{\mathcal{S},T} \longrightarrow H^0(F(S)/F^+, \mathfrak{g}_n\overline{r}) \qquad (1)\\ &\longrightarrow \oplus_{v\in T} H^0(F_v^+, \mathfrak{g}_n\overline{r}) \longrightarrow H^1_{\mathcal{S},T} \longrightarrow H^1(F(S)/F^+, \mathrm{ad}\,\overline{r})^{\eta} \\ &\longrightarrow \begin{matrix} \oplus_{v\in T} H^1(F_v^+, \mathrm{ad}\,\overline{r})^{\eta} \\ \oplus_{v\in S-T} H^1(F_v^+, \mathrm{ad}\,\overline{r})/\mathcal{L}_v \end{matrix} \longrightarrow H^2_{\mathcal{S},T} \longrightarrow H^2(F(S)/F^+, \mathrm{ad}\,\overline{r}) \\ &\longrightarrow \oplus_{v\in S} H^2(F_v^+, \mathrm{ad}\,\overline{r}) \longrightarrow H^3_{\mathcal{S},T} \longrightarrow H^3(F(S)/F^+, \mathrm{ad}\,\overline{r}) \\ &\longrightarrow \cdots \end{aligned}$$

(the superscript $^{\eta}$ indicates that we take the image of cohomology with coefficients in $\operatorname{ad}\overline{r}$ in cohomology with coefficients in $\mathfrak{g}_n\overline{r}$. A similar notation is used in [21, § 4.1.4]). We define $h^i_{\mathcal{S},T} = \dim_k H^i_{\mathcal{S},T}$.

Lemma 2.12 *Let notation be as above. We have* $\dim_k \mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}_{R^{loc}_{\mathcal{S},T}}, \mathfrak{m}^2_{R^T_{\mathcal{S}}}) = h^1_{\mathcal{S},T}$. *Consequently, there is a surjection* $R^{loc}_{\mathcal{S},T}[\![X_1, \ldots, X_g]\!] \to R^T_{\mathcal{S}}$ *of* $R^{loc}_{\mathcal{S},T}$*-algebras with* $g = h^1_{\mathcal{S},T}$.

Proof We observe that there is an isomorphism

$$\operatorname{Hom}_k\left(\mathfrak{m}_{R^T_{\mathcal{S}}}/(\mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, \mathfrak{m}^2_{R^T_{\mathcal{S}}}), k\right) \cong \operatorname{Hom}\left(R^T_{\mathcal{S}}/\mathfrak{m}_{R^{\mathrm{loc}}_{\mathcal{S},T}}, k[\epsilon]\right),$$

and the latter space corresponds to equivalence classes of tuples $(r, (\alpha_v)_{v\in T})$, where r is a lifting of $\overline{r}$ to $k[\epsilon]$ of type $\mathcal{S}$, $\alpha_v \in \widehat{\mathcal{G}}_n(k[\epsilon])$, and for each $v \in T$, $\alpha_v^{-1} r|_{G_{F_v^+}} \alpha_v$ is the trivial lifting. We can write $r = (1+\epsilon\phi)\overline{r}$ and $\alpha_v = 1+\epsilon_v a_v$ with $\phi \in Z^1(F_v^+, \operatorname{ad}\overline{r})$ and $a_v \in \mathfrak{g}_n\overline{r}$, and $\partial(\phi, (a_v)_{v\in T}) = 0$. The pairs $(\phi, (a_v)_{v\in T})$ and $(\phi', (a'_v)_{v\in T})$ define equivalent T-framed deformations if and only if there exists $b \in \mathfrak{g}_n\overline{r}$ such that $\phi - \phi' = \partial b$ and $a_v - a'_v = b$ for each $v \in T$. It is now easy to check that the space of such equivalence classes of cocycles is canonically identified with $H^1_{\mathcal{S},T}$. □

If M is any $k[G_{F^+,S}]$-module, then we define the Euler characteristics

$$\chi(F_v^+, M) = \sum_{i=0}^{2}(-1)^i h^i(F_v^+, M),$$

$$\chi(F(S)/F^+, M) = \sum_{i=0}^{2}(-1)^i h^i(F(S)/F^+, M)$$

and

$$\chi_{\mathcal{S},T} = \sum_{i=0}^{3}(-1)^i h^i_{\mathcal{S},T}.$$

Lemma 2.13 *With assumptions as above, we have an equality*

$$\chi_{\mathcal{S},T} = 1 - \#T + \chi(F(S)/F^+, \operatorname{ad}\overline{r}) - \sum_{v\in S}\chi(F_v^+, \operatorname{ad}\overline{r}) - \sum_{v\in S-T}\left(\ell^1_v - n^2\right)$$

Proof Use the above long exact sequence and the exact sequences ($v \in S$):

$$0\to H^0(F_v^+, \operatorname{ad}\overline{r})\to H^0(F_v^+, \mathfrak{g}_n)\to H^0(F_v^+, k)\to H^1(F_v^+, \operatorname{ad}\overline{r})\to H^1(F_v^+, \operatorname{ad}\overline{r})^{\eta} \to 0.$$

We also note that for $i \geq 3$, the map

$$H^i(F(S)/F^+, \operatorname{ad}\overline{r}) \to \prod_{v\in S_\infty} H^i(F_v^+, \operatorname{ad}\overline{r})$$

is bijective (and so $H^i_{\mathcal{S},T} = 0$ if $i \geq 4$). □

Corollary 2.14 *Suppose that* $S = T \sqcup Q \sqcup S_\infty$ *(a disjoint union) and that* $S_p \subset T$. *Then we have*

$$\chi_{\mathcal{S},T} = 1 - \#T - \sum_{v\in Q\sqcup S_\infty}(\ell^1_v - n^2).$$

Proof We evaluate each term of the formula of Lemma 2.13 in turn. We have

$$\chi(F(S)/F^+, \operatorname{ad}\overline{r}) = \sum_{v \in S_\infty} \left(h^0(F_v^+, \operatorname{ad}\overline{r}) - n^2\right)$$

(by [23, Theorem 5.1]), and

$$\sum_{v \in S_\infty} \chi(F_v^+, \operatorname{ad}\overline{r}) = \sum_{v \in S_\infty} h^0(F_v^+, \operatorname{ad}\overline{r})$$

(as $h^1(F_v^+, \operatorname{ad}\overline{r}) = h^2(F_v^+, \operatorname{ad}\overline{r})$), and

$$\sum_{v \in T \sqcup Q} \chi(F_v^+, \operatorname{ad}\overline{r}) = -n^2[F^+ : \mathbb{Q}]$$

(by [23, Theorem 2.8]). Summing these up now gives

$$\begin{aligned}\chi_{\mathcal{S},T} &= 1 - \#T - n^2[F^+ : \mathbb{Q}] + \sum_{v \in S_\infty} h^0(F_v^+, \operatorname{ad}\overline{r}) + n^2[F^+ : \mathbb{Q}] - \sum_{v \in S_\infty} h^0(F_v^+, \operatorname{ad}\overline{r}) \\ &\quad - \sum_{v \in Q \sqcup S_\infty} (\ell_v^1 - n^2) \\ &= 1 - \#T - \sum_{v \in Q \sqcup S_\infty} (\ell_v^1 - n^2),\end{aligned}$$

as desired. □

If $v \in S$, then we define

$$\mu_v = \ker\left(H^1(F_v^+, \operatorname{ad}\overline{r}) \to H^1(F_v^+, \mathfrak{g}_n\overline{r})\right).$$

If v is split in F, then $\mu_v = 0$, but in general it can be non-trivial. We always have $\mu_v \subset \mathcal{L}_v$. If $T \subset S$, then we define a 'dual Selmer group'

$$H^1_{\mathcal{S}^\perp,T} = \left\{x \in H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \mid \forall v \in T, \langle x, \mu_v\rangle = 0; \forall v \in S - T, \langle x, \mathcal{L}_v\rangle = 0\right\}.$$

We define $h^1_{\mathcal{S}^\perp,T} = \dim_k H^1_{\mathcal{S}^\perp,T}$.

Lemma 2.15 *Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\mathcal{D}_v\}_{v \in S})$ be a global deformation problem, and let $T \subset S$. Then $h^2_{\mathcal{S},T} = h^1_{\mathcal{S}^\perp,T}$ and $h^3_{\mathcal{S},T} = h^0(F(S)/F^+, \operatorname{ad}\overline{r}(1))$.*

Proof We use the Poitou–Tate exact sequence. More precisely, we have two exact sequences of k-vector spaces:

$$\begin{tikzcd} & H^1(F(S)/F^+, \operatorname{ad}\overline{r})^\eta \arrow[r] & \begin{matrix}\oplus_{v\in T} H^1(F_v^+, \operatorname{ad}\overline{r})^\eta \\ \oplus_{v \in S-T} H^1(F_v^+, \operatorname{ad}\overline{r})/\mathcal{L}_v\end{matrix} \arrow[d] \\ \oplus_{v \in S} H^2(F_v^+, \operatorname{ad}\overline{r}) \arrow[d] & H^2(F(S)/F^+, \operatorname{ad}\overline{r}) \arrow[l] & H^2_{\mathcal{S},T} \arrow[l] \\ H^3_{\mathcal{S},T} \arrow[r] & 0, & \end{tikzcd} \quad (2)$$

and

$$\begin{tikzcd}
 & H^1(F(S)/F^+, \operatorname{ad}\overline{r})^\eta \arrow[r] & \begin{smallmatrix} \oplus_{v \in T} H^1(F_v^+, \operatorname{ad}\overline{r})^\eta \\ \oplus_{v \in S-T} H^1(F_v^+, \operatorname{ad}\overline{r})/\mathcal{L}_v \end{smallmatrix} \arrow[d] \\
\oplus_{v \in S} H^2(F_v^+, \operatorname{ad}\overline{r}) \arrow[d] & H^2(F(S)/F^+, \operatorname{ad}\overline{r}) \arrow[l] & (H^1_{\mathcal{S}^\perp, T})^\vee \arrow[l] \\
H^0(F(S)/F^+, \operatorname{ad}\overline{r}(1))^\vee \arrow[r] & 0. &
\end{tikzcd} \tag{3}$$

The first of these is part of the long exact sequence (1), while the second arises from the Poitou–Tate exact sequence (see [23, Theorem 4.10]). The lemma now follows immediately on comparing (2) and (3). $\square$

2.3 Local deformation problems

We continue with the notation of Sect. 2.1, and define some local deformation problems.

2.3.1 Deformations at infinity

Suppose that $v \in S_\infty$ and $\chi(c_v) = -1$ (we will always assume this in applications below). In this section, we study the unrestricted deformation functor $\operatorname{Lift}_v^\square$ and its tangent space $\ell_v^1 = \operatorname{Lift}_v^\square(k[\epsilon])$.

Lemma 2.16 *Suppose that $p = 2$.*

(i) *Suppose that n is odd. Then $\overline{r}(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(1_n, 1)\jmath$.*
(ii) *Suppose that n is even. Then $\overline{r}(c_v)$ is $\mathrm{GL}_n(k)$-conjugate either to $(1_n, 1)\jmath$ or $(\Psi_n, 1)\jmath$, where $\Psi_n \in \mathrm{GL}_n(k)$ is the matrix with 1's on the antidiagonal and 0's everywhere else.*

Proof Let $\overline{r}(c_v) = (\overline{A}, 1)\jmath$. We calculate

$$\overline{r}(c_v)^2 = (\overline{A}{}^t\overline{A}^{-1}, 1) = 1,$$

hence $\overline{A} = {}^t\overline{A}$. Conjugation by $g \in \mathrm{GL}_n(k)$ replaces $\overline{A}$ by $g\overline{A}{}^t g$, so the problem comes down to the classification of $\mathrm{GL}_n(k)$-conjugacy classes of symmetric matrices.

Rephrasing the problem slightly, we must show that for any $m \geq 1$, any k-vector space V of dimension m with non-degenerate symmetric bilinear pairing is isomorphic to one of A_m or B_m, where $A_m = k^m$ with the pairing of Gram matrix 1_n, and $B_m = k^m$ with the pairing of Gram matrix Ψ_m. This follows by induction, together with the easily checked observations $A_m \cong A_1^m$, $B_m \cong B_2^{m/2}$ or $B_2^{(m-1)/2} \oplus A_1$ (according to whether m is even or odd, respectively) and $B_2 \oplus A_1 \cong A_3$. $\square$

Lemma 2.17 (i) *We have $\ell_v^1 = n(n+1)/2$.*
(ii) *Suppose that $p \neq 2$ or that $p = 2$ and $\overline{r}(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(1_n, 1)\jmath$. Then $\dim_k \mu_v = 1$ and the natural map $H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r})$ is injective.*
(iii) *Suppose that $p = 2$, n is even, and $\overline{r}(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(\Psi_n, 1)\jmath$. Then $\dim_k \mu_v = 0$ and the natural map $H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r})$ is 0.*

Proof Let $r \in \mathrm{Lift}_v^{\square}(k[\epsilon])$. Then we can write $r(c_v) = ((1+\epsilon X)\overline{A}, 1)j$, and $X \in \mathrm{ad}\,\overline{r}$ is arbitrary, subject to the condition

$$\begin{aligned} r(c_v)^2 &= \left((1+\epsilon X)\overline{A}, 1\right) j \left((1+\epsilon X)\overline{A}, 1\right) j \\ &= \left((1+\epsilon X)\overline{A}, 1\right)\left({}^t\overline{A}^{-1}(1-\epsilon {}^t X), 1\right) = (1+\epsilon(X - {}^t X), 1) = 1. \end{aligned}$$

In other words, X must be symmetric. This proves the first part of the lemma. The second part is clear if $p \neq 2$. If $p = 2$, we have

$$\mu_v = \ker(H^1(F_v^+, \mathrm{ad}\,\overline{r}) \to H^1\left(F_v^+, \mathfrak{g}_n\overline{r})\right) = \mathrm{im}\left(H^0(F_v^+, k) \to H^1(F_v^+, \mathrm{ad}\,\overline{r})\right),$$

where the boundary map is the one attached to the short exact sequence

$$0 \longrightarrow \mathrm{ad}\,\overline{r} \longrightarrow \mathfrak{g}_n\overline{r} \longrightarrow k \longrightarrow 0.$$

A calculation with cocycles reveals that we can identify μ_v with the image of the map $H^1(F_v^+, k) \to H^1(F_v^+, \mathrm{ad}\,\overline{r})$ which is attached to the short exact sequence

$$0 \longrightarrow k \longrightarrow \mathrm{ad}\,\overline{r} \longrightarrow \mathrm{ad}_0\,\overline{r} \longrightarrow 0.$$

Let us assume that $\overline{r}(c_v)$ is equal either to $(1_n, 1)j$ or $(\Psi_n, 1)j$. Let $\mathfrak{t} \subset \mathrm{ad}\,\overline{r}$ denote the diagonal Cartan subalgebra, and consider the exact sequence of $k[G_{F_v^+}]$-modules

$$0 \longrightarrow k \longrightarrow \mathfrak{t} \longrightarrow \mathfrak{t}/k \longrightarrow 0.$$

Then $\mathfrak{t} \subset \mathrm{ad}\,\overline{r}$ admits a $G_{F_v^+}$-stable complement (consisting of matrices with 0's on the diagonal), so we can finish the proof of the lemma by analysing what happens inside $\mathfrak{t}$.

If $\overline{r}(c_v) = (1_n, 1)j$ (so we are in the second case of the lemma), then $G_{F_v^+}$ acts trivially on each term in this sequence, which is therefore a split exact sequence of $k[G_{F_v^+}]$-modules. The desired assertions follow immediately from this. If $\overline{r}(c_v) = (\Psi_n, 1)j$ (so we are in the third case of the lemma), then c_v acts on $\mathfrak{t} \cong k^{2n}$ by reversing the order of coordinates. The usual calculation of cohomology of cyclic groups shows that $H^1(F_v^+, \mathfrak{t}) = 0$, which implies the desired statement in this case also. □

2.3.2 *Taylor–Wiles deformations*

Let v be a finite place of F^+ which splits in $F(\zeta_p)$ and at which $\overline{r}$ is unramified, and suppose that $\overline{r}(\mathrm{Frob}_v)$ is semi-simple. Let $\alpha_v \in k$ be an eigenvalue of $\overline{r}(\mathrm{Frob}_v)$ of multiplicity n_1, say. We can decompose

$$\overline{\rho}|_{G_{F_v^+}} = \overline{A}_v \oplus \overline{B}_v, \tag{4}$$

where $A_v(\mathrm{Frob}_v) = \alpha_v \cdot 1_{n_1}$. We define a subfunctor $\mathcal{D}_v^{\mathrm{TW}} \subset \mathrm{Lift}_v^{\square}$ as follows: if $R \in \mathrm{CNL}_{\mathcal{O}}$ and $r \in \mathrm{Lift}_v^{\square}(R)$, then we say that $r \in \mathcal{D}_v^{\mathrm{TW}}(R)$ if there is a decomposition

$$\rho = A_v \oplus B_v \tag{5}$$

lifting the decomposition (4), and such that B_v is unramified and $A_v|I_{F_v^+} = \psi_v \cdot 1_{n_1}$ for some character $\psi_v \colon I_{F_v^+} \to R^\times$. The deformation problem $\mathcal{D}_v^{\mathrm{TW}}$ depends on the choice of α_v, although we do not include it in the notation. It is known (see [32, Lemma 4.2]) that $\mathcal{D}_v^{\mathrm{TW}}$ is a local deformation problem.

Let $\Delta_v = k(v)^\times(p)$ denote the p-part of the finite abelian group $k(v)^\times$. We observe that if $\rho \in \mathcal{D}_v^{\mathrm{TW}}(R)$, then there is a canonical homomorphism $\Delta_v \to R^\times$, given by $\psi_v \circ \mathrm{Art}_{F_v^+}$.

2.3.3 Potentially crystalline deformations

Now suppose that $v \in S_p$; then v splits in F, by assumption. We now recall, following [5, § 1.4], some local deformation problems whose existence and basic properties have been established by Kisin [17]. Let $\mathbb{Z}^n_+ \subset \mathbb{Z}^n$ denote the set of tuples $(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n$, and let $\boldsymbol{\lambda}_v \in (\mathbb{Z}^+_n)^{\mathrm{Hom}_{\mathbb{Q}_p}(F_v^+, \overline{\mathbb{Q}}_p)}$. Let K be a finite extension of F_v^+ inside $\overline{F}_v^+$.

According to [5, § 1.4], there is a reduced, p-torsion free quotient $R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}}$ of $R_v^\square$ with the following properties:

- The quotient $R_v^\square \to R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}}$ defines a local deformation problem. We write $\mathcal{D}_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}}$ for the corresponding set-valued functor in $\mathrm{CNL}_\mathcal{O}$. In the case $K = F_v^+$, we omit the superscript from the notation and simply write $\mathcal{D}_v^{\boldsymbol{\lambda}_v}$.
- Let $\rho^\square : G_{F_v^+} \to \mathrm{GL}_n(R_v^\square)$ denote the universal lifting, and fix a homomorphism $f : R_v^\square \to \overline{\mathbb{Q}}_p$. Then f factors through the quotient $R_v^\square \to R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}}$ if and only if $f \circ \rho^\square|_{G_K}$ is crystalline, of Hodge–Tate type $\boldsymbol{\lambda}_v$.
- The ring $R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}}[1/p]$ is formally smooth over E, of dimension $1+n^2+[F_v^+ : \mathbb{Q}_p]n(n-1)/2$.

The ring $R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}} = R_v^{\boldsymbol{\lambda}_v, K-\mathrm{cr}} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p$ is independent of the choice of coefficient field E (see [5, Lemma 1.2.1]). This fact will be used in Sect. 3.2.

2.3.4 A useful local deformation problem

Now suppose that $v \in S - (S_p \cup S_\infty)$ splits in F. We write R_v^{fl} for the maximal reduced, p-torsion free quotient of $R_v^\square[1/p]$, and $\mathcal{D}_v^{\mathrm{fl}}$ for the corresponding set-valued functor on $\mathrm{CNL}_\mathcal{O}$. It is easy to see that $\mathcal{D}_v^{\mathrm{fl}}$ is a local deformation problem.

The ring R_v^{fl} has the following useful property (see [5, Lemma 1.3.2]). Let $\rho^\square : G_{F_v^+} \to \mathrm{GL}_n(R_v^\square)$ be the universal lifting, and let $f : R_v^\square \to \overline{\mathbb{Q}}_p$ be a homomorphism (that therefore factors through the quotient R_v^{fl}). Then there is an irreducible admissible representation π of $\mathrm{GL}_n(F_v^+)$ over $\overline{\mathbb{Q}}_p$ such that $\mathrm{rec}^T_{F_v^+}(\pi) \cong (f \circ \rho^\square)^{\text{F-ss}}$. Suppose that π is generic. Then the ring $R_v^{\mathrm{fl}}[1/p]$ is formally smooth over E at the closed point corresponding to f.

The ring $R_v^{\mathrm{fl}} = R_v^{\mathrm{fl}} \otimes_\mathcal{O} \overline{\mathbb{Q}}_p$ is independent of the choice of coefficient field E (see [5, Lemma 1.2.1]). This fact will be used in Sect. 3.2.

2.4 Taylor–Wiles systems

Consider a global deformation problem

$$\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\mathcal{D}_v\}_{v \in S}) .$$

Let $T = S - S_\infty$, and $\overline{\rho} = \overline{r}|_{G_{F,S}}$.

Definition 2.18 Let $N \geq 1$ be an integer. A Taylor–Wiles datum of level N is a pair $(Q, (\alpha_v)_{v \in Q})$ satisfying the following conditions:

(i) Q is a finite set of finite places of F^+.
(ii) For each $v \in Q$, $v \notin S$ and v splits in $F(\zeta_{p^N})$.
(iii) For each $v \in Q$, $\overline{\rho}(\mathrm{Frob}_v)$ is semi-simple and $\alpha_v \in k$ is one of its eigenvalues.

If $(Q, (\alpha_v)_{v\in Q})$ is a Taylor–Wiles datum, then we define the augmented global deformation problem

$$\mathcal{S}_Q = \left(F, \overline{r}, \mathcal{O}, \chi, S \cup Q, \{\mathcal{D}_v\}_{v\in S} \cup \{\mathcal{D}_v^{\mathrm{TW}}\}_{v\in Q}\right),$$

where for each $v \in Q$ the local deformation problem $\mathcal{D}_v^{\mathrm{TW}}$ is defined with respect to the eigenvalue $\alpha_v \in k$, as in Sect. 2.3.2. Let $\Delta_Q = \prod_{v\in Q} k(v)^\times(p)$. Then there is a canonical homomorphism $\mathcal{O}[\Delta_Q] \to R_{\mathcal{S}_Q}$, and a canonical identification $R_{\mathcal{S}_Q} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} = R_{\mathcal{S}}$.

Lemma 2.19 *Let $(Q, (\alpha_v)_{v\in Q})$ be a Taylor–Wiles datum. Then there is an exact sequence*

$$0 \longrightarrow H^1_{\mathcal{S}_Q^\perp, T} \longrightarrow H^1_{\mathcal{S}^\perp, T} \longrightarrow \oplus_{v\in Q}\, k,$$

the last arrow being given by $[\psi] \mapsto (\operatorname{tr} e_{\mathrm{Frob}_v, \alpha_v} \psi(\mathrm{Frob}_v))_{v\in Q}$. (Here we write $e_{\mathrm{Frob}_v,\alpha_v} \in \operatorname{ad}\overline{r} = M_n(k)$ for the unique idempotent in $k[\overline{\rho}(\mathrm{Frob}_v)]$ with image equal to the α_v-eigenspace of $\overline{\rho}(\mathrm{Frob}_v)$).

Proof If $v \in Q$, let $\mathcal{L}_v \subset H^1(F_v^+, \operatorname{ad}\overline{r})$ be the subspace corresponding to $\mathcal{D}_v^{\mathrm{TW}}(k[\epsilon])$, and let $\mathcal{L}_v^{\mathrm{ur}}$ be the subspace corresponding to the functor of unramified deformations. Then $\mathcal{L}_v^{\mathrm{ur}} \subset \mathcal{L}_v$. By definition, we have

$$\begin{aligned} H^1_{\mathcal{S}^\perp, T} = \ker \Big[& H^1(F(S\cup Q)/F^+, \operatorname{ad}\overline{r}(1)) \to \\ & \prod_{v\in T} H^1(F_v^+, \operatorname{ad}\overline{r}(1))/\mu_v^\perp \times \prod_{v\in S-T} H^1(F_v^+, \operatorname{ad}\overline{r}(1))/\mathcal{L}_v^\perp \\ & \times \prod_{v\in Q} H^1(F_v^+, \operatorname{ad}\overline{r}(1))/(\mathcal{L}_v^{\mathrm{ur}})^\perp \Big], \end{aligned}$$

and

$$H^1_{\mathcal{S}_Q^\perp, T} = \ker H^1_{\mathcal{S}^\perp, T} \to \prod_{v\in Q} (\mathcal{L}_v^{\mathrm{ur}})^\perp/\mathcal{L}_v^\perp.$$

To show the lemma, it is therefore enough to show that for each $v \in Q$, there is an isomorphism $(\mathcal{L}_v^{\mathrm{ur}})^\perp/\mathcal{L}_v^\perp \cong k$, given at the level of cocycles by the formula $[\psi] \mapsto \operatorname{tr} e_{\mathrm{Frob}_v,\alpha_v}\psi(\mathrm{Frob}_v)$. Writing $\overline{r}|_{G_{F_v^+}} = \overline{A}_v \oplus \overline{B}_v$ as in the definition of $\mathcal{D}_v^{\mathrm{TW}}$, we calculate $\operatorname{ad}\overline{r} \cong \operatorname{End}(\overline{A}_v) \oplus \operatorname{Hom}(\overline{A}_v, \overline{B}_v) \oplus \operatorname{Hom}(\overline{B}_v, \overline{A}_v) \oplus \operatorname{End}(\overline{B}_v)$, hence

$$H^1(F_v^+, \operatorname{ad}\overline{r}) \cong H^1(F_v^+, \operatorname{ad}\overline{A}_v) \oplus H^1(F_v^+, \operatorname{ad}\overline{B}_v)$$

(as the other two summands have trivial cohomology). The representation $\overline{A}_v$ is assumed to be scalar, so $\operatorname{ad}\overline{A}_v$ is a trivial Galois module, and we have a canonical isomorphism $H^1(F_v^+, \operatorname{ad}\overline{A}_v) \cong H^1(F_v^+, k) \otimes_k \operatorname{ad}\overline{A}_v$. The group $H^1(F_v^+, k)$ is a 2-dimensional k-vector space, endowed with a perfect duality; the subspace of unramified classes is 1-dimensional, and equal to its own orthogonal complement. Writing $Z \subset \operatorname{ad}\overline{A}_v$ for the subspace of scalar endomorphisms, we can therefore identify

$$\begin{aligned} \mathcal{L}_v^\perp &= \big\{(x, y) \in \mathcal{L}_v^{\mathrm{ur}} \subset H^1(F_v^+, \operatorname{ad}\overline{A}_v) \oplus H^1(F_v^+, \operatorname{ad}\overline{B}_v) \\ &\cong H^1(F_v^+, \operatorname{ad}\overline{r}) \mid \langle x, H^1(F_v^+, Z)\rangle = 0\big\} \\ &= H^1_{\mathrm{ur}}(F_v^+, k) \otimes_k \operatorname{ad}^0\overline{A}_v \oplus H^1_{\mathrm{ur}}(F_v^+, \overline{B}_v). \end{aligned}$$

This expression is equivalent to the desired result. ◻

The following definition of adequate subgroups is taken from [13].

Definition 2.20 Let K be a field. We say that a subgroup $H \subset \mathrm{GL}_n(K)$ is adequate if it satisfies the following conditions:

(i) We have $H^1(H, K) = 0$ and $H^1(H, \mathrm{ad}_0) = 0$.
(ii) For each simple $K[H]$-submodule $W \subset \mathrm{ad}$, there exists a semi-simple element $\sigma \in H$ with an eigenvalue $\alpha \in K$ such that $\mathrm{tr}\, e_{\sigma,\alpha} W \neq 0$.

The second condition implies that $M_n(K)$ is spanned as a K-vector space by the semi-simple elements $\sigma \in H \subset \mathrm{ad}$. In particular, an adequate subgroup acts absolutely irreducibly in its tautological representation on K^n.

Proposition 2.21 *Let $\mathcal{S} = (F, \overline{r}, \mathcal{O}, \chi, S, \{\mathcal{D}_v\}_{v\in S})$ be a global deformation problem, and let $T = S - S_\infty$. We make the following assumptions:*

(i) *For each $v \in S_\infty$, $\mu(c_v) = -1$ and $\mathcal{D}_v = \mathrm{Lift}_v^\square$.*
(ii) *If $p \neq 2$, then $F = F^+(\zeta_p)$. If $p = 2$, then $F = F^+(\sqrt{-1})$.*
(iii) *If $p = 2$ and n is even, then there exists $v \in S_\infty$ such that $\overline{r}(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(1_n, 1)_J$ (cf. Sect. 2.3.1).*
(iv) *The group $\overline{\rho}(G_F) \subset \mathrm{GL}_n(k)$ is adequate.*

Let $q = h^1_{\mathcal{S}^\perp,T} - 1$ and $g = q + |T| - 1 - [F^+ : \mathbb{Q}]n(n-1)/2$. Then for each $N \geq 1$, we can find infinitely many Taylor–Wiles data $(Q, (\alpha_v)_{v\in Q})$ of level N such that $\#Q = q$ and the map $R^{loc}_{\mathcal{S},T} \to R^T_{\mathcal{S}_Q}$ can be extended to a surjection $R^{loc}_{\mathcal{S},T}[[X_1, \ldots, X_g]] \twoheadrightarrow R^T_{\mathcal{S}_Q}$.

Proof We first note that $R^{\mathrm{loc}}_{\mathcal{S},T} = R^{\mathrm{loc}}_{\mathcal{S}_Q,T}$ by definition, and that $R^T_{\mathcal{S}_Q}$ can be topologically generated as an $R^{\mathrm{loc}}_{\mathcal{S}_Q,T}$-algebra by $h^1_{\mathcal{S}_Q,T}$ elements. To prove the proposition, it is therefore enough to show that for each $N \geq 1$, we can find infinitely many Taylor–Wiles data $(Q, (\alpha_v)_{v\in Q})$ of level N such that $|Q| = q$ and $h^1_{\mathcal{S}_Q,T} = g$.

On the other hand, we have by Corollary 2.14 and the local calculations of Sect. 2.3 an equality

$$\chi_{\mathcal{S}_Q,T} = 1 - |T| - |Q| + [F^+ : \mathbb{Q}]n(n-1)/2.$$

Combining this with Lemma 2.15, we obtain an equality

$$\begin{aligned} h^1_{\mathcal{S}_Q,T} &= h^2_{\mathcal{S}_Q,T} - h^3_{\mathcal{S}_Q,T} - 1 + |T| + |Q| - [F^+ : \mathbb{Q}]n(n-1)/2 \\ &= h^1_{\mathcal{S}_Q^\perp,T} - 2 + |T| + |Q| - [F^+ : \mathbb{Q}]n(n-1)/2. \end{aligned}$$

It is therefore enough to show that for each $N \geq 1$, we can find infinitely many Taylor–Wiles data $(Q, (\alpha_v)_{v\in Q})$ of level N such that $h^1_{\mathcal{S}_Q^\perp,T} = 1 = h^1_{\mathcal{S}^\perp,T} - |Q|$. We will do this by killing cohomology classes in $H^1_{\mathcal{S}^\perp,T}$ in the usual manner, using Lemma 2.19.

Fix now a choice of $N \geq 2$, large enough so that $F_N = F(\zeta_{p^N})$ strictly contains F. Let $[\psi] \in H^1_{\mathcal{S}^\perp,T} \subset H^1(F(S)/F^+, \mathrm{ad}\, \overline{r}(1))$ be a cohomology class with non-zero image in $H^1(F(S)/F_N, \mathrm{ad}\, \overline{r}(1))$. We claim that we can find infinitely many Taylor–Wiles data $(\{w\}, \alpha_w)$ of level N such that $[\psi] \notin H^1_{\mathcal{S}_{\{w\}}^\perp,T}$.

We first show that this claim implies the proposition. Indeed, let s denote the dimension of the image of $H^1_{\mathcal{S}^\perp,T}$ in $H^1(F(S)/F_N, \mathrm{ad}\, \overline{r}(1))$. By applying the claim repeatedly we can find infinitely many Taylor–Wiles data $(Q, \{\alpha_v\}_{v\in Q})$ of level N such that $|Q| = s$, $h^1_{\mathcal{S}_Q^\perp,T} = h^1_{\mathcal{S}^\perp,T} - s$ and the map $H^1_{\mathcal{S}_Q^\perp,T} \to H^1(F(S)/F_N, \mathrm{ad}\, \overline{r}(1))$ is trivial.

It follows that $H^1_{\mathcal{S}_Q^\perp,T}$ is contained inside

$$\ker\left[H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1)) \to H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))\right] = H^1\left(F_N/F^+, \operatorname{ad}\overline{r}(1)^{G_{F_N}}\right).$$

Our assumption that $\overline{\rho}(G_F)$ is adequate implies that $\overline{\rho}(G_{F_N}) = \overline{\rho}(G_F)$ and $\operatorname{ad}\overline{r}(1)^{G_{F_N}} = k$ (i.e. the subspace of scalar matrices). At this point, we split into cases according to the parity of p. If p is odd, then it is easy to see that $H^1_{\mathcal{S}_Q^\perp,T} = H^1(F_N/F^+, k)$ is 1-dimensional. It follows that $s = q = |Q|$, and the proof of the proposition is completed in this case. If $p = 2$, then $H^1(F_N/F^+, k)$ is 2-dimensional. However, it follows from our hypotheses and the second part of Lemma 2.17 that there exists a place $v \in S_\infty$ such that the map $H^1(F_N/F^+, k) \to H^1(F_v^+, k) \to H^1(F_v^+, \operatorname{ad}\overline{r}(1))$ is injective, and hence that $H^1_{\mathcal{S}_Q^\perp,T}$ is again 1-dimensional (we observe that classes in $H^1_{\mathcal{S}_Q^\perp,T}$ are locally trivial at $v \in S_\infty$, by definition, because $v \notin T$). We again see that $s = q = |Q|$, and the proof of the proposition is completed in this case also.

We now return to the proof of the claim. Let $[\psi] \in H^1_{\mathcal{S}^\perp,T} \subset H^1(F(S)/F^+, \operatorname{ad}\overline{r}(1))$ be a cohomology class with non-zero image in $H^1(F(S)/F_N, \operatorname{ad}\overline{r}(1))$. By Lemma 2.19, it is enough to find a place w of F^+ and an element $\alpha_w \in k$ satisfying the following conditions:

- w splits in F_N and $\overline{\rho}(\mathrm{Frob}_w)$ is semi-simple.
- $\alpha_w \in k$ is an eigenvalue of $\overline{\rho}(\mathrm{Frob}_w)$ such that $\operatorname{tr} e_{\mathrm{Frob}_w,\alpha_w}\psi(\mathrm{Frob}_w) \neq 0$.

By the Chebotarev density theorem, it is even enough to find elements $\sigma \in G_{F_N}$ and $\alpha \in k$ such that $\overline{\rho}(\sigma)$ is semi-simple, and α is an eigenvalue of $\overline{\rho}(\sigma)$ such that $\operatorname{tr} e_{\sigma,\alpha}\psi(\sigma) \neq 0$.

Let K/F be the extension cut out by $\operatorname{ad}\overline{\rho}$, and let $K_N = K \cdot F_N$. Let $f = \psi|_{K_N}$, an element of $H^1(F(S)/K_N, \operatorname{ad}\overline{r}(1))^{G_F} \subset H^1(F(S)/K_N, \operatorname{ad}\overline{\rho})^{G_{F_N}}$. The image of $[\psi]$ in $H^1(F(S)/F_N, \operatorname{ad}\overline{r})$ is non-zero. On the other hand, the group $H^1(K_N/F_N, \operatorname{ad}\overline{r}) \cong H^1(K/F, \operatorname{ad}\overline{\rho}) = 0$, as follows from the assumption that $\overline{\rho}$ has adequate image. It follows that f is non-zero, as a homomorphism $f\colon \mathrm{Gal}(F(S)/K_N) \to \operatorname{ad}\overline{r}$.

Let $V \subset \operatorname{ad}\overline{\rho}$ denote the k-span of the image of f, a $k[G_{F_N}]$-module. We can find a simple $k[G_{F_N}]$-submodule $W \subset V$, an element $\sigma_0 \in G_{F_N}$ such that $\overline{\rho}(\sigma_0)$ is semi-simple, and an eigenvalue $\alpha_0 \in k$ of $\overline{\rho}(\sigma_0)$ such that $\operatorname{tr} e_{\sigma_0,\alpha_0} W \neq 0$. If $\operatorname{tr} e_{\sigma_0,\alpha_0}\psi(\sigma_0) \neq 0$, then we are done on taking $\sigma = \sigma_0$ and $\alpha = \alpha_0$.

Let us suppose instead that $\operatorname{tr} e_{\sigma_0,\alpha_0}\psi(\sigma_0) = 0$, and choose an element $\tau \in K_N$ such that $\operatorname{tr} e_{\sigma_0,\alpha_0} f(\tau) \neq 0$. We claim that we are now done on taking $\sigma = \tau\sigma_0$. Indeed, there is an eigenvalue α of $\overline{\rho}(\sigma)$ such that $e_{\sigma_0,\alpha_0} = e_{\sigma,\alpha}$, and we calculate using the cocycle relation

$$\operatorname{tr} e_{\sigma,\alpha}\psi(\sigma) = \operatorname{tr} e_{\sigma_0,\alpha_0}(\psi(\tau) + \psi(\sigma_0)) = \operatorname{tr} e_{\sigma_0,\alpha_0} f(\tau) \neq 0.$$

This completes the proof. □

3 Galois representations and automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$

In this section, we recall the class of automorphic representations with which we work and make some preliminary observations about their attached Galois representations.

Definition 3.1 Let $n \geq 1$ be an integer.

(i) Let F be an imaginary CM number field. A RACSDC automorphic representation is an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ satisfying the following conditions:

(a) π is regular algebraic.
(b) π is conjugate self-dual, i.e. $\pi^c \cong \pi^\vee$, $c \in \mathrm{Gal}(F/F^+)$ the non-trivial element.
(c) π is cuspidal.

(ii) Let F be a totally real number field. A RAESDC automorphic representation is a pair (π, χ) consisting of an automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ and a continuous character $\chi: \mathbb{A}_F^\times \to \mathbb{C}^\times$, satisfying the following conditions:

(a) π is regular algebraic.
(b) π is essentially self-dual, i.e. $\pi \cong \pi^\vee \otimes (\chi \circ \det)$.
(c) π is cuspidal.

It is usual to require in the definition of a RAESDC automorphic representation that the value $\chi_v(-1)$ is independent of the choice of place $v|\infty$ of F; however, Patrikis [24, Theorem 2.0.1] has recently shown that this follows from the other conditions.

If F is an imaginary CM field or a totally real field, and π [resp. (π, χ)] is a RACSDC (resp. RAESDC) automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, and $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$ is an isomorphism, then there exists a continuous semi-simple representation $r_\iota(\pi): G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, which satisfies the following local-global compatibility condition at each finite place v of F:

$$\mathrm{WD}(r_\iota(\pi)|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{rec}^T_{F_v}(\iota^{-1}\pi_v).$$

In particular, if $v|p$ then $r_\iota(\pi)|_{G_{F_v}}$ is de Rham, and there is a recipe for the Hodge–Tate weights of π in terms of ι and π_∞; see [9, § 2.1] for a precise statement and list of references.

3.1 The sign of a conjugate self-dual Galois representation

Let F be an imaginary CM number field with maximal totally real subfield F^+, and let $c \in G_{F^+}$ be a fixed choice of complex conjugation. Let k be a field, $\rho: G_F \to \mathrm{GL}_n(k)$ an absolutely irreducible representation, and $\mu: G_{F^+} \to k^\times$ a character.

Suppose that there is an isomorphism $\rho^c \cong \rho^\vee \otimes \mu$; equivalently, that there exists a pairing $\langle\cdot,\cdot\rangle: k^n \times k^n \to k$ such that for all $\delta \in G_F$, $x, y \in k^n$, we have $\langle \rho(\delta)x, \rho(\delta^c)y\rangle = \mu(\delta)\langle x, y\rangle$. The pairing $\langle\cdot,\cdot\rangle$ is then uniquely determined up to scalar, and (after possibly replacing μ by $\mu\delta_{F/F^+}$, as in the proof of Lemma 2.2) we have

$$\langle x, y\rangle = -\mu(c)\langle y, x\rangle \tag{6}$$

for all $x, y \in k^n$. Following [5, § 2], we say that a pair (ρ, μ) satisfying the condition (6) is *polarized*. By Lemma 2.1, this is equivalent to asking that ρ extend to a homomorphism $r: G_{F^+} \to \mathcal{G}_n(k)$ such that $\nu \circ r = \mu$. This condition is independent of the choice of c. We have the following result, due to Bellaiche–Chenevier [2].

Theorem 3.2 *Let π be a RACSDC automorphic representation of* $\mathrm{GL}_n(\mathbb{A}_F)$, $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$ *an isomorphism, and suppose that $r_\iota(\pi)$ is irreducible. Then the pair $(r_\iota(\pi), \epsilon^{1-n}\delta^n_{F/F^+})$ is polarized.*

Now suppose that k is a perfect field of characteristic 2 and that n is even, and that the pair (ρ, μ) is polarized. We can attach a discrete invariant to ρ at each infinite place v of F^+ as follows. Let $c_v \in G_{F^+}$ be a choice of complex conjugation at this place, and let $r: G_{F^+} \to \mathcal{G}_n(k)$ be a choice of extension of ρ such that $\nu \circ r = \mu$.

According to Lemma 2.16, there are two distinct possibilities for the $\mathrm{GL}_n(k)$-conjugacy class of $r(c_v) \in \mathcal{G}_n(k)$: it is conjugate either to $(1_n, 1)\jmath$, or to $(\Psi_n, 1)\jmath$. We observe that the matrix Ψ_n admits skew-symmetric lifts in $\mathrm{GL}_n(W(k))$, while the matrix 1_n does not (here $W(k)$ denotes the ring of Witt vectors of k).

Definition 3.3 Let k be a perfect field of characteristic 2, and suppose that n is even. Let (ρ, μ) be polarized, and let v be an infinite place of F^+. We say that the pair (ρ, μ) is strongly residually odd at v if $r(c_v)$ is $\mathrm{GL}_n(k)$-conjugate to $(1_n, 1)_J$.

To motivate this definition, we have the following simple lemma.

Lemma 3.4 *Let $p = 2$, and let n be an even integer. Let E be a finite extension of $\mathbb{Q}_p$, and let $r\colon G_{F^+} \to \mathcal{G}_n(\mathcal{O})$ be a homomorphism such that (setting $\rho = r|_{G_F}$, $\mu = \nu \circ r$) $\overline{\rho}$ is absolutely irreducible. Then the pair (ρ, μ) is polarized. If $v|\infty$ is a place of F^+, and $(\overline{\rho}, \overline{\mu})$ is strongly residually odd at v, then $\mu(c_v) = -1$.*

Lemma 3.5 *Let k be a finite field, and let $\sigma\colon G_{F^+} \to \mathrm{GL}_2(k)$ be a continuous representation such that $\sigma|_{G_F}$ is absolutely irreducible. Let $\chi = \det \sigma$, and let $\psi\colon G_F \to k^\times$ be a character such that $\psi\psi^c = \epsilon\chi$, and set $\rho = \sigma|_{G_F} \otimes \psi^{-1}$. Then:*

(i) *The character χ is totally odd if and only if the pair (ρ, ϵ^{-1}) is polarized.*
(ii) *Suppose that k has characteristic 2, and let v be an infinite place of F^+. Then (ρ, ϵ^{-1}) is strongly residually odd at v if and only if $\sigma(c_v) \in \mathrm{GL}_2(k)$ is non-trivial.*

Proof The proof is by explicit calculation. Fix a place $v|\infty$ of F^+ and a choice $c = c_v \in G_{F^+}$ of complex conjugation at v. Note first that ρ is absolutely irreducible and $\rho^c \cong \rho^\vee \otimes \epsilon^{-1}$. Define

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

so that ${}^t\sigma(\gamma)J\sigma(\gamma) = \chi(\gamma)J$ for all $\gamma \in G_{F^+}$. An easy calculation then shows that the relation

$${}^t\rho(\delta)A^{-1}\rho(\delta^c) = \epsilon^{-1}(\delta)A^{-1}$$

holds if we take $A = \sigma(c)J^{-1}$, in which case ${}^tA = -\chi(c)A$. We can therefore define an extension $r\colon G_{F^+} \to \mathcal{G}_2(k)$ of ρ by setting $r(c) = (A, \epsilon^{-1}\chi(c))_J$. This shows the first part of the lemma.

For the second part, we observe that the only two possibilities for $\sigma(c_v)$ (up to conjugation in $\mathrm{GL}_2(k)$) are $\sigma(c_v) = 1$ or

$$\sigma(c_v) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In the first case, we obtain $A = J^{-1} = \Psi_2$. In the second case, we obtain

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

and this symmetric matrix is $\mathrm{GL}_2(k)$-conjugate to 1_2. This completes the proof. □

3.2 Irreducible components of Galois deformation rings

In this section we recall some important ideas from [5], which formalize the idea of two global Galois representations having the same local properties (compare the statement of Theorem 1.1). Fix a prime p. Let l be a prime, and let K be a finite extension of $\mathbb{Q}_l$ inside $\overline{\mathbb{Q}}_l$.

Definition 3.6 Fix continuous representations $\rho_1, \rho_2 \colon G_K \to \mathrm{GL}_n(\overline{\mathbb{Z}}_p)$, and let $r_1, r_2 \colon G_K \to \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ be the corresponding residual representations $r_1 = \rho_1 \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_p}$, $r_2 = \rho_2 \bmod \mathfrak{m}_{\overline{\mathbb{Z}}_p}$. Suppose that $r_1 \cong r_2$.

(i) Suppose that $l \neq p$. We say that ρ_1 connects with ρ_2, and write $\rho_1 \sim \rho_2$, if ρ_1, ρ_2 define points on a common irreducible component of $\mathrm{Spec}(R_v^{\mathrm{fl}} \otimes \overline{\mathbb{Q}}_p)$, where $R_v^{\mathrm{fl}} \otimes \overline{\mathbb{Q}}_p$ is the ring defined with respect to r_1 in Sect. 2.3.4.

(ii) Suppose that $l = p$. We say that ρ_1 connects with ρ_2, and write $\rho_1 \sim \rho_2$, if there exists a finite extension K'/K and $\boldsymbol{\lambda}_v \in (\mathbb{Z}_+^n)^{\mathrm{Hom}_{\mathbb{Q}_p}(K, \overline{\mathbb{Q}}_p)}$ such that each of $\rho_1|_{G_{K'}}$ and $\rho_2|_{G_{K'}}$ is crystalline of Hodge–Tate type $\boldsymbol{\lambda}_v$, and ρ_1, ρ_2 define points on a common irreducible component of $\mathrm{Spec}(R_v^{\boldsymbol{\lambda}_v, K'-\mathrm{cr}} \otimes \overline{\mathbb{Q}}_p)$, where $R_v^{\boldsymbol{\lambda}_v, K'-\mathrm{cr}} \otimes \overline{\mathbb{Q}}_p$ is the ring defined with respect to r_1 in Sect. 2.3.3.

(We use the notation r_1, r_2 instead of $\overline{\rho}_1, \overline{\rho}_2$ because of our convention that $\overline{\rho}_i$ is the semi-simplified residual representation of $\overline{\rho}$). It follows from [5, Lemma 1.2.2] that these definitions make sense, independently of the choice of isomorphism $r_1 \cong r_2$.

Now let F be a number field, and suppose that $\rho \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ is a continuous representation such that $\overline{\rho}$ is absolutely irreducible. Then after conjugating, we can assume that ρ takes values in $\mathrm{GL}_n(\overline{\mathbb{Z}}_p)$, and ρ is then unique up to $\mathrm{GL}_n(\overline{\mathbb{Z}}_p)$-conjugation.

Suppose $\rho_1, \rho_2 \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ are continuous representations such that $\overline{\rho}_1$ and $\overline{\rho}_2$ are absolutely irreducible and $\overline{\rho}_1 \cong \overline{\rho}_2$, and let v be a finite place of F. We will say that $\rho_1|_{G_{F_v}}$ connects to $\rho_2|_{G_{F_v}}$, and write $\rho_1|_{G_{F_v}} \sim \rho_2|_{G_{F_v}}$, if this relation holds as in Definition 3.6 with respect to these choices of integral lattice.

4 Automorphic forms on definite unitary groups

In this section we define spaces of algebraic modular forms on definite unitary groups, and use them to prove an $R = \mathbb{T}$ type result. Apart from the Galois-theoretic ingredients of Sect. 2, there is nothing in the current section that will be new to an expert. In order to avoid repetition, we therefore refer to [32, § 6] for detailed definitions and proofs, and merely recall notation and basic properties here. We first fix a prime p and an integer $n \geq 2$. At the beginning of [32, § 6], it is assumed that p is odd, but this plays no role in the parts recalled here. We fix a coefficient field $E \subset \overline{\mathbb{Q}}_p$ with ring of integers $\mathcal{O}$, maximal ideal λ, and residue field k.

We begin with an imaginary CM number field F with maximal totally real subfield F^+, satisfying the following conditions:

- The extension F/F^+ is everywhere unramified, and each prime of F^+ above p splits in F.
- We have $n[F^+ \colon \mathbb{Q}] \equiv 0 \bmod 4$.

Let $c \in \mathrm{Gal}(F/F^+)$ denote the non-trivial element. We can then find a unitary group G over F^+, split by F, and satisfying the following conditions:

- For each finite place v of F^+, $G(F_v^+)$ is quasi-split.
- The group $G(F^+ \otimes_{\mathbb{Q}} \mathbb{R})$ is compact.

We can extend G to an affine group scheme over $\mathcal{O}_{F^+}$, still denoted G, with the following property: for each place v of F^+ split as $v = ww^c$ in F, there is an isomorphism $\iota_w \colon G(\mathcal{O}_{F^+}) \cong \mathrm{GL}_n(\mathcal{O}_{F_w})$. We use the following notation:

- S_p is the set of places of F^+ above p. For each $v \in S_p$, we choose a place $\widetilde{v}$ of F above v and write $\widetilde{S}_p = \{\widetilde{v} \mid v \in S_p\}$. We write I_p for the set of embeddings $F^+ \hookrightarrow \overline{\mathbb{Q}}_p$, and $\widetilde{I}_p$ for the set of embeddings $F \hookrightarrow \overline{\mathbb{Q}}_p$ inducing a place of $\widetilde{S}_p$.
- Let $\mathbb{Z}_n^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \cdots \geq \lambda_n\}$. For each $\boldsymbol{\lambda} = (\lambda_\tau)_\tau \in (\mathbb{Z}_n^+)^{\widetilde{I}_p}$, there is a finite free $\mathcal{O}$-module $M_{\boldsymbol{\lambda}}$, with a continuous action of the group $\prod_{v \in S_p} G(\mathcal{O}_{F_v^+}) \cong \prod_{w \in \widetilde{S}_p} \mathrm{GL}_n(\mathcal{O}_{F_w})$. It is constructed from the tensor product (over $\tau \in \widetilde{I}_p$) of the algebraic representations of GL_n of highest weights λ_τ.
- If $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$ is an open compact subgroup such that $U_v \subset G(\mathcal{O}_{F_v^+})$ for each $v \in S_p$, and $\boldsymbol{\lambda} \in (\mathbb{Z}_n^+)$, and A is a $\mathcal{O}$-module, then we write $S_{\boldsymbol{\lambda}}(U, A)$ for the set of all functions
$$f: G(F^+)\backslash G(\mathbb{A}_{F^+}^\infty) \to M_{\boldsymbol{\lambda}} \otimes_{\mathcal{O}} A$$
satisfying the following condition: for all $u \in U$, $g \in G(\mathbb{A}_{F^+}^\infty)$, we have $f(gu) = u_p^{-1} f(g)$, where u_p denotes projection to the p-component.
- If v is a finite place of F^+ and $U_v \subset G(F_v^+)$ is an open compact subgroup, then we write $\mathcal{H}(G(F_v^+), U_v)$ for the convolution algebra of U_v-biinvariant functions $f: G(F_v^+) \to \mathbb{Z}$ (with the Haar measure giving U_v total measure 1). If $v = ww^c$ is split in F and $U_v = \iota_w^{-1}(\mathrm{GL}_n(\mathcal{O}_{F_w}))$, then we write T_w^i, $i = 1, \dots, n$ for the standard unramified Hecke operators given as the characteristic functions of the double cosets
$$U_v \iota_w^{-1}\left(\mathrm{diag}(\underbrace{\varpi_w, \dots, \varpi_w}_{i}, \underbrace{1, \dots, 1}_{n-i})\right) U_v.$$
If $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$ is an open compact subgroup, and A is an $\mathcal{O}$-module, then for each finite place v of F^+, the algebra $\mathcal{H}(G(F_v^+), U_v)$ acts on $S_{\boldsymbol{\lambda}}(U, A)$ in a natural way.
- If $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$ is an open compact subgroup and T is a finite set of finite places of F^+ such that for each $v \notin T$, U_v is a hyperspecial maximal compact subgroup and v is prime to p, then we define $\mathbb{T}_{\boldsymbol{\lambda}}^T(U, \mathcal{O})$ to be the $\mathcal{O}$-subalgebra of $\mathrm{End}_{\mathcal{O}}(S_{\boldsymbol{\lambda}}(U, \mathcal{O}))$ generated by the operators T_w^i, $i = 1, \dots, n$, as w ranges through the set of all finite places of F which are split over F^+ and prime to T.
- We define
$$\mathcal{A}_{\boldsymbol{\lambda}} = \varinjlim_{U} S_{\boldsymbol{\lambda}}(U, \overline{\mathbb{Q}}_p),$$
the direct limit running through all open compact subgroups $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$. Then $\mathcal{A}_{\boldsymbol{\lambda}}$ is an semi-simple admissible $\overline{\mathbb{Q}}_p[G(\mathbb{A}_{F^+}^\infty)]$-module. If $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$ is a fixed choice of isomorphism, then for each irreducible submodule $\pi \subset \mathcal{A}$, $\pi \otimes_{\overline{\mathbb{Q}}_p, \iota} \mathbb{C}$ is the finite part of an automorphic representation of $G(\mathbb{A}_{F^+})$, with an infinite part that can be described explicitly in terms of $\boldsymbol{\lambda}$.

Theorem 4.1 *Let $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$ be an open compact subgroup, and let T be a finite set of finite places of F^+ such that $\mathbb{T}_{\boldsymbol{\lambda}}^T(U, \mathcal{O})$ is defined.*

(i) *Let $\pi \subset \mathcal{A}_{\boldsymbol{\lambda}}$ be an irreducible submodule such that $\pi^U \neq 0$. Then there exists a continuous semi-simple representation $\rho_p(\pi): G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, satisfying the following conditions:*

  (a) $\rho_p(\pi)^c \cong \rho_p(\pi)^\vee \epsilon^{1-n}$.

(b) *If v is a finite place of F^+ such that U_v is a hyperspecial maximal compact subgroup, and w is a place of F dividing v, then $\rho_p(\pi)|_{G_{F_w}}$ is unramified.*
(c) *If $v \in S_p$, then $\rho_p(\pi)|_{G_{F_{\widetilde{v}}}}$ is de Rham, and for each $\tau \in \widetilde{I}_p$, we have*

$$\mathrm{HT}_\tau(r_p(\pi)) = \{\lambda_{\tau,1} + (n-1), \lambda_{\tau,2} + (n-2), \ldots, \lambda_{\tau,n}\}.$$

(d) *If v is a finite place of F^+ split in F as $v = ww^c$, then there is an isomorphism $\mathrm{WD}(\rho_p(\pi)|_{G_{F_w}})^{F\text{-}ss} \cong \mathrm{rec}^T_{F_w}(\pi_w \circ \iota_w^{-1})$. In particular, if $v \in S_p$ and π_v is unramified, then $\rho_p(\pi)|_{G_{F_{\widetilde{v}}}}$ is crystalline.*
(e) *If $\rho_p(\pi)$ is irreducible, then the pair $(\rho_p(\pi), \epsilon^{1-n}\delta^n_{F/F^+})$ is polarized.*

(ii) *Let $\mathfrak{m} \subset \mathbb{T}^T_\lambda(U, \mathcal{O})$ be a maximal ideal with residue field k. Then there exists a continuous semi-simple representation $\overline{\rho}_\mathfrak{m} \colon G_F \to \mathrm{GL}_n(\mathbb{T}_\lambda(U, \mathcal{O})/\mathfrak{m})$ satisfying the following conditions:*

(a) $\overline{\rho}^c_\mathfrak{m} \cong \overline{\rho}^\vee_\mathfrak{m}\epsilon^{1-n}$.
(b) *If $v \notin T$ is a finite place of F^+ and w is a place of F above v, then $\overline{\rho}_\mathfrak{m}|_{G_{F_w}}$ is unramified. If moreover w is split over F^+, then the characteristic polynomial of $\overline{\rho}_\mathfrak{m}(\mathrm{Frob}_w)$ is equal to*

$$X^n + \cdots + (-1)^j q_w^{j(j-1)/2} T^j_w X^{n-j} + \cdots + (-1)^n q_w^{n(n-1)/2} T^n_w \in (\mathbb{T}_\lambda(U, \mathcal{O})/\mathfrak{m})[X].$$

(c) *If $\overline{\rho}_\mathfrak{m}$ is irreducible, then the pair $(\overline{\rho}_\mathfrak{m}, \epsilon^{1-n}\delta^n_{F/F^+})$ is polarized.*

(iii) *Let $\mathfrak{m} \subset \mathbb{T}^T_\lambda(U, \mathcal{O})$ be a maximal ideal with residue field k, and suppose that $\overline{\rho}_\mathfrak{m}$ is irreducible. Let $\overline{r}_\mathfrak{m} \colon G_{F^+} \to \mathcal{G}_n(\mathbb{T}_\lambda(U, \mathcal{O})/\mathfrak{m})$ be a choice of extension of $\overline{\rho}_\mathfrak{m}$ (which exists, by Lemma 2.2, and satisfies $\nu \circ \overline{r}_\mathfrak{m} = \epsilon^{1-n}\delta^n_{F/F^+}$, by the second part of the theorem). Then there exists a lifting $r_\mathfrak{m} \colon G_{F^+} \to \mathcal{G}_n(\mathbb{T}_\lambda(U, \mathcal{O})_\mathfrak{m})$ of $\overline{r}_\mathfrak{m}$ satisfying the following conditions:*

(a) $\nu \circ r_\mathfrak{m} = \epsilon^{1-n}\delta^n_{F/F^+}$.
(b) *If $v \notin T$ is a finite place of F^+ and w is a place of F above v, then $r_\mathfrak{m}|_{G_{F_w}}$ is unramified. If moreover w is split over F^+, then the characteristic polynomial of $r_\mathfrak{m}|_{G_F}(\mathrm{Frob}_w)$ is equal to*

$$X^n + \cdots + (-1)^j q_w^{j(j-1)/2} T^j_w X^{n-j} + \cdots + (-1)^n q_w^{n(n-1)/2} T^n_w \in \mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}[X].$$

Moreover, $r_\mathfrak{m}$ is uniquely determined by these conditions, up to strict equivalence.

Proof The first part follows by the argument in the proof of [14, Theorem 2.3], together with local-global compatibility for RACSDC automorphic representations of $\mathrm{GL}_n(\mathbb{A}_F)$, in its strong form (see [6,7]), and Theorem 3.2. The second part follows from the first part by reduction modulo p.

To prove the third part, we use the Galois representations constructed in the first. The algebra $\mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}$ is $\mathcal{O}$-flat and reduced, of dimension 1. For each minimal prime $\mathfrak{p} \subset \mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}$, we can find an $\mathcal{O}$-embedding $\mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}/\mathfrak{p} \hookrightarrow \overline{\mathbb{Q}}_p$ and an irreducible subrepresentation $\pi \subset \mathcal{A}_\lambda$ such that $\pi^U \neq 0$ and for each unramified Hecke operator $T^i_w \in \mathbb{T}^T_\lambda(U, \mathcal{O})$, the image of T^i_w in $\mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}/\mathfrak{p} \subset \overline{\mathbb{Q}}_p$ equals the eigenvalue of T^i_w on π^U.

In particular, we can find (by combining the first part of the lemma, Lemmas 2.2 and 2.3) a finite extension $K_\mathfrak{p}$ of $\mathbb{T}^T_\lambda(U, \mathcal{O})_\mathfrak{m}/\mathfrak{p}[1/p]$ inside $\overline{\mathbb{Q}}_p$ with ring of integers $\mathcal{O}_{K_\mathfrak{p}}$, together with a representation $r_\mathfrak{p} \colon G_{F^+} \to \mathcal{G}_n(\mathcal{O}_{K_\mathfrak{p}})$ satisfying the following conditions:

- $r_{\mathfrak{p}}|_{G_F} \otimes_{\mathcal{O}_{K_{\mathfrak{p}}}} \overline{\mathbb{Q}}_p \cong r_p(\pi)$.
- If $v \notin T$ is a finite place of F^+ and w is a place of F above v, then $r_{\mathfrak{p}}|_{G_{F_w}}$ is unramified. If moreover w is split over F^+, then the characteristic polynomial of $r_{\mathfrak{p}}|_{G_F}(\mathrm{Frob}_w)$ is equal to
$$X^n + \cdots + (-1)^j q_w^{j(j-1)/2} T_w^j X^{n-j} + \cdots + (-1)^n q_w^{n(n-1)/2} T_w^n \in (\mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}}/\mathfrak{p})[X].$$
- We have $r_{\mathfrak{p}} \bmod \mathfrak{m}_{\mathcal{O}_{K_{\mathfrak{p}}}} = \overline{r}_{\mathfrak{m}}$ (i.e. equality, not just $\mathcal{G}_n^0(k)$-conjugacy).

Let $\mathcal{O}_{K_{\mathfrak{p}}}^0 \subset \mathcal{O}_{K_{\mathfrak{p}}}$ denote the subring consisting of elements whose image in the residue field $\mathcal{O}_{K_{\mathfrak{p}}}/\mathfrak{m}_{\mathcal{O}_{K_{\mathfrak{p}}}}$ in fact lies in k. Then $r_{\mathfrak{p}}$ is valued in $\mathcal{G}_n(\mathcal{O}_{K_{\mathfrak{p}}}^0)$, and we can form the lifting $r_{\mathfrak{m}}^0 = \times_{\mathfrak{p}} r_{\mathfrak{p}}$, valued in the ring

$$A = \times_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^0 = \left\{ (x_{\mathfrak{p}})_{\mathfrak{p}} \in \prod_{\mathfrak{p}} \mathcal{O}_{K_{\mathfrak{p}}}^0 \mid x_{\mathfrak{p}} \bmod \mathfrak{m}_{\mathcal{O}_{K_{\mathfrak{p}}}} \text{ independent of } \mathfrak{p} \right\}.$$

(thus the pullback is taken relative to k). There is a natural embedding $\mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}} \hookrightarrow A$, and for each $\delta \in G_F$, we have $\mathrm{tr}\, r_{\mathfrak{m}}^0|_{G_F}(\delta) \in \mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}}$. Indeed, by the Chebotarev density theorem it suffices to check that $\mathrm{tr}\, r_{\mathfrak{m}}^0|_{G_F}(\mathrm{Frob}_w)$ lies in $\mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}}$, when w is a finite place of F split over F^+ and not dividing T; but this follows from our knowledge of the characteristic polynomial of $r_{\mathfrak{m}}^0|_{G_F}(\mathrm{Frob}_w)$.

By Lemma 2.4, therefore, we can find a $\mathcal{G}_n^0(A)$-conjugate $r_{\mathfrak{m}}$ of $r_{\mathfrak{m}}^0$ which is valued in $\mathcal{G}_n(\mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}})$. Then $r_{\mathfrak{m}}$ is a lifting of $\overline{r}_{\mathfrak{m}}$ which clearly has the desired properties. By Lemma 2.5, $r_{\mathfrak{m}}$ is uniquely determined by these properties, up to strict equivalence. This completes the proof. □

We now specialize to our situation of interest. Fix a finite set T of finite places of F^+, split in F, and containing S_p, and let $U = \prod_v U_v \subset G(\mathbb{A}_{F^+}^\infty)$ be an open compact subgroup satisfying the following conditions:

- If $v \in S_p$, then $U_v = \iota_{\widetilde{v}}^{-1}\, \mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}})$.
- If $v \notin T$ is a finite place of U_v, then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$.
- For all $g \in G(\mathbb{A}_{F^+}^\infty)$, the (a priori finite) group $U \cap gG(F^+)g^{-1}$ is trivial.

Fix a choice of $\lambda \in (\mathbb{Z}_+^n)^{\widetilde{I}_p}$, and let $\pi \subset \mathcal{A}_\lambda$ be an irreducible submodule such that $\pi^U \neq 0$ and $\overline{\rho_p(\pi)}$ is irreducible. Let $\mathfrak{m} \subset \mathbb{T}_\lambda^T(U, \mathcal{O})$ be the maximal ideal associated to the Hecke eigenvalues of π. Then there is an isomorphism $\overline{r_p(\pi)} \cong \overline{\rho}_{\mathfrak{m}}$. We fix a choice of extension $\overline{r}_{\mathfrak{m}}$ and lifting $r_{\mathfrak{m}}$, as in the statement of Theorem 4.1. Let $S = T \cup S_\infty$, and consider the global deformation problem

$$\mathcal{S} = \left(F, \overline{r}_{\mathfrak{m}}, \mathcal{O}, \epsilon^{1-n}\delta_{F/F^+}^n, S, \{\mathcal{D}_v^{\lambda_v, \mathrm{cr}}\}_{v \in S_p} \cup \{\mathcal{D}_v^{\mathrm{fl}}\}_{v \in T - S_p} \right).$$

(the local deformation problems have been defined in Sect. 2.3). It follows immediately from the construction of Theorem 4.1 that the lifting $r_{\mathfrak{m}}$ is of type $\mathcal{S}$, so is classified by a surjective homomorphism $R_{\mathcal{S}} \to \mathbb{T}_\lambda^T(U, \mathcal{O})_{\mathfrak{m}}$.

Theorem 4.2 *Let $f \colon R_{\mathcal{S}} \to \mathcal{O}$ be a homomorphism corresponding to a lifting $r \colon G_{F^+} \to \mathcal{G}_n(\mathcal{O})$ of $\overline{r}_{\mathfrak{m}}$. Suppose that:*

(i) *The subgroup $\overline{\rho}_{\mathfrak{m}}(G_F) \subset \mathrm{GL}_n(k)$ is adequate, in the sense of Definition 2.20.*
(ii) *For each $v \in T$, we have $\rho|_{G_{F_v^+}} \sim r_p(\pi)|_{G_{F_v^+}}$.*

(iii) *If p is odd, then $F = F^+(\zeta_p)$. If $p = 2$, then $F = F^+(\sqrt{-1})$.*
(iv) *If $p = 2$ and n is even, then there exists a place $v|\infty$ such that $(\overline{\rho}_{\mathfrak{m}}, \epsilon^{1-n}\delta^n_{F/F^+})$ is strongly residually odd, in the sense of Definition 3.3.*

Then there exists an irreducible submodule $\sigma \subset \mathcal{A}_\lambda$ such that $\sigma^U \neq 0$ and $\rho_p(\sigma) \cong r|_{G_F}$.

Proof The proof is essentially the same as the proof of [32, Theorem 6.8], with references to Proposition 4.4 of *op. cit.* replaced with references to Proposition 2.21 of this article. We therefore feel free to only sketch parts of the proof that are substantially the same as the proof of [32, Theorem 6.8]. After possibly enlarging $\mathcal{O}$, we can find a homomorphism $f'\colon \mathbb{T}^T_\lambda(U, \mathcal{O})_{\mathfrak{m}} \to \mathcal{O}$ such that $(f' \circ r_{\mathfrak{m}})|_{G_F} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \rho_p(\pi)$.

Let $q = h^1_{\mathcal{S}^\perp,T} - 1$ and $g = q + |T| - 1 - [F^+\colon\mathbb{Q}]n(n-1)/2$. By Proposition 2.21, we can find for each $N \geq 1$ a Taylor–Wiles datum $(Q_N, (\alpha_v)_{v\in Q})$ of level N such that $\#Q_N = q$ and the map $R^{\mathrm{loc}}_{\mathcal{S},T} \to R^T_{\mathcal{S}_{Q_N}}$ can be extended to a surjection $R^{\mathrm{loc}}_{\mathcal{S},T}[\![X_1, \ldots, X_g]\!] \twoheadrightarrow R^T_{\mathcal{S}_{Q_N}}$ (we write here $\mathcal{S}_{Q_N}$ for the auxiliary deformation problem constructed in Sect. 2.4. We remark that in applying Proposition 2.21, we are using hypotheses (i), and (iii) and (iv) of the theorem). We define $\Delta_{Q_N} = \prod_{v\in Q_N} k(v)^\times(p)$, as in Sect. 2.4. Then $R_{\mathcal{S}_{Q_N}}$ is an $\mathcal{O}[\Delta_{Q_N}]$-algebra, and there is a canonical isomorphism $R_{\mathcal{S}_{Q_N}} \otimes_{\mathcal{O}[\Delta_{Q_N}]} \mathcal{O} \cong R_{\mathcal{S}}$.

Let $H = S_\lambda(U, \mathcal{O})_{\mathfrak{m}}$. Then H is a faithful $\mathbb{T}_\lambda(U, \mathcal{O})_{\mathfrak{m}}$-module, and becomes an $R_{\mathcal{S}}$-module via the surjective homomorphism $R_{\mathcal{S}} \to \mathbb{T}_\lambda(U, \mathcal{O})_{\mathfrak{m}}$ constructed above. Arguing exactly as in [32, § 5] and the proof of [32, Theorem 6.8], we can construct for each $N \geq 1$ an $R_{\mathcal{S}_{Q_N}}$-module H_{Q_N}, free over $\mathcal{O}[\Delta_{Q_N}]$, together with an isomorphism $H_{Q_N} \otimes_{\mathcal{O}[\Delta_{Q_N}]} \mathcal{O} \cong H$ of $R_{\mathcal{S}_{Q_N}}$-modules (we remark that it is assumed at some point in [32, § 5] that the residue characteristic is odd; however, this assumption plays no role in the proof).

Let $\mathcal{T} = \mathcal{O}[\![X_1, \ldots, X_{(n^2+1)|T|-1}]\!]$. Choose a representative $r_{\mathcal{S}}\colon G_{F^+} \to \mathcal{G}_n(R_{\mathcal{S}})$ of the universal deformation, and for each $N \geq 1$ a representative $r_{\mathcal{S}_{Q_N}}\colon G_{F^+} \to \mathcal{G}_n(R_{\mathcal{S}_{Q_N}})$ lifting $R_{\mathcal{S}}$. These choices determine canonical isomorphisms $R^T_{\mathcal{S}} \cong R_{\mathcal{S}} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$ and $R^T_{\mathcal{S}_{Q_N}} \cong R_{\mathcal{S}_{Q_N}} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$, and give $H^T = H \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$ and $H^T_{Q_N} = H_{Q_N} \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$ the structure of $R^T_{\mathcal{S}}$ and $R^T_{\mathcal{S}_{Q_N}}$-module, respectively.

We choose for each $N \geq 1$ a surjection $R^{\mathrm{loc}}_{\mathcal{S},T}[\![X_1, \ldots, X_g]\!] \twoheadrightarrow R^T_{\mathcal{S}_{Q_N}}$. We let $R_\infty = R^{\mathrm{loc}}_{\mathcal{S},T}[\![X_1, \ldots, X_g]\!]$ and $S_\infty = \mathcal{O}[\![\mathbb{Z}_p^q]\!] \widehat{\otimes}_{\mathcal{O}} \mathcal{T}$, and choose for each $N \geq 1$ a surjection $\mathbb{Z}_p^q \to \Delta_{Q_N}$ (which gives rise to a surjection $S_\infty \to \mathcal{T}[\Delta_{Q_N}]$).

After patching exactly as in the proof of [32, Theorem 6.8], we obtain the following objects:

- A ring homomorphism $S_\infty \to R_\infty$, together with an R_∞-module H_∞, free over S_∞.
- A surjection $R_\infty \to R_{\mathcal{S}}$ of S_∞-algebras, together with an isomorphism $H_\infty \otimes_{S_\infty} \mathcal{O} \cong H$ of R_∞-modules.

In particular, we have

$$\begin{aligned}\operatorname{depth}_{R_\infty} H_\infty &\geq \operatorname{depth}_{S_\infty} H_\infty = \dim S_\infty = 1 + q + (n^2+1)|T| - 1\\ &= q + (n^2+1)|T| = \dim R_\infty.\end{aligned}$$

It follows that $\operatorname{depth}_{R_\infty} H_\infty = \dim R_\infty$, and $\operatorname{Supp}_{R_\infty} H_\infty \subset \operatorname{Spec} R_\infty$ is a union of irreducible components, by [31, Lemma 2.3]. For each $v \in T$, let $\mathcal{C}_v$ be the unique irreducible component of $\operatorname{Spec} R_v^{\lambda_v,\mathrm{cr}}$ or $\operatorname{Spec} R_v^{\mathrm{fl}}$ containing the point corresponding to $r_p(\pi)$ (if $v \in S_p$, it is unique because $R_v^{\lambda_v,\mathrm{cr}}[1/p]$ is formally smooth over E. If $v \in T - S_p$, it is unique because π_v is generic, cf. Sect. 2.3.4). Let $R_v^{\mathcal{C}_v}$ denote the affine ring of $\mathcal{C}_v$, endowed with its reduced subscheme structure, and let $R'_\infty = (\widehat{\otimes}_{v \in T} R_v^{\mathcal{C}_v})[\![X_1, \dots, X_g]\!]$. Then $\operatorname{Spec} R'_\infty \subset \operatorname{Spec} R_\infty$ is an irreducible component, which lies in the support of H_∞ as R_∞-module; indeed, the closed point x' of $\operatorname{Spec} R_\infty[1/p]$ corresponding to the composite

$$R_\infty \to R_{\mathcal{S}} \to \mathbb{T}_{\boldsymbol{\lambda}}^T(U, \mathcal{O})_{\mathfrak{m}} \to \mathcal{O}$$

given by f' lies in $\operatorname{Spec} R'_\infty[1/p]$, and $\operatorname{Spec} R'_\infty[1/p]$ is (by hypothesis 2 of the theorem) the unique irreducible component of $\operatorname{Spec} R_\infty[1/p]$ containing this closed point. It follows that the closed point x of $\operatorname{Spec} R_\infty[1/p]$ corresponding to f is also in the support of H_∞, and hence that

$$H_\infty \otimes_{R_\infty, f} \mathcal{O}[1/p] \cong H \otimes_{R_{\mathcal{S}}, f} \mathcal{O}[1/p] \neq 0.$$

This is only possible if f factors through the quotient $R_{\mathcal{S}} \to \mathbb{T}_{\boldsymbol{\lambda}}^T(U, \mathcal{O})_{\mathfrak{m}}$ or, equivalently, if there exists an irreducible submodule $\sigma \subset \mathcal{A}_{\boldsymbol{\lambda}}$ such that $\sigma^U \neq 0$ and $r|_{G_F} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \rho_p(\sigma)$. This completes the proof. □

We now state as a corollary a consequence of Theorem 4.2 for Galois representations valued in GL_n.

Corollary 4.3 *Let F be an imaginary CM number field with totally real subfield F^+, and let T be a finite set of finite places of F^+ containing S_p and split in F. We assume that F satisfies the following conditions:*

(i) *The extension F/F^+ is everywhere unramified, and $n[F^+ : \mathbb{Q}] \equiv 0 \bmod 4$.*
(ii) *If p is odd, then $F = F^+(\zeta_p)$. If $p = 2$, then $F = F^+(\sqrt{-1})$.*

Let $\rho \colon G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation satisfying the following conditions:

(i) *The group $\overline{\rho}(G_F)$ is adequate. In particular, $\overline{\rho}$ is absolutely irreducible.*
(ii) *We have $\rho^c \cong \rho^\vee \epsilon^{1-n}$.*
(iii) *If w is a finite place of F not dividing T, then $\rho|_{G_{F_w}}$ is unramified.*
(iv) *If w is a finite place of F dividing p, then $\rho|_{G_{F_w}}$ is crystalline.*
(v) *There exists a RACSDC automorphic representation Π of weight $\boldsymbol{\lambda}$ and an isomorphism $\iota \colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$ satisfying the following conditions:*
   (a) *There is an isomorphism $\overline{r_\iota(\Pi)} \cong \overline{\rho}$.*
   (b) *If w is a finite place of F not dividing T, then Π_w is unramified.*
   (c) *If w is a finite place of F dividing p, then Π_w is unramified.*
   (d) *If w is a finite place of F dividing T, then $r_\iota(\Pi)|_{G_{F_w}} \sim \rho|_{G_{F_w}}$.*
   (e) *If $p = 2$ and n is even, then there exists a place $v | \infty$ such that $\overline{\rho}$ is strongly residually odd at v.*

Then ρ is automorphic: there exists a RACSDC automorphic representation Σ of weight $\boldsymbol{\lambda}$ and an isomorphism $\rho \cong r_\iota(\Sigma)$.

Proof After possibly enlarging T, we can assume that it contains a place v_0, prime to p, such that both Π and ρ are unramified above v_0. We introduce a unitary group G as at the beginning of Sect. 4. By [22, Théorème 5.4], there exists an isomorphism $\iota \colon \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and an irreducible submodule $\pi \subset \mathcal{A}_{\boldsymbol{\lambda}}$ satisfying the following conditions:

- If v is an inert place of F^+, then π_v is unramified.
- If $v = ww^c$ is a split place of F^+, then there is an isomorphism $\pi_v \cong \iota^{-1}\Pi_w \circ \iota_w^{-1}$.

Then $\rho_p(\pi) \cong r_\iota(\Pi)$. By Lemmas 2.3 and 2.2, together with the fact that $\overline{\rho_p(\pi)} \cong \overline{\rho}$, we can find a coefficient field E and a homomorphism $r: G_{F^+} \to \mathcal{G}_n(\mathcal{O})$ satisfying the following conditions:

- $r|_{G_F} \otimes_{\mathcal{O}} \overline{\mathbb{Q}}_p \cong \rho$.
- $\nu \circ r = \epsilon^{1-n}\delta^n_{F/F^+}$.

Let $U = \prod_v U_v \subset G(\mathbb{A}^\infty_{F^+})$ be the open compact subgroup defined as follows:

- If $v \notin T$, then U_v is a hyperspecial maximal compact subgroup of $G(F_v^+)$.
- If $v \in S_p$, then $U_v = \iota_{\widetilde{v}}^{-1}(\mathrm{GL}_n(\mathcal{O}_{F_{\widetilde{v}}}))$.
- If $v \in T - S_p$, then U_v is any torsion-free subgroup of $G(F_v^+)$ such that $\pi_v^{U_v} \neq 0$.

Then $\pi^U \neq 0$, and for all $g \in G(\mathbb{A}^\infty_{F^+})$, the group $U \cap gG(F^+)g^{-1}$ is trivial. We can then apply Theorem 4.2 to r, to deduce the existence of an irreducible subrepresentation $\sigma \subset \mathcal{A}_\lambda$ such that $\rho \cong r_p(\sigma)$. We then deduce from [22, Corollaire 5.3] the existence of a RACDSC automorphic representation Σ of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\Sigma)$. This completes the proof. □

5 An automorphy lifting theorem

Theorem 5.1 *Let $n \geq 2$. Let F be an imaginary CM number field with totally real subfield F^+. Fix a prime p and an isomorphism $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$, and consider a continuous representation*

$$\rho: G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p).$$

Suppose that ρ satisfies the following conditions:

(i) *There is an isomorphism $\rho^c \cong \rho^\vee \epsilon^{1-n}$.*
(ii) *The group $\overline{\rho}(G_{F(\zeta_p)}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_p)$ is adequate, in the sense of Definition 2.20.*
(iii) *The representation ρ is almost everywhere unramified.*
(iv) *There exists a RACSDC automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that:*

   (a) *There is an isomorphism $\overline{r_\iota(\pi)} \cong \overline{\rho}$.*
   (b) *For each finite place v of F, we have $r_\iota(\pi)|_{G_{F_v}} \sim \rho|_{G_{F_v}}$ (this condition is automatic if π_v and $\rho|_{G_{F_v}}$ are both unramified). In particular, if $v|p$, then $\rho|_{G_{F_v}}$ and $r_\iota(\pi)|_{G_{F_v}}$ are potentially crystalline.*

(v) *If $p = 2$ and n is even, then there exists a place $v|\infty$ of F^+ at which the pair $(\overline{\rho}, \epsilon^{1-n}\delta^n_{F/F^+})$ is strongly residually odd, in the sense of Definition 3.3.*

Then ρ is automorphic: there exists a RACSDC automorphic representation Π of $\mathrm{GL}_n(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\Pi)$.

Remark Before giving the proof of Theorem 5.1, we compare it with [32, Theorem 7.1], in the case that the character μ in *loc. cit.* is taken to be trivial. Then the hypotheses and conclusions of the two theorems are essentially the same, except for the following:

(i) Here we do not assume that the residue characteristic p is odd (the main point of this paper).
(ii) We do not assume that $\zeta_p \notin F$.
(iii) The definition of 'adequate subgroup' used here (i.e. Definition 2.20) is more general than that of [32, Definition 2.3] in the case that p is odd and $p|n$.

Proof of Theorem 5.1 We reduce the theorem to Corollary 4.3, using the technique of soluble base change (i.e. using [5, Lemma 2.2.2]). First, we can replace F by $F(\zeta_p)$ (if p is odd) or $F(\sqrt{-1})$ (if $p = 2$). If $p = 2$ and n is even, then it is necessary to check that this preserves the condition that $(\overline{\rho}, \epsilon^{1-n}\delta^n_{F/F^+})$ is strongly residually odd at some infinite place, but this follows easily from the definitions.

It now suffices to observe that we can find a soluble extension L/F satisfying the following conditions:

- $n[L^+ : \mathbb{Q}] \equiv 0 \bmod 4$ and L/L^+ is everywhere unramified.
- $\overline{\rho}(G_L) = \overline{\rho}(G_F)$.
- If v is a place of L dividing p, then $\rho|_{G_{F_v}}$ is crystalline and v is split over L^+.
- If v is a finite place of L not dividing p, then $\rho|_{G_{F_v}}$ is unipotently ramified and π_v has an Iwahori-fixed vector. If either of $\rho|_{G_{F_v}}$ or π_v is ramified, then v is split over L^+.

We can now apply Corollary 4.3 to deduce that $\rho|_{G_L}$ is automorphic, and hence that ρ is automorphic. This concludes the proof. □

6 Application: the case of GL_2

In this section, we apply Theorem 5.1 to improve a modularity lifting theorem of Kisin [18] for 2-adic potentially Barsotti-Tate representations. More precisely, we remove the local conditions at p but add a local condition at ∞. The theorem we prove is the following:

Theorem 6.1 *Let F be a totally real number field, and let $\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ be a continuous representation satisfying the following conditions:*

(i) *ρ is almost everywhere unramified.*
(ii) *For each place $v|2$ of F, $\rho|_{G_{F_v}}$ is potentially crystalline. For each embedding $\tau : F_v \hookrightarrow \overline{\mathbb{Q}}_p$, $\mathrm{HT}_\tau(\rho) = \{0, 1\}$.*
(iii) *$\overline{\rho}$ is absolutely irreducible, and has non-soluble image. There exists a place $v|\infty$ of F such that $\overline{\rho}(c_v)$ is non-trivial.*
(iv) *There exists a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_F)$ and an isomorphism $\iota : \overline{\mathbb{Q}}_2 \cong \mathbb{C}$ such that $\overline{r_\iota(\pi)} \cong \overline{\rho}$.*

Then ρ is automorphic: there exists a RAESDC automorphic representation (σ, ψ) of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $\rho \cong r_\iota(\sigma)$.

(We remind the reader that the condition that $\overline{\rho}$ has non-soluble image is equivalent, by the classification of finite subgroups of $\mathrm{PGL}_2(\overline{\mathbb{F}}_2)$, to asking that the projective image of $\overline{\rho}(G_F)$ is a conjugate of $\mathrm{PSL}_2(\mathbb{F}_{2^a})$ for some $a > 1$. In particular, the projective image is a simple group.) By [18, Proposition 3.2.9], to prove Theorem 6.1, it is enough to show the following result:

Theorem 6.2 *Let F be a totally real number field, and let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$. Fix an isomorphism $\iota : \overline{\mathbb{Q}}_2 \cong \mathbb{C}$. Suppose that $\overline{r_\iota(\pi)}$ is absolutely irreducible, with non-soluble image, and that there exists a place $v|\infty$ of F such that $\overline{r_\iota(\pi)}(c_v)$ is non-trivial. Then there exists a soluble totally real extension F'/F and a RAESDC automorphic representation (σ, ψ) of $\mathrm{GL}_2(\mathbb{A}_{F'})$ satisfying the following conditions:*

(i) *σ has weight 0 and $\overline{r_\iota(\sigma)} \cong \overline{r_\iota(\pi)}|_{G_{F'}}$.*
(ii) *For each place $v|2$ of F', σ_v is unramified and ι-ordinary.*

The idea of proving a result like Theorem 6.2, and applying it to improve Kisin's results is due to Barnet-Lamb, Gee, and Geraghty, and has been carried out by them in the case that the residue characteristic p is odd [3,4]. Here we will use our main theorem to generalize their technique to the case $p = 2$.

The main idea, as in [3, Introduction], is to construct lifts with prescribed local properties by using tensor products. However, we cannot use automorphic induction from GL_2 to GL_4, as the induction of a representation from an even degree extension can never be adequate in characteristic 2. Instead, we use the $\mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GL}_4$ tensor product functoriality, and check that adequacy is preserved in our case of interest.

We begin with some preliminary results about tensor products.

Lemma 6.3 *Let F be a number field, and let k be a finite field of characteristic 2. Let $r_1, r_2 : G_F \to \mathrm{GL}_2(k)$ be absolutely irreducible representations with non-soluble image, and suppose that the extensions $F(\mathrm{ad}\, r_1)$ and $F(\mathrm{ad}\, r_2)$ are disjoint over F. Then the group $(r_1 \otimes r_2)(G_F) \subset \mathrm{GL}_4(k)$ is adequate.*

Proof Let $H_1 = r_1(G_F)$, $H_2 = r_2(G_F)$, and $r = r_1 \otimes r_2$ and $H = r(G_F) \subset \mathrm{GL}_4(k)$. We recall that we must show the following:

- We have $H^1(H, k) = 0$ and $H^1(H, \mathrm{ad}_0\, r) = 0$.
- For each simple $k[H]$-submodule $W \subset \mathrm{ad}\, r$, there exists a semi-simple element $\sigma \in H$ with an eigenvalue $\alpha \in K$ such that $\mathrm{tr}\, e_{\sigma,\alpha} W \neq 0$.

We first note that the groups H_1, H_2 are adequate. Indeed, they have no 2-power quotients, and the group $H^1(H, \mathrm{ad}_0\, r)$ is 0, by [10, Lemma 42]. The unique simple $k[H_1]$-submodule of $\mathrm{ad}\, r_1$ is the subspace Z_1 of scalar matrices, and we have $\mathrm{tr}\, e_{\sigma,\alpha} Z_1 \neq 0$ for any element $\sigma \in H$ with two distinct eigenvalues $\alpha, \beta \in k$.

It follows immediately that the group H has no 2-power quotients, since it is itself a quotient of $H_1 \times H_2$. To show that $H^1(H, \mathrm{ad}_0\, r) = 0$, it is enough (by inflation-restriction) to show that $H^1(H_1 \times H_2, \mathrm{ad}_0\, r) = 0$. By the Künneth formula, we have

$$H^1(H_1 \times H_2, \mathrm{ad}\, r) = H^1(H_1 \times H_2, \mathrm{ad}\, r_1 \otimes_k \mathrm{ad}\, r_2) = 0,$$

so it is enough to show that the map $H^2(H_1 \times H_2, k) \to H^2(H_1 \times H_2, \mathrm{ad}\, r)$ arising from the short exact sequence

$$0 \longrightarrow k \longrightarrow \mathrm{ad}\, r \longrightarrow \mathrm{ad}_0\, r \longrightarrow 0$$

is injective. However, this map is identified under the Künneth isomorphism with a map

$$\begin{aligned} H^2(H_1 \times H_2, k) &\cong H^2(H_2, k) \oplus H^2(H_1, k) \to H^2(H_2, \mathrm{ad}\, r_2) \oplus H^2(H_1, \mathrm{ad}\, r_1) \\ &\cong H^2(H_1 \times H_2, \mathrm{ad}\, r), \end{aligned}$$

since $H^1(H_1, k) = H^1(H_2, k) = 0$. It is therefore a sum of the maps $H^2(H_i, k) \to H^2(H_i, \mathrm{ad}\, r_i)$, which are injective because the groups H_i are adequate.

The space $\mathrm{ad}\, r$ has a unique simple submodule $Z_1 \otimes_k Z_2 = Z$, the subspace of scalar matrices. To establish the second point in the definition of adequacy, it is therefore enough to show that H contains a semi-simple element σ with an eigenvalue of multiplicity 1. Since the projective images of H_1 and H_2 are assumed equal to conjugates of $\mathrm{PGL}_2(\mathbb{F}_{2^{a_i}})$ for some $a_i > 1$, it is easy to see that this can be achieved. □

Lemma 6.4 *Let F'/F be a Galois extension of totally real number fields, let (π, χ) be a RAESDC automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F'})$ $(n \geq 2)$, let p be a prime, and let*

$\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$ be an isomorphism such that $r_\iota(\pi)$ is irreducible. Let $\rho: G_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$ be a continuous representation such that $\rho|_{G_{F'}} \cong r_\iota(\pi)$. Finally, let S be a finite set of places of F, containing the infinite places, the places above p, and any place above which ρ or F' is ramified. Then the L-function

$$L^S(\iota\rho, s) = \prod_{v \notin S} \det(1 - \iota\rho(\mathrm{Frob}_v) q_v^{-s})^{-1}$$

has a meromorphic continuation to $\mathbb{C}$ and is analytic and non-vanishing at the point $s = \frac{1}{2}(1 + n - \alpha)$, where $\alpha \in \mathbb{Z}$ is the unique integer such that $\chi \| \cdot \|_{F'}^{-\alpha}$ has finite order.

Proof By Brauer's theorem, we can find subfields $F'/F_i/F$, $i = 1, \ldots, m$, with $\mathrm{Gal}(F'/F_i)$ soluble, integers n_i, and finite order characters $\psi_i: G_{F_i} \to \overline{\mathbb{Q}}_p^\times$, such that

$$\rho = \sum_i n_i \operatorname{Ind}_{G_F}^{G_{F_i}} \left(\rho|_{G_{F_i}} \otimes \psi_i \right)$$

in the Grothendieck group of continuous representations of G_F. Let S_i denote the set of places of F_i above S; we then have a similar identity of L-functions (valid a priori in a right half-plane where the relevant Euler products converge absolutely)

$$L^S(\iota\rho, s) = \prod_{i=1}^m L^S \left(\operatorname{Ind}_{G_F}^{G_{F_i}} (\iota\rho|_{G_{F_i}} \otimes \psi_i), s \right)^{n_i} = \prod_{i=1}^m L^{S_i} \left(\iota\rho|_{G_{F_i}} \otimes \psi_i, s \right)^{n_i}.$$

For each $i = 1, \ldots, m$, the representation π descends to a cuspidal automorphic representation π_i of $\mathrm{GL}_n(\mathbb{A}_{F_i})$. We can therefore rewrite the above product as

$$\begin{aligned} L^S(\iota\rho, s) &= \prod_{i=1}^m L^{S_i}(\pi_i \| \cdot \|_{F_i}^{(1-n)/2} \otimes \iota\psi_i, s)^{n_i} \\ &= \prod_{i=1}^m L^{S_i} \left(\pi_i \| \cdot \|_{F_i}^{-\alpha/2} \otimes \iota\psi_i, s + \frac{1}{2}(\alpha + 1 - n) \right)^{n_i}. \end{aligned}$$

Each $\pi_i \| \cdot \|_{F_i}^{-\alpha/2} \otimes \iota\psi_i$ is a unitary cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{F_i})$. It is known (see [16, Theorem 1.3]) that each of the L-functions $L^{S_i}(\pi_i \| \cdot \|_{F_i}^{-\alpha/2} \otimes \iota\psi_i, s)$ has a meromorphic continuation to the whole complex plane, and is holomorphic and non-vanishing at the point $s = 1$. It follows that $L^S(\iota\rho, s)$ has a meromorphic continuation to the whole complex plane, and is holomorphic and non-vanishing at the point $s = \frac{1}{2}(1 + n - \alpha)$. This completes the proof. □

Proposition 6.5 *Let F be a totally real number field, let p be a prime, and let $\rho_1, \rho_2: G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be continuous representations such that $\rho_1 \otimes \rho_2$, $\mathrm{Sym}^2 \rho_1$ and $\mathrm{Sym}^2 \rho_2$ are irreducible. Suppose moreover that each of ρ_1 and ρ_2 is unramified outside finitely many places, and de Rham with distinct Hodge–Tate weights. Fix an isomorphism $\iota: \overline{\mathbb{Q}}_p \cong \mathbb{C}$.*

(i) *Suppose there exist RAESDC automorphic representations (π_1, χ_1) and (π_2, χ_2) of $\mathrm{GL}_2(\mathbb{A}_F)$ such that $r_\iota(\pi_1) \cong \rho_1$ and $r_\iota(\pi_2) \cong \rho_2$. Suppose moreover that $\rho_1 \otimes \rho_2$ is Hodge–Tate regular (i.e. for each embedding $\tau: F \hookrightarrow \overline{\mathbb{Q}}_p$, the set $\mathrm{HT}_\tau(\rho_1 \otimes \rho_2)$ contains 4 distinct elements). Then there exists a RAESDC automorphic representation (σ, ψ) of $\mathrm{GL}_4(\mathbb{A}_F)$ such that $r_\iota(\sigma) \cong \rho_1 \otimes \rho_2$, where $\psi = \chi_1\chi_2 \| \cdot \|_F$.*

(ii) *Suppose that there exists a RAESDC automorphic representation* (σ, ψ) *of* $\mathrm{GL}_4(\mathbb{A}_F)$ *such that* $r_\iota(\sigma) \cong \rho_1 \otimes \rho_2$ *and* $r_\iota(\psi) = (\det \rho_1)(\det \rho_1)\epsilon^3$. *Suppose moreover that there exist Galois totally real extensions* F_1/F, F_2/F *and RAESDC automorphic representations* (Π_1, ξ_1) *and* (Π_2, ξ_2) *of* $\mathrm{GL}_2(\mathbb{A}_{F_1})$ *and* $\mathrm{GL}_2(\mathbb{A}_{F_2})$, *respectively, such that* $r_\iota(\Pi_1) \cong \rho_1|_{G_{F_1}}$ *and* $r_\iota(\Pi_2) \cong \rho_2|_{G_{F_2}}$. *Then there exists a RAESDC automorphic representation* (π_1, χ_1) *of* $\mathrm{GL}_2(\mathbb{A}_F)$ *such that* $r_\iota(\pi_1) \cong \rho_1$.

Proof The first part follows from [27, Theorem M] (take $\sigma = \pi_1 \boxtimes \pi_2$). For the second part, let S be a finite set of places of F containing the infinite places, the places dividing p, and any places at which F_1, F_2, ρ_1, ρ_2, or σ is ramified. Then we have an equality

$$L^S(\sigma \times \sigma^\vee, s) = L^S(\sigma \times \sigma\psi^{-1}, s) = L^S(\sigma, \mathrm{Sym}^2 \otimes \psi^{-1}, s) L^S(\sigma, \wedge^2 \otimes \psi^{-1}, s).$$

Both sides have a meromorphic continuation to the entire complex plane. The left-hand side has a simple pole at $s = 1$, because σ is cuspidal. On the other hand, we have

$$L^S(\sigma, \wedge^2 \otimes \psi^{-1}, s) = L^S(\iota[\wedge^2 r_\iota(\sigma)] \otimes r_\iota(\psi)^{-1}\epsilon^3, s), \tag{7}$$

and decomposing the representation $\wedge^2 r_\iota(\sigma)$ allows us to factorize this L-function as

$$= L^S(\iota \,\mathrm{Sym}^2 \rho_1 \otimes \det(\rho_2) r_\iota(\psi)^{-1}\epsilon^3, s) L^S(\iota\,\mathrm{Sym}^2 \rho_2 \otimes \det(\rho_1) r_\iota(\psi)^{-1}\epsilon^3, s). \tag{8}$$

It follows from our assumptions, Lemma 6.4 and the existence of the symmetric square lifting for GL_2 that the function in (8) is analytic and non-vanishing at the point $s = 1$. We deduce that the function $L^S(\sigma, \mathrm{Sym}^2 \otimes \psi^{-1}, s)$ has a simple pole at $s = 1$. It then follows from [15, Corollary 3.2.1] (see also [1, Theorem 4.26]) that there exists a cuspidal automorphic representation Σ of $\mathrm{GSpin}_4(\mathbb{A}_F)$ with weak transfer equal to σ (here we write GSpin_4 for the split general spin group in 4 variables).

Let us now write $H = \mathrm{GSpin}_4$, $G = \mathrm{GL}_2 \times \mathrm{GL}_2$, and Z_H, Z_G for the centres of these groups. There is a short exact sequence

$$0 \longrightarrow H \longrightarrow G \longrightarrow \mathbb{G}_m \longrightarrow 0,$$

the map $G \to \mathbb{G}_m$ being given by the formula $(g_1, g_2) \mapsto \det(g_1)/\det(g_2)$. This identifies Z_H with the group $\{\lambda_1, \lambda_2 \in \mathbb{G}_m \times \mathbb{G}_m \mid \lambda_1^2 = \lambda_2^2\} \subset Z_G$.

We have $G(F) \cap (H(\mathbb{A}_F)Z_G(\mathbb{A}_F)) = H(F)Z_G(F)$. The argument of [25, Proposition 3.1.4] then allows us to construct a cuspidal automorphic representation $\pi' = \pi_1' \otimes \pi_2'$ of $G(\mathbb{A}_F) = \mathrm{GL}_2(\mathbb{A}_F) \times \mathrm{GL}_2(\mathbb{A}_F)$ with the property that $\pi'|_{H(\mathbb{A}_F)}$ contains Σ. The representations π_1', π_2' are necessarily essentially square-integrable at infinity, and it follows from [25, Proposition 2.6.7] that there exist continuous homomorphisms $R_1, R_2 \colon G_F \to \mathrm{PGL}_2(\overline{\mathbb{Q}}_l)$ with the following properties:

- The representations R_1, R_2 are unramified at all finite places $v \notin S$ of F.
- For all $v \notin S$, we have $\mathrm{WD}(R_1|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{ad}^0 \mathrm{rec}^T_{F_v}(\iota^{-1}\pi'_{1,v})$ and $\mathrm{WD}(R_2|_{G_{F_v}})^{\text{F-ss}} \cong \mathrm{ad}^0 \mathrm{rec}^T_{F_v}(\iota^{-1}\pi'_{2,v})$.

Combining [25, Corollary 3.1.6] with the Chebotarev density theorem, we see that there is an isomorphism $R_1 \oplus R_2 \cong \mathrm{ad}^0 \rho_1 \oplus \mathrm{ad}^0 \rho_2$, hence (after possibly swapping R_1 and R_2) we can identify $R_1 \cong \mathrm{ad}^0 \rho_1$ and $R_2 \cong \mathrm{ad}^0 \rho_2$.

The existence of the representation ρ_1 implies that π_1' has a twist π_1'' which is regular algebraic, hence RAESDC (see [25, Proposition 2.6.7]). The representations ρ_1 and $r_\iota(\pi_1'')$ satisfy $\mathrm{ad}^0 \rho_1 \cong \mathrm{ad}^0 r_\iota(\pi_1'')$, which implies that there is a character $\eta \colon G_F \to \overline{\mathbb{Q}}_l^\times$ such that $\rho_1 \otimes \eta \cong r_\iota(\pi_1'')$. To complete the proof of the proposition, we just need to show that the character η is geometric (so that $\rho_1 = r_\iota(\pi_1'') \otimes \eta^{-1}$ is itself automorphic). However, this follows from the existence of an inclusion $\eta \subset \rho_1^\vee \otimes r_\iota(\pi_1'')$ and the fact that the property of being de Rham is stable under tensor products. □

We now construct an auxiliary automorphic Galois representation which has the right local properties at 2.

Proposition 6.6 *Let F be a totally real number field, and let K/F be a Galois extension. Then there exists a soluble totally real extension F'/F, an isomorphism $\iota: \overline{\mathbb{Q}}_2 \cong \mathbb{C}$, and a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_{F'})$ satisfying the following conditions:*

(i) *π is ι-ordinary and unramified at every finite place.*
(ii) *π has weight 0 and is of trivial central character.*
(iii) *The residual representation $\overline{r_\iota(\pi)}$ is absolutely irreducible, with projective image conjugate to $\mathrm{PGL}_2(\mathbb{F}_4)$. For each place $v|\infty$ of F', $\overline{r_\iota(\pi)}(c_v)$ is non-trivial.*
(iv) *The extension $F'(\mathrm{ad}\,\overline{r_\iota(\pi)})/F$ is linearly disjoint from K.*

Proof After possibly replacing F by a preliminary soluble extension, we can find an everywhere unramified and totally odd representation $\rho: G_F \to \mathrm{GL}_2(\mathbb{C})$ with projective image isomorphic to A_5. It then follows from [26, Théorème 0.3] that ρ is automorphic (i.e. associated to a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ of limit of discrete series type at infinity). The representation $\overline{\rho}$ is absolutely irreducible, with projective image a conjugate of $A_5 \cong \mathrm{PGL}_2(\mathbb{F}_4)$ (of course, $\overline{\rho}$ makes sense because of our fixed identification $\iota: \overline{\mathbb{Q}}_2 \cong \mathbb{C}$). By Hida theory, in the guise of [33, § 1], we can find an ι-ordinary RAESDC automorphic representation (σ, ψ) of weight 0 such that $\overline{r_\iota(\pi)} \cong \overline{\rho}$. Finally, passing to a further extension F'/F and arguing as in [12, Lemma 5.1.7], we can replace the base change of π to F' by another RAESDC automorphic representation which satisfies all the requirements of the proposition. □

In the next proposition, we use potential automorphy techniques as in [30] to construct automorphic ordinary lifts of our residual Galois representation of interest.

Proposition 6.7 *Let k be a finite field of characteristic 2, let F be a totally real field, and let $\overline{\rho}: G_F \to \mathrm{GL}_2(k)$ be an absolutely irreducible representation with non-soluble image. Let K/F be a Galois extension. Then we can find a Galois totally real extension F'/F, an isomorphism $\iota: \overline{\mathbb{Q}}_2 \cong \mathbb{C}$, and a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_{F'})$ satisfying the following conditions:*

(i) *π is ι-ordinary and unramified at every finite place.*
(ii) *π has weight 0 and is of trivial central character.*
(iii) *There is an isomorphism $\overline{r_\iota(\pi)} \cong \overline{\rho}|_{G_{F'}}$.*
(iv) *The extension F'/F is linearly disjoint from K.*

Proof Let $N/\mathbb{Q}$ be a totally real number field of degree $[k: \mathbb{Q}]$ in which 2 is totally inert. Let x be the unique place of N above 2, and fix an identification $k(x) \cong k$. We recall (following [30, § 1]) that a N-HBAV over a field K is a triple (A, i, j), where:

- A is an abelian variety over K of dimension $[N: \mathbb{Q}]$.
- ι is an embedding $\iota: \mathcal{O}_N \hookrightarrow \mathrm{End}_K(A)$.
- j is an isomorphism $\mathcal{O}_N^+ \cong \mathcal{P}(A, i)$ of ordered invertible $\mathcal{O}_N$-modules.

(an ordered invertible $\mathcal{O}_N$-module is an invertible $\mathcal{O}_N$-module M with choice of connected component of $M \otimes_{\mathbb{Q}} \mathbb{R}$; $\mathcal{P}(A, i)$ is the module of $\mathcal{O}_N$-linear polarizations of A; $\mathcal{O}_N^+$ is the standard invertible $\mathcal{O}_N$-module given by the identity component of $\mathcal{O}_N \otimes_{\mathbb{Q}} \mathbb{R}$).

Let $p > 5$ be a prime split in N and unramified in F, and let E/F be an elliptic curve such that the mod p Galois representation $\overline{\rho}_{E,p}$ of E has image $\mathrm{GL}_2(\mathbb{F}_p)$, the mod 3 Galois representation $\overline{\rho}_{E,3}$ has image $\mathrm{GL}_2(\mathbb{F}_3)$, and E has ordinary reduction at each place $v|3$ of F. Let y be a place of N above p, so that $k(y) = \mathbb{F}_p$. Let V_x be the F-scheme in $k(x)$-vector spaces corresponding to $\overline{\rho}$, and fix a choice of non-degenerate, Galois equivariant, symplectic pairing $V_x \times V_x \to k(x)(1)$. Let V_y be the F-scheme in $k(y)$-vector spaces corresponding to $E[p]$, and fix a choice of pairing $V_y \times V_y \to k(y)(1)$ (for example, the Weil pairing of E).

There is a fine moduli space X/F for tuples $(A, i, j, \eta_x, \eta_y)$, where (A, i, j) is an N-HBAV, and $\eta_x\colon V_x \to A[x]$ and $\eta_y\colon V_y \to A[y]$ are isomorphisms which take our fixed pairings to the $j(1)$-Weil pairings on $A[x]$ and $A[y]$, respectively; see [28, § 1]. Using the complex uniformization, one sees that X is smooth and $X(\mathbb{C})$ is connected, hence X is geometrically connected.

We now claim the following:

- For each place $v|2$ of F, there exists a Galois extension L_v/F_v and a Galois-invariant subset $\Omega_v \subset X(L_v)$ such that for each $P \in \Omega_v$, the corresponding N-HBAV A over L_v has good ordinary reduction.
- For each place $v|\infty$ of F, $X(F_v)$ is non-empty.

We establish each point in turn. We first observe that if E'/F is any elliptic curve, then $E' \otimes_{\mathbb{Z}} \mathcal{O}_N$ has a canonical structure of N-HBAV. In particular, there exists an N-HBAV (A, i, j) over F which has good ordinary reduction at each place $v|2$ of F. Let L_v/F_v be a Galois extension such that $A[2p]$, $\overline{\rho}$, and $E[p]$ are all trivial as G_{F_v}-modules; then A defines a point P of $X(L_v)$. Having good ordinary reduction is an open condition, so any sufficiently small open neighbourhood $\Omega_v \subset X(L_v)$ of P satisfies the first point above.

For the second point, it is enough to show that there exist N-HBAV $(A, i, j)/F_v$ such that c_v acts trivially on $A[x]$ (if $\overline{\rho}(c_v)$ is trivial) or that c_v acts non-trivially on $A[x]$ (if $\overline{\rho}(c_v)$ is non-trivial). Indeed, if $v|\infty$ then there are exactly 2 isomorphism classes of representations $G_{F_v} \to \mathrm{GL}_2(k)$ and exactly 1 isomorphism class of representations $G_{F_v} \to \mathrm{GL}_2(k(y))$ with odd determinant. However, this follows from the above observation and the existence of elliptic curves E'/F with $E'[2]$ either trivial or non-trivial as G_{F_v}-module.

By the theorem of Moret–Bailly (see e.g. [5, Proposition 3.1.1]), we can find a Galois totally real extension F'/F and a point $P \in X(F')$ such that for each place $v|2$ of F and each place $w|v$ of F', we have $F'_w \cong L_v$ as F_v-algebras and $P \in \Omega_v$. In particular, writing (A, i, j) for the N-HBAV over F' corresponding to P, we see that A has good ordinary reduction at each place $w|2$ of F', $A[x] \cong \overline{\rho}|_{G_{F'}}$, and $A[y] \cong \overline{\rho}_{E,p}|_{G_{F'}}$. We can moreover choose F'/F be disjoint from K/F. By first enlarging K/F to contain the extension of F cut out by $\overline{\rho}$ and $E[3p]$, we can therefore assume that $\overline{\rho}(G_{F'}) = G_F$, $\overline{\rho}_{E,p}(G_{F'}) = \overline{\rho}_{E,p}(G_F)$, and $\overline{\rho}_{E,3}(G_{F'}) = \overline{\rho}_{E,3}(G_F)$.

It follows from the theorem of Langlands–Tunnell that $\overline{\rho}_{E,3}|_{G_{F'}}$ is modular, hence E/F' is modular by [11, Theorem 1.1], hence A/F' is modular by [5, Theorem 4.2.1], i.e. there exists a RAESDC automorphic representation (π, χ) of $\mathrm{GL}_2(\mathbb{A}_{F'})$ and an isomorphism $\iota\colon \overline{\mathbb{Q}}_2 \cong \mathbb{C}$ such that $r_\iota(\pi) \cong T_x A$. We observe that π has weight 0, is ι-ordinary and unramified at the places above 2, and has trivial central character. The proposition now follows on replacing F' by a soluble totally real extension and applying the main theorem of [29]. □

Lemma 6.8 *Let F be a totally real number field, and let $\overline{\rho}\colon G_F \to \mathrm{SL}_2(\overline{\mathbb{F}}_2)$ be a continuous, absolutely irreducible representation with insoluble image. Suppose that $\overline{\rho}$ is everywhere unramified, and that for each place $v|2$ of F, $\overline{\rho}|_{G_{F_v}}$ is trivial. Then there exists a continuous lifting $\rho\colon G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ of $\overline{\rho}$ satisfying the following conditions:*

(i) *For each finite place* $v \nmid 2$ *of* F, $\rho|_{G_{F_v}}$ *is unramified.*
(ii) *For each place* $v|2$ *of* F *and for each embedding* $\tau: F_v \hookrightarrow \overline{\mathbb{Q}}_2$, $\rho|_{G_{F_v}}$ *is crystalline ordinary and* $\mathrm{HT}_\tau(\rho) = \{0, 1\}$.
(iii) $\det \rho = \epsilon^{-1}$.
(iv) *There is a totally real Galois extension* F'/F, *a RAESDC automorphic representation* (π, χ) *of* $\mathrm{GL}_2(\mathbb{A}_{F'})$ *and an isomorphism* $\iota: \overline{\mathbb{Q}}_2 \cong \mathbb{C}$ *such that* $\rho|_{G_{F'}} \cong r_\iota(\pi)$.

Proof This follows from Proposition 6.7 and the method of Khare–Wintenberger, as we now sketch. Fix a coefficient field E with ring of integers $\mathcal{O}$ and residue field k, such that $\overline{\rho}$ takes values in $\mathrm{GL}_2(k)$. Let us say that a totally real extension F'/F is allowable if F' and $F(\overline{\rho})$ are linearly disjoint over F. If F'/F is allowable, we write $\mathcal{D}^\square_{F'}: \mathrm{CNL} \to \mathrm{Sets}$ for the functor of liftings $\rho: G_{F'} \to \mathrm{GL}_2(R)$ of $\overline{\rho}|_{G_{F'}}$ satisfying the following conditions:

- If $v \nmid 2$ is a place of F', then $\rho|_{G_{F'_v}}$ is unramified.
- If $v|2$ is a place of F', then $\rho|_{G_{F'_v}}$ is crystalline and ordinary with Hodge–Tate weights $\{0, 1\}$. More precisely, $\rho|^\vee_{G_{F'_v}}$ defines a point of the ring $R^{\mathrm{ord},1,\square}_{\overline{\rho}}$ which is defined in [18, Proposition 2.4.6]. We note that it is shown in *loc. cit.* that under our hypotheses, the ring $R^{\mathrm{ord},1,\square}_{\overline{\rho}}$ is a domain.
- $\det \rho = \epsilon^{-1}$.

We write $\mathcal{D}_{F'}$ for the corresponding functor of deformations (i.e. quotient of $\mathcal{D}^\square_{F'}$ by the free action of $\widehat{\mathrm{PGL}_2}$). We write $R_{F'} \in \mathrm{CNL}_\mathcal{O}$ for the representing object of $\mathcal{D}_{F'}$ (which exists). A standard calculation in obstruction theory (cf. [21, Proposition 4.5]) shows that $\dim R_{F'} \geq 1$ for any choice of F' as above.

By Proposition 6.7, we can find an allowable extension F'/F such that $R_{F'}$ has an automorphic $\overline{\mathbb{Q}}_2$-point. It then follows from a patching argument (cf. [21, Proposition 9.3]) that $R_{F'}$ is a finite $\mathcal{O}$-algebra, and every $\overline{\mathbb{Q}}_2$-point of $R_{F'}$ corresponds to an automorphic Galois representation. The map $R_{F'} \to R_F$ is finite (cf. [19, Lemma 3.6]), so R_F is a finite $\mathcal{O}$-algebra, so has a $\overline{\mathbb{Q}}_2$-point (as $\dim R_F \geq 1$) corresponding to a representation $\rho: G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$. We see that $\rho|_{G_{F'}}$ is automorphic. This completes the proof of the lemma. □

6.1 The proof of Theorem 6.2

We can now finish the proof of Theorem 6.2. Recall that in the situation of the theorem, F is a totally real number field and $\overline{\rho}_1: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$ is a continuous, absolutely irreducible representation with insoluble image. Moreover, there is an isomorphism $\iota: \overline{\mathbb{Q}}_2 \cong \mathbb{C}$ and a RAESDC automorphic representation (π_1, χ_1) such that $\overline{r_\iota(\pi_1)} \cong \overline{\rho}_1$.

After replacing F by a soluble totally real extension and π_1 by another automorphic representation, we can assume the following:

- The representation π_1 has weight 0 and trivial central character. For each finite place v of F, $\pi_{1,v}$ is unramified. For each place $v|2$ of F, $\pi_{1,v}$ is not ι-ordinary and $\overline{\rho}_1|_{G_{F_v}}$ is trivial. Let $\rho_1 = r_\iota(\pi_1)$.
- There exists a continuous lift $\rho'_1: G_F \to \mathrm{GL}_2(\overline{\mathbb{Q}}_2)$ of $\overline{\rho}_1$, unramified outside 2, crystalline-ordinary above 2, and with $\mathrm{HT}_\tau(\rho'_1) = \{0, 2\}$ for each embedding $\tau: F \hookrightarrow \overline{\mathbb{Q}}_2$. Moreover, ρ'_1 is potentially automorphic (use Lemma 6.8, and Hida theory to change the weight).

We can also assume given the following:

- A continuous, everywhere unramified representation $\overline{\rho}_2: G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_2)$, with projective image conjugate to $\mathrm{PGL}_2(\mathbb{F}_4)$. For each place $v|2$, $\overline{\rho}_2|_{G_{F_v}}$ is trivial and for each place $v|\infty$, $\overline{\rho}_2(c_v)$ is non-trivial. Moreover, the extensions $F(\mathrm{ad}\,\overline{\rho}_1)$ and $F(\mathrm{ad}\,\overline{\rho}_2)$ are linearly disjoint over F (use Proposition 6.6).

- A RAESDC automorphic representation (π_2, χ_2) of $\mathrm{GL}_2(\mathbb{A}_F)$ which is everywhere unramified, of trivial central character, ι-ordinary, and of weight $\boldsymbol{\lambda} = (\lambda_\tau)_{\tau \in \mathrm{Hom}(F,\mathbb{C})}$, $\lambda_\tau = (1, 0)$ for all τ. Moreover, we have $\overline{r_\iota(\pi_2)} \cong \overline{\rho}_2$. We write $\rho_2 = r_\iota(\pi_2)$ (use Proposition 6.6, and Hida theory to change the weight).
- A RAESDC automorphic representation (π_2', χ_2') of $\mathrm{GL}_2(\mathbb{A}_F)$ which is everywhere unramified, of trivial central character, and of weight 0. Moreover, we have $\overline{r_\iota(\pi_2')} \cong \overline{\rho}_2$ and for each place $v|2$ of F, $\pi_{2,v}'$ is not ι-ordinary. We write $\rho_2' = r_\iota(\pi_2')$ (the existence of π_2' is can be deduced from that of π_2, together with our freedom to make a base change).

Let $r = \rho_1 \otimes \rho_2'$, and $r' = \rho_1' \otimes \rho_2$. Then we have $\overline{r} \cong \overline{r}' \cong \overline{\rho}_1 \otimes \overline{\rho}_2$. To prove the theorem, it is enough to show that the representation r' is automorphic. Indeed, it then follows from the second part of Proposition 6.5 that ρ_1' is automorphic, and another application of Hida theory implies the existence of an ordinary weight 0 automorphic lift of $\overline{\rho}_1$.

On the other hand, we know by the first part of Proposition 6.5 that the representation r is automorphic. We will now apply Theorem 5.1 to deduce the automorphy of r' from that of r. Let E/F be a totally imaginary quadratic extension in which every place $v|2$ of F splits. Let S_2 denote the set of places of F above 2, and choose for each place $v \in S_2$ a place $\widetilde{v}$ of E above v. Let $\widetilde{S}_2 = \{\widetilde{v} \mid v \in S_2\}$. By [5, Lemma A.2.5], we can find a crystalline character $\mu \colon G_E \to \overline{\mathbb{Q}}_2^\times$ such that $\mu\mu^c = \epsilon^{-1}$ and for each embedding $\tau \colon E \hookrightarrow \overline{\mathbb{Q}}_2$, we have $\mathrm{HT}_\tau(\mu) = 0$ if τ induces a place of $\widetilde{S}_2$, and $\mathrm{HT}_\tau(\mu) = 1$ otherwise. We now claim the following:

- There exists a RACSDC automorphic representation σ of $\mathrm{GL}_4(\mathbb{A}_E)$ such that $r_\iota(\sigma) \cong r|_{G_E} \otimes \mu$. Indeed, we can take $\sigma = (\pi_1 \boxtimes \pi_2')_E \otimes (|\cdot|^{1/2}\iota^{-1}\mu \circ \det)$, where $(?)_E$ denotes quadratic base change to E.
- The group $\overline{r}(G_E) \subset \mathrm{GL}_4(\overline{\mathbb{F}}_2)$ is adequate. This is the content of Lemma 6.3.
- If v is a place of E, then $r|_{G_E} \otimes \mu \sim r'|_{G_E} \otimes \mu$ (see Sect. 3.2 for the definition of $\sim$). If $v \nmid 2$, then this is clear (as $r|_{G_{E_v}}$ and $r'|_{G_{E_v}}$ are unramified, so $r|_{G_{E_v}} \sim r'|_{G_{E_v}}$, and this relation is preserved under character twists, cf. the remarks after [5, Lemma 1.3.3]). If $v|2$, then we observe that $\rho_1|_{G_{F_v}} \sim \rho_2|_{G_{F_v}}$ (by [18, Corollary 2.3.13]) and $\rho_1'|_{G_{F_v}} \sim \rho_2'|_{G_{F_v}}$ (by [12, Lemma 3.4.3]). The remarks in [5, § 1.4] then imply that $(\rho_1 \otimes \rho_2')|_{G_{F_v}} \sim (\rho_1' \otimes \rho_2)|_{G_{F_v}}$, hence $(r \otimes \mu)|_{G_{E_{\widetilde{v}}}} \sim (r' \otimes \mu)|_{G_{E_{\widetilde{v}}}}$.
- Let $v|\infty$ be a place such that $\overline{\rho}_1(c_v)$ is non-trivial (which exists, by hypothesis). The representation $(\overline{r}, \epsilon^{-3})$ is polarized, and is strongly residually odd at v (the proof is the same as the proof of Lemma 3.5, given that $\overline{r}(c_v)$ is the tensor product of two regular unipotent matrices).

The hypotheses of Theorem 5.1 are therefore satisfied, and we deduce that the representation $r'|_{G_E} \otimes \mu$ is automorphic, hence $r' = \rho_1' \otimes \rho_2$ is automorphic, by soluble descent. As indicated above, this implies the desired result.

7 An erratum to [32]

In this section we correct a mistake in [32]. There is a gap in the reduction of [32, Theorem 7.1] to [32, Theorem 6.8]. The same gap appears in the reduction of [32, Theorem 9.1] to [32, Theorem 8.6]. The problem arises as follows. Let l be a prime, let F be an imaginary CM number field with maximal totally real subfield F^+, and let $r \colon G_{F^+} \to \mathcal{G}_n(\overline{\mathbb{F}}_l)$ be a continuous representation such that $r^{-1}(\mathcal{G}_n^0(\overline{\mathbb{F}}_l)) = G_F$, and $\overline{\rho} = \overline{r}|_{G_F}$ is absolutely irreducible. If $F \subset F^+(\zeta_l)$, then $G_{F^+(\zeta_l)} \subset G_F$, and it is not possible for the subgroup $\overline{r}(G_{F^+(\zeta_l)}) \subset \mathcal{G}_n(\overline{\mathbb{F}}_l)$ to be adequate in the sense of [32, Definition 2.3] (as this definition requires in particular that the subgroup surject onto the component group of $\mathcal{G}_n$).

The condition that $\overline{r}(G_{F^+(\zeta_l)}) \subset \mathcal{G}_n(\overline{\mathbb{F}}_l)$ be an adequate subgroup is needed in the proof of [32, Theorem 6.8] in order to be able to invoke [32, Proposition 4.4]. The hypotheses of [32, Theorem 7.1] do not rule out the possibility that $F \subset F^+(\zeta_l)$. This problem does not arise in e.g. the proof of [8, Theorem 4.4.2] since it is assumed there that the prime l is unramified in F, so in particular $F \cap \mathbb{Q}(\zeta_l)$ must be trivial.

In order to correct the mistake, we therefore state and prove a revised version of [32, Proposition 4.4] which holds under more general conditions. This proposition is very similar to Proposition 2.21. We use notation and deformation-theoretic definitions as in [32]. In particular, l is a fixed *odd* prime.

Proposition 7.1 *Let $q_0 \in \mathbb{Z}_{\geq 0}$ and suppose that $\zeta_l \notin F$. Suppose given a deformation problem*

$$\mathcal{S} = (F/F^+, S, \widetilde{S}, \mathcal{O}, \overline{r}, \chi, \{\mathcal{D}_v\}_{v \in S}).$$

Suppose that $\overline{\rho} = \overline{r}|_{G_F}$ is absolutely irreducible and that $\overline{\rho}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is adequate, in the sense of Definition 2.20. Suppose also that for $v \in S - T$ we have

$$\dim_k L_v - \dim_k H^0(G_{F_{\widetilde{v}}}, \operatorname{ad}\overline{r}) = \begin{cases} [F_v^+ : \mathbb{Q}_l]n(n-1)/2 & \text{if } v|l, \\ 0 & \text{if } v \nmid l. \end{cases}$$

Let q be the larger of $\dim_k H^1_{\mathcal{L}^\perp, T}(G_{F^+,S}, \operatorname{ad}\overline{r}(1))$ and q_0.

Then for any $N \in \mathbb{Z}_{\geq 1}$, we can find $(Q, \widetilde{Q}, \{\overline{\psi}_v\}_{v \in Q})$ as in [32, Definition 4.1]*, such that*

- *$\#Q = q \geq q_0$.*
- *if $v \in Q$ then $\mathbb{N}v \equiv 1 \bmod l^N$.*
- *$R^{\square_T}_{\mathcal{S}_Q}$ can be topologically generated over $R^{loc}_{\mathcal{S},T} = R^{loc}_{\mathcal{S}_Q,T}$ by*

$$\#Q - \sum_{v \in T, v|l} [F_v^+ : \mathbb{Q}_l]n(n-1)/2 - n \sum_{v|\infty} (1 + \chi(c_v))/2$$

elements.

Proof Fix $N \geq 1$ and let $\overline{\rho} = \overline{r}|_{G_F}$. Just as in the proof of [32, Proposition 4.4], we can reduce to showing the following claim: for any element $[\phi] \in H^1_{\mathcal{L}^\perp, T}(G_{F^+,S}, \operatorname{ad}\overline{r}(1))$, we can find an element $\sigma \in G_{F(\zeta_{l^N})}$ such that $\overline{\rho}(\sigma)$ is semi-simple, together with an eigenvalue α of $\overline{\rho}(\sigma)$ such that $\operatorname{tr} e_{\sigma,\alpha}\phi(\sigma) \neq 0$.

Let L be the extension of $F^+(\zeta_{l^N})$ cut out by $\overline{r}$; equivalently, the extension of $F(\zeta_{l^N})$ cut out by $\overline{\rho}$. We first show that

$$H^1(\operatorname{Gal}(L/F^+), \operatorname{ad}\overline{r}(1)) = 0. \tag{9}$$

There is a short exact sequence

$$\begin{aligned} 0 \longrightarrow\, & H^1(\operatorname{Gal}(F(\zeta_{l^N})/F^+), \operatorname{ad}\overline{r}(1)^{G_{F(\zeta_{l^N})}}) \\ \longrightarrow\, & H^1(\operatorname{Gal}(L/F^+), \operatorname{ad}\overline{r}(1)) \longrightarrow H^1(\operatorname{Gal}(L/F(\zeta_{l^N})), \operatorname{ad}\overline{r}(1)). \end{aligned} \tag{10}$$

Since $\overline{\rho}(G_{F(\zeta_l)})$ is adequate, it has no l-power order quotients. We find that $\overline{\rho}(G_{F(\zeta_{l^N})}) = \overline{\rho}(G_{F(\zeta_l)})$ and $\operatorname{ad}\overline{r}(1)^{G_{F(\zeta_{l^N})}} \cong k(\epsilon\delta_{F/F^+})$. The first term of (10) can therefore be identified with

$$H^1(\mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)), k)^{\epsilon\delta_{F/F^+}} = \big\{ f \in \mathrm{Hom}(\mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)), k) \mid \\ \forall x \in \mathrm{Gal}(F(\zeta_{l^N})/F(\zeta_l)),\, y \in \mathrm{Gal}(F(\zeta_{l^N})/F^+),\, f(yxy^{-1}) = \epsilon\delta_{F/F^+}(y) f(x) \big\}.$$

Since the extension $F(\zeta_{l^N})/F^+$ is abelian, this group can be non-zero if and only if the character $\epsilon\delta_{F/F^+}$ is trivial on G_{F^+}, if and only if $F = F^+(\zeta_l)$. However, we have assumed that $\zeta_l \notin F$, so we find that this group is zero. The third term of (10) is also zero, because $\mathrm{Gal}(L/F(\zeta_{l^N})) \cong \overline{\rho}(G_{F(\zeta_l)})$ and this last subgroup is assumed to be adequate.

There is another short exact sequence

$$0 \longrightarrow H^1(\mathrm{Gal}(L/F^+), \operatorname{ad}\overline{r}(1)) \longrightarrow H^1(G_{F^+}, \operatorname{ad}\overline{r}(1)) \longrightarrow H^1(G_L, \operatorname{ad}\overline{r}(1))^{G_{F^+}}. \tag{11}$$

We have shown that the first term of (11) is 0, and hence the image of the cohomology class $[\phi]$ in $H^1(G_L, \operatorname{ad}\overline{r}(1))^{G_{F^+}}$ is non-zero. There is an inclusion

$$H^1(G_L, \operatorname{ad}\overline{r}(1))^{G_{F^+}} \subset H^1(G_L, \operatorname{ad}\overline{\rho})^{G_{F(\zeta_l)}},$$

so we can identify this restriction with a non-zero, $G_{F(\zeta_l)}$-equivariant homomorphism $f: G_L \to \operatorname{ad}\overline{\rho}$.

By assumption, we can find an element $\sigma_0 \in G_{F(\zeta_{l^N})}$ such that $\overline{\rho}(\sigma_0)$ is semi-simple, together with an eigenvalue $\alpha \in k$ of $\overline{\rho}(\sigma_0)$ such that $\operatorname{tr} e_{\sigma_0,\alpha} f(G_L) \neq 0$. If $\operatorname{tr} e_{\sigma_0,\alpha}\phi(\sigma_0) \neq 0$, then we're done on taking $\sigma = \sigma_0$. Suppose instead that $\operatorname{tr} e_{\sigma_0,\alpha}\phi(\sigma_0) = 0$, and choose $\tau \in G_L$ such that $\operatorname{tr} e_{\sigma_0,\alpha}\phi(\tau) \neq 0$. We then take $\sigma = \tau\sigma_0$ so that $\overline{\rho}(\sigma) = \overline{\rho}(\sigma_0)$, and calculate with the cocycle relation that

$$\operatorname{tr} e_{\sigma,\alpha}\phi(\sigma) = \operatorname{tr} e_{\sigma_0,\alpha}\phi(\tau) + \operatorname{tr} e_{\sigma_0,\alpha}\phi(\sigma_0) = \operatorname{tr} e_{\sigma_0,\alpha}\phi(\tau) \neq 0.$$

This concludes the proof. □

We then deduce:

Proposition 7.2 *The results* [32, Theorem 6.8] *and* [32, Theorem 8.6] *hold with the assumption '$\overline{r}(G_{L^+(\zeta_l)})$ is adequate, in the sense of* [32, Definition 2.3]*' replaced with the following two assumptions:*

(i) $\zeta_l \notin L$.
(ii) *Let $\overline{\rho} = \overline{r}|_{G_L}$. Then $\overline{\rho}(G_{L(\zeta_l)}) \subset \mathrm{GL}_n(k)$ is adequate, in the sense of Definition 2.20.*

Proof As indicated above, the assumption '$\overline{r}(G_{L^+(\zeta_l)})$ is adequate, in the sense of [32, Definition 2.2]' is used only to invoke [32, Proposition 4.4]. To prove the more general result, it is therefore enough to replace references to [32, Proposition 4.4] in the proofs with references to Proposition 7.1 above. □

Corollary 7.3 *The results [32, Theorem 7.1] and [32, Theorem 9.1] hold with the assumption '$\overline{\rho}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(k)$ is adequate, in the sense of [32, Definition 2.3]' replaced with the following assumption:*

(i) *The group $\overline{\rho}(G_{F(\zeta_l)}) \subset \mathrm{GL}_n(k)$ is adequate, in the sense of Definition 2.20.*

Proof The same proofs apply verbatim, except that the results [32, Theorem 6.8] and [32, Theorem 8.6] should be replaced by the modified versions in Proposition 7.2. □

We observe that if $l \nmid n$, then [32, Definition 2.3] is equivalent to Definition 2.20. However, if $l|n$ then a subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ is never adequate in the sense of [32, Definition 2.3] (because $H^0(\mathrm{GL}_n(\overline{\mathbb{F}}_l), \mathrm{ad}^0) \neq 0$), but there are many subgroups of $\mathrm{GL}_n(\overline{\mathbb{F}}_l)$ which are adequate in the sense of Definition 2.20, as follows from [13, Theorem 11.5].

It follows that in the case $l \nmid n$, we have merely fixed a gap in the proofs of [32, Theorem 7.1] and [32, Theorem 9.1]. However, in the case $l|n$, Corollary 7.3 is a new result, which relies upon the new definition of adequacy given in [13].

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References

1. Asgari, M., Shahidi, F.: Image of functoriality for general spin groups. Manuscr. Math. **144**(3–4), 609–638 (2014)
2. Bellaïche, J., Chenevier, G.: The sign of Galois representations attached to automorphic forms for unitary groups. Compos. Math. **147**(5), 1337–1352 (2011)
3. Barnet-Lamb, T., Gee, T., Geraghty, D.: Congruences between Hilbert modular forms: constructing ordinary lifts. Duke Math. J. **161**(8), 1521–1580 (2012)
4. Barnet-Lamb, T., Gee, T., Geraghty, D.: Congruences between Hilbert modular forms: constructing ordinary lifts, II. Math. Res. Lett. **20**(1), 67–72 (2013)
5. Barnet-Lamb, T., Gee, T., Geraghty, D., Taylor, R.: Potential automorphy and change of weight. Ann. Math. (2) **179**(2), 501–609 (2014)
6. Caraiani, A.: Local-global compatibility and the action of monodromy on nearby cycles. Duke Math. J. **161**(12), 2311–2413 (2012)
7. Caraiani, A.: Monodromy and local-global compatibility for $l = p$. Algebra Number Theory **8**(7), 1597–1646 (2014)
8. Clozel, L., Harris, M., Taylor, R.: Automorphy for some l-adic lifts of automorphic mod l Galois representations. Publ. Math. Inst. Hautes Études Sci. **108**, 1–181 (2008). With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras
9. Clozel, L., Thorne, J.A.: Level-raising and symmetric power functoriality, I. Compos. Math. **150**(5), 729–748 (2014)
10. Dickinson, M.: On the modularity of certain 2-adic Galois representations. Duke Math. J. **109**(2), 319–382 (2001)
11. Gee, T.: Erratum—a modularity lifting theorem for weight two Hilbert modular forms. Math. Res. Lett. **16**(1), 57–58 (2009)
12. Geraghty, D.J.: Modularity lifting theorems for ordinary Galois representations. Ph.D. Thesis, Harvard University, pp. 131 (2010)
13. Guralnick, R., Herzig, F., Tiep, P.H.: Adequate groups and indecomposable modules. To appear in Journal of the European Mathematical Society. arXiv:1405.0043 [math.RT]
14. Guerberoff, L.: Modularity lifting theorems for Galois representations of unitary type. Compos. Math. **147**(4), 1022–1058 (2011)
15. Hundley, J., Sayag, E.: Descent construction for GSpin groups: main results and applications. Electron. Res. Announc. Math. Sci. **16**, 30–36 (2009)

16. Jacquet, H., Shalika, J.A.: A non-vanishing theorem for zeta functions of GL_n. Invent. Math. **38**(1), 1–16 (1976/77)
17. Kisin, M.: Potentially semi-stable deformation rings. J. Am. Math. Soc. **21**(2), 513–546 (2008)
18. Kisin, M.: Modularity of 2-adic Barsotti–Tate representations. Invent. Math. **178**(3), 587–634 (2009)
19. Khare, C., Wintenberge, J.-P.: On Serre's conjecture for 2-dimensional mod p representations of $\mathrm{Gal}(\overline{\mathcal{Q}/\mathcal{Q}})$. Ann. Math. (2) **169**(1), 229–253 (2009)
20. Khare, C., Wintenberger, J.-P.: Serre's modularity conjecture. I. Invent. Math. **178**(3), 485–504 (2009)
21. Khare, C., Wintenberger, J.-P.: Serre's modularity conjecture. II. Invent. Math. **178**(3), 505–586 (2009)
22. Labesse, J.-P.: Changement de base CM et séries discrètes. In: On the Stabilization of the Trace Formula, Volume 1 of Stab. Trace Formula Shimura Var. Arith. Appl., pp. 429–470. International Press, Somerville, MA (2011)
23. Milne, J.S.: Arithmetic Duality Theorems, 2nd edn. BookSurge, LLC, Charleston, SC (2006)
24. Patrikis, S.: On the sign of regular algebraic polarizable automorphic representations. Math. Ann. **362**(1–2), 147–171 (2015)
25. Patrikis, S.: Variations on a Theorem of Tate. Preprint. arXiv:1207.6724 [math.NT]
26. Pilloni, V., Stroh, B.: Surconvergence, Ramification et modularité. To appear in Astérisque
27. Ramakrishnan, D.: Modularity of the Rankin–Selberg L-series, and multiplicity one for SL(2). Ann. Math. (2) **152**(1), 45–111 (2000)
28. Rapoport, M.: Compactifications de l'espace de modules de Hilbert–Blumenthal. Compos. Math. **36**(3), 255–335 (1978)
29. Skinner, C.M., Wiles, A.J.: Base change and a problem of Serre. Duke Math. J. **107**(1), 15–25 (2001)
30. Taylor, R.: Remarks on a conjecture of Fontaine and Mazur. J. Inst. Math. Jussieu **1**(1), 125–143 (2002)
31. Taylor, R.: Automorphy for some l-adic lifts of automorphic mod l Galois representations. II. Publ. Math. Inst. Hautes Études Sci. **108**, 183–239 (2008)
32. Thorne, J.A.: On the automorphy of l-adic Galois representations with small residual image. J. Inst. Math. Jussieu **11**(4), 855–920: With an appendix by Robert Guralnick. Florian Herzig, Richard Taylor and Jack Thorne (2012)
33. Wiles, A.: On ordinary λ-adic representations associated to modular forms. Invent. Math. **94**(3), 529–573 (1988)