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ERRATUM

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Erratum to: Modular properties of nodal curves on $K3$ surfaces

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In the original publication [3], the Theorem 3.1 and Theorem 4.2 are erroneous.

(i) First, the proof of Theorem 3.1 is incorrect: The fault is at the Step 2 of the proof. In the meantime, the result has been proved in [4] with better bounds.
(ii) Second, I correct Theorem 4.2. At p. 884, the last row of the diagram A.1 should be tensored by $\mathscr{O}_E(-2E)$. This error affects the subsequent computations from Lemma A.2 onward, which are used in the proof of the Theorem.

The corrected version is provided below.

1 The modified Wahl map

Recall that $\hat{C} \in |\mathscr{L} = \mathscr{A}^d|$ is a nodal curve with nodes $\mathcal{N} := \{\hat{x}_1, \ldots, \hat{x}_\delta\}$ on the polarized $K3$ surface $(S, \mathscr{A})$, such that $\mathscr{A} \in \mathrm{Pic}(S)$ is not divisible, $\mathscr{A}^2 = 2(n-1)$. (Note that the article [3] deals only with $K3$ surfaces with cyclic Picard group). Let $\sigma : \tilde{S} \to S$ be the blow-up of S at $\mathcal{N}$, and denote by E^a, $a = 1, \ldots, \delta$, the exceptional divisors, and $E := E^1 + \cdots + E^\delta$. The normalization C of $\hat{C}$ fits into

$$\begin{array}{ccc} (C, \Delta) & \overset{\tilde{u}}{\hookrightarrow} & \tilde{S} \\ {\scriptstyle \nu}\downarrow & \overset{u}{\searrow} & \downarrow{\scriptstyle \sigma} \\ (\hat{C}, \mathcal{N}) & \underset{j}{\hookrightarrow} & S, \end{array} \tag{1}$$

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$\tilde{u}$ is an embedding, and $K_C = \sigma^*\mathscr{L}(-E) \otimes \mathscr{O}_C$. The curve C carries the divisor

$$\Delta := x_{1,1} + x_{1,2} + \cdots + x_{\delta,1} + x_{\delta,2},$$

where $\{x_{a,1}, x_{a,2}\} = E^a \cap C$ is the pre-image of $\hat{x}_a \in \hat{C}$ by ν.

In general, if V' is a subscheme of some variety V, $\mathscr{I}_V(V')$ or $\mathscr{I}(V')$ stands for its sheaf of ideals, and $DV' \subset V \times V$ denotes the diagonally embedded V'.

Let (X, Δ_X) be an arbitrary smooth, irreducible curve together with δ pairwise disjoint pairs of points $\Delta_X = \{\{x_{1,1}, x_{1,2}\}, \ldots, \{x_{\delta,1}, x_{\delta,2}\}\} \subset X$. The exact sequence $0 \to \mathscr{I}(DX)^2 \to \mathscr{I}(DX) \to K_X \to 0$ yields the Wahl map

$$w_X : H^0\left(X \times X, \mathscr{I}(DX) \otimes K_{X\times X}\right) \to H^0\left(X, K_X^3\right).$$

The vector space $H^0(\mathscr{I}(DX) \otimes K_{X\times X})$ splits into

$$H^0(\mathscr{I}(DX) \otimes K_{X\times X}) \cap \operatorname{Sym}^2 H^0(K_X) \oplus \bigwedge^2 H^0(K_X),$$

and w_X vanishes on the first direct summand, as it is skew-symmetric. Denote

$$P_{\Delta_X} := \bigcup_{a=1}^{\delta} \{x_{a,1}, x_{a,2}\} \times \{x_{a,1}, x_{a,2}\} \subset X \times X, \tag{2}$$

and let w_{X,Δ_X} be the restriction of w_X to $H^0\big(\mathscr{I}(P_{\Delta_X}) \cdot \mathscr{I}(DX) \otimes K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X)$. (Thus, w_{X,Δ_X} is a punctual modification of the usual Wahl map.) With this notation, we replace [3, Theorem4.2] by the following.

Theorem 1 (i) *Let $(S, \mathscr{A})$, $\mathscr{A}^2 \geqslant 6$, be as above. Consider a nodal curve $\hat{C} \in |d\mathscr{A}|$ with*

$$\delta \leqslant \min\left\{\frac{d^2\mathscr{A}^2}{3(d+4)}, \delta_{\max}(n, d)\right\} \tag{3}$$

nodes and let (C, Δ) be as above ($\delta_{\max}(n, d)$ is defined in [3, p. 872]; the minimum is the first expression, except a finite number of cases). Then, the homomorphism $w_{C,\Delta}$ is not surjective.
(ii) *For generic a generic curve X of genus $g \geqslant 12$ with generic markings Δ_X, such that $\delta \leqslant \frac{g-1}{2}$, the homomorphism w_{X,Δ_X} is surjective.*

This is a nonsurjectivity property for *the pair* (C, Δ), rather than for C itself. I conclude the note (see Sect. 4) with some evidence toward the nonsurjectivity of the Wahl map w_C itself, and comment on related work in [4].

2 Relationship between the Wahl maps of C and $\tilde{S}$

Lemma 2 (i) *The following diagram has exact rows and columns:*

$$\begin{array}{ccccc}
0 \to \mathscr{I}_{C\times C}(DC)^2 & \longrightarrow & \mathscr{I}_{C\times C}(D\Delta)\cdot\mathscr{I}_{C\times C}(DC) & \xrightarrow{w'_C} & K_C(-\Delta) \to 0 \\
\| & & \downarrow & & \downarrow \\
0 \to \mathscr{I}_{C\times C}(DC)^2 & \longrightarrow & \mathscr{I}_{C\times C}(DC) & \xrightarrow{w_C} & K_C \to 0 \\
 & & \downarrow{\scriptstyle \mathrm{ev}_\Delta} & & \downarrow{\scriptstyle \mathrm{ev}_\Delta} \\
 & & \mathscr{I}_{C\times C}(DC)_{D\Delta} & = & K_{C,\Delta}
\end{array} \tag{4}$$

In other words, we have $\mathscr{I} := \mathscr{I}_{C\times C}(D\Delta)\cdot\mathscr{I}_{C\times C}(DC) = w_C^{-1}(K_C(-\Delta))$. Also, the involution τ_C which interchanges the factors of $C \times C$ leaves $\mathscr{I}$ invariant.

(ii) $H^0\big(C \times C, \mathscr{I} \otimes K_{C\times C}\big) = w_C^{-1}\big(\, H^0(C, K_C^3(-\Delta))\,\big)$.
(iii) *For* $\Lambda := \big\{s - \tau_C^*(s) \mid s \in H^0\big(\mathscr{I} \otimes K_{C\times C}\big)\big\}$ *holds*

$$\Lambda \overset{(\star)}{=} H^0\big(\mathscr{I} \otimes K_{C\times C}\big) \cap \bigwedge^2 H^0(K_C) \overset{(\star\star)}{\subset} H^0\big(\mathscr{I}_{C\times C}(DC) \otimes K_{C\times C}\big).$$

(iv) $w'_C(\Lambda) = w'_C\big(\, H^0\big(\mathscr{I} \otimes K_{C\times C}\big)\big)$.

Proof (i) The middle column is exact because $\mathscr{I}_{C\times C}(DC)$ is locally free. We check the exactness of the first row around each point $(o, o) \in D\Delta$. Let u be a local (analytic) coordinate on C such that $o = 0$, and u_1, u_2 be the corresponding coordinates on $C \times C$. Then, the first row becomes $0 \to \langle u_2 - u_1\rangle^2 \to \langle u_2 - u_1\rangle \cdot \langle u_1, u_2\rangle \to \langle u\rangle \cdot \mathrm{d}u \to 0$, with $\mathrm{d}u := (u_2 - u_1)\mathrm{mod}(u_2 - u_1)^2$, which is exact. The second statement is obvious.

(ii) We tensor (4) by $K_{C\times C}$, and take the sections in the last two columns. An elementary diagram chasing yields the claim.

(iii) Let us prove $(\star)$. The vector space Λ is contained in $\bigwedge^2 H^0(K_C)$ by the very definition, and also in $H^0\big(\mathscr{I}\otimes K_{C\times C}\big)$ because the sheaf $\mathscr{I}(C, \Delta)$ is τ_C-invariant. For the inclusion in the opposite direction, take s in the intersection. As $s \in \bigwedge^2 H^0(K_C)$, it follows $\tau_C^*(s) = -s$, so $s = 1/2 \cdot (s - \tau_C^*(s)) \in \Lambda$. The inclusion $(\star\star)$ is obvious.

(iv) Indeed, the Wahl map is anti-commutative: $w_C(\sum_i s_i \otimes t_i) = -w_C(\sum_i t_i \otimes s_i)$.

□

Lemma 3 *Let* $\Xi := \{(x_{a,1}, x_{a,2}), (x_{a,2}, x_{a,1}) \mid a = 1, \ldots, \delta\} \subset C \times C$, *and consider the sheaf of ideals* $\mathscr{I}(C, \Delta) := \mathscr{I}(\Xi) \cdot \mathscr{I} \overset{(2)}{=} \mathscr{I}(P_\Delta) \cdot \mathscr{I}(DC) \subset \mathscr{I}$. *Furthermore, denote*

$$\Lambda(\Delta) := H^0(\mathscr{I}(C, \Delta) \otimes K_{C\times C}) \cap \bigwedge^2 H^0(K_C). \tag{5}$$

Then, the following statements hold:

(i) $\mathscr{I}(C, \Delta)$ *is* τ_C*-invariant, so* $w'_C\big(\Lambda(\Delta)\big) = w'_C\big(\, H^0(\mathscr{I}(C, \Delta) \otimes K_{C\times C})\,\big)$.
(ii) $\mathscr{I}(C, \Delta) + \mathscr{I}_{C\times C}(DC)^2 = \mathscr{I}$, *and* $\mathscr{I}(C, \Delta) \cap \mathscr{I}(DC)^2 = \mathscr{I}(\Xi) \cdot \mathscr{I}(DC)^2$. *Therefore, the various sheaves introduced so far fit into the commutative diagram*

$$\begin{array}{ccccccccc}
0 & \to & \mathscr{I}(\Xi)\cdot\mathscr{I}(DC)^2 & \to & \mathscr{I}(C,\Delta) & \xrightarrow{w'_{C,\Delta}} & K_C(-\Delta) & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathscr{I}(DC)^2 & \to & \mathscr{I} & \xrightarrow{w'_C} & K_C(-\Delta) & \to & 0 \\
 & & \downarrow \mathrm{ev}''_\Xi & & \downarrow \mathrm{ev}'_\Xi & & & & \\
 & & \mathscr{O}_\Xi & \xrightarrow{\cong} & \mathscr{O}_\Xi & & & &
\end{array} \tag{6}$$

(The homomorphism $w_{C,\Delta}$ in the introduction equals $H^0(w'_{C,\Delta})$, defined after tensoring by $K_{C\times C}$.)

Proof (i) The proof is identical to Lemma 2(iv).

(ii) The inclusion $\subset$ is clear. For the reverse, notice that $\mathscr{O}_{C\times C} = \mathscr{I}(\Xi) + \mathscr{I}$, so $\mathscr{I} \subset \mathscr{I}(C, \Delta) + \mathscr{I}^2 \subset \mathscr{I}(C, \Delta) + \mathscr{I}(DC)^2$. The second claim is analogous.

□

Now, we compare the Wahl maps of C and $\tilde{S}$. Let $\rho : \mathscr{I}_{\tilde{S}\times\tilde{S}}(D\tilde{S}) \to \mathscr{I}_{C\times C}(DC)$ be the restriction homomorphism, and $\mathscr{M} := \sigma^*\mathscr{L}(-E)$. The diagram below relates various objects involved in the definition of w_C and $w_{\tilde{S}}$:

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{O}_{\tilde{S}}(-E) & \to & \mathscr{O}_{\tilde{S}}(E) & \to & \mathscr{O}_{2E}(E) & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & \mathscr{M}(-2E) & \to & \mathscr{M} & \to & \mathscr{M}\otimes\mathscr{O}_{2E} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & K_C(-2\Delta) & \to & K_C & \to & K_C\otimes\mathscr{O}_{2\Delta} & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ & 0 & & 0 & & 0 & \end{array} \qquad \begin{array}{ccccccc} 0 & \longrightarrow & H^0(\mathscr{O}_{\tilde{S}}(E)) = \mathbb{C}\tilde{s}_E & \to & H^0(\mathscr{O}_{2E}(E)) & \to & H^1(\mathscr{O}_{\tilde{S}}(-E)) \to 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^0(\mathscr{M}(-2E)) & \hookrightarrow & H^0(\mathscr{M}) & \longrightarrow & H^0(\mathscr{M}\otimes\mathscr{O}_{2E}) & \to & H^1(\mathscr{M}(-2E)) \to \cdots \\ \downarrow & & \downarrow & & \downarrow & & \\ H^0(K_C(-2\Delta)) & \hookrightarrow & H^0(K_C) & \longrightarrow & \bigoplus_{x\in\Delta} K_{C,2x} & \longrightarrow & \\ \downarrow & & & & & & \\ H^1(\mathscr{O}_{\tilde{S}}(-E)) & & & & & & \end{array} \tag{7}$$

The rightmost column corresponds to the first-order expansions of the sections along E and at Δ. By using $0\to\mathscr{O}_E(1)\to\mathscr{O}_{2E}\to\mathscr{O}_E\to 0$, we deduce that it fits into:

$$\begin{array}{ccccc} H^0(\mathscr{O}_E)\cong\mathbb{C}^{\delta} & \hookrightarrow & H^0(\mathscr{O}_E(2))\cong\mathbb{C}^{3\delta} & \twoheadrightarrow & H^0(\mathscr{O}_\Delta)\cong\mathbb{C}^{2\delta} \\ \downarrow\cong & & \downarrow & & \downarrow \\ H^0(\mathscr{O}_{2E}(E)) & \hookrightarrow & H^0(\mathscr{M}\otimes\mathscr{O}_{2E}) & \twoheadrightarrow & H^0(K_C\otimes\mathscr{O}_{2\Delta}) \\ & & \downarrow & & \downarrow \\ & & H^0(\mathscr{O}_E(1))\cong\mathbb{C}^{2\delta} & \xrightarrow{\cong} & H^0(\mathscr{O}_\Delta)\cong\mathbb{C}^{2\delta} \end{array} \tag{8}$$

Along each $E^a\subset\tilde{S}$, we consider local coordinates u, v as follows: v is the coordinate along E^a, and u is a coordinate in the normal direction to E^a (so E^a is given by $\{u=0\}$). Moreover, we assume that C is given by $\{v=0\}$ around the intersection points $\{x_{a,1}, x_{a,2}\}=E^a\cap C$. Then, any element $\tilde{s}\in H^0(\mathscr{M})$ can be expanded as

$$\tilde{s}=\tilde{s}_0^a(v)+u\tilde{s}_1^a(v)+O\left(u^2\right), \tag{9}$$

and its image in $H^0(\mathscr{M}\otimes\mathscr{O}_{2E^a})$ is $\tilde{s}_0^a(v)+u\tilde{s}_1^a(v)$. Finally, observe that the values of $w_{\tilde{S}}$ are sections of $\Omega^1_{\tilde{S}}\otimes\mathscr{M}^2$, and the restriction of this latter to E fits into

$$0\to\underbrace{\mathscr{O}_E(3)}_{\text{normal component}}\to\Omega^1_{\tilde{S}}\otimes\mathscr{M}^2|_E\to\underbrace{\Omega^1_E\otimes\mathscr{M}^2_E=\mathscr{O}_E}_{\text{tangential component}}\to 0.$$

Lemma 4 *Let the notation be as in* (9). *We consider* $\tilde{e}=\sum_i\tilde{s}_i\wedge\tilde{t}_i\in\bigwedge^2 H^0(\mathscr{M})$, *and let* $e:=\rho(\tilde{e})=\sum_i s_i\wedge t_i$. *Then, the following statements hold:*

(i) $w_{\tilde{S}}(\tilde{e})\in H^0(\Omega^1_{\tilde{S}}(-E)\otimes\mathscr{M}^2)$ *if and only if:*

$$\begin{cases} (\star) & \sum_i\left(s_i(x_{a,1})t_i(x_{a,2})-t_i(x_{a,1})s_i(x_{a,2})\right)=0, \quad \textit{and} \\ (\star\star) & \sum_i\left(\tilde{s}^a_{i,0}\tilde{t}^a_{i,1}-\tilde{t}^a_{i,0}\tilde{s}^a_{i,1}\right)=0, \quad \forall a=1,\ldots,\delta. \end{cases}$$

(ii) $\Lambda(E):=w_{\tilde{S}}^{-1}\left(H^0(\Omega^1_{\tilde{S}}(-E)\otimes\mathscr{M}^2)\right)\cap\bigwedge^2 H^0(\mathscr{M})$ *has the property*

$$w_{\tilde{S}}(\Lambda(E))=w_{\tilde{S}}\left(w_{\tilde{S}}^{-1}\left(H^0(\Omega^1_{\tilde{S}}(-E)\otimes\mathscr{M}^2)\right)\right).$$

(iii) $\rho\big(\Lambda(E)\big)\subset\Lambda(\Delta)$, *where the right hand side is defined by* (5).

Proof (i) The element $w_{\tilde{S}}(\tilde{e})$ vanishes along E if and only if both its tangential and normal components along each $E^a\subset E$ vanish. A short computation shows that the normal component is $(\star\star)$. The tangential component is $\sum_i\left(\tilde{s}^a_{i,0}\left(\tilde{t}^a_{i,0}\right)'-\tilde{t}^a_{i,0}\left(\tilde{s}^a_{i,0}\right)'\right)$. But $\tilde{s}^a_{i,0}, \tilde{t}^a_{i,0}\in H^0(\mathscr{O}_{E^a}(1))$, that is they are linear polynomials in v, so

$$\tilde{s}^a_{i,0}\left(\tilde{t}^a_{i,0}\right)'-\tilde{t}^a_{i,0}\left(\tilde{s}^a_{i,0}\right)'=\tilde{s}^a_{i,0}\left(x_{a,1}\right)\tilde{t}^a_{i,0}\left(x_{a,2}\right)-\tilde{t}^a_{i,0}\left(x_{a,1}\right)\tilde{s}^a_{i,0}\left(x_{a,2}\right),$$

up to a constant factor. Also, we have $\tilde{s}^a_{i,0}(x_{a,j}) = \tilde{s}^a(x_{a,j}) = s(x_{a,j})$, and ($\star$) follows.

(ii) The vector space $w_{\tilde{S}}^{-1}\big(H^0(\Omega^1_{\tilde{S}}(-E) \otimes \mathscr{M}^2)\big)$ is invariant under the involution $\tau_{\tilde{S}}$ of $\tilde{S} \times \tilde{S}$ which switches the two factors. As $w_{\tilde{S}}$ is anti-commutative, the claim follows as in Lemma 2.

(iii) Take $\tilde{e} \in \Lambda(E)$ and $e := \rho(\tilde{e})$. Then, $e(x_{a,1}, x_{a,2}) = -e(x_{a,2}, x_{a,1}) \overset{(\star)}{=} 0$, and also $w_C(e)(x_{a,j})$ equals the expression ($\star\star$) at $x_{a,j}$ (so it vanishes), for $j = 1, 2$. □

Now. we consider the commutative diagram:

$$\begin{array}{ccccc} \Lambda(E) & \xrightarrow{w_{\tilde{S}}} & H^0\big(\tilde{S}, \Omega^1_{\tilde{S}}(-E) \otimes \mathscr{M}^2\big) & \xrightarrow{\mathrm{res}_C} & H^0\big(C, \Omega^1_{\tilde{S}}|_C \otimes K_C^2(-\Delta)\big) \\ \rho_\Delta \big\downarrow \ \text{cf. 4(iii)} & & \big\downarrow & \swarrow b & \\ \Lambda(\Delta) & \xrightarrow{w_{C,\Delta}} & H^0\big(C, K_C^3(-\Delta)\big) & & \end{array} \tag{10}$$

It is the substitute in the case of nodal curves for [3, diagram (4.2)].

Lemma 5 *Assume* $\mathrm{Pic}(S) = \mathbb{Z}\mathscr{A}$. *Then,* $\rho_\Delta : \Lambda(E) \to \Lambda(\Delta)$ *is surjective.*

Proof The restriction $H^0(\tilde{S}, \mathscr{M}) \to H^0(C, K_C)$ is surjective (see [3, LemmaA.1]), and the kernel of $\bigwedge^2 H^0(\tilde{S}, \mathscr{M}) \to \bigwedge^2 H^0(C, K_C)$ consists of elements of the form $\tilde{t} \wedge (\tilde{s}_C \tilde{s}_E)$, where $\tilde{t} \in H^0(\mathscr{M})$ and $\tilde{s}_C, \tilde{s}_E$ are the canonical sections of $\mathscr{O}_{\tilde{S}}(C)$ and $\mathscr{O}_{\tilde{S}}(E)$, respectively. (See the middle column of (7).)

Consider $e = \sum_i (s_i \otimes t_i - t_i \otimes s_i) \in \Lambda(\Delta)$, and let $\tilde{e} = \sum_i (\tilde{s}_i \otimes \tilde{t}_i - \tilde{t}_i \otimes \tilde{s}_i) \in \bigwedge^2 H^0(\mathscr{M})$ be such that $\rho(\tilde{e}) = e$. The proof of 4(i) shows that, for all a, the tangential component of $w_{\tilde{S}}(\tilde{e})|_{E^a}$ equals $e(x_{a,1}, x_{a,2}) = 0$, so $w_{\tilde{S}}(\tilde{e})|_E$ is a section of $\Omega^1_{E/\tilde{S}} \otimes \mathscr{M}_E^2 \cong \mathscr{O}_E(3)$. Since $w_{\tilde{S}}(\tilde{e})|_E$ vanishes at the points of Δ, it is actually determined up to an element in $H^0(\mathscr{O}_E(1))$. We claim that this latter can be canceled by adding to $\tilde{e}$ a suitable element of the form $\tilde{t} \wedge (\tilde{s}_C \tilde{s}_E)$. A short computation yields

$$w_{\tilde{S}}\left(\tilde{t} \wedge (\tilde{s}_C \tilde{s}_E)\right)|_E = \tilde{t}_E \cdot (\tilde{s}_C|_E) \cdot (\mathrm{d}\tilde{s}_E)|_E \in \mathscr{O}_E(3),$$

where $\tilde{t}_E \in H^0(\mathscr{O}_E(1))$, $\tilde{s}_C|_E \in H^0(\mathscr{O}_E(2))$ vanishes at $\Delta = E \cap C$, and $(\mathrm{d}\tilde{s}_E)|_E \in H^0(\mathscr{O}_E)$ (it is a section of $\Omega^1_{\tilde{S}}|_E$ with vanishing tangential component). Thus, these two latter factors are actually (nonzero) scalars.

The previous discussion shows that $\tilde{e} + \tilde{t} \wedge (\tilde{s}_C \tilde{s}_E) \in \Lambda(E)$ as soon as $\tilde{t} \in H^0(\mathscr{M})$ satisfies $\tilde{t}_E = -w_{\tilde{S}}(\tilde{e})|_E \in H^0(\mathscr{M}_E)$. According to Corollary 8, such an element $\tilde{t}$ exists because the restriction $H^0(\mathscr{M}) \to H^0(\mathscr{M}_E)$ is surjective. □

Proof of Theorem 1 (i) *Case* $\mathrm{Pic}(S) = \mathbb{Z}\mathscr{A}$. If $w_{C,\Delta}$ is surjective; then, the homomorphism b in the diagram (10) is surjective too. Now, we follow the same pattern as in [3, p. 884,top]: b is the restriction homomorphism at the level of sections of

$$0 \to K_C \to \Omega^1_{\tilde{S}}\big|_C \otimes K_C^2(-\Delta) \to K_C^3(-\Delta) \to 0,$$

and its surjectivity implies that this sequence splits. This contradicts [3, Lemma4.1].

General case. It is a deformation argument. We consider

$$\mathscr{K}_n := \left\{(S, \mathscr{A}) \mid \mathscr{A} \in \mathrm{Pic}(S) \text{ is ample, not divisible, } \mathscr{A}^2 = 2(n-1)\right\},$$
$$\mathscr{V}^d_{n,\delta} := \left\{\left((S, \mathscr{A}), \hat{C}\right) \mid (S, \mathscr{A}) \in \mathscr{K}_n,\ \hat{C} \in |d\mathscr{A}| \text{ nodal curve with } \delta \text{ nodes}\right\}.$$

Then, the natural projection $\kappa : \mathscr{V}^d_{n,\delta} \to \mathscr{K}_n$ is submersive onto an open subset of $\mathscr{K}_n$. (See [3, Theorem 1.1(iii)] and the reference therein.)

Hence, for any $((S, \mathscr{A}), \hat{C}) \in \mathscr{V}^d_{n,\delta}$ there is a smooth deformation $((S_t, \mathscr{A}_t), \hat{C}_t)$ parameterized by an open subset $T \subset \mathscr{K}_n$. The points $t \in T$ such that $\mathrm{Pic}(S_t) = \mathbb{Z}\mathscr{A}_t$ are dense; for these w_{C_t,Δ_t} are nonsurjective. Since the nonsurjectivity condition is closed, we deduce that $w_{C,\Delta}$ is nonsurjective too.

(ii) Now let (X, Δ_X) be a generic marked curve of genus at least 12. By [1], the Wahl map $w_X : \bigwedge^2 H^0(K_X) \to H^0(K_X^3)$ is surjective; thus, $\widetilde{w}'_X := H^0(w'_X \otimes K_{X\times X})$ in (6) is surjective as well (see lemma 2(ii)). As $\delta \leqslant \frac{g-1}{2}$, the evaluation homomorphism $H^0(K_X) \to K_X \otimes \mathscr{O}_{\Delta_X}$ is surjective for generic markings, so the same holds for

$$H^0(K_X)^{\otimes 2} \to \bigoplus_{a=1}^{\delta} \big(K_{X,x_{a,1}} \oplus K_{X,x_{a,2}}\big)^{\otimes 2}.$$

The restriction to the anti-symmetric part (on both sides) yields the surjectivity of

$$\mathrm{ev}_{\Xi} : \bigwedge^2 H^0(K_X) \to \bigoplus_{a=1}^{\delta} K_{X,x_{a,1}} \otimes K_{X,x_{a,2}} = \bigoplus_{a=1}^{\delta} K_{X\times X,(x_{a,1},\, x_{a,2})}.$$

(For $s \in \bigwedge^2 H^0(K_X)$, $\mathrm{ev}_{\Xi}(s)$ takes opposite values at $(x_{a,1}, x_{a,2})$ and $(x_{a,2}, x_{a,1})$.)

The diagram (6) yields

$$\begin{array}{ccccc}
H^0\big(\mathscr{I}(\Xi)\cdot\mathscr{I}(DX)^2 \otimes K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X) & \hookrightarrow & H^0\big(\mathscr{I}(\Xi)\cdot\mathscr{I} \otimes K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X) & \xrightarrow{w_{X,\Delta_X}} & H^0(K_X^3(-\Delta_X)) \\
\downarrow & & \downarrow & & \| \\
H^0\big(\mathscr{I}(DX)^2 \otimes K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X) & \longrightarrow & H^0\big(\mathscr{I} \otimes K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X) & \xrightarrow{\widetilde{w}'_X} & H^0(K_X^3(-\Delta_X)) \\
\downarrow \mathrm{ev}'_{\Xi} & & \downarrow \mathrm{ev}_{\Xi'} & & \\
\bigoplus_{a=1}^{\delta} K_{X,x_{a,1}} \otimes K_{X,x_{a,2}} & = & \bigoplus_{a=1}^{\delta} K_{X,x_{a,1}} \otimes K_{X,x_{a,2}}. & &
\end{array} \tag{11}$$

A straightforward diagram chasing shows that w_{X,Δ_X} is surjective if

$$\mathrm{ev}''_{\Xi} : \underbrace{H^0\big(\mathscr{I}(DX)^2 \cdot K_{X\times X}\big) \cap \bigwedge^2 H^0(K_X)}_{:=G} \to \underbrace{\bigoplus_{a=1}^{\delta} K_{X\times X,(x_{a,1},\, x_{a,2})}}_{:=H_{\Xi}}$$

is so, or equivalently when the induced $h_{\Xi} : \bigwedge^{\delta} G \to \bigwedge^{\delta} H_{\Xi}$ is nonzero. This is indeed the case for generic markings.

Claim $\bigcap_{\Delta_X} \mathrm{Ker}(h_{\Xi}) = 0$. ($h_{\Xi}$ depends on Δ_X.) Indeed, since $\dim G \geqslant \delta$, we have

$$\begin{array}{ccccc}
H^0(\mathscr{I}(DX)^2 \otimes K_{X\times X}) \cap \bigwedge^2 H^0(K_X) & \hookrightarrow & H^0(\mathscr{I} \otimes K_{X\times X}) \cap \bigwedge^2 H^0(K_X) & \hookrightarrow & \bigwedge^2 H^0(K_X) \\
\downarrow \mathrm{ev}''_{\Xi} & & \downarrow \mathrm{ev}_{\Xi'} & & \downarrow \mathrm{ev}_{\Xi} \\
\bigoplus_{a=1}^{\delta} K_{X\times X,(x_{a,1},\, x_{a,2})} & = & \bigoplus_{a=1}^{\delta} K_{X\times X,(x_{a,1},\, x_{a,2})} & = & \bigoplus_{a=1}^{\delta} K_{X\times X,(x_{a,1},\, x_{a,2})},
\end{array}$$

$$0 \neq \bigwedge^{\delta} G \subset \bigwedge^{\delta}\Big(\bigwedge^2 H^0(K_X)\Big) \subset H^0(K_{X\times X})^{\otimes\delta} = H^0\big((X^2)^{\delta}, K_{X\times X} \boxtimes \cdots \boxtimes K_{X\times X}\big).$$

The wedge is a direct summand of the tensor product (appropriate skew-symmetric sums), and h_Ξ is induced by the evaluation map

$$\mathrm{ev}^\delta : H^0\big((X^2)^\delta, K_{X\times X}\boxtimes\cdots\boxtimes K_{X\times X}\big)\otimes\mathscr{O}\to K_{X\times X}\boxtimes\cdots\boxtimes K_{X\times X}$$

at $\big((x_{1,1},x_{1,2}),\ldots,(x_{\delta,1},x_{\delta,2})\big)\in(X^2)^\delta$. If $e\in\bigwedge^{\delta} G$ belongs to the intersection above, then $e\in H^0\big(\mathrm{Ker}(\mathrm{ev}^\delta)\big)=\{0\}$. Hence, for any $e_1,\ldots,e_\delta\in G$ with $e_1\wedge\cdots\wedge e_\delta\neq 0$, there are markings Δ_X such that $\mathrm{ev}''_\Xi(e_1),\ldots,\mathrm{ev}''_\Xi(e_\delta)$ are linearly independent in H_Ξ (thus, they span it). □

3 Multiple point Seshadri constants of $K3$ surfaces with cyclic Picard group

This section is independent of the rest. Here we determine a lower bound for the multiple point Seshadri constants of $\mathscr{A}$, which is necessary for proving Lemma 5.

Definition 6 (See [2, Section 6] for the original definition) The multiple point Seshadri constant of $\mathscr{A}$ corresponding to $\hat{x}_1,\ldots,\hat{x}_\delta\in S$ is defined as

$$\varepsilon=\varepsilon_{S,\delta}(\mathscr{A}):=\inf_Z\frac{Z\cdot\mathscr{A}}{\sum_{a=1}^{\delta}\mathrm{mult}_{\hat{x}_a}(Z)}=\sup\big\{c\in\mathbb{R}\mid\sigma^*\mathscr{A}-cE\text{ is ample on }\tilde{S}\big\}.\quad(12)$$

The infimum is taken over all integral curves $Z\subset S$ which contain at least one of the points $\hat{x}_a$ above. Throughout this section, we assume that $Z\in|z\mathscr{A}|$, with $z\geqslant 1$.

As the self-intersection number of any ample line bundle is positive, the upper bound $\varepsilon\leqslant\frac{\sqrt{\mathscr{A}^2}}{\sqrt{\delta}}$ is automatic. We are interested in finding a lower bound.

Theorem 7 *Assume that* $\mathrm{Pic}(S)=\mathbb{Z}\mathscr{A}$, $\mathscr{A}^2=2(n-1)\geqslant 4$, *and* $\delta\geqslant 1$. *Then, the Seshadri constant* (12) *satisfies* $\varepsilon\geqslant\frac{2\mathscr{A}^2}{\delta+\sqrt{\delta^2+4\delta(2+\mathscr{A}^2)}}$, *for any points* $\hat{x}_1,\ldots,\hat{x}_\delta\in S$.

Our proof is inspired from [5], which treats the case $\delta=1$.

Proof We may assume that the points are numbered such that

$$\mathrm{mult}_{\hat{x}_a}(Z)\geqslant 2,\ \text{for } a=1,\ldots,\alpha,\quad \mathrm{mult}_{\hat{x}_a}(Z)=1,\ \text{for } a=\alpha+1,\ldots,\beta,\quad(\beta\leqslant\delta).$$

We denote $p:=\sum_{a=1}^{\alpha}\mathrm{mult}_{\hat{x}_a}(Z)\geqslant 2\alpha$ and $m:=\sum_{a=1}^{\delta}\mathrm{mult}_{\hat{x}_a}(Z)\leqslant p+\delta-\alpha$.

If $\alpha=0$, then $\frac{z\cdot\mathscr{A}^2}{m}\geqslant\frac{\mathscr{A}^2}{\delta}$ satisfies the inequality, so we may assume $\alpha\geqslant 1$. A point of multiplicity m lowers the arithmetic genus of Z by at least $\binom{m}{2}$; hence,

$$p_a(Z)=\frac{z^2\mathscr{A}^2}{2}+1\geqslant\frac{1}{2}\sum_{a=1}^{\alpha}\big(\mathrm{mult}_{\hat{x}_a}(Z)^2-\mathrm{mult}_{\hat{x}_a}(Z)\big)\underset{\text{inequality}}{\overset{\text{Jensen}}{\geqslant}}\frac{1}{2}\Big(\frac{p^2}{\alpha}-p\Big),$$

so $p \leqslant \frac{\alpha+\sqrt{\alpha^2+4\alpha(2+z^2\mathscr{A}^2)}}{2}$. We deduce the following inequalities:

$$\frac{z\mathscr{A}^2}{m} \geqslant \frac{z\mathscr{A}^2}{p-\alpha+\delta} \geqslant \underbrace{\frac{z\mathscr{A}^2}{\delta+\frac{\sqrt{\alpha^2+4\alpha(2+z^2\mathscr{A}^2)}-\alpha}{2}}}_{\text{decreasing in}\,\alpha} \geqslant \underbrace{\frac{z\mathscr{A}^2}{\delta+\frac{\sqrt{\delta^2+4\delta(2+z^2\mathscr{A}^2)}-\delta}{2}}}_{\text{increasing in}\,z}$$
$$\geqslant \frac{2\mathscr{A}^2}{\delta+\sqrt{\delta^2+4\delta\left(2+\mathscr{A}^2\right)}}.$$

□

Corollary 8 *$H^0(\mathscr{M}) \to H^0(\mathscr{M}_E)$ is surjective, for $\mathscr{A}^2 \geqslant 6$ and $\delta \leqslant \frac{d^2\mathscr{A}^2}{3(d+4)}$.*

Proof Indeed, it is enough to check that $H^1(\tilde{S}, \mathscr{M}(-E)) = H^1(\tilde{S}, K_{\tilde{S}} \otimes \mathscr{M}(-2E))$ vanishes. By the Kodaira vanishing theorem, this happens as soon as $\mathscr{M}(-2E) = \sigma^*\mathscr{A}^d(-3E)$ is ample. The previous theorem implies that, in order to achieve this, is enough to impose $\frac{3}{d} \leqslant \frac{2\mathscr{A}^2}{\delta+\sqrt{\delta^2+4\delta(2+\mathscr{A}^2)}}$, which yields $\delta \leqslant \frac{d^2(\mathscr{A}^2)^2}{3(d\mathscr{A}^2+3\mathscr{A}^2+6)}$. □

4 Concluding remarks

(I) Evidence for the nonsurjectivity of w_C Theorem 1 is a nonsurjectivity property for the Wahl map of the *pointed curve* $(C, \varDelta)$, rather than that of the curve C itself.

Claim. In order to prove the nonsurjectivity of the Wahl map w_C, is enough to have the surjectivity of the evaluation homomorphism

$$H^0(\mathscr{I}(DC)^2 \otimes K_{C\times C}) \to \bigoplus_{a=1}^{\delta} K_{C\times C,(x_{a,1},x_{a,2})} \oplus K_{C\times C,(x_{a,2},x_{a,1})}. \tag{13}$$

(For δ in the range (3), corollary 8 implies that $K_C = \mathscr{M}_C$ separates $\varDelta$, consequently $\bigwedge^2 H^0(K_C) \to \bigoplus_{a=1}^{\delta} K_{C\times C,(x_{a,1},x_{a,2})}$ is surjective. The surjectivity of (13) yields that of $\mathrm{ev}''_{\varXi}$ in (11), which is relevant for us.)
For the claim, observe that one has the following implications (see (4), (11)):

$$w_C \text{ surjective } \Rightarrow w'_C \text{ surjective } \underset{\text{surj.}}{\overset{(13)}{\Rightarrow}} w_{C,\varDelta} \text{ surjective, a contradiction.} \tag{14}$$

The surjectivity of (13) is clearly a positivity property for $\mathscr{I}(DC)^2 \otimes K_{C\times C}$. We use again the Seshadri constants to argue why this is likely to hold. The $\varXi$-pointed Seshadri constants of the self-product of a *very general curve* X at *very general points* $\varXi$ (as in Lemma 3) satisfy (see [6, p. 65 below Theorem 1.6, and Lemma 2.6]):

$$\varepsilon_{X\times X,\,\Xi}(\mathscr{I}(DX)^2\otimes K_{X\times X}) \geqslant 2(g-2)\varepsilon_{\mathbb{P}^2,\,g+\delta}(\mathscr{O}_{\mathbb{P}^2}(1)) > \frac{2(g-2)}{\sqrt{g+\delta}}\sqrt{1-\frac{1}{8(g+\delta)}}, \tag{15}$$

$$\varepsilon_{X\times X,\,\Xi}(\mathscr{I}(DX)^4\otimes K_{X\times X}) \geqslant 4\cdot\frac{g-3}{2}\cdot\varepsilon_{\mathbb{P}^2,\,g+\delta}(\mathscr{O}_{\mathbb{P}^2}(1)) > \frac{2(g-3)}{\sqrt{g+\delta}}\sqrt{1-\frac{1}{8(g+\delta)}} =: \varphi(g,\delta). \tag{16}$$

The equation (16) implies (see [2, Proposition6.8]) that $\left(\mathscr{I}(DX)^2\otimes K_{X\times X}\right)^2$ generates the jets of order $\lfloor\varphi(g,\delta)\rfloor-2$ at $\Xi\subset X\times X$. (We only need the generation of jets of order zero for $\mathscr{I}(DX)^2\otimes K_{X\times X}$; also, note that $\varphi(g,\delta)$ grows linearly with $\sqrt{g}$ as long as δ is small compared with g (see (3)).) This discussion suggests that $\mathscr{I}(DX)^2\otimes K_{X\times X}$ is 'strongly positive/generated.' However, the passage to (13) above requires even more control.

(II) Related work In [4], the author extensively studies the properties of nodal curves on $K3$ surfaces. Among several other results, he proves the nonsurjectivity of a marked Wahl map (different from the one introduced in here) for nodal curves on $K3$ surfaces.

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