The Borel map in locally integrable structures

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Abstract

Given a locally integrable structure $\mathcal{V}$ over a smooth manifold Ω and given $p \in \Omega$ we define the *Borel map of $\mathcal{V}$ at p* as the map which assigns to the germ of a smooth solution of $\mathcal{V}$ at p its formal Taylor power series at p. In this work we continue the study initiated in Barostichi et al. (Math. Nachr. 286(14–15):1439–1451, 2013), Della Sala and Lamel (Int J Math 24(11):1350091, 2013) and present new results regarding the Borel map. We prove a general necessary condition for the surjectivity of the Borel map to hold and also, after developing some new devices, we study some classes of CR structures for which its surjectivity is valid. In the final sections we show how the Borel map can be applied to the study of the algebra of germs of solutions of $\mathcal{V}$ at p.

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1 Introduction

The purpose of this paper is to discuss our recent results on the *Borel map* and the *Borel property* for locally integrable structures. If one thinks about an integrable structure as a system of (linear, first order) PDEs with the right number of basic solutions, it becomes an intriguing question to study the relationship between *formal solutions* (i.e. formal power series in the solutions of the structure) and *solutions*. The relationship between the two comes, of course, from associating to a smooth solution its formal Taylor series at a distinguished point (e.g. the origin) in the structure. The Taylor series of a solution can be written as a series in the elements of a set of basic first integrals $\{Z_1, \ldots, Z_m\}$ defined near the origin; we refer to this map, defined by

$$\mathfrak{b} : \mathfrak{S}_0 \to \mathbb{C}[\![Z_1, \ldots, Z_m]\!], \quad \mathfrak{b}(u) = \sum_{\alpha \in \mathbb{N}^m} u_\alpha Z^\alpha,$$

where the u_α are appropriate derivatives evaluated at 0 of u, as the *Borel map* (at the origin).

We have started the study of this map, in particular the natural question of when it is surjective (the *Borel property*), in a series of papers of the second author with Barostichi and Petronilho [3] and of the first and the third author in the context of CR structures [6]. In our current paper, we can give important insights into the nature of the geometric properties of the structure determining whether the Borel property holds or not, and we find relationships with interesting open questions in the analysis of locally integrable structures.

Before we begin with the discussion of our results, we refer the reader to Sect. 2 for thorough definitions of locally integrable structures, the Borel map (which associates to any smooth solution its formal solution), and the Borel property (meaning that the Borel map is surjective). The Borel property can be used to understand, and, in some circumstances, bridge the gap between the local algebra of power series spaces and the analysis of properties of smooth solutions.

In Sect. 3 we use functional analytic methods in order to characterize (abstractly) when the Borel property holds in Proposition 3.2: roughly stated, the Borel property holds if and only if the following is true: when one can uniformly control the action of a sequence of differential operators on the solutions of the structure by the C^ℓ-norm on some compact set, then the operators in the sequence need to be of bounded order. We use this fact to provide some conceptually simpler and, in view of later results, cleaner proofs of the fact that the existence of peak functions of finite type (or in the locally integrable case, the fact that property ($\mathfrak{B}$) holds) is sufficient for the Borel property to hold, and for the fact that the existence of a flat solution is necessary for the Borel property to hold.

However, the results in our current paper show that geometric properties of this form are far too rough to understand the Borel map. We hope that this means that understanding the Borel map is more feasible than understanding whether e.g. a peak function exists (which is a very hard undertaking, see e.g. the survey by Noell [11]), as it turns out that the Borel map is a very subtle instrument which feels a lot of the intrinsic geometry of the integrable structure. In particular, the present results give

hope (and lead to some actual conjectures) that one can reach a satisfactory geometric characterization of the Borel property, and show that its application to e.g. the structure of ideals of solutions gives important insights into the behaviour of solutions.

There are also structural aspects of the Borel map which make its study very appealing: We encounter one such aspect when we study *partial Borel maps* in Sect. 4. Partial Borel maps are defined as restrictions of the Borel map to solutions which are flat in a number of the basic solutions, giving rise to formal series only depending on the other basic solutions. It turns out (Theorem 4.1) that the Borel map is surjective if and only if the partial Borel maps associated to a choice of a set of basic solutions and to its complementary set are both surjective.

Our main new necessary condition (Theorem 6.1) for the surjectivity of the Borel map is that the polynomial hull of $Z(K)$, where $Z = (Z_1, \ldots, Z_m)$ is the embedding of the structure into $\mathbb{C}^m$ by means of a set of basic solutions, does not contain any analytic discs. It is tempting to conjecture (especially when considering the proof of that statement) that this condition is not only necessary but also sufficient.

Hence one of the remaining objectives of the paper is a discussion of the possible gap between the necessity of the condition and the stronger conditions known to be sufficient. A particular case in question is an application of the result on partial Borel maps to structures whose characteristic set is of maximal dimension; in that case, we see that the Borel map is surjective if none of the solutions of the structure is open (Theorem 7.3).

This result highlights yet another interesting problem to which the Borel property has a curious connection, namely the question whether there is a solution (with nontrivial differential in a noncharacteristic direction) which is actually open; it also, therefore connects with the question of whether a maximum principle is valid for solutions of the given structure. We shall, however, in this paper not follow these lines of inquiry further.

Instead, we have decided to focus on the study of what we think is the main geometric question left over in our approach here in a special model case of tube structures. We obtain a rather complete picture in that case, which is discussed in Sect. 8. We show in Theorem 8.1 that if neither the known condition for surjectivity (property ($\mathfrak{B}$)), nor the condition for failure of surjectivity (open mapping property) hold, that we can reduce the problem to studying sets which are in some sense "characteristic" for property $\mathfrak{B}$. and it is in many cases the geometry of these sets which allows us to determine whether the Borel map is onto or not (Theorems 8.2 and 8.3).

In the last two sections of the paper, Sects. 9 and 10, we study two particular algebraic aspects of the ring of solutions: we first show that its maximal ideal is finitely generated by a set of basic solutions if property ($\mathfrak{B}$) holds (Theorem 9.1). There are also other situations in which we can guarantee this basic property, but we would definitely like to know whether the maximal ideal in the ring of solutions is always generated by a set of basic solutions (or not). In the other extreme, we also show that principal manifold ideals automatically (without further assumptions on the structure) satisfy the Nullstellensatz (Lemma 10.1).

We would like to note that the current paper leaves open a number of fascinating problems concerning the behaviour of the Borel map and the relation between the

algebra of formal solutions and the algebra of solutions; we discuss a number of them in section 11.

2 The Borel property in locally integrable structures

A. Let Ω be a smooth (paracompact) manifold of dimension N over which we assume given a locally integrable structure $\mathcal{V}$ of rank n. Thus $\mathcal{V}$ is a vector subbundle of $\mathbb{C}T\Omega$ of rank n whose orthogonal bundle $\mathcal{V}^{\perp} \subset \mathbb{C}T^*\Omega$ is locally spanned by the differentials of $m = N - n$ smooth functions.

If $p \in \Omega$ we set

$$\mathfrak{S}_p \doteq \{f \in C_p^{\infty} : \mathrm{L}f = 0, \ \forall \text{ sections L of } \mathcal{V} \text{ near } p\},$$

where we are denoting by C_p^{∞} the ring of germs of smooth functions at p. It is clear that $\mathfrak{S}_p$ is also a ring.

For each $k \geq 0$ let $\mathfrak{m}_p^k$ denote the ideal of C_p^{∞} formed by all $f \in C_p^{\infty}$ for which there is a constant $C > 0$ such that $|f(q)| \leq Cd(q, p)^{k+1}$ for q in a neighborhood of p.[1] It is also clear that $\mathfrak{m}_p^{k+1} \subset \mathfrak{m}_p^k$ for every $k \geq 0$ and that $\mathfrak{m}_p^k \cap \mathfrak{S}_p$ is an ideal of $\mathfrak{S}_p$. We can then form the quotient ring $\mathcal{J}(\mathcal{V})_p^k \doteq \mathfrak{S}_p/(\mathfrak{m}_p^k \cap \mathfrak{S}_p)$, which is called the *ring of k- jets of solutions at* p. We have well defined homomorphisms $\iota_k : \mathcal{J}(\mathcal{V})_p^k \to \mathcal{J}(\mathcal{V})_p^{k-1}, k \geq 1$, induced by the inclusions $\mathfrak{m}_p^k \subset \mathfrak{m}_p^{k-1}$. Furthermore $\iota_k \circ \pi_k = \pi_{k-1}, k \geq 1$, where π_k stands for the quotient map $\mathfrak{S}_p \to \mathcal{J}(\mathcal{V})_p^k$. We can form the projective limit

$$\mathcal{J}(\mathcal{V})_p^{\infty} \doteq \varprojlim \mathcal{J}(\mathcal{V})_p^k$$

which is then called the ring of *formal solutions for* $\mathcal{V}$ *at* p. Recall that $\mathcal{J}(\mathcal{V})_p^{\infty}$ is the set of all sequences $(s_k)_{k \geq 0}$ with $s_k \in \mathcal{J}_p^k$ and $s_{k-1} = \iota_k(s_k)$ for every $k \geq 1$. Finally we define

$$\mathfrak{b}_{\mathcal{V},p} : \mathfrak{S}_p \to \mathcal{J}(\mathcal{V})_p^{\infty}, \quad \mathfrak{b}_{\mathcal{V},p}(u) = (\pi_k(u))_{k \geq 0}$$

Definition 2.1 We shall refer to the ring homomorphism $\mathfrak{b}_{\mathcal{V},p}$ as the Borel map for $\mathcal{V}$ at p. We shall also say that $\mathcal{V}$ satisfies the Borel property at p if $\mathfrak{b}_{\mathcal{V},p}$ is surjective.

B. Let $\mathcal{V}$ be a smooth, locally integrable structure defined on a smooth manifold Ω and let $p \in \Omega$. According to [4] we can assert the following: p is the center of a smooth coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$, which can be assumed defined in a product $U = B \times \Theta$, where B (respectively Θ) is an open ball centered at the origin in $\mathbb{R}_x^m$ (respectively $\mathbb{R}_t^n$), over which there is defined a smooth, real vector-valued function $\Phi(x, t) = (\Phi_1(x, t), \ldots, \Phi_m(x, t))$ satisfying $\Phi(0, 0) = 0, D_x\Phi(0, 0) = 0$, such

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[1] Here d is any distance function defined near p by using local coordinates. It is easily seen that the definition of the ideals $\mathfrak{m}_p^k$ is invariant.

that the differential of the functions

$$Z_k(x, t) = x_k + i\Phi_k(x, t), \qquad k = 1, \ldots, m,$$

span $\mathcal{V}^\perp$ over U.

Moreover $dZ_1, \ldots, dZ_m, dt_1, \ldots, dt_n$ span $\mathbb{C}T^*\Omega$ over U.

Over U we can define smooth vector fields

$$M_k = \sum_{k'=1}^{m} \mu_{kk'}(x, t) \frac{\partial}{\partial x_{k'}}, \quad k = 1, \ldots, m$$

characterized by the rule

$$M_k Z_{k'} = \delta_{k,k'}, \quad k, k' = 1, \ldots, m.$$

It follows that the complex vector fields

$$L_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^{m} \frac{\partial \Phi_k}{\partial t_j}(x, t) M_k, \quad j = 1, \ldots, n,$$

span $\mathcal{V}|_U$. Moreover, $L_1, \ldots, L_n, M_1, \ldots, M_m$ span $\mathbb{C}T\Omega|_U$.

The following relations are easily checked, for every $j, j' = 1, \ldots, n, k, k' = 1, \ldots, m$:

$$dZ_k(L_j) = 0, \ dZ_k(M_{k'}) = \delta_{kk'}, \ dt_j(L_{j'}) = \delta_{jj'}, \ dt_j(M_k) = 0,$$

from which we conclude that $L_1, \ldots, L_n, M_1, \ldots, M_m$ are pairwise commuting.

Set, for $W \subset U$ open,

$$\mathfrak{S}(W) \doteq \{u \in C^\infty(W) : L_j u = 0, \quad j = 1, \ldots, n\};$$

it follows, according to the previously established, that

$$\mathfrak{S}_0 = \lim_{W \to \{0\}} \mathfrak{S}(W).$$

We are now ready to give a concrete representation of the Borel map for $\mathcal{V}$ at the origin using this basic set of generators $\{Z_1, \ldots, Z_m\}$. Firstly we observe that if $u \in \mathfrak{S}_0$ then all derivatives up to order k of the solution

$$v_k \doteq u - \sum_{|\alpha| \le k} \frac{M^\alpha u(0)}{\alpha!} Z(x, t)^\alpha$$

vanish at the origin; this can be easily seen for $(M^\beta L^\gamma v_k)(0) = 0$ if $\beta \in \mathbb{Z}_+^m, \gamma \in \mathbb{Z}_+^n$, $|\beta| + |\gamma| \le k$. In particular $v_k \in \mathfrak{m}_0^k \cap \mathfrak{S}_0$ and hence the class of u in $\mathcal{J}(\mathcal{V})_0^k$ equals

that of $u - v_k$, which gives rise to an isomorphism

$$\eta_k : \mathcal{J}_0^k(\mathcal{V})_0 \longrightarrow \mathbb{C}_k[Z_1, \ldots, Z_m]$$

where the latter denotes the vector space of all polynomials in $Z_1, \ldots, Z_m$ of order $\leq k$. Furthermore, for each $k \geq 1$ we have commutative diagrams

$$
\begin{array}{ccc}
\mathcal{J}_0^k(\mathcal{V})_0 & \xrightarrow{\eta_k} & \mathbb{C}_k[Z_1, \ldots, Z_m] \\
\downarrow{\iota_k} & & \downarrow \\
\mathcal{J}_0^{k-1}(\mathcal{V})_0 & \xrightarrow{\eta_{k-1}} & \mathbb{C}_{k-1}[Z_1, \ldots, Z_m]
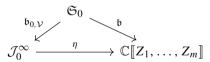\end{array}
$$

where the vertical arrows at the right stand for the natural projections. If we recall that the ring of formal power series $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ equals the projective limit $\lim_{\leftarrow} \mathbb{C}_k[Z_1, \ldots, Z_m]$ we finally obtain an isomorphism

$$\eta : \mathcal{J}_0^\infty \longrightarrow \mathbb{C}[\![Z_1, \ldots, Z_m]\!].$$

For the representation of the Borel map for $\mathcal{V}$ at the origin in terms of $\{Z_1, \ldots, Z_m\}$ we must just observe that the map $\mathfrak{b} : \mathfrak{S}_0 \longrightarrow \mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ given by

$$\mathfrak{b}(u) = \sum_{\alpha \in \mathbb{Z}^m} \frac{(M^\alpha u)(0)}{\alpha!} Z(x, t)^\alpha$$

makes the diagram

$$
\begin{array}{ccc}
 & \mathfrak{S}_0 & \\
\mathfrak{b}_{0,\mathcal{V}} \swarrow & & \searrow \mathfrak{b} \\
\mathcal{J}_0^\infty \xrightarrow{\quad \eta \quad} & & \mathbb{C}[\![Z_1, \ldots, Z_m]\!]
\end{array}
$$

commutative. In particular we conclude that the Borel property for $\mathcal{V}$ holds at the origin if and only if $\mathfrak{b}$ is surjective. Moreover the image of $\mathfrak{b}_{0,\mathcal{V}}$ and $\mathfrak{b}$ are isomorphic.

3 General properties of the Borel map

A. It is our goal in this work to study not only conditions to ensure the surjectivity of $\mathfrak{b}$ but also to analyze its algebraic properties and apply them to the study of the properties of the algebra $\mathfrak{S}_0$.

We first recall a result proved in [3], Lemma 3.2: $\mathfrak{b}$ is surjective if and only if there exists an open neighborhood of the origin $V \subset U$ such that

$$\mathfrak{b}_V : \mathfrak{S}(V) \longrightarrow \mathbb{C}[\![Z_1, \ldots, Z_m]\!] \tag{1}$$

is surjective. Here $\mathfrak{b}_V = \mathfrak{b} \circ \sigma_V$, where $\sigma_V : \mathfrak{S}(V) \to \mathfrak{S}_0$ asssociates to $u \in \mathfrak{S}(V)$ its germ at the origin.

Both $\mathfrak{S}(V)$ and $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ can be endowed with natural Fréchet algebra structures. Indeed the first is a closed subalgebra of $C^\infty(V)$ whereas for the second we consider its usual algebra structure endowed with its Fréchet topology defined by the seminorms $\sum_\alpha a_\alpha Z^\alpha \mapsto |a_\beta|$, $\beta \in \mathbb{Z}_+^m$. Furthermore $\mathfrak{b}_V$ is a homomorphism of Fréchet algebras, a consequence of the Leibniz rule, and has dense image since it contains $\mathbb{C}[Z_1, \ldots, Z_m]$.

Let $\mathfrak{S}_\infty(V, 0)$ denote the ideal of $\mathfrak{S}(V)$ formed by all $v \in \mathfrak{S}(V)$ which vanish to infinite order at the origin. Thus $\ker \mathfrak{b}_V = \mathfrak{S}_\infty(V; 0)$ and hence if $\mathfrak{b}_V$ is surjective we obtain an isomorphism of Fréchet algebras

$$\mathfrak{S}(V)/\mathfrak{S}_\infty(V, 0) \simeq \mathbb{C}[\![Z_1, \ldots, Z_m]\!].$$

Notice that $\mathfrak{b}_V$ can never be an isomorphism for $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ is a local algebra whereas the spectrum of $\mathfrak{S}(V)$ is not an unitary set: if $(x_0, t_0) \in V$ is such that $Z_j(x_0, t_0) \neq 0$ for some $j \in \{1, \ldots, m\}$ then the Dirac measure at (x_0, t_0) defines continuous homomorphism of $\mathfrak{S}(V)$ which is different from the Dirac measure at the origin (cf. Theorem 3.1 in [3]). In general the spectrum of the Fréchet algebra $\mathfrak{S}(V)/\mathfrak{S}_\infty(V, 0)$ equals the set of all nonzero continuous homomorphisms $\mathfrak{S}(V) \to \mathbb{C}$ that vanish on $\mathfrak{S}_\infty(V; 0)$ [7, pp. 81–82]. Hence when $\mathfrak{b}_V$ is surjective the only homomorphism $\mathfrak{S}(V) \to \mathbb{C}$ that vanishes on $\mathfrak{S}_\infty(V; 0)$ is the Dirac measure at the origin.

B. Both $\mathfrak{S}(V)$ and $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ are also Fréchet-Montel spaces. Indeed the former is a closed subspace of the Fréchet-Montel space $C^\infty(V)$ and the latter is isomorphic to a countable product of copies of the complex field, which is easily seen to be Fréchet-Montel (Tychonoff theorem). We will make use the following result:

Proposition 3.1 *Let E, F be Fréchet-Montel spaces and let $A : E \to F$ be a continuous linear map with $A(E)$ dense in F. The following properties are equivalent:*

1. *$A(E) = F$;*
2. *${}^t A(F')$ is strongly closed;*
3. *$\forall \mathcal{B} \subset F'$, ${}^t A(\mathcal{B}) \subset E'$ strongly bounded $\Rightarrow \mathcal{B}$ strongly bounded;*
4. *$\forall \{y_j'\} \subset F'$, $\{{}^t A(y_j')\}$ strongly bounded $\Rightarrow \{y_j'\}$ is strongly bounded;*
5. *${}^t A(F')$ is strongly sequentially closed in E'.*

Proof The equivalence of (1) and (2) follows from [9], p. 22. The equivalence of (1) and (5) follows from [9], p. 18. Now, since ${}^t A$ is injective, (2) implies that $({}^t A)^{-1} : {}^t A(F') \to F'$ is continuous with respect to the strong topologies and then it maps strongly bounded sets into strongly bounded sets, which gives (3). It is clear that (3) implies (4). Assume now that (4) holds and let $\{{}^t A(y_j')\} \subset {}^t A(F')$, ${}^t A(y_j') \to x'$ strongly in E'. By (4) $\{y_j'\}$ is strongly bounded in E'. Since E', endowed with the strong topology, is also a Montel space, it follows that $\overline{\{y_j'\}}$ is compact, which in particular implies that $\overline{\{{}^t A(y_j')\}} = {}^t A\overline{\{y_j'\}}$. Then $x' \in {}^t A(E')$, which proves (5).

We apply Proposition 3.1 with $A = \mathfrak{b}_V$, $E = \mathfrak{S}(V)$, $F = \mathbb{C}[\![Z_1, \ldots, Z_m]\!]$. The dual of $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ is the space $\mathbb{C}[Z, \ldots, Z_m]$ under the duality

$$\mathbb{C}[\![Z_1, \ldots, Z_m]\!] \times \mathbb{C}[Z_1, \ldots, Z_m] \to \mathbb{C}, \quad \left(\sum_\alpha a_\alpha Z^\alpha, \sum_{\text{finite}} b_\alpha Z^\alpha \right) \mapsto \sum a_\alpha b_\alpha.$$

Hence the transpose of $\mathfrak{b}_V$ is the map $\mathbb{C}[Z, \ldots, Z_m] \ni P \mapsto \lambda_P \in \mathfrak{S}(V)'$,

$$\lambda_P(f) = (\tilde{P}(M)f)(0), \quad f \in \mathfrak{S}(V),$$

where $\tilde{P}$ is the polynomial obtained from P after dividing its coefficient b_α by $\alpha!$. Thus the Borel map $\mathfrak{b}_V$ is surjective if and only if given any sequence of polynomials $P_j \in \mathbb{C}[Z_1, \ldots, Z_m]$ with λ_{P_j} bounded in $\mathfrak{S}(V)'$ then P_j is bounded in $\mathbb{C}[Z_1, \ldots, Z_m]$.

Now a sequence P_j is bounded in $\mathbb{C}[Z_1, \ldots, Z_m]$ if and only if there is k such that $\text{degree}(P_j) \leq k$ for every j and the sequences of the corrresponding coefficients are bounded in $\mathbb{C}$. On the other hand the sequence λ_{P_j} is bounded in $\mathfrak{S}(V)'$ if and only if it is equicontinuous, that is

$$\begin{gathered} \textit{There are an open set } 0 \in W \subset\subset V, \ \ell \in \mathbb{Z}_+ \textit{ and } C > 0 \textit{ such that} \\ |(\tilde{P}_j(M)f)(0)| \leq C\|f\|_{C^\ell(\bar{W})}, \quad f \in \mathfrak{S}(V), \ j \geq 1. \end{gathered} \tag{3.1}$$

Notice that applying (3.1) to the monomials $f = Z^\beta$ implies that the sequence of corresponding coefficients of P_j is bounded in $\mathbb{C}$. We summarize:

Proposition 3.2 $\mathfrak{b}_V$ *is surjective if and only if the following holds: given any sequence of polynomials* $P_j \in \mathbb{C}[Z_1, \ldots, Z_m]$ *satisfying (3.1) then* $\sup\{\text{degree}(P_j)\} < \infty$.

4 The partial Borel maps

A. We keep the notation established in the previous section and start with a digression regarding the theory of tensor products in the category of Fréchet spaces.

Let $1 \leq p < m$ and consider the natural inclusions

$$\mathbb{C}[\![Z_1, \ldots, Z_p]\!] \hookrightarrow \mathbb{C}[\![Z_1, \ldots, Z_m]\!], \quad \mathbb{C}[\![Z_{p+1}, \ldots, Z_m]\!] \hookrightarrow \mathbb{C}[\![Z_1, \ldots, Z_m]\!].$$

Then $\mathbb{C}[\![Z_1, \ldots, Z_p]\!] \otimes \mathbb{C}[\![Z_{p+1}, \ldots, Z_m]\!]$ can be identified to the (dense) subspace of $\mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ formed by all power series of the form

$$\sum_{j=1}^M S_{1,j}(Z_1, \ldots, Z_p) S_{2,j}(Z_{p+1}, \ldots, Z_m).$$

Recall that $\mathbb{C}[[Z_1, \ldots, Z_p]] \widehat{\otimes}_\pi \mathbb{C}[[Z_{p+1}, \ldots, Z_m]]$ is the completion of this space endowed with the strongest locally convex topology which makes the natural map

$$B : \mathbb{C}[[Z_1, \ldots, Z_p]] \times \mathbb{C}[[Z_{p+1}, \ldots, Z_m]] \to \mathbb{C}[[Z_1, \ldots, Z_p]] \otimes \mathbb{C}[[Z_{p+1}, \ldots, Z_m]]$$

continuous. On the other hand since the space of formal power series is nuclear [12], p. 526, Corollary 1, it follows from [12], p. 511, Theorem 50.1 that the canonical map of

$$\mathbb{C}[[Z_1, \ldots, Z_p]] \widehat{\otimes}_\pi \mathbb{C}[[Z_{p+1}, \ldots, Z_m]] \longrightarrow \mathbb{C}[[Z_1, \ldots, Z_p]] \widehat{\otimes}_\varepsilon \mathbb{C}[[Z_{p+1}, \ldots, Z_m]]$$

is an isomorphism (cf. the definition of the ε topology in [12], page 434). In other words both π and ε topologies coincide. If we apply the same reasoning as in the proof of [12], p. 531, Theorem 51.6, it follows that $\mathbb{C}[[Z_1, \ldots, Z_p]] \widehat{\otimes} \mathbb{C}[[Z_{p+1}, \ldots, Z_m]] \cong \mathbb{C}[[Z_1, \ldots, Z_m]]$.

By a property of the π-topology [14], Theorem 6.4, p. 63, it then follows that every element $S \in \mathbb{C}[[Z_1, \ldots, Z_m]]$ can be represented in the form

$$S = \sum_{j=1}^\infty S_{1,j}(Z_1, \ldots, Z_p) S_{2,j}(Z_{p+1}, \ldots, Z_m), \tag{4.1}$$

where

$$\sum_{j=1}^\infty q_k(S_{1,j}) q_k(S_{2,j}) < 1$$

and $q_1 < q_2 < \ldots$ is a sequence of continuous seminorms that define the Fréchet topology in $\mathbb{C}[[Z_1, \ldots, Z_m]]$.

B. Denote by $\mathfrak{S}_0^{(1)}$ (resp. $\mathfrak{S}_0^{(2)}$) the space of all $u \in \mathfrak{S}_0$ such that $M^\alpha u(0) = 0$ if $\alpha \not\subset \{1, \ldots, p\}$ (resp. $\alpha \not\subset \{p+1, \ldots, n\}$). We then obtain homomorphisms induced by $\mathfrak{b}$:

$$\mathfrak{b}_1 : \mathfrak{S}_0^{(1)} \to \mathbb{C}[[Z_1, \ldots, Z_p]], \quad \mathfrak{b}_2 : \mathfrak{S}_0^{(2)} \to \mathbb{C}[[Z_{p+1}, \ldots, Z_m]].$$

We shall refer to the maps $\mathfrak{b}_\ell$ as the *partial Borel maps* for $\mathcal{V}$ at the origin with the respect to the decomposition $\{1, \ldots, m\} = \{1, \ldots, p\} \cup \{p+1, \ldots, m\}$.

Theorem 4.1 *The Borel map $\mathfrak{b}$ is surjective if and only if each $\mathfrak{b}_\ell$ is surjective, $\ell = 1, 2$.*

Proof If $\mathfrak{b}$ is surjective and if $S \in \mathbb{C}[[Z_1, \ldots, Z_p]] \subset \mathbb{C}[[Z_1, \ldots, Z_m]]$ then there is $u \in \mathfrak{S}_0$ such that $\mathfrak{b}(u) = S$. But a fortiori $u \in \mathfrak{S}_0^{(1)}$ by the definition of $\mathfrak{b}$ and thus $\mathfrak{b}_1(u) = \mathfrak{b}(u) = S$, which shows that $\mathfrak{b}_1$ is surjective. An analogous argument shows the surjectivity of $\mathfrak{b}_2$.

We show the converse. Firstly we remark that if V is an open neighborhood of the origin and if we denote by $\mathfrak{S}^{(\ell)}(V)$, $j = 1, 2$, the space of all $u \in \mathfrak{S}(V)$ such that

the germ of u at the origin belongs the $\mathfrak{S}_0^{(\ell)}$ then each $\mathfrak{S}^{(\ell)}(V)$ is a closed subspace of $\mathfrak{S}(V)$ and hence also a Fréchet space.

By a Baire category argument (cf. Lemma 3.2 in [3]) there is an open neighborhood V of the origin such that both induced maps

$$\mathfrak{b}_{1,V} : \mathfrak{S}^{(1)}(V) \to \mathbb{C}[\![Z_1, \ldots, Z_p]\!], \quad \mathfrak{b}_{2,V} : \mathfrak{S}^{(2)}(V) \to \mathbb{C}[\![Z_{p+1}, \ldots, Z_m]\!]$$

are surjections between Fréchet spaces. From [14], Theorem 6.6, p. 65, it follows that

$$\mathfrak{b}_{1,V} \widehat{\otimes} \mathfrak{b}_{2,V} : \mathfrak{S}^{(1)}(V) \otimes_\pi \mathfrak{S}^{(2)}(V) \longrightarrow \mathbb{C}[\![Z_1, \ldots, Z_p]\!] \widehat{\otimes}_\pi \mathbb{C}[\![Z_{p+1}, \ldots, Z_m]\!]$$

is a surjection between Fréchet spaces.

Thus by [14], Theorem 6.5, p. 63, given S as in (4.1) there are $u_j \in \mathfrak{S}^{(1)}(V)$, $v_j \in \mathfrak{S}^{(2)}(V)$ such that $\sum_{j=1}^\infty u_j(x,t)v_j(y,t)$ converges in $C^\infty(V \times V)$ and such that

$$S = \sum_{j=1}^\infty \mathfrak{b}_1(u_j)\mathfrak{b}_2(v_j).$$

Now since each $\mathfrak{b}_j$ is defined as the restriction of $\mathfrak{b}$ we can further write

$$S = \sum_{j=1}^\infty \mathfrak{b}(u_j)\mathfrak{b}(v_j) = \sum_{j=1}^\infty \mathfrak{b}(u_j v_j),$$

since $\mathfrak{b}$ is an algebra homomorphism. But then if we set $u(x,t) \doteq \sum_{j=1}^\infty u_j(x,t)v_j(x,t)$ then $u \in \mathfrak{S}(V)$ and $\mathfrak{b}(u) = S$, which completes the proof.

Still keeping the notation previously established we consider the locally integrable structure $\mathcal{V}_1$ over U defined as $\mathcal{V}_1^\perp = \text{span}\{dZ_1, \ldots, Z_p\}$. Notice that a u is a solution for $\mathcal{V}_1$ if and only if

$$\mathrm{L}_j u = 0, \ \mathrm{M}_k u = 0, \quad j = 1, \ldots, n, \ k = p+1, \ldots, n.$$

In particular $\mathrm{M}^\alpha u = 0$ in a full neighborhood of the origin if $\alpha \not\subset \{1, \ldots, p\}$ and consequently the following statement is immediate:

Proposition 4.1 *If the Borel map for $\mathcal{V}_1$ at the origin is surjective then the same is true for the partial Borel map $\mathfrak{b}_1$.*

5 Partial hypocomplexity

In this section we continue to write $Z(x,t) = (Z_1(x,t), \ldots, Z_m(x,t)) \in \mathbb{C}^m$ and remark that for a fixed structure $\mathcal{V}$ all concepts below are independent of a particular choice of such map.

A. In the first paragraph of this section we recall the concept of hypocomplexity and some results presented in [12]. Denote by $\mathcal{O}^{(m)}$ the sheaf of germs of holomorphic functions at the origin in $\mathbb{C}^m$. We say that $\mathcal{V}$ is *hypocomplex at the origin* if every germ of (weak) solution u for $\mathcal{V}$ at the origin can be written as $u = H \circ Z$ for some $H \in \mathcal{O}^{(m)}$. In this case given any solution u for $\mathcal{V}$ defined near the origin we have, for some constant $C > 0$, $|\mathrm{M}^\alpha u(0)| \leq C^{|\alpha|+1}\alpha!$, $\alpha \in \mathbb{Z}_+^m$, and consequently hypocomplexity at the origin implies the non surjectivity of the Borel map.

The following theorem gives a complete characterization of hypocomplexity in terms of the compact neighborhoods of the origin in U. If we recall that for a compact set $P \subset \mathbb{C}^m$ its rational hull can be characterized as the set all $z \in \mathbb{C}^m$ having the following property: *every algebraic hypersurface through z intersects P*, we can state Theorem III.5.1 in [12] in the following form:

Theorem 5.1 *The following properties are equivalent:*

1. *$\mathcal{V}$ is hypocomplex at the origin;*
2. *For every compact neighborhood $K_0 \subset\subset U$ of the origin in $\mathbb{R}^N$ the rational hull of $Z(K_0)$ is a neighborhood of the origin in $\mathbb{C}^m$;*
3. *For every compact neighborhood $K_0 \subset\subset U$ of the origin in $\mathbb{R}^N$ the polynomial hull of $Z(K_0)$ is a neighborhood of the origin in $\mathbb{C}^m$.*

As a consequence we obtain:

Corollary 5.1 *If $\mathcal{V}$ is hypocomplex at the origin then any non constant solution near the origin is open at the origin.*

For a proof see ([12], Corollary III.5.2).

Corollary 5.2 *Assume $m = 1$. Then $\mathcal{V}$ is hypocomplex at the origin if and only if Z is open at the origin.*

Proof The rational hull of any compact set in $\mathbb{C}$ is the compact itself.

B. Recall that if $\mathcal{V}$ is a locally integrable structure over Ω its characteristic set is the subset of $\mathrm{T}^*\Omega$ defined by $\mathrm{T}^\circ = \mathcal{V}^\perp \cap \mathrm{T}^*\Omega$.

Taking into account Corollary 5.1, and for further reference, we conclude this section introducing a weakened version of hypocomplexity:

Definition 5.1 We shall say that $\mathcal{V}$ is *partially hypocomplex at the origin* if there is a smooth solution W for $\mathcal{V}$ near the origin, with $dW|_0 \in \mathrm{T}_0^\circ \setminus 0$, such that W is open at the origin.

Remark 5.1 Write the coordinates in $\mathbb{C}^2$ as $z = x + iy$, $w = s + it$ and consider the hypersurface Σ defined by $t = s|z|^2$. The CR structure $\mathcal{V}$ on Σ is such that its orthogonal is spanned by the differentials of the functions $Z_1 = x + iy$, $Z_2 = s + is|z|^2$. The characteristic set at the origin is spanned by $ds|_0$ and the function $W = Z_2 + iZ_1^2$ is a solution with $dW(0) = ds|_0$. Moreover introducing $s' = s - 2xy$ as a new variable we have

$$W(x, y, s') = s' + i(x^2 - y^2 + (s' + 2xy)(x^2 + y^2))$$

and then $(\mathrm{Im} W)(x, y, 0)$ changes sign at the origin in $\mathbb{R}^2$. Hence W is open at the origin and consequently this CR structure is partially hypocomplex (but not hypocomplex) at the origin. $\qquad\square$

6 A necessary condition for the surjectivity of the Borel map

In the preceding section we have seen that when $\widehat{Z(K)}$ is a neighborhood of the origin $\mathbb{C}^m$ ($K \subset U$ a compact neighborhood of the origin) the Borel map is not surjective. We now prove a much stronger statement:

Theorem 6.1 *Suppose that for every $K \subset U$ compact neighborhood of the origin the polynomial hull $\widehat{Z(K)}$ of $Z(K)$ in $\mathbb{C}^m$ contains a non constant complex curve through the origin. Then the Borel map for $\mathcal{V}$ at the origin is not surjective.*

Proof Let u be a solution for $\mathcal{V}$ defined near the origin. There are a compact neighborhood K of the origin in $\mathbb{R}^N$ and a sequence of polynomials $P_\nu \in \mathbb{C}[z_1, \ldots, z_m]$ such that $P_\nu \circ Z$ converges to u over K in the C^∞ topology (the Baouendi–Treves approximation theorem). In particular P_ν converges uniformly over $Z(K)$. Now by hypothesis there is a non constant complex curve $\tau \mapsto \gamma(\tau) \in \widehat{Z(K)}$, defined near the origin in the complex plane and such that $\gamma(0) = 0$. Hence $P_\nu(\gamma(\tau))$ converges uniformly to a holomorphic function $\alpha(\tau)$ in a neighborhood of the origin in $\mathbb{C}$. In particular

$$\frac{\mathrm{d}^k}{\mathrm{d}\tau^k} P_\nu(\gamma(\tau))|_{\tau=0} \to \alpha^{(k)}(0) \tag{6.1}$$

for every k. On the other hand, the Faà di Bruno formula gives

$$\frac{\mathrm{d}^k}{\mathrm{d}\tau^k} P_\nu(\gamma(\tau))|_{\tau=0} = \sum_{1 \leq |\alpha| \leq k} \Lambda_{\alpha,k} (\partial_z^\alpha P_\nu)(0),$$

$$\Lambda_{\alpha,k} \doteq \sum_{s=1}^{k} \sum_{p_s(\alpha,k)} k! \prod_{j=1}^{s} \frac{[\gamma^{(\ell_j)}(0)]^{\alpha_j}}{\alpha_j! \ell_j!^{|\alpha_j|}}$$

where $p_s(\alpha, k)$ is the set of all $(\alpha_1, \ldots, \alpha_s, \ell_1, \ldots, \ell_s) \in (\mathbb{Z}_+^m)^s \times \mathbb{Z}_+^s$ satisfying $|\alpha_j| > 0$, $\sum \alpha_j = \alpha$ and $\sum |\alpha_j| \ell_j = k$.

By hypothesis there is $r \geq 1$ such that

$$\gamma(\tau) = \tau^r \gamma_\bullet(\tau)/r!, \quad \zeta \doteq \gamma_\bullet(0) \neq 0.$$

Thus $\gamma^{(j)}(0) = 0$ if $j \leq r - 1$ and $\gamma^{(r)}(0) = \zeta \neq 0$.

We assume $k = rq$, where $q = 1, 2, \ldots$ and consider two cases:

– Case 1: $|\alpha| > q$. If $(\alpha_1, \ldots, \alpha_s, \ell_1, \ldots, \ell_s) \in p_s(\alpha, rq)$ we have $\sum_j |\alpha_j|\ell_j < |\alpha|r$. Hence $\ell_j < r$ for some j and thus $\Lambda_{\alpha,rq} = 0$. $\quad\square$
– Case 2: $|\alpha| = q$. f $(\alpha_1, \ldots, \alpha_s, \ell_1, \ldots, \ell_s) \in p_s(\alpha, rq)$ we have $\sum_j |\alpha_j|\ell_j = |\alpha|r$. Hence if $\ell_j \geq r$ for every j we necessarily must have $\ell_j = r$ for every j $\quad\square$

Summing up when $k = rq$ we conclude that $\Lambda_{\alpha,rq} = 0$ if $|\alpha| > q$ and

$$\Lambda_{\alpha,rq} = (rq)! \sum_{s=1}^{rq} \sum_{\substack{\sum_{j=1}^{s}\alpha_j=\alpha \\ \alpha_j\neq 0}} \prod_{j=1}^{s} \frac{[\gamma^{(r)}(0)]^{\alpha_j}}{\alpha_j!r!^{|\alpha_j|}} \doteq A_{\alpha,q}\zeta^{\alpha} \quad \text{if } |\alpha| = q,$$

where $A_{\alpha,q}$ is a positive constant. Thus

$$\frac{d^{rq}}{d\tau^{rq}} P_\nu(\gamma(\tau))|_{\tau=0} = \sum_{|\alpha|=q} A_{\alpha,q}\zeta^{\alpha}(\partial_z^{\alpha} P_\nu)(0) + Q_q(\partial_z) P_\nu(0)$$

where $Q_q(X) = \sum_{|\beta|\leq q-1} Q_{q,\beta}X^{\beta}/\beta! \in \mathbb{C}[X_1, \ldots, X_m]$ has degree $\leq q - 1$.
   Now since

$$(\partial^{\alpha} P_\nu/\partial z^{\alpha})(0) = M^{\alpha}\{P_\nu \circ Z\}|_{(x,t)=(0,0)}$$

from (6.1) we obtain

$$\alpha^{(rq)}(0) = \sum_{|\alpha|=q} A_{\alpha,q}\zeta^{\alpha}(M^{\alpha}u)(0) + Q_{rq}(M)u(0)$$

and consequently for some constant $C > 0$ we have

$$\left| \sum_{|\alpha|=q} A_{\alpha,q}\zeta^{\alpha}(M^{\alpha}u)(0) + Q_q(M)u(0) \right| \leq C^{q+1}(rq)!.$$

In particular, if $\sum_{\beta} a_\beta Z(x,t)^{\beta}/\beta! \in \mathbb{C}[\![Z_1, \ldots, Z_m]\!]$ belongs to the image of the Borel map for $\mathcal{V}$ at the origin then

$$\left| \sum_{|\alpha|=q} A_{\alpha,q}a_\alpha\zeta^{\alpha} + \sum_{|\beta|\leq q-1} Q_{q,\beta}a_\beta \right| \leq C^{q+1}(rq)!, \quad k \geq 0,$$

for some $C > 0$. Since it is easy to construct indutively a sequence (a_β) for which this property is not satisfied for any $C > 0$ our proof is complete.

Remark 6.1 Our argument in the proof of Theorem 6.1 can be enlightened by the following discussion. Given a formal curve $\gamma(t) \in \mathbb{C}[\![t]\!]^m$, with $\gamma(0) = 0$, the map

$$\gamma^* : \mathbb{C}[\![z_1, \ldots, z_m]\!] \longrightarrow \mathbb{C}[\![t]\!], \quad u \mapsto u \circ \gamma,$$

is onto if $\gamma'(0) \neq 0$. More generally, if $\gamma(t) = t^d \delta(t)$ with $\delta(0) \neq 0$, and if we consider the projection map

$$\pi : \mathbb{C}[\![t]\!] \longrightarrow \mathbb{C}[\![t^d]\!], \quad \pi\left(\sum_j \alpha_j t^j\right) = \sum_k \alpha_{dk} t^{dk},$$

then

$$\pi \circ \gamma^* : \mathbb{C}[\![z_1, \ldots, z_m]\!] \longrightarrow \mathbb{C}[\![t^d]\!]$$

is onto, since by the Faà di Bruno formula, for each k there exists a polynomial p_k such that the coefficient of t^{dk} in $(\pi \circ \gamma^*)(\sum_\alpha a_\alpha Z^\alpha)$ can be written as

$$\sum_{|\alpha|=k} a_\alpha \delta(0)^\alpha + p_k(a_\beta, \delta_j : |\beta| < k, j \leq dk).$$

Theorem 6.1 shows that if γ happens to be an analytic curve contained in $\widehat{Z(K)}$, then by the Baouendi–Treves approximation theorem,

$$(\pi \circ \gamma^*)\,(\mathfrak{b}(\mathfrak{S}_0)) \subset \mathbb{C}\{t^d\}$$

and hence the Borel property must fail. $\qquad\square$

Remark 6.2 For the CR structure defined in Remark 5.1 the Borel map at the origin is not surjective since the complex curve $w = 0$ is contained Σ.

Remark 6.3 Write the coordinates in $\mathbb{C}^3$ as $z_j = x_j + iy_j$, $j = 1, 2$, and $w = s + it$ and consider the hypersurface Σ defined by

$$t = \left|z_1^2 - z_2^3\right|^2.$$

Let $\mathcal{V}$ be the CR structure on Σ induced by the complex structure in $\mathbb{C}^3$. Since Σ contains the germ of the curve $\zeta \mapsto (\zeta^3, \zeta^2, 0)$ it follows from Theorem 6.1 that the Borel map for $\mathcal{V}$ at the origin is not surjective. We do conjecture that the polynomial hull of a compact neighbourhood of 0 in M also does not contain any regular curve. For such a compact neighborhood of the origin $K \subset \Sigma$ in Σ it can be shown (see [5]) that the the analogous question for the *holomorphic hull* of K has an affirmative answer, that is, the holomorphic hull of K does not contain any germ of a *regular* curve curve through the origin.

7 Sufficient conditions for the surjectivity of the Borel map

In this section we recall two conditions which imply the surjectivity of the Borel map.

A. Here we assume that $\mathcal{V}$ defines on Ω a CR structure of the hypersurface type. Hence we have $\mathcal{V} \cap \overline{\mathcal{V}} = 0$ and T$^{\mathrm{o}}$ is a real line subbundle of T$^*\Omega$. Let $p \in \Omega$, let $V \subset \Omega$ be an open neighborhood of p and let $\psi \in \mathfrak{S}(V)$. We say that ψ is a *peak function at* p if $\psi(p) = 0$, $\psi(q) \neq 0$ for $q \neq p$ and $\arg \psi \neq -\pi$ in $V \backslash \{p\}$. Furthermore, we say that a peak function if of *finite type* if $|\psi(q)| \geq Cd(q, p)^\alpha$ for positive constants C and α.

The following theorem is the main result in [6] :

Theorem 7.1 *If $\mathcal{V}$ is a CR structure of the hypersurface type in Ω which admits a peak function of finite type at $p \in \Omega$ then the Borel map for $\mathcal{V}$ at p is surjective.*

We shall present a sketch of a proof of this result based on the characterization given in Proposition 3.2. For this we shall show that given any sequence of polynomials $P_j \in \mathbb{C}[Z_1, \ldots, Z_m]$ such that degree$(P_j) \to \infty$, given $0 \in W \subset U$ a neighborhood of the origin and $\ell \in \mathbb{Z}_+$ there is a sequence $f_j \in \mathfrak{S}(W)$ such that

$$|(\tilde{P}_j(\mathrm{M}) f_j)(0)| = 1, \quad \|f_j\|_{C^\ell(\bar{W})} \to 0.$$

We can assume that Ω is a hypersurface embedded in $\mathbb{C}^m$ and that $Z_j = z_j|_\Omega$, $j = 1, \ldots, m$, where $(z_1, \ldots, z_m)$ are the holomorphic coordinates in $\mathbb{C}^m$. We can also assume that the peak function ψ is defined in in V.

For any j let $b_{\alpha_j} z^{\alpha_j}$ be a non-vanishing monomial of P_j of maximal degree, and define $C_{\alpha_j} = 1/b_{\alpha_j}$. For any multiindex $\beta \in \mathbb{Z}_+^m$ we put $d_\beta = 2^{-|\beta|}$. We define a sequence $f_j \in \mathfrak{S}(V)$ by $f_j = C_{\alpha_j} Z^{\alpha_j} \varphi_{\alpha_j}$, where $\varphi_{\alpha_j} \in \mathfrak{S}(V)$ is the function constructed in [6], Lemma 4.2. We have that $\varphi_{\alpha_j}(0) = 1$ for all $j \in \mathbb{N}$, and all its derivative vanish at 0 (see [6] Corollary 4.3).

By [6] Lemma 5.1 and more in particular from equation (5.4) in [6], we have the following: for fixed $\beta \in \mathbb{Z}_+^m$ there exists $j_0(\beta) \in \mathbb{Z}_+$ such that $|\mathrm{M}^\beta f_j(q)| \leq A|\alpha_j|^{|J|} d_{\alpha_j}$ for all $q \in V$ and $j \geq j_0(\beta)$ (here we are using the fact that $|\alpha_j| \to \infty$ as $j \to \infty$), where the constant A depends on $|\beta|$ but not on α_j. Using these inequalities for all $\beta \in \mathbb{Z}_+^m$ with $|\beta| \leq \ell$, it follows that there exist $j_1(\ell) \in \mathbb{Z}_+$ and $A_1 = A_1(\ell) > 0$ such that $\|f_j\|_{C^\ell(V)} \leq A_1 |\alpha_j|^\ell d_{\alpha_j}$. Thus $\|f_j\|_{C^\ell(\bar{W})} \to 0$ as $j \to \infty$ for any neighborhood $W \subset\subset V$ of the origin.

On the other hand, let us consider $(\tilde{P}_j(\mathrm{M}) f_j)(0)$. Since $(\mathrm{M}_k \varphi_{\alpha_j})(0) = 0$ for all k, it follows that $(\tilde{P}_j(\mathrm{M}) f_j)(0) = C_{\alpha_j}(\tilde{P}_j(\mathrm{M}) Z^{\alpha_j})(0) \varphi_{\alpha_j}(0) = C_{\alpha_j}(\tilde{P}_j(\mathrm{M}) Z^{\alpha_j})(0)$. Using that $\mathrm{M}_k Z_{k'} = \delta_{kk}$ it is clear that $\mathrm{M}^\beta Z^{\alpha_j}(0) = 0$ for all $\beta \neq \alpha_j$, hence $C_{\alpha_j}(\tilde{P}_j(\mathrm{M}) Z^{\alpha_j})(0) = C_{\alpha_j} b_{\alpha_j}(\mathrm{M}^{\alpha_j} Z^{\alpha_j})(0)/\alpha_j! = C_{\alpha_j} b_{\alpha_j} = 1$, which completes the proof. □

B. Next we introduce a very similar condition stated in [3] which now applies to an arbitrary locally integrable structure $\mathcal{V}$. We say that $\mathcal{V}$ satisfies condition ($\mathfrak{B}$) at $p \in \Omega$ if there is a smooth solution W for $\mathcal{V}$ near p such that the following conditions holds:

1. $W(p) = 0$, $\mathrm{d}W(p) \in \mathrm{T}_p^{\mathrm{o}} \backslash 0$ and $\arg W \neq -\pi/2$ near p;
2. There are smooth solutions $W_1, \ldots, W_{m-1}$ defined in a neighborhood of p, $W_j(p) = 0$, such that $\mathrm{d}W_1, \ldots, \mathrm{d}W_{m-1}, \mathrm{d}W$ are linearly independent and positive constants μ and C such that $(|W_1| + \cdots + |W_{m-1}|)^\mu \leq C|W|$ near p.

The main result in [3] is the following:

Theorem 7.2 *Property ($\mathfrak{B}$) at p implies the surjectivity of the Borel map for $\mathcal{V}$ at p.*

It is an easy corollary of Theorem 7.2 the fact that when $\mathcal{V}$ has rank $N - 1$, that is when $\mathcal{V}^\perp$ is locally spanned by the differential of a single function, *the surjectivity of the Borel map at $p \in \Omega$ is equivalent to the fact that $\mathcal{V}$ is not hypocomplex at p* ([3], Corollary 6.2).

The conjunction of Theorem 4.1 and this result allows us to obtain the following statement:

Theorem 7.3 *Assume that the characteristic set for the locally integrable $\mathcal{V}$ over Ω at $p \in \Omega$ has maximum dimension (= m). If $\mathcal{V}$ is not partially hypocomplex at p then the Borel map for $\mathcal{V}$ at p is surjective.*

Proof Since $\dim \mathrm{T}_p^o = m$ by ([4] Theorem I.10.1) we can find smooth solutions $Z_1, \ldots, Z_m$ for $\mathcal{V}$ near p with $dZ_1, \ldots, dZ_m$ linearly independent and $dZ_j(p) \in \mathrm{T}_p^o$ for all $j = 1, \ldots, m$. By hypothesis none of the functions Z_j is open at p and hence by Corollary 4.1 and the result just stated we conclude that the Borel maps for the structures $\mathcal{V}_j = \mathrm{span}\,\{dZ_j\}^\perp$ are surjective at p. Hence Proposition 4.1 in conjunction with Theorem 4.1 gives the sought conclusion.

8 A class of tubular structures

A. We recall (cf. [13], p. 308) that a locally integrable structure $\mathcal{V}$ over Ω of rank n is *tubular* if given any point $p \in \Omega$ there are an open neighborhood U of p and an abelian finite dimensional subalgebra $\mathfrak{g}$ of $C^\infty(U; T\Omega)$ such that $[\mathfrak{g}, \mathcal{V}|_U] \subset \mathcal{V}|_U$, $\dim \mathfrak{g}_q = \dim \mathfrak{g}$ and $\mathbb{C}T\Omega_q = \mathcal{V}_q + \mathfrak{g}_q$ for all $q \in U$. Here

$$\mathfrak{g}_q = \{X_q : X \in \mathfrak{g}\} \subset \mathrm{T}_q\Omega, \quad q \in U.$$

It is proved in ([12], p. 308) that $\mathcal{V}$ is tubular if and only if given any point $p \in \Omega$ there are, as in section 1(B), a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ centered at p ($N = m + N$) and defined in an open neighborhood $U = B \times \Theta$ of the origin in $\mathbb{R}^N$ and a smooth map $\Phi = (\Phi_1, \ldots, \Phi_m) : \Theta \to \mathbb{R}^m$ satisfying $\Phi(0) = 0$ such that $\mathcal{V}^\perp$ is spanned over U by the differential of the functions

$$Z_j(x, t) = x_j + \Phi_j(t), \quad j = 1, \ldots, m.$$

Observe that a set of n linearly independent vector fields which span $\mathcal{V}|_U$ is given by

$$\mathrm{L}_j = \frac{\partial}{\partial t_j} - i \sum_{k=1}^m \frac{\partial \Phi_k}{\partial t_j}(t) \frac{\partial}{\partial x_k}, \quad j = 1, \ldots, n.$$

Moreover since that in this particular case the vector fields M_k equal $\partial/\partial x_k$ the Borel map at the origin for $\mathcal{V}$ is given by

$$\mathfrak{S}_0 \ni u \mapsto \mathfrak{b}(u) = \sum_{\alpha \in \mathbb{Z}^m} \frac{(\partial_x^\alpha u)(0,0)}{\alpha!} Z(x,t)^\alpha.$$

From now on we shall assume that

Ω and $\mathcal{V}$ are real-analytic.

In particular Φ is a real-analytic map. The main reason for assuming such a hypothesis is that in this case hypocomplexity for $\mathcal{V}$ at the origin is perfectly determined: by a result due to Baouendi and Treves [2] this structure $\mathcal{V}$ is hypocomplex at the origin if and only if for every $\xi \in \mathbb{R}^m \backslash \{0\}$ the map $t \mapsto \Phi(t) \cdot \xi$ is open at the origin.

B. Assume that $m = n + 1$ and suppose that Φ has the special form

$$\Phi(t) = (t, \phi(t))$$

where $\phi : \mathbb{R}^n \to \mathbb{R}$ is real analytic, $\phi(0) = 0$, $d\phi(0) = 0$. Such structure $\mathcal{V}_\bullet$ is CR of the hypersurface type: indeed in this case it is the CR structure induced by the complex structure on $\mathbb{C}^{n+1}$, where the complex coordinates are written as $(z_1, \ldots, z_{n+1})$, on the hypersurface defined by $\operatorname{Im} z_{m+1} = \phi(\operatorname{Im} z_1, \ldots, \operatorname{Im} z_n)$. Notice that for this structure the CR vector fields read

$$L_j = \frac{\partial}{\partial t_j} - i\frac{\partial}{\partial x_j} - i\frac{\partial\phi}{\partial t_j}(t)\frac{\partial}{\partial x_m}, \quad j = 1, \ldots, m.$$

Notice also that in this case

$$\xi \cdot \Phi(t) = \sum_{j=1}^{m-1} \xi_j t_j + \xi_m \phi(t)$$

which is open at the origin if $\xi_j \neq 0$ for some $j = 1, \ldots, m-1$. Hence $\mathcal{V}_\bullet$ is hypocomplex at the origin if and only if $\phi(t)$ is open at the origin.

We first study the case $n = 1$, which is very simple. If $\mathcal{V}$ is not hypocomplex at the origin then either ϕ has a zero of even order at the origin or else ϕ vanishes identically. In the latter case we are in the Levi flat case in which case the Borel map at the origin is not surjective whereas that in the former case the argument in the proof of Theorem 8.1 below shows the existence of a peak function for $\mathcal{V}_\bullet$ at the origin and hence the surjectivity of the Borel map at the origin follows from Theorem 7.1.

In what follows we then assume that $n \geq 2$ and that ϕ does not vanish identically.

Our discussion of the surjectivity of the Borel map for this particular CR structure will be given in terms of the (germ of the) variety $V \doteq \phi^{-1}\{0\}$. We start with the following result:

Theorem 8.1 *Let $\mathcal{V}_\bullet$, ϕ and V be as before.*

1. *If ϕ is open at the origin then the Borel map for $\mathcal{V}_\bullet$ is not surjective;*
2. *If $V = \{0\}$ then the Borel map for $\mathcal{V}_\bullet$ is surjective.*

Proof We have already seen that if ϕ is open then $\mathcal{V}_\bullet$ is hypocomplex at the origin and hence (1) follows.

For (2) we can assume without loss of generality that $\phi > 0$ outside the origin. Hence from the analiticity of ϕ we conclude that $\phi(t) \geq c|t|^{2q}$ if $|t| \leq r$, where $c > 0$, $r > 0$ are small constants and $q \in \mathbb{N}$. We set

$$\psi(x,t) = -i(x_m + i\phi(t)) + (x_m + i\phi(t))^2 + \kappa \sum_{j=1}^{m-1} (x_j + it_j)^{2q},$$

where κ is positive small constant. It is clear that $\psi \in \mathfrak{S}(U)$. Furthermore if $r > 0$ is chosen such that $\phi(t) \leq 1/2$ if $|t| \leq r$ then

$$\operatorname{Re} \Psi(x,t) \geq x_m^2 + \phi(t)/2 + \kappa \sum_{j=1}^{m-1} \operatorname{Re}\{(x_j + it_j)^{2q}\} \geq x_m^2 + c|t|^{2q}/2$$

$$+\kappa \sum_{j=1}^{m-1} \operatorname{Re}\{(x_j + it_j)^{2q}\}$$

If we now use the elementary fact that for every $0 < \varepsilon < 1$ there is $C_\varepsilon > 0$ (depending on q) such that

$$\operatorname{Re}(z^{2q}) \geq (1 - \varepsilon)(\operatorname{Re} z)^{2q} - C_\varepsilon(\operatorname{Im} z)^{2q}, \quad z \in \mathbb{C},$$

choosing $\varepsilon = 1/2$ and κ small enough gives

$$\operatorname{Re} \Psi(x,t) \geq x_m^2 + c|t|^{2q}/4 + \kappa(x_1^{2q} + \ldots + x_{m-1}^{2q})/2, \quad |t| \leq r.$$

Hence ψ is a peak function of finite type for $\mathcal{V}_\bullet$ at the origin and then (2) follows from Theorem 7.1.

We have now to face the situation when $V \neq \{0\}$ and say $\phi \geq 0$. The former is equivalent to the existence of a (germ of a) non trivial real analytic curve $\gamma(s)$ through the origin in t-space over which ϕ vanishes identically. Notice that $\phi = 0$ implies $d\phi = 0$ (because $\phi \geq 0$) and hence also $d\phi$ vanishes on γ.

Theorem 8.2 *Let $\mathcal{V}_\bullet$, ϕ and V be as before. Assume that V contains the (germ of) a non trivial real analytic curve $\gamma(s)$ through the origin such that each of its components has a zero of odd order at the origin. Then the Borel map for $\mathcal{V}_\bullet$ at the origin is not surjective.*

Proof Write $\gamma(s) = (\gamma_1(s), \ldots, \gamma_n(s))$ and consider the tube structure $\mathcal{V}_\sharp$ on the (x, s)-space defined by the first integrals

$$Z_j^\sharp(x, t) = x_j + i\gamma_j(s), \quad j = 1, \ldots, n.$$

This structure is defined by a single vector field, namely:

$$L^\sharp = \frac{\partial}{\partial s} - \sum_{j=1}^n \gamma_j'(s) \frac{\partial}{\partial x_j} .$$

The point for considering this new tube structure is the following key observation: *if $u(x, t)$ is a smooth solution for $\mathcal{V}_\bullet$ near the origin then $v(x, s) \doteq u(x_1, \ldots, x_n, 0, \gamma(s))$ is a smooth solution for $\mathcal{V}_\sharp$ near the origin, that is, $L^\sharp v = 0$.* This follows from a simple computation.

Now if each γ_j has an odd order zero at the origin then the map

$$s \mapsto \sum_{j=1}^n \gamma_j(s)\xi_j$$

is open at the origin in $\mathbb{R}$ for any $\xi \in \mathbb{R}^n \backslash 0$, and consequently by the Baouendi–Treves [2] result alluded to above, it follows that $\mathcal{V}_\sharp$ is hypocomplex. Hence if u is any smooth solution for $\mathcal{V}_\bullet$ near the origin and if v is defined as above then we obtain the bounds

$$|\partial_x^\alpha u(0, 0)| = |\partial_x^\alpha v(0, 0)| \le C^{|\alpha|+1}\alpha!, \quad \alpha \in \mathbb{Z}_+^m,$$

which imply that the Borel map for $\mathcal{V}_\bullet$ at the origin is not surjective.

C. In the rest of this section we shall focus on the case when $n = 2$ and $\phi(t) = (t_1^p - t_2^q)^2$, $p, q \in \mathbb{N}$. Write $p/q = \alpha/\beta$, with α and β without common factors. By Theorem 8.2 the Borel map for $\mathcal{V}_\bullet$ at the origin is not surjective if both α and β are odd since ϕ vanishes on the curve $\gamma(s) = (s^\beta, s^\alpha)$.

We shall now study some of the cases when $\alpha \ne \beta$ and either α or β is even. We are able to settle the following situations:

Theorem 8.3 *Let ϕ be as before:*

- *if $q = 2$ and p is odd then the Borel map is surjective;*
- *if $q = 2$ and p is even then the Borel map is not surjective.*

Remark 8.1 In each one of the cases where the Borel map is not surjective, the necessary condition established in Theorem 6.1 is not satisfied (indeed, we prove the non-surjectivity precisely by applying Theorem 6.1).

We will first concentrate on the second statement.

C1. Given $k \in \mathbb{N}$, consider the following hypersurface of $\mathbb{C}^3$, which is equivalent to the ones introduced in subsection **B.** up to a complex linear change of coordinates:

$$\Sigma = \{x_3 = (x_1^2 - x_2^{2k})^2\}.$$

We also put $\Sigma_0 = \Sigma \cap \{z_3 = 0\}$. Then Σ_0 can be seen as the union of the two hypersurfaces $S^+ = \{x_1 = x_2^k\}$ and $S^- = \{x_1 = -x_2^k\}$, biholomorphic to each other. We want to show that the polynomial hull $\widehat{\Sigma_0}$ of Σ_0 in $\mathbb{C}^2$ (and thus the polynomial hull $\widehat{\Sigma}$ of Σ in $\mathbb{C}^3$) contains a complex line passing through 0.

To this aim, we define $\Sigma'_0 = \Sigma_0 \cap \{x_2 \geq 0\}$, $\Sigma''_0 = \Sigma_0 \cap \{x_2 \leq 0\}$: then we can write $\Sigma'_0 = \{x_2 = \sqrt[k]{|x_1|}\}$ and $\Sigma''_0 = \{x_2 = -\sqrt[k]{|x_1|}\}$. We claim that $\widehat{\Sigma'_0}$ contains (a neighborhood of 0 in) $\{(0, z_2) \in \mathbb{C}^2 : x_2 \geq 0\}$, and similarly $\widehat{\Sigma''_0}$ contains (a neighborhood of 0 in) $\{(0, z_2) \in \mathbb{C}^2 : x_2 \leq 0\}$.

Choose then $c \in \mathbb{C}$, $c = a + ib$ with $0 < a < (2k-1)/(2k)^{\frac{2k}{2k-1}}$ and define $f : \mathbb{C} \to \mathbb{C}^2$ as $f(\zeta) = (\zeta, c + \zeta^2)$; furthermore define $\rho : \mathbb{C}^2 \to \mathbb{R}$ as $\rho(z_1, z_2) = x_2 - \sqrt[k]{|x_1|}$. Writing $\zeta = u + iv$ we can express the composition $\rho \circ f : \mathbb{C} \to \mathbb{R}$ as $\rho \circ f(\zeta) = a - v^2 + u^2 - \sqrt[k]{|u|}$.

Let now $\varphi : \mathbb{R}^+ \to \mathbb{R}$ be defined as $\varphi(t) = \sqrt[k]{t} - t^2$. A simple computation shows that φ is strictly increasing on the interval $[0, 1/(2k)^{\frac{k}{2k-1}}]$ and $\varphi(1/(2k)^{\frac{k}{2k-1}}) = (2k-1)/(2k)^{\frac{2k}{2k-1}}$. We can thus set $d' = \varphi^{-1}(a)$ and, choosing $d' < d < 1/(2k)^{\frac{k}{2k-1}}$, define the rectangle $R = \{u + iv : |u| < d, |v| < \sqrt{a+1}\}$.

With this choice of R we have that $\rho \circ f|_{\partial R} < 0$. Indeed, whenever $|v| = \sqrt{a+1}$ we can write $\rho \circ f(\zeta) \leq a - v^2 = -1$, while for $|u| = d$ one has $\rho \circ f(\zeta) \leq a - \varphi(|u|) \leq a - \varphi(d) < 0$ by the choice of d. On the other hand $\rho \circ f(0) = a > 0$. It follows that the open set $U' = R \cap \{\rho \circ f > 0\}$ is non-empty and relatively compact in R. The open set $\mathbb{C} \backslash \overline{U'}$ has a unique unbounded connected component V. Putting $U = \mathbb{C} \backslash \overline{V}$, it follows that U is simply connected, $0 \in U$ and $\partial U \subset \{\rho \circ f = 0\}$.

We can thus consider $f : U \to \mathbb{C}^2$ as an analytic disc attached to Σ'_0 because $f(\partial U) \subset \Sigma'_0$. Since $f(0) = (0, c)$, it follows that $(0, c) \in \widehat{\Sigma'_0}$, which verifies the claim. By Theorem 6.1 we conclude that the Borel map is not surjective, which proves the second statement in Theorem 8.3.

C2. We are now going to treat the first claim in Theorem 8.3.

In order to do so we are going to study the properties of some particular domains of $\mathbb{C}^2$. Fix a small enough $\tau > 0$ (to be specified later) and $k_0 \in \mathbb{N}$. We define $\Omega \subset \mathbb{C}^2(z_1, z_3)$ to be the set

$$\Omega = \{(z_1, z_3) \in \mathbb{C}^2 : x_3 \geq 0, \ |z_3| < \tau, \ |z_1| < 1 + \sqrt[k_0]{x_3}\}$$

and put $\Omega_0 = \Omega \cap \{z_3 = 0\}$ (i.e. the unit disc in $\mathbb{C}(z_1)$). We denote by $A^\infty(\Omega)$, $A^\infty(\Omega_0)$ the subspace of $C^\infty(\overline{\Omega})$, $C^\infty(\overline{\Omega}_0)$ given by the functions which are holomorphic in the interior of Ω, Ω_0.

Proposition 8.1 *The restriction map $A^\infty(\Omega) \to A^\infty(\Omega_0)$ is surjective. More precisely, for all $f \in A^\infty(\Omega_0)$ there is $\widetilde{f} \in A^\infty(\Omega)$ such that $\widetilde{f}|_{\Omega_0} = f$ and $\frac{\partial^k \widetilde{f}}{\partial z_3^k}|_{\Omega_0} = 0$ for all $k \geq 1$.*

To achieve the proof of the Proposition, we modify the construction in [6], and sometimes refer to lemmas in there without further mention. First, we need to prove an estimate which will be useful later:

Lemma 8.1 *Fixed $r > 0$, we have*

$$\frac{1}{2} \log(j^2) \leq \frac{1}{\sin(1/j^{r+2})} (1 - (j^r \sin(1/j^{r+2}))^{\frac{1}{j^{r+2}}}) \leq 2 \log(2j^2)$$

for all large enough $j \in \mathbb{N}$.

Proof Put $x_j = 1 - (j^r \sin(1/j^{r+2}))^{\frac{1}{j^{r+2}}}$; then $x_j > 0$ and

$$(1 - x_j)^{j^{r+2}} = j^r \sin(1/j^{r+2}).$$

Moreover, since $(j^r \sin(1/j^{r+2}))^{\frac{1}{j^{r+2}}} \geq (1/2j^2)^{\frac{1}{j^{r+2}}} \to 1$ as $j \to \infty$, we have $x_j \to 0$ as $j \to \infty$. From the expression above we get

$$\log(1 - x_j) = \frac{1}{j^{r+2}} \log(j^r \sin(1/j^{r+2}));$$

since for j large enough we have $-2x_j \leq \log(1 - x_j) \leq -x_j$ and $\frac{1}{2j^{r+2}} \leq \sin(1/j^{r+2}) \leq \frac{1}{j^{r+2}}$, we can write

$$-x_j \geq \frac{\log(1/2j^2)}{j^{r+2}} \Rightarrow x_j \leq \frac{\log(2j^2)}{j^{r+2}}, \quad -x_j \leq \frac{\log(1/j^2)}{2j^{r+2}} \Rightarrow x_j \geq \frac{\log(j^2)}{2j^{r+2}}$$

for large j. The conclusion follows from these inequalities and again from the fact that $\frac{1}{2j^{r+2}} \leq \sin(1/j^{r+2}) \leq \frac{1}{j^{r+2}}$ for large enough j.

Fix an increasing sequence $\{m_\ell\}$ of positive integers such that $m_\ell \geq e^\ell$. Define sequences of functions $\{\psi_j\}, \{\xi_j\}, \{\varphi_j\}$ by putting

$$\psi_j(z_1, z_3) = -\frac{\lambda(1/j^{k_0\ell+2})}{z^{\ell/j^{k_0\ell+2}}} + B(1/j^{k_0\ell+2}), \quad \xi_j = e^{\psi_j}, \quad \varphi_j = e^{-\xi_j}.$$

for all $m_\ell \leq j < m_{\ell+1}$, where $B(y) = 1/\sin(y)$ and $\lambda(y) = B(y)^{1-y}$. Note that ψ_j is well-defined on $\Omega \setminus \Omega_0$, and Re $\psi_j(z_1, z_3) \to -\infty$ as $z_3 \to 0$. Furthermore the function φ_j extends continuously to $\overline{\Omega}$ and $\varphi_j \equiv 1$ on Ω_0. Put $D_j = \{(z_1, z_3) \in \Omega : |z_3| \leq 1/j^{k_0})\}$, and fix $p \in \Omega \setminus D_j$. With the same computations as in Lemma 4.1 (choosing $A_\beta = j^2$) we have $|\text{Im } \psi_j(p)| \leq \frac{1}{\cos(\ell/j^{2\ell+2})} =: d_j$ for $m_\ell \leq j < m_{\ell+1}$ and

$$\text{Re } \psi_j(p) \geq B\left(1/j^{k_0\ell+2}\right)\left(1 - \left(\frac{(j^{k_0})^\ell}{B\left(1/j^{k_0\ell+2}\right)}\right)^{\frac{1}{j^{k_0\ell+2}}}\right) =$$

$$= \frac{1}{\sin(1/j^{k_0\ell+2})}(1 - (j^{k_0\ell}\sin(1/j^{k_0\ell+2}))^{\frac{1}{j^{k_0\ell+2}}}) \geq \log(j^2)/2 = \log(j)$$

by Lemma 8.1. Choose $1 < d < \pi/2$ such that $d \geq d_j$ for all $j \in \mathbb{N}$ large enough (indeed $d_j \to 1$ as $j \to \infty$). From the expression above follows that

$$\mathrm{Re}\,\xi_j(p) \geq |\xi_j(p)|\cos(d_j) = e^{\mathrm{Re}\,\psi_j(p)}\cos(d_j) \geq j\cos(d_j) \geq j\cos(d)$$

and thus

$$|\varphi_j(p)| = e^{-\mathrm{Re}\,\xi_j(p)} \leq e^{-j\cos(d)}.$$

On the other hand we have $|\varphi_j(p)| \leq e$ for all $p \in D_j$ (same proof as in Lemma 4.2).

Lemma 8.2 *For all $p = (z_1, z_3) \in \Omega$ we have $|\varphi_j(p)| \leq \frac{e^2}{(1+\sqrt[k_0]{|z_3|})^j}$.*

Proof Suppose first that $p \in D_j$, i.e. $|z_3| \leq 1/j^{k_0}$. Then $\frac{e^2}{(1+\sqrt[k_0]{|z_3|})^j} \geq \frac{e^2}{(1+1/j)^j} \geq e \geq |\varphi_j(p)|$. If instead $p \in \Omega\backslash D_j$ we can write $\frac{e^2}{(1+\sqrt[k_0]{|z_3|})^j} \geq \frac{1}{(1+\sqrt[k_0]{\tau})^j} = \frac{1}{(e^{\cos(d)})^j} \geq |\varphi_j(p)|$ if $\tau > 0$ is small enough.

The next statement is an immediate consequence of the chain rule.

Lemma 8.3 *Fix $k \in \mathbb{N}$. There is a polynomial $P_k(X_1, X_2, \ldots, X_k)$ such that*

$$\frac{\partial^k \varphi_j}{\partial z_3^k} = \varphi_j P_k\left(\frac{\partial\xi_j}{\partial z_3}, \frac{\partial^2\xi_j}{\partial z_3^2}, \ldots, \frac{\partial^k\xi_j}{\partial z_3^k}\right)$$

for all $j \in \mathbb{N}$. Furthermore, P_k is weighted homogeneous of degree k (where the variable X_j has weight j).

Thus, to obtain an estimate for $\frac{\partial^k \varphi_j}{\partial z_3^k}$ we need to give one for $\frac{\partial^h\xi_j}{\partial z_3^h}$, $h \leq k$. In the next lemma we show that $|\frac{\partial^h\xi_j}{\partial z_3^h}(p)|$ grows as a polynomial in j if $p \in \Omega \cap D_j$, while if $p \in \Omega\backslash D_j$ its growth is compensated by the exponential decay of $|\varphi_j(p)|$, resulting in the following statement:

Lemma 8.4 *Let $k \in \mathbb{N}$, $k \geq 1$. There exist $N_k > 0$, $\tau' > 0$ such that*

$$\left|\frac{\partial^k \varphi_j}{\partial z_3^k}(p)\right| \leq N_k j^{3k_0 k}\frac{1}{(1+\sqrt[k_0]{|z_3|})^j}$$

for all $p = (z_1, z_3) \in \Omega$ with $|z_3| \leq \tau'$ and all $j \in \mathbb{N}$.

Proof In the following we always consider $\ell, j \in \mathbb{N}$ such that $m_\ell \leq j < m_{\ell+1}$, and fix $h \in \mathbb{N}$. Moreover we put $y_j = 1/j^{k_0\ell+2}$. The following expression for $\frac{\partial^h \xi_j}{\partial z_3^h}$ can be checked inductively:

$$\frac{\partial^h \xi_j}{\partial z_3^h} = \xi_j \sum_{a=1}^h \beta_{a,j} \frac{\ell^a}{z_3^{a\ell y_j + h}}$$

where $\beta_{a,j}$ is bounded in j for all $1 \leq a \leq h$. Thus we have

$$\left| \frac{\partial^h \xi_j}{\partial z_3^h} \right| \leq |\xi_j| \sum_{a=1}^h |\beta_{a,j}| \frac{\ell^a}{|z_3|^{a\ell y_j + h}} \leq C_h \ell^h |\xi_j| \frac{1}{|z_3|^{h\ell y_j + h}}$$

for some constant $C_h > 0$ (independent of j). Taking in account the definition of ξ_j, we can write

$$\left| \frac{\partial^h \xi_j}{\partial z_3^h} \right| \leq C_h \ell^h \frac{1}{|z_3|^{h\ell y_j + h}} e^{-\mathrm{Re}\frac{\lambda(y_j)}{z_3^{\ell y_j}}} e^{B(y_j)}$$

$$\leq C_h \ell^h \frac{1}{|z_3|^{h\ell y_j + h}} e^{-\frac{\lambda(y_j)\cos(\pi \ell y_j)}{|z_3|^{\ell y_j}}} e^{B(y_j)}.$$

Define the function $\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\kappa(r) = \frac{1}{r^{h\ell y_j + h}} e^{-\frac{\lambda(y_j)\cos(\pi \ell y_j)}{r^{\ell y_j}}} ;$$

clearly $\kappa(r) \to 0$ as $r \to 0^+$ and as $r \to +\infty$. Computing the first derivative

$$\kappa'(r) = \left(-\frac{h\ell y_j + h}{r^{h\ell y_j + h + 1}} + \frac{\ell y_j \lambda(y_j) \cos(\pi \ell y_j)}{r^{(h+1)\ell y_j + h + 1}} \right) e^{-\frac{\lambda(y_j)\cos(\pi \ell y_j)}{r^{\ell y_j}}}$$

we see that it vanishes only at

$$\widetilde{r} = \left(\frac{\ell y_j \lambda(y_j) \cos(\pi \ell y_j)}{h\ell y_j + h} \right)^{1/\ell y_j}$$

hence κ is increasing for $0 \leq r < \widetilde{r}$ and decreasing for $r > \widetilde{r}$. Furthermore

$$(y_j \lambda(y_j))^{1/\ell y_j} = \left(\frac{y_j \sin(y_j)^{y_j}}{\sin(y_j)} \right)^{1/\ell y_j} = \left(\frac{y_j}{\sin(y_j)} \right)^{1/\ell y_j} \sin(y_j)^{1/\ell}$$

$$= \left(\frac{1}{(1 - y_j^2/6 + O(y_j^4))^{1/y_j}} \sin(y_j) \right)^{1/\ell}$$

and

$$\cos(\pi\ell y_j)^{1/\ell y_j} = (1 - (\pi\ell y_j)^2/2 + O((\pi\ell y_j)^4))^{1/\ell y_j}$$

are bounded (above and below) independently of j, so that for some $K > 0$ we can write

$$\widetilde{r} \geq K\left(\frac{\ell}{h\ell y_j + h}\right)^{1/\ell y_j} = K\left(\frac{\ell}{h}\right)^{j^{k_0\ell+2}/\ell}\left(\frac{1}{1 + \ell/j^{k_0\ell+2}}\right)^{j^{k_0\ell+2}/\ell}$$

$$\geq \frac{K}{e}\left(\frac{\ell}{h}\right)^{j^{k_0\ell+2}/\ell}.$$

If $h \leq \ell$ we obtain $\widetilde{r} \geq \frac{K}{e}$. In fact, if $h < \ell$ we have $\widetilde{r} \to \infty$ as $j \to \infty$, so we can assume $\widetilde{r} \geq \tau$. Thus the function κ is increasing on the interval $[0, \tau]$.

Let $p \in \Omega \cap D_j$, $p = (z_1, z_3)$. Since $|z_3| \leq 1/j^{k_0}$ we have

$$\left|\frac{\partial^h \xi_j}{\partial z_3^h}(p)\right| \leq C_h \ell^h \kappa(|z_3|) e^{B(1/j^{k_0\ell+2})}$$

$$\leq C_h \ell^h \kappa(1/j^{k_0}) e^{B(1/j^{k_0\ell+2})}$$

$$\leq C_h \ell^h j^{k_0 h(1+\ell/j^{k_0\ell+2})} \frac{\exp\left(B(1/j^{k_0\ell+2})\right)}{\exp\left(j^{k_0\ell/j^{k_0\ell+2}}\lambda(1/j^{k_0\ell+2})\cos(\pi\ell/j^{k_0\ell+2})\right)}$$

$$= C_h \ell^h j^{k_0 h(1+\ell/j^{k_0\ell+2})} \exp(\alpha_j).$$

We can rewrite the argument of the exponential as follows:

$$\alpha_j = \frac{1}{\sin(1/j^{k_0\ell+2})}\left(1 - \cos(\pi\ell/j^{k_0\ell+2})(j^{k_0\ell}\sin(1/j^{k_0\ell+2}))^{\frac{1}{j^{k_0\ell+2}}}\right)$$

$$= \frac{1}{\sin(1/j^{k_0\ell+2})}\left(1 - (j^{k_0\ell}\sin(1/j^{k_0\ell+2}))^{\frac{1}{j^{k_0\ell+2}}}\right)$$

$$+ \frac{1 - \cos(\pi\ell/j^{k_0\ell+2})}{\sin(1/j^{k_0\ell+2})}(j^{k_0\ell}\sin(1/j^{k_0\ell+2}))^{\frac{1}{j^{k_0\ell+2}}}.$$

The second summand in the expression above is bounded (in fact it can be seen to be $O(\ell^2/j^{k_0\ell+2})$), while the first one is estimated by $2\log(2j^2)$ by Lemma 8.1. We deduce that

$$\left|\frac{\partial^h \xi_j}{\partial z_3^h}(p)\right| \leq C_h \ell^h j^{k_0 h(1+\ell/j^{k_0\ell+2})} \exp(\log(j^4) + O(1)) \leq C_h' j^{3k_0 h}$$

for a large enough $C_h' > 0$ (here we are using the fact that $\ell \leq \log(j)$ by the choice of m_ℓ).

The estimate above, together with Lemma 8.3, show that there exists $N_k' > 0$ such that

$$\left| \frac{\partial^k \varphi_j}{\partial z_3^k}(p) \right| \le N_k' j^{3k_0 k} \tag{8.1}$$

for all $p \in \Omega \cap D_j$.

Consider now $p \in \Omega \backslash D_j$, $p = (z_1, z_3)$. Since $|z_3| \ge 1/j^{k_0}$ we have

$$\left| \frac{\partial^h \xi_j}{\partial z_3^h}(p) \right| \le C_h \ell^h \frac{1}{|z_3|^{h\ell/j^{k_0\ell+2}+h}} |\xi_j(p)| \le C_h \ell^h j^{k_0 h(1+\ell/j^{k_0\ell+2})} |\xi_j(p)|$$

$$\le C_h (\log(j))^h j^{k_0 h(1+\ell/j^{k_0\ell+2})} |\mathrm{Re}\,\xi_j| \cos(d_j) \le C_h'' j^{2k_0 h}(-\log(|\varphi_j(p)|)).$$

As before, using Lemma 8.3 we get that there exists $N_k'' > 0$ such that

$$\left| \frac{\partial^k \varphi_j}{\partial z_3^k}(p) \right| \le N_k'' j^{2k_0 k} |\varphi_j(p)| (-\log(|\varphi_j(p)|))^k$$

for all $p \in \Omega \backslash D_j$. Since $|\varphi_j(p)| \le e^{-j\cos(d)} \to 0$ as $j \to \infty$ we have that

$$(-\log(|\varphi_j(p)|))^k \le 1/\sqrt{|\varphi_j(p)|}$$

for all $p \in \Omega \backslash D_j$ and all large enough j, and thus

$$\left| \frac{\partial^k \varphi_j}{\partial z_3^k}(p) \right| \le N_k'' j^{2k_0 k} \sqrt{|\varphi_j(p)|} \le N_k'' j^{2k_0 k} e^{-j\cos(d)/2}. \tag{8.2}$$

Using that $1/(1 + \sqrt[k_0]{|z_3|})^j \ge 1/e$ if $|z_3| \le 1/j^{k_0}$ and $1/(1 + \sqrt[k_0]{|z_3|})^j \ge e^{-j\cos(d)/2}$ if $|z_3| \le \tau'$ small enough, we can put together (8.1) and (8.2) as in Lemma 8.2 to conclude that there exists $N_k > 0$ such that

$$\left| \frac{\partial^k \varphi_j}{\partial z_3^k}(p) \right| \le N_k j^{3k_0 k} \frac{1}{(1 + \sqrt[k_0]{|z_3|})^j}$$

for all $p \in \Omega$, $j \in \mathbb{N}$.

Proof of Proposition 8.1: For $t > 0$, define the dilation $\psi_t : \mathbb{C}^2 \to \mathbb{C}^2$ as $\psi_t(z_1, z_3) = (z_1, t z_3)$, and let $\Omega^t = \psi_t^{-1}(\Omega)$. We have

$$\Omega^t = \{(z_1, z_3) \in \mathbb{C}^2 : x_3 \ge 0,\ |z_3| < \tau/t,\ |z_1| < 1 + \sqrt[k_0]{t}\sqrt[k_0]{x_3}\}$$

so that $\Omega^t \cap \{z_3 = 0\} = \Omega_0$ and $\Omega^t \cap \{|z_3| < \tau\} \subset \Omega$ if $t < 1$. For a given $f \in A^\infty(\Omega_0)$, we will construct $\widetilde{f}$, holomorphic in the interior of Ω, such that $\frac{\partial^{h+k}\widetilde{f}}{\partial z_1^h \partial z_3^k}$

extends continuously to Ω_0 for all h, $k \geq 0$. Then it is clear that $\widetilde{f}|_{\Omega'} \in A^\infty(\Omega')$, and thus $\widetilde{f} \circ \psi_t^{-1} \in A^\infty(\Omega)$; furthermore $\widetilde{f} \circ \psi_t^{-1}|_{\Omega_0} = f$ since ψ_t is the identity on Ω_0.

Let $f \in A^\infty(\Omega_0)$, $f(z_1) = \sum_{j=0}^\infty a_j z_1^j$. Since f is smooth up to $b\Omega_0$ the sequence a_j goes to 0 faster than any polynomial, that is for all $k \in \mathbb{N}$ there is $A_k > 0$ such that $|a_j| \leq A_k/j^k$ for all $j \geq 1$.

We define now $\widetilde{f}(z_1, z_3) = \sum_j a_j z_1^j \varphi_j(z_3)$. By Lemma 8.2 follows that the series $\sum_j a_j z_1^j \varphi_j$ converges uniformly on compact sets of the interior of Ω, hence $\widetilde{f}$ is a well-defined holomorphic function in the interior of Ω. We will show now that, for all $k \geq 1$, $\sup_{\Omega_c} |\frac{\partial^k \widetilde{f}}{\partial z_3^k}| \to 0$ as $c \to 0$, where $\Omega_c = \Omega \cap \{z_3 = c\}$. This will imply that $\widetilde{f}$ (as well as $\frac{\partial^k \widetilde{f}}{\partial z_3^k}$) extends continuously to Ω_0, and $\widetilde{f}|_{\Omega_0} = f$. The same argument, applied to $\frac{\partial^h \widetilde{f}}{\partial z_1^h}$, proves that $\frac{\partial^{h+k} \widetilde{f}}{\partial z_1^h \partial z_3^k}$ extends continuously to Ω_0.

Fix then $k \in \mathbb{N}$, and let $A_{3k_0k+2} > 0$ such that $|a_j| \leq A_{3k_0k+2}/j^{3k_0k+2}$ for all $j \in \mathbb{N}$. Given $\epsilon > 0$, let $j_0 \in \mathbb{N}$ such that $N_k A_{3k_0k+2} \sum_{j>j_0} \frac{1}{j^2} < \epsilon$. For any $p = (z_1, z_3) \in \Omega$ we get

$$
\begin{aligned}
\left| \frac{\partial^k \widetilde{f}}{\partial z_3^k}(p) \right| &= \left| \sum_{j \leq j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) + \sum_{j > j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) \right| \\
&\leq \left| \sum_{j \leq j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) \right| + \sum_{j > j_0} |a_j||z_1|^j \left| \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) \right| \\
&\leq \left| \sum_{j \leq j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) \right| + \sum_{j > j_0} \frac{A_{3k_0k+2}}{j^{3k_0k+2}} N_k j^{3k_0k} \left(\frac{|z_1|}{1 + \sqrt[k_0]{|z_3|}} \right)^j \\
&\leq \left| \sum_{j \leq j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3) \right| + \epsilon
\end{aligned}
$$

where we used Lemma 8.4 and the fact that $|z_1| \leq (1 + \sqrt[k_0]{x_3}) \leq (1 + \sqrt[k_0]{|z_3|})$. Since $\sum_{j \leq j_0} a_j z_1^j \frac{\partial^k \varphi_j}{\partial z_3^k}(z_3)$ is a finite sum and φ_j is flat at 0 for all j, we conclude that $\left| \frac{\partial^k \widetilde{f}}{\partial z_3^k}(p) \right| < 2\epsilon$ for $|z_3|$ small enough.

Corollary 8.1 *Define $\Gamma \subset \mathbb{C}^3(z_1, z_2, z_3)$ as the set*

$$
\Gamma = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \ |z_3| < \tau, \ |z_1|^2 + |z_2|^2 < 1 + \sqrt[k_0]{x_3}\}
$$

and let $\Gamma_0 = \Gamma \cap \{z_3 = 0\}$ be the unit ball in $\mathbb{C}^2$. Then the restriction map $A^\infty(\Gamma) \to A^\infty(\Gamma_0)$ is surjective. More precisely, for all $f \in A^\infty(\Gamma_0)$ there is $\widetilde{f} \in A^\infty(\Gamma)$ such that $\widetilde{f}|_{\Gamma_0} = f$ and $\frac{\partial^k \widetilde{f}}{\partial z_3^k}|_{\Gamma_0} = 0$ for all $k \geq 1$.

Proof Given $f \in A^\infty(\Gamma_0)$, we can apply the construction of Proposition 8.1 on the slices $\Gamma \cap \{\alpha z_1 = \beta z_2\}$ to define an extension $\tilde{f}$ of f to Γ, holomorphic on each slice. Since the sequence of "cut off" functions φ_j is independent of α, β, $\tilde{f}$ is in fact globally holomorphic in (z_1, z_2, z_3).

Corollary 8.2 *Define* $\Sigma'' \subset \mathbb{C}^3(z_1, z_2, z_3)$ *as the set*

$$\Sigma'' = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \ x_1 \geq x_2^2 - \sqrt[k_0]{x_3}\}$$

and let $(S, 0)$ *be a germ of smooth real hypersurface of* $\mathbb{C}^3$ *such that* $0 \in S$ *and* $S \subset \Sigma''$. *Furthermore let* $S_0 = S \cap \{z_3 = 0\}$. *Then for any formal series* $\sigma = \sum_{j_1, j_2} a_{j_1 j_2} z_1^{j_1} z_2^{j_2}$ *there is a (germ of a) solution* $g \in \mathfrak{S}(S)$ *whose Taylor series at 0 is given by* σ *and* $\frac{\partial^k g}{\partial z_3^k}|_{S_0} = 0$ *for all* $k \geq 1$.

Proof Define the Cayley transformation $\Phi : \mathbb{C}^3 \backslash \{z_1 = 1\} \to \mathbb{C}^3$ as

$$\Phi(z_1, z_2, z_3) = \left(\frac{1 + z_1}{1 - z_1}, \frac{z_2}{1 - z_1}, z_3 \right);$$

we have that Φ maps Γ' to Σ'', where

$$\Gamma' = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \ |z_3| < \tau, \ |z_1|^2 + |z_2|^2 < 1 + |1 - z_1|^2 \sqrt[k_0]{x_3}\}$$

on the other hand, since $|1 - z_1|^2$ is bounded we have (locally) $\Gamma' \subset \Gamma''$ with

$$\Gamma'' = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \ |z_3| < \tau, \ |z_1|^2 + |z_2|^2 < 1 + C \sqrt[k_0]{x_3}\}$$

for some large enough $C > 0$. However Γ'' is biholomorphic to the set Γ of Corollary 8.1 via a rescaling of the z_3 coordinate, so the conclusion of Corollary 8.1 holds for Γ''. Since $\Phi(-1, 0, 0) = (0, 0, 0)$ we can consider $\sigma' = \sigma \circ \Phi$ as a formal power series centered at the point $p_0 = (-1, 0, 0) \in \Gamma_0''$. Since $\psi = -i(z_1 + 1)$ is a (global) peak function of finite order for Γ_0'' at p_0, there exists a smooth CR solution $f \in \mathfrak{S}(\Gamma_0'')$ whose Taylor expansion at p_0 is σ'. By Corollary 8.1 there exists $\tilde{f} \in A^\infty(\Gamma'')$ such that $\tilde{f}|_{\Gamma_0''} = f$ and $\frac{\partial^k \tilde{f}}{\partial z_3^k}|_{\Gamma_0''} = 0$ for all $k \geq 1$. Putting $g = \tilde{f} \circ \Phi^{-1}$, we have that g is defined on a neighborhood of 0 in Σ'' and smooth up to the boundary. By construction $g|_S$ satisfies the requirements of the Corollary.

Consider now for $\ell \geq 0$ the hypersurface

$$\Sigma = \{x_3 = (x_1^{2\ell+1} - x_2^2)^2\} \subset \mathbb{C}^3$$

and put $\Sigma_0 = \Sigma \cap \{z_3 = 0\}$.

Using the notation of section 3 with $m = 3$, $p = 2$, we consider the partial Borel maps

$$\mathfrak{b}_1 : \mathfrak{S}_0^{(1)} \to \mathbb{C}[\![z_1, z_2]\!], \quad \mathfrak{b}_2 : \mathfrak{S}_0^{(2)} \to \mathbb{C}[\![z_3]\!].$$

We have that $\mathfrak{b}_2$ is surjective because $z_{3|\Sigma} = (x_1^{2\ell+1} - x_2^2)^2 + iy_3$ is a peak function at 0, which implies that the corank 1 structure induced on Σ by the function z_3 satisfies the Borel property.

In view of Theorem 4.1, the first claim of Theorem 8.3 is proved if $\mathfrak{b}_1$ is also surjective. This is the content of the following statement:

Proposition 8.2 *Let $\sum_{j_1,j_2} a_{j_1 j_2} z_1^{j_1} z_2^{j_2}$ be any formal series in (z_1, z_2). Then there is a neighborhood U of 0 in Σ and a solution $g \in \mathfrak{S}(U)$ such that*

- *the Taylor expansion of g at 0 is given by $\sum_{j_1,j_2} a_{j_1 j_2} z_1^{j_1} z_2^{j_2}$;*
- $\frac{\partial^k g}{\partial z_3^k}|_{\Sigma_0} = 0$ *for all $k \geq 1$.*

Proof Define the domains

$$\Sigma' = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \; x_1 \geq \sqrt[2\ell+1]{x_2^2 - \sqrt{x_3}}\},$$

$$\Sigma'' = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \; x_1 \geq x_2^2 - 2\sqrt[4\ell+2]{x_3}\}.$$

It is clear that $\Sigma \subset \Sigma'$; we claim that, if $\epsilon > 0$ is small enough, $\Sigma' \cap B_\epsilon(0) \subset \Sigma'' \cap B_\epsilon(0)$ (where $B_\epsilon(0) \subset \mathbb{C}^3$ is the ball of radius ϵ centered at 0). Indeed, for (small) fixed $x_3 \geq 0$ consider the function $\gamma : \mathbb{R}^+ \to \mathbb{R}$

$$\gamma(t) = \sqrt[2\ell+1]{\sqrt{x_3} - t} - (2\sqrt[4\ell+2]{x_3} - t).$$

Looking at the interval $[0, 2\sqrt[4\ell+2]{x_3}]$ we note that $\gamma(0) = -\sqrt[4\ell+2]{x_3} \leq 0$ and moreover $\gamma(2\sqrt[4\ell+2]{x_3}) = \sqrt[2\ell+1]{\sqrt{x_3} - 2\sqrt[4\ell+2]{x_3}} \leq \sqrt[2\ell+1]{\sqrt{x_3} - 2\sqrt{x_3}} = -\sqrt[4\ell+2]{x_3} \leq 0$. On the other hand we have

$$\gamma'(t) = -\frac{1}{2\ell+1} \cdot \frac{1}{(\sqrt{x_3} - t)^{2\ell/(2\ell+1)}} + 1$$

hence γ' vanishes exactly at $t = \sqrt{x_3} \pm \frac{1}{\sqrt[2\ell]{(2\ell+1)^{2\ell+1}}}$. If x_3 is small enough, neither of these values lies in the interval $[0, 2\sqrt[4\ell+2]{x_3}]$, showing that γ is monotone on that interval. Since $\gamma(0) \leq 0$ and $\gamma(2\sqrt[4\ell+2]{x_3}) \leq 0$ we must have $\gamma \leq 0$ on $[0, 2\sqrt[4\ell+2]{x_3}]$, i.e.

$$\sqrt[2\ell+1]{x_2^2 - \sqrt{x_3}} \geq x_2^2 - 2\sqrt[4\ell+2]{x_3} \text{ for } 0 \leq x_2^2 \leq 2\sqrt[4\ell+2]{x_3}.$$

If instead $\sqrt{x_3} \leq x_2^2 \leq 1$ we have $0 \leq x_2^2 - \sqrt{x_3} \leq 1$, so we can write (for small x_3)

$$\sqrt[2\ell+1]{x_2^2 - \sqrt{x_3}} \geq x_2^2 - \sqrt{x_3} \geq x_2^2 - 2\sqrt[4\ell+2]{x_3} \text{ for } \sqrt{x_3} \leq x_2^2 \leq 1.$$

Since $\sqrt{x_3} \leq \sqrt[4\ell+2]{x_3}$ we conclude that $\sqrt[2\ell+1]{x_2^2 - \sqrt{x_3}} \geq x_2^2 - 2\sqrt[4\ell+2]{x_3}$ for x_3 small enough and $-1 \leq x_2 \leq 1$, which proves the claimed inclusion $\Sigma' \cap B_\epsilon(0) \subset \Sigma'' \cap B_\epsilon(0)$.

Using a suitable change of coordinates we can map Σ'' biholomorphically to the domain $\{x_1 \geq |z_2|^2 - \sqrt[4\ell+2]{x_3}\}$, so that Σ_0'' is a one-sided neighborhood of the Lewy hypersurface $\{x_1 = |z_2|^2\}$. We denote again by $\sigma = \sum_{j_1, j_2} a_{j_1 j_2} z_1^{j_1} z_2^{j_2}$ the formal series obtained by transforming the one in the statement through this coordinate change. The conclusion of the Proposition follows then by applying Corollary 8.2 with $S = \Sigma$ and $k_0 = 4\ell + 2$.

By using the methods above, we can deduce directly the following (apparently more general) consequence:

Theorem 8.4 *With the notation of Theorem 8.1, suppose that $n = 2$ and $\phi(t) = (f(t_1, t_2))^2$ where the differential of f does not vanish at 0 and the domain $\{f(t_1, t_2) > 0\}$ is strictly convex (or concave) around 0. Then the Borel map is surjective.*

Proof Let us consider the tube manifold $\Sigma = \{x_3 = (f(x_1, x_2))^2\}$. Up to a linear change of coordinates, we can suppose that the tangent line of $\{f = 0\}$ at 0 is $\frac{\partial}{\partial x_1}$ and $\{f > 0\}$ is (locally) strictly convex. Then it is easy to show that there exists $C > 0$ such that $f(x_1, x_2) \geq x_1 - Cx_2^2$ for all x_1, x_2 around 0. This implies that Σ is locally contained in the domain

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 : x_3 \geq 0, \quad x_1 \geq Cx_2^2 - \sqrt{x_3}\}.$$

From Corollary 8.2 follows that the partial Borel map $\mathfrak{b}_1$ is surjective, which implies the Borel property just as in the proof of the first claim in Theorem 8.3.

9 The structure of the maximal ideal of $\mathfrak{S}_p$

A. In most of this section we shall assume that the locally integrable structure $\mathcal{V}$ over Ω satisfies condition $(\mathfrak{B})$ at $p \in \Omega$ (cf. Section 6B).

According to ([3], proof of Theorem 6.1) we can assert the following: p is the center of a smooth coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$, which can be assumed defined in a product $U = B \times \Theta$, where B (respectively Θ) is an open ball centered at the origin in $\mathbb{R}_x^m$ (respectively $\mathbb{R}_t^n$), over which there is defined a smooth, real vector-valued function $\Phi(x, t) = (\Phi_1(x, t), \ldots, \Phi_m(x, t))$ satisfying $\Phi(0, 0) = 0$, $D_x \Phi(0, 0) = 0$, in such a way that the differential of the functions

$$Z_k(x, t) = x_k + i\Phi_k(x, t), \qquad k = 1, \ldots, m,$$

span $\mathcal{V}^\perp$ over U. Contracting U even more around the origin we may achieve:

1. $dZ_m(0, 0) \in T_{(0,0)}^0$ and $\arg Z_m \neq -\pi$ in U;
2. There are constants $C, M > 0$ so that $(|Z_1| + \ldots + |Z_{m-1}|)^M \leq C|Z_m|$ in U.

As before we can consider the corresponding vector fields L_j, M_k satisfying the standard orthogonality conditions.

B. $\mathfrak{S}_0$ is a commutative local ring with maximal ideal

$$\mathfrak{m} = \{u \in \mathfrak{S}_0 : u(0) = 0\}.$$

Our goal now is to give sufficient conditions in order to insure that $\mathfrak{m}$ is a finitely generated $\mathfrak{S}_0$-module. This is of course true when $\mathcal{V}$ is hypocomplex at the origin. On the other hand we also have the following result:

Theorem 9.1 *Assume that $\mathcal{V}$ satisfies condition $(\mathfrak{B})$ at the origin. If either $\mathcal{V}$ is minimal at the origin or if $\mathcal{V}$ is a real-analytic locally integrable structure then the following holds: if $W_1, \ldots W_m \in \mathfrak{m}$ are such that $dW_1(0), \ldots, dW_m(0)$ are linearly independent then*

$$\mathfrak{m} = \langle W_1, \ldots W_m \rangle$$

as a $\mathfrak{S}_0$-module

We start by proving:

Lemma 9.1 *If $\mathcal{V}$ satisfies condition $(\mathfrak{B})$ given $u \in \mathfrak{m}$ there are $v_j \in \mathfrak{S}_0$ such that*

$$u - \sum_{j=1}^{m} v_j Z_j \in \ker \mathfrak{b}.$$

Proof Since $u(0) = 0$ we can write $\mathfrak{b}(u) = \sum_{j=1}^{m} g_j Z_j$, where $g_j \in \mathbb{C}[\![Z_1, \ldots, Z_m]\!]$. By the surjectivity of $\mathfrak{b}$ we can find $v_j \in \mathfrak{S}_0$ such that $\mathfrak{b}(v_j) = g_j$, $j = 1, \ldots, m$. Then $\mathfrak{b}\left(u - \sum_{j=1}^{m} v_j Z_j\right) = \mathfrak{b}(u) - \sum_{j=1}^{m} g_j Z_j = 0$.

We also have:

Lemma 9.2 *Assume that condition $(\mathcal{B})$ holds and also that $\mathcal{V}$ is minimal at 0. Then*

$$\ker \mathfrak{b} = \bigcap_{k=1}^{\infty} \langle Z_m^k \rangle.$$

Before we embark in the proof of Lemma 9.2 we show how it leads to the proof of Theorem 9.1. Indeed let $W_1, \ldots, W_m$ be as in its statement. By Lemmas 9.1 and 9.2 we can write

$$W_k = \sum_{r=1}^{m} \gamma_{kr} Z_r, \quad \gamma_{kr} \in \mathfrak{S}_0.$$

Since $Z_k(0) = 0$ for every k we have

$$dW_k(0) = \sum_{r=1}^{m} \gamma_{kr}(0) dZ_r(0)$$

and hence the matrix $(\gamma_{kr}(0))_{1 \le k, r \le m}$ is invertible. By continuity it follows that the matrix of germs $(\gamma_{kr})_{1 \le k, r \le m}$ is invertible and that its inverse $(\gamma^{kr})_{1 \le k, r \le m}$ is such

that γ^{kr} belongs to $\mathfrak{S}_0$, since the latter is a ring. Furthermore we have

$$Z_k = \sum_{r=1}^{m} \gamma^{kr} W_r$$

and this concludes the proof of Theorem 9.1.

Proof of Lemma 9.2. Let $V \subset U$ be an open neighborhood of the origin and let $u \in \mathfrak{S}(V)$ vanish to infinite order at 0. Assume first that $\mathcal{V}$ is minimal at the origin. By [10] there are an open set $\mathcal{U}$ in $\mathbb{C}^m$, a compact neighborhood of the origin $K \subset V$ (both indeed independent of u) and $h \in \mathcal{O}(\mathcal{U})$ such that the following is true:

– $Z(K) \subset \bar{\mathcal{U}}$, for every $\alpha \in \mathbb{Z}_+^m$ the holomorphic function $\partial^\alpha h$ extends continuously up to $\mathcal{U} \cup Z(K)$ and

$$(\partial^\alpha h) \circ Z = \mathrm{M}^\alpha u \text{ on } K. \tag{9.1}$$

$\square$

Notice in particular that if we consider the continuous functions on $\mathcal{U} \times \mathcal{U}$

$$U_\alpha(z, w) = |\partial^\alpha h(z) - \sum_{|\beta| \le k - |\alpha|} \partial^{\alpha+\beta} h(w)(z-w)^\beta / \beta! | / |z - w|^{|\alpha-k|},$$

defined as zero when $z = w$, they all extend continuously to $Z(K) \times Z(K)$. Consequently the family $\{u_{\alpha,\beta}\}_{(\alpha,\beta)\in\mathbb{Z}_+^m\times\mathbb{Z}_+^m}$, defined as

$$u_{\alpha,\beta}(z, \bar{z}) = \begin{cases} (\partial^\alpha h)|_{Z(K)} & \text{if } \beta = 0, \\ 0 & \text{if } \beta \ne 0 \end{cases}$$

is a Whitney family on $Z(K)$.

By the Whitney extension theorem ([8], Theorem 2.3.6) for every p there is $H_p \in C^p(\mathbb{C}^m)$ such that

$$(\partial_z^\alpha \partial_{\bar{z}}^\beta H_p)|_{Z(K)} = \begin{cases} (\partial_z^\alpha h)|_K & \text{if } \beta = 0, \\ 0 & \text{if } \beta \ne 0 \end{cases}$$

and $|\alpha| + |\beta| \le p$. By (9.1) all the derivatives of H_p of order $\le p - 1$ vanish at the origin and hence we must have $|H_p(z)| = \mathrm{O}(|z|^p)$, for z near the origin in $\mathbb{C}^m$. In particular

$$|u(x, t)| = |h(Z(x, t))| = |H_p(Z(x, t))| = \mathrm{O}(|Z(x, t)|^p).$$

Hence, by ($\mathfrak{B}$), we obtain

$$|u(x, t)| = \mathrm{O}(|Z_m(x, t)|^p), \quad p \ge 0. \tag{9.2}$$

Repeating the argument with $M^\alpha u$ replacing u we further obtain

$$|(M^\alpha u)(x, t)| = O(|Z_m(x, t)|^p), \quad p \geq 0. \tag{9.3}$$

Define $v_k(x, t) = u(x, t)/Z_m^k(x, t)$, if $Z_m(x, t) \neq 0$, $v_k(x, t) = 0$ when $Z_m(x, t) = 0$. Then (9.2) implies that $v_k(x, t)$ is continuous and is smooth when $x_m \neq 0$. By a standard result in distribution theory ([8], Theorem 3.1.3) we have $L_j v_k = 0$ and

$$M_\ell v_k = (M_\ell u)/Z_m^k - ku/Z_m^{k+1} \tag{9.4}$$

in the distribution sense, $j = 1, \ldots, n$, $\ell = 1, \ldots, m$. By (9.3) it follows that the right hand side of (9.4) is continuous (if defined as zero when $Z_m = 0$) and then by ([8], Theorem 3.1.7) it follows that $v_k \in C^1$ and that $L_j v_k = 0$ in the classical sense, $j = 1, \ldots, n$. If we iterate the argument it follows that v_k is smooth for every $k \in \mathbb{Z}_+$ and also that $L_j v_k = 0$ for all $j = 1, \ldots, n$ and all $k \in \mathbb{Z}_+$.

Next we assume that $Z_1, \ldots, Z_m$ are real-analytic functions and let V be an open neighborhood of the origin in U. By the Baouendi–Treves approximation theorem the following can be said: there is an open ball $W \subset\subset V$ centered the origin such that every element in $\mathfrak{S}(V)$ is constant on the set

$$F_0 = \{(x, t) \in W : Z(x, t) = 0\}.$$

Let $u \in \mathfrak{S}(V)$ vanish to infinite order at the origin. Then $M^\alpha u \in \mathfrak{S}(V)$ ($\alpha \in \mathbb{Z}_+^m$) vanish at the origin and consequently vanish on F_0. Consequently all derivatives of u vanish on F_0 and hence Taylor's formula gives, for every $q \in \mathbb{Z}_+$,

$$|u(x, t)| \leq A_q |(x, t) - (x', t')|^q, \quad (x, t) \in W, \ (x', t') \in F_0,$$

where A_q only depends on bounds for the derivatives of u on $\bar{W}$ of order q. Taking the infimum over $(x', t') \in F_0$ we obtain

$$|u(x, t)| \leq A_q \mathrm{dist}((x, t), F_0)^q, \quad (x, t) \in W.$$

Let $K \subset W$ be a compact neighborhood of the origin. Since F_0 is the zero set of the real-analytic function $f \doteq |Z_1|^2 + \ldots + |Z_m|^2$ by Lojasiewicz inequality (cf. [M], Theorem 4.1) there are constants $C > 0$ and $\gamma > 0$ such that

$$\mathrm{dist}((x, t), F_0)^\gamma \leq C|Z(x, t)|^2, \quad (x, t) \in K.$$

Hence

$$|u(x, t)| \leq C^{1/\gamma} A_q |Z(x, t)|^{2q/\gamma}, \quad (x, t) \in K,$$

for every $q \in \mathbb{Z}_+$. Again by ($\mathfrak{B}$) we derive the validity of (9.2) in this case and the preceding argument applies without modifications. The proof of Lemma 9.2 is complete. $\qquad\square$

Corollary 9.1 *Assume that $\mathcal{V}$ is a real analytic locally integrable structure of rank $N - 1$ (that is, $\mathcal{V}^\perp$ is a complex fiber subbundle of $\mathbb{C}T^*\Omega$). Then the conclusion of Theorem 8.1 holds at every point in Ω.*

Indeed when the rank of $\mathcal{V}$ is $N - 1$ and $p \in \Omega$ then $\mathcal{V}$ is not hypocomplex at p if and only if property $(\mathfrak{B})$ holds at p ([3], Corollary 6.2).

C. Besides the hypocomplex case, the conclusion of Theorem 9.1 holds in some cases when it is not known whether condition $(\mathfrak{B})$ is valid or not. As in section 6(A) we assume that $\mathcal{V}$ is the locally integrable structure associated to a smooth, minimal, (weakly) convex hypersurface $\Omega \subset \mathbb{C}^m$. Assume $0 \in \Omega$. We claim that

$$\mathfrak{m} = \langle z_1|_\Omega, \ldots, z_m|_\Omega \rangle.$$

Indeed let V be an open neighborhood of the origin in Ω and let $f \in \mathfrak{S}(V)$ satisfy $f(0) = 0$. Then there is a weakly convex smooth domain $\mathcal{U}$ in $\mathbb{C}^m$ such that $\partial \mathcal{U} \cap \Omega \doteq W \subset V$ is an open neighborhood of the origin in Ω and there is $F \in \mathcal{O}(\mathcal{U}) \cap C^\infty(\overline{\mathcal{U}})$ such that $F = f$ in W. Since $F(0) = 0$ we can write, for $z \in \overline{\mathcal{U}}$,

$$F(z) = F(z_1, \ldots, z_n) = \int_0^1 \frac{\partial}{\partial t}(F(tz_1, \ldots, tz_n))dt$$

where the integral is well-defined because, by convexity, $(tz_1, \ldots, tz_n) \in \overline{\mathcal{U}}$ for $0 \leq t \leq 1$. By the chain rule we get

$$F(z) = \sum_{j=1}^m \int_0^1 z_j \frac{\partial F}{\partial z_j}(tz_1, \ldots, tz_m)dt = \sum_{j=1}^m z_j F_j(z),$$

where

$$F_j(z) \doteq \int_0^1 \frac{\partial F}{\partial z_j}(tz_1, \ldots, tz_n)dt$$

is holomorphic on $\mathcal{U}$ and smooth up to the boundary for all $1 \leq j \leq m$, so that

$$f = \sum_j (F_j|_W)(z_j|_W).$$

Such argument applies for instance to the hypersurface

$$\Omega_\sharp = \{(z, w) \in \mathbb{C}^2 : \operatorname{Im} w = e^{-1/|z|}\}$$

which is convex, minimal but not of finite type. Note that we do not currently know whether the Borel property holds for the CR structure induced on $\Omega_\sharp$.

10 Principal manifold ideals

We continue to work under the notation established in the last section. Let $f_1, \ldots, f_\ell \in \mathfrak{m}$ and consider the ideal $I = \langle f_1, \ldots, f_\ell \rangle \subset \mathfrak{m}$. We say that I is a *manifold ideal* if

$$\left(\mathrm{d}f_1 \wedge \ldots \wedge \mathrm{d}f_\ell \wedge \mathrm{d}\overline{f_1} \wedge \ldots \wedge \mathrm{d}\overline{f_\ell}\right)(0) \neq 0. \tag{10.1}$$

We denote by $V(I)$ the germ $\{f_1 = \ldots = f_\ell = 0\}$, and call it the *variety* of I.

Lemma 10.1 *If I is a manifold ideal then $V(I)$ is the germ of a regular submanifold of real codimension 2ℓ of $\mathbb{R}^N$ around 0. Moreover, we can find a coordinate system $(x_1, \ldots, x_m, t_1, \ldots, t_n)$ centered at the origin in $\mathbb{R}^N$ and solutions $Z_1, \ldots, Z_m$ satisfying the properties listed in Section 1B such that $I = \langle Z_1, \ldots, Z_\ell \rangle$.*

Proof The first claim is an immediate consequence of (10.1) whereas the second follows from the arguments in ([4], Theorem I.10.1) as done in ([3], Section 4).

We shall now restrict our attention *principal maximal ideals*, that is the ones generated by a single element $f \in \mathfrak{S}_0$ such that $(df \wedge d\overline{f})(0) \neq 0$. For any submanifold germ V of $\mathbb{R}^N$ around 0, we denote by $\mathcal{I}(V)$ the *ideal* of V, i.e. the ideal of $\mathfrak{S}_0$ consisting of those germs vanishing on V. It is clear that $I \subset \mathcal{I}(V(I))$. Our aim is to show that the opposite inclusion also holds:

Theorem 10.1 *Let $I \subset \mathfrak{S}_0$ be a principal manifold ideal. Then $\mathcal{I}(V(I)) = I$.*

We remark that in the previous statement no assumption is made about the minimality of $\mathcal{V}$ nor on the validity of property $(\mathfrak{B})$. In order to prove Theorem 10.1 we first prove a simple lemma.

Lemma 10.2 *Let $k \geq 2$. Then the function $\mathbb{C} \ni z \to \phi_k(z) = \overline{z}^k/z \in \mathbb{C}$ is of class C^{k-2}.*

Proof Clearly ϕ_k extends continuously to 0 since the function $\overline{z}/z$ is bounded. Choose $j, \ell \in \mathbb{N}$ such that $j + \ell \leq k - 2$. Then

$$\frac{\partial^{j+\ell}}{\partial z^j \partial \overline{z}^\ell} \phi_k(z) = (-1)^j \frac{j! k!}{\ell!} \frac{\overline{z}^{k-\ell}}{z^{j+1}} = (-1)^j \frac{j! k!}{\ell!} \left(\frac{\overline{z}}{z}\right)^{j+1} \overline{z}^{k-(j+\ell+1)}$$

is again continuous around 0 by the boundedness of $\overline{z}/z$, since $k - (j + \ell + 1) \geq 1$.

Proof of Theorem 10.1. We can assume that we are in the situation described in Section 1B in such a way $I = \langle Z_1 \rangle$ (cf. Lemma 10.1). Moreover since I is a principal maximal ideal we can even assume that $\phi_1(x, t) = t_1$.

Let $g \in \mathfrak{S}_0$ vanish on $V(I) = \{Z_1 = 0\} = \{x_1 = t_1 = 0\}$. Our goal is to show that $g = g_\bullet Z_1$ for some $g_\bullet \in \mathfrak{S}_0$.

We start by setting

$$h_k = \sum_{j=1}^{k} \frac{(-1)^j}{j!} Z_1^j \, \mathrm{M}_1^j g, \quad k \geq 1. \tag{10.2}$$

Notice that $h_k \in \mathfrak{S}_0$. We claim that

$$M^\alpha(g + h_k)|_{V(I)} = 0, \quad \alpha \in \mathbb{Z}_+^m, \ |\alpha| \leq k. \tag{10.3}$$

In order to prove (10.3) we first note that if $j \geq 2$ then $0 = M_j Z_1 = M_j x_1$ and hence M_j only involves $\partial/\partial x_2, \dots, \partial/\partial x_m$. Thus if $v \in C_0^\infty$ vanishes on $V(I)$ the same is true for $M^\alpha v$ if $\alpha = (0, \alpha_2, \dots, \alpha_m) \in \mathbb{Z}_+^m$. Thus (10.3) follows if we show that $M_1^\ell(g + h_k) = 0$ on $V(I)$ if $\ell \leq k$. By Leibniz rule we have

$$M_1^\ell(g + h_k) = M_1^\ell g + M_1^\ell \left\{ \sum_{j=1}^k \frac{(-1)^j}{j!} Z_1^j M_1^j g \right\}$$

$$= M_1^\ell g + \sum_{j=1}^k \sum_{r=0}^{\min\{j,\ell\}} \frac{(-1)^j}{j!} \binom{\ell}{r} \frac{j!}{(j-r)!} Z_1^{j-r} M_1^{j+\ell-r} g$$

If we restrict this last sum to $V(I)$ and recall that $\ell \leq k$ we obtain

$$\sum_{j=1}^\ell (-1)^j \binom{\ell}{j} (M^\ell g)|_{Z_1=0} = -(M^\ell g)|_{Z_1=0},$$

which completes the proof of (10.3).

Let $g' = g/Z_1$. Then g' is defined – and is a solution of $\mathcal{V}$ – on the complement of $\{Z_1 = 0\}$. It is enough to prove that for any $k \geq 2$ the germ g' extends across $\{Z_1 = 0\}$ as a function of class C^{k-2}. If h_{k-1} is as in (10.2) then $h_{k-1}/Z_1 \in \mathfrak{S}_0$ and hence we are left to showing that $(g + h_{k-1})/Z_1$ extends accross $Z_1 = 0$ as a function of class C^{k-2}.

We take advantage of (10.3). By Taylor's formula we can write

$$g + h_{k-1} = A_k Z_1 + B_k \overline{Z_1}^k,$$

where $A_k, B_k \in C_0^\infty$. Consequently by Lemma 10.2 we can write

$$(g + h_{k-1})/Z_1 = A_k + B_k \phi_k(Z_1)$$

is of class C^{k-2}, which completes the proof. $\qquad\square$

Example 1 If the assumption that I is a manifold ideal is not satisfied, the conclusion of Theorem 10.1 can fail to hold. For instance, let $\mathcal{V}$ be the locally integrable structure on $\mathbb{R}^3$, with coordinates written as (x, y, s), whose orthogonal $\mathcal{V}^\perp$ is spanned by the differential of the functions

$$Z = x + iy, \quad W = s + i(x^2 + y^2)$$

(this is the standard Hans Lewy structucture on $\mathbb{C}^2$), and define $I = \langle W \rangle$. Then I is not a manifold ideal, and we have that $V(I) = \{0\}$ and $\mathcal{I}(V(I)) = \langle Z, W \rangle = \mathfrak{m} \supsetneq I$.

Also note that $\mathfrak{m}$ does not coincide with the radical of the ideal I, since there is no $k \in \mathbb{Z}_+$ such that Z^k/W is of class C^∞ around 0. It follows that the *Nullstellensatz* does not hold for a (general) ideal of $\mathfrak{S}_0$.

Example 2 On the other hand, consider the structure $\mathcal{V}$ on $\mathbb{R}^5$, with coordinates written as (x_1, x_2, y_1, y_2, s) whose orthogonal $\mathcal{V}^\perp$ is spanned by the differential of the functions

$$Z_1 = x_1 + iy_1, \quad Z_2 = x_2 + iy_2, \quad W = s + i(x_1^2 + y_1^2 - x_2^2 - y_2^2).$$

Once again we have that $I = \langle W \rangle$ is not a manifold ideal, but in this case we have $\mathcal{I}(V(I)) = I$. Indeed it is well known that $\mathcal{V}$ is hypocomplex at the origin [1]. On the other hand, writing the complex coordinates in $\mathbb{C}^3$ as (z_1, z_2, w) we see that that if $H \in \mathcal{O}^{(3)}$ vanishes on $V(I)$ then $H(z_1, z_2, 0)$ vanishes on $|z_1|^2 = |z_2|^2$, and consequently $H = wH_1$, with $H_1 \in \mathcal{O}^{(3)}$. This proves our claim.

11 Some open questions

A certain number of questions arise, in our opinion, naturally from the results presented in the previous sections. Despite the quite elementary nature of some of them (the topic of the algebraic properties of the ring $\mathfrak{S}_p$ appears to be to some extent unexplored) their treatment seems to lead to delicate analytic issues. The following is an (incomplete) list of the problems which are for us most natural and interesting:

Question 1 Is the necessary condition found in Theorem 6.1 also sufficient for the surjectivity of the Borel map?

We conjecture that this should be the case, at least when the structure $\mathcal{V}$ is real-analytic.

Question 2 Does the conclusion of Theorem 10.1 hold for a non principal manifold ideal?

The method used in the proof of Theorem 10.1 does not extend easily to ideals generated by more than one solution.

Question 3 Is there an example in which the maximal ideal $\mathfrak{m}$ is *not* generated by the basic solutions $Z_1, \ldots, Z_m$?

The results in Sect. 9 show that this property in various situations, far apart from each other. The knowledge of the behavior of the Borel map seems to be important in most of the proofs, with the exception of the argument in Sect. 9C.

Question 4 For what values of p and q does the structure in Theorem 8.3 satisfy the Borel property?

We expect that the Borel property should hold precisely when p and q have different parity.

Question 5 Is the image of $\flat$ always a suitable quotient of a ring of the form $(\mathbb{C}\{Z_1, \ldots, Z_p\})[\![Z_{p+1}, \ldots, Z_m]\!]$?

In other words, in the cases settled so far the image of $\flat$ consists of formal series in a subset of variables whose coefficients are holomorphic functions in the other variables (more precisely, these coefficients must have a common radius of convergence).

Question 6 Suppose that two integrable structures $\mathcal{V}_1$, $\mathcal{V}_2$ are not hypocomplex (e.g. correspond to pseudoconvex hypersurfaces M_1, $M_2 \subset \mathbb{C}^n$), and the solutions rings $\mathfrak{S}_0^1$, $\mathfrak{S}_0^2$ are isomorphic. Does it follow that $\mathcal{V}_1$, $\mathcal{V}_2$ are locally equivalent (i.e. that M_1 and M_2 are locally CR diffeomorphic)?

It is clear that the answer to the previous question is negative if $\mathcal{V}_1$, $\mathcal{V}_2$ are hypocomplex, since both rings of solutions will always be isomorphic to the ring of convergent power series in n variables: if, however, there are enough solutions, one might hope that the ring $\mathfrak{S}_0$ contains enough information.

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