AAECC (2017) 28:497–525
DOI 10.1007/s00200-017-0316-0

ORIGINAL PAPER

Spaces of directed paths on pre-cubical sets

Krzysztof Ziemiański[1]

Received: 22 May 2016 / Revised: 11 December 2016 / Accepted: 20 February 2017 /
Published online: 2 March 2017

Abstract The spaces of directed paths on the geometric realizations of pre-cubical sets, called also □-sets, can be interpreted as the spaces of possible executions of Higher Dimensional Automata, which are models for concurrent computations. In this paper we construct, for a sufficiently good pre-cubical set K, a CW-complex $W(K)_v^w$ that is homotopy equivalent to the space of directed paths between given vertices v, w of K. This construction is functorial with respect to K, and minimal among all functorial constructions. Furthermore, explicit formulas for incidence numbers of the cells of $W(K)_v^w$ are provided.

Keywords Directed space · Pre-cubical set · Path space · Higher dimensional automaton

Mathematics Subject Classification 55U05 · 68Q99

1 Introduction

In recent years, much effort was made to understand spaces of directed paths on d-spaces. Particularly interesting examples of d-spaces are geometric realizations of pre-cubical sets [3,4], thanks to their applications in concurrency—their directed path spaces can be interpreted as the executions spaces of Higher Dimensional Automata [8]. Raussen [9] proved that such spaces, under certain conditions, have the homotopy types of CW-complexes. In this paper we construct a combinatorial model of the space

✉ Krzysztof Ziemiański
ziemians@mimuw.edu.pl

1 Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland

of directed paths between two fixed vertices of a pre-cubical set, assuming it satisfies a certain mild condition, i.e., its non-looping length covering being proper (cf. 1.3, 1.4).

The special cases of Higher Dimensional Automata are cubical complexes which are state spaces of PV-programs, i.e., concurrent programs using Dijkstra's semaphores [2] for synchronization. Models for the execution spaces of PV-programs were constructed in [10–12]. It is also known that homotopy types of such spaces may be quite complicated; in fact, any finite homotopy type can be obtained as a connected component of such a space [18], and the Betti numbers of execution spaces may grow exponentially with respect to the length of a PV-program [13]. Since cubical complexes coming from PV-programs are proper, the results of this paper can be applied in this case. The model of the execution space of a PV-program constructed here has a nice (though complicated) structure of a "permutahedral complex". This allows to use methods of combinatorial topology to understand these spaces. On the other hand, Theorem 1.4 allows to implement an algorithm which calculates the homology of the execution state of a PV-program which could be more efficient than previously known ones.

A pre-cubical set K, called also a $\square$-set, is a sequence of disjoints sets $K[n]$, for $n \geq 0$, equipped with *face maps* $d_i^\varepsilon \colon K[n] \to K[n-1]$, for $\varepsilon \in \{0,1\}$ and $i \in \{1,\dots,n\}$, that satisfy pre-cubical relations; namely, $d_i^\varepsilon d_j^\eta = d_{j-1}^\eta d_i^\varepsilon$ for $i < j$. *A* $\square$*-map* $f \colon K \to K'$ between $\square$-sets is a sequence of maps $f[n] \colon K[n] \to K[n']$ that commute with the face maps. Elements of $K[n]$ will be called *n-cubes* of K; in particular 0-cubes will be called *vertices*. A *bi-pointed* $\square$-set is a triple (K, v, w), where K is a $\square$-set, and $v, w \in K[0]$ are vertices in K. Let $\square\mathbf{Set}$ and $\square\mathbf{Set}_*^*$ denote the category of $\square$-sets and $\square$-maps and the category of bi-pointed $\square$-sets and $\square$-maps preserving the distinguished vertices, respectively.

Let us introduce a notation for arbitrary compositions of face maps. For a function $f \colon \{1,\dots,n\} \to \{0,1,*\}$ such that $|f^{-1}(*)| = m$, define a map $d_f \colon K[n] \to K[m]$ by

$$d_f = d_1^{f(1)} d_2^{f(2)} \dots d_n^{f(n)}, \tag{1.1}$$

where d_i^* is, by convention, the identity map; we will also write $d_{f^{-1}(0), f^{-1}(1)}$ for d_f. Finally, let $d_A^0 := d_{A,\emptyset}, d_A^1 = d_{\emptyset,A}$ and $d^\varepsilon = d_{\{1,\dots,n\}}^\varepsilon \colon K[n] \to K[0]$. If $f \colon \{1,\dots,n\} \to \{0,1,*\}, g \colon \{1,\dots,m\} \to \{0,1,*\}$ are functions such that $|f^{-1}(*)| = m$, $|g^{-1}(*)| = k$, then $d_g d_f = d_h$, where

$$h \colon \{1,\dots,n\} \ni i \mapsto \begin{cases} f(i) & \text{for } i \notin f^{-1}(*) \\ g(\alpha(i)) & \text{for } i \in f^{-1}(*) \end{cases} \in \{0,1,*\}, \tag{1.2}$$

and $\alpha \colon f^{-1}(*) \to \{1,\dots,m\}$ is the unique increasing bijection.

The results of this paper apply to finite $\square$-sets that have proper non-looping length coverings. We say that a $\square$-set K is *proper*, if the map

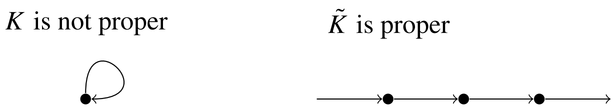
$$\coprod_{n\geq 0} K[n]\colon c \mapsto \{d^0(c), d^1(c)\} \in 2^{K[0]} \tag{1.3}$$

is an injection, i.e., the cubes of K can be distinguished by their extreme vertices. Every proper $\square$-set is non-self-linked in the sense of [3], i.e., for every $c \in K[n]$ and $f, g\colon \{1, \ldots, n\} \to \{0, 1, *\}$, the equality $d_f(c) = d_g(c)$ implies that $f = g$. *The non-looping length covering* [11, 5.1] of a $\square$-set K is the $\square$-set $\tilde{K}$ such that $\tilde{K}[n] = K[n] \times \mathbb{Z}$ and

$$d_i^\varepsilon(c, k) = \left(d_i^\varepsilon(c), k + \varepsilon\right). \tag{1.4}$$

Clearly, if a $\square$-set is proper, then its non-looping length covering is also proper. Let $\mathbf{pc\square Set}_*^* \subseteq \mathbf{\square Set}_*^*$ be the full subcategory of finite bi-pointed $\square$-sets having proper non-looping coverings.

Remark There are $\square$-sets which are not proper but their non-looping length coverings are proper; the simplest example is the directed circle, see (1.5). In many cases, the barycentric subdivision of a non-proper $\square$-set is proper (1.6). Unfortunately, if $K = (\square^2 \sqcup \square^2)/\partial\square^2$ is the union of two squares glued along their boundaries, then the non-looping length covering of the barycentric subdivision of K (even iterated) is not proper.



(1.5)

(1.6)

Definition 1.1 Let K be a $\square$-set and let $v, w \in K[0]$ be two of its vertices. *A cube chain in K from v to w* is a sequence of cubes $\mathbf{c} = (c_1, \ldots, c_l)$, where $c_k \in K[n_k]$ and $n_k > 0$, such that

- $d^0(c_1) = v$,
- $d^1(c_l) = w$,
- $d^1(c_i) = d^0(c_{i+1})$ for $i = 1, \ldots, l-1$.

The sequence $(n_1, \ldots, n_l)$ will be called *the type* of a cube chain $\mathbf{c}$, $\dim(\mathbf{c}) = n_1 + \cdots + n_l - l$ *the dimension* of $\mathbf{c}$, and $n_1 + \cdots + n_l$ *the length* of $\mathbf{c}$. The set of all cube chains in K from v to w will be denoted by $\mathrm{Ch}(K)_v^w$, and the set of cube chains of dimension equal to m (resp. less than m, less or equal to m) by $\mathrm{Ch}^{=m}(K)_v^w$ (resp. $\mathrm{Ch}^{<m}(K)_v^w$, $\mathrm{Ch}^{\leq m}(K)_v^w$). Note that a cube chain has dimension 0 if and only if it contains 1-cubes only.

For a cube chain $\mathbf{c} = (c_1, \dots, c_l) \in \mathrm{Ch}(K)_v^w$ of type $(n_1, \dots, n_l)$, an integer $k \in \{1, \dots, l\}$ and a subset $A \subsetneq \{1, \dots, n_k\}$ having r elements, where $0 < r < n_k$, define a cube chain

$$d_{k,A}(\mathbf{c}) = \Big(c_1, \dots, c_{k-1}, d^0_{\bar{A}}(c_k), d^1_A(c_k), c_{k+1}, \dots, c_l\Big) \in \mathrm{Ch}(K)_v^w, \qquad (1.7)$$

where $\bar{A} = \{1, \dots, n_k\} \setminus A$. The cube chain $\mathbf{c}$ is thus subdivided once at one of the inner vertices of c_k. Let $\leq$ be the partial order on $\mathrm{Ch}(K)$ spanned by all relations having the form $d_{k,A}(\mathbf{c}) \leq \mathbf{c}$. Clearly $\dim(d_{k,A}(\mathbf{c})) = \dim(\mathbf{c}) - 1$; thus, the relation $\leq$ is antisymmetric.

For a $\square$-set K let $|K|$ denote its geometric realization (3.1) and, for a bi-pointed $\square$-set (K, v, w), let $\vec{P}(K)_v^w$ be the space of directed paths on $|K|$ from v to w (cf. 2.1). Clearly, $(K, v, w) \mapsto \mathrm{Ch}(K)_v^w$ and $(K, v, w) \mapsto \vec{P}(K)_v^w$ are functors from $\square\mathbf{Set}^*_*$ into the categories of posets and topological spaces, respectively.

Let **hTop** be the homotopy category of the category of topological spaces. We prove the following

Theorem 1.2 *The functors*

$$\vec{P}\colon \mathbf{pc}\square\mathbf{Set}^*_* \ni (K, v, w) \mapsto \vec{P}(K)_v^w \in \mathbf{hTop}$$

and

$$|\mathrm{Ch}|\colon \mathbf{pc}\square\mathbf{Set}^*_* \ni (K, v, w) \mapsto |\mathrm{Ch}(K)_v^w| \in \mathbf{hTop}$$

are naturally equivalent. In other words, for every finite bi-pointed $\square$-set (K, v, w) having a proper non-looping covering, there exists a homotopy equivalence $\varepsilon_{(K,v,w)}$: $\vec{P}(K)_v^w \to |\mathrm{Ch}(K)_v^w|$ such that, for every $\square$-map $f\colon K \to K'$, the diagram

$$\begin{tikzcd}
\vec{P}(K)_v^w \arrow[r, "\alpha \mapsto |f| \circ \alpha"] \arrow[d, "\varepsilon_{(K,v,w)}"'] & \vec{P}(K')_{f(v)}^{f(w)} \arrow[d, "\varepsilon_{(K', f(v), f(w))}"] \\
|\mathrm{Ch}(K)_v^w| \arrow[r, "Ch(f)\colon \mathbf{c} \mapsto f(\mathbf{c})"] & |\mathrm{Ch}(K')_{f(v)}^{f(w)}|
\end{tikzcd}$$

commutes up to homotopy.

Next, we prove that the spaces $\mathrm{Ch}(K)_v^w$ have a natural CW-structure, which is coarser than the simplicial one. For a poset P and $x \in P$, let $P_{\leq x}$ (resp. $P_{<x}$) be the subposet of P containing all elements that are less or equal to x (resp. less than x).

Theorem 1.3 *Let (K, v, w) be a finite bi-pointed $\square$-set having a proper non-looping covering (i.e., $(K, v, w) \in \mathbf{pc}\square\mathbf{Set}$). The space $|\mathrm{Ch}(K)_v^w|$ is a regular CW-complex with d-dimensional cells having the form $|\mathrm{Ch}_{\leq \mathbf{c}}(K)|$ for $\mathbf{c} \in \mathrm{Ch}^{=d}(K)_v^w$. Furthermore, for every $\square$-map $f\colon K \to K'$, the induced map $f_*\colon |\mathrm{Ch}(K)_v^w| \to |\mathrm{Ch}(K')_{f(v)}^{f(w)}|$ is cellular.*

We also calculate the incidence numbers between the cells of this CW-complex. In Sect. 8 we construct, for every $\mathbf{c} \in \mathrm{Ch}^{=n}(K)_v^w$, a cycle $g_{\mathbf{c}}$ that represents a generator in the simplicial homology group

$$[g_{\mathbf{c}}] \in H_n(|\mathrm{Ch}_{\leq \mathbf{c}}(K)|_v^w, |\mathrm{Ch}_{<\mathbf{c}}(K)|_v^w), \tag{1.8}$$

As a consequence of Theorem 1.3, the group $H_n(|\mathrm{Ch}^{\leq n}(K)_v^w|, |\mathrm{Ch}^{<n}(K)_v^w|)$ is a free group generated by $g_{\mathbf{c}}$ for $\mathbf{c} \in \mathrm{Ch}^{=n}(K)_v^w$.

Theorem 1.4 *Let (K, v, w) be a finite bi-pointed $\square$-set having a proper non-looping covering, and let*

$$\partial_n \colon H_n(|\mathrm{Ch}^{\leq n}(K)_v^w|, |\mathrm{Ch}^{<n}(K)_v^w|) \to H_{n-1}(|\mathrm{Ch}^{\leq n-1}(K)_v^w|, |\mathrm{Ch}^{<n-1}(K)_v^w|)$$

be the differential in the cellular chain complex of $|\mathrm{Ch}(K)_v^w|$ (cf. [7, p. 139]). For a cube chain $\mathbf{c} \in \mathrm{Ch}^{=n}(K)_v^w$ of type $(n_1, \ldots, n_l)$ we have

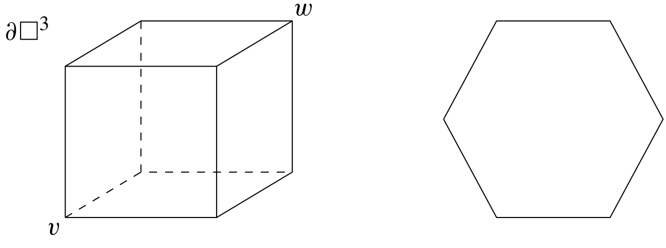
$$\partial(g_{\mathbf{c}}) = \sum_{k=1}^{l} \sum_{r=1}^{n_k-1} \sum_{A \subseteq \{1,\ldots,n_k\}\colon |A|=r} (-1)^{n_1+\cdots+n_{k-1}+k+r+1} \operatorname{sgn}(A) g_{d_{k,A}(\mathbf{c})},$$

where

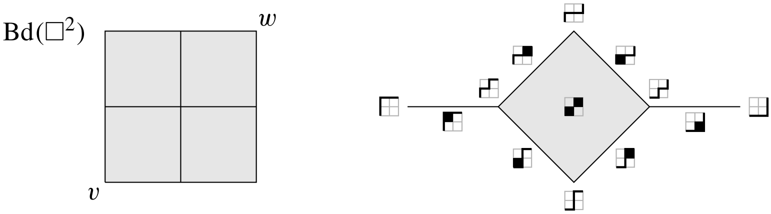
$$\operatorname{sgn}(A) = \begin{cases} 1 & \textit{if } \sum_{i \in A} i \equiv \sum_{i=1}^{r} i \mod 2, \\ -1 & \textit{otherwise.} \end{cases}$$

As a consequence, the homology of $\vec{P}(K)_v^w$ can be calculated using the formula above.

We conclude with two examples. For $K = \partial\square^3$ (1.9), $|\mathrm{Ch}(K)_v^w|$ is a hollow hexagon; its vertices correspond to directed paths from v to w running along vertices, i.e., cube chains of type $(1, 1, 1)$, and its edges correspond to cube chains of types $(1, 2)$ or $(2, 1)$. For K being the barycentric subdivision of $\square^2$, $|\mathrm{Ch}(K)_v^w|$ is presented in (1.10).



(1.9)



(1.10)

2 Directed spaces

In this section, we recall the notion of d-space and introduce complete d-spaces [5], which are necessary to define the d-structure on the geometric realization of a d-simplicial complex. For a complete exposition of this topic see for example [6].

Definition 2.1 Let X be a topological space and let $P(X)$ be its path space, i.e. the space of continuous maps $[0, 1] \to X$ with compact-open topology.

- *A d-structure* on X is a subset $\mathfrak{D} \subseteq P(X)$ which contains all constant paths, and is closed with respect to concatenations and non-decreasing reparametrizations of $[0, 1]$, not necessarily surjective.
- *A d-space* is a pair $(X, \mathfrak{D})$, where X is a topological space and $\mathfrak{D}$ is a d-structure on X.
- Let $(X, \mathfrak{D})$, $(X', \mathfrak{D}')$ be d-spaces. A continuous map $f\colon X \to X'$ is *a d-map* if it preserves d-structure, i.e. $f \circ \alpha \in \mathfrak{D}'$ for each $\alpha \in \mathfrak{D}$.
- *A d-homeomorphism* is an invertible d-map.

The category of d-spaces and d-maps is complete and cocomplete. For an arbitrary family of paths $\mathfrak{S} \subseteq P(X)$ there exists a smallest d-structure $\overline{\mathfrak{S}}$ on X containing $\mathfrak{S}$; it can be constructed as the intersection of all d-structures containing $\mathfrak{S}$, or by adding to $\mathfrak{S}$ all constant paths and all non-decreasing reparametrizations of concatenations of paths in $\mathfrak{S}$.

The following d-spaces play an important role in this paper:

- *the directed real line* $\vec{\mathbb{R}} = (\mathbb{R}$, non-decreasing paths),
- *the directed n-cube* $\vec{I}^n = ([0, 1]^n$, paths with non-decreasing coordinates),
- *the directed n-simplex* $\vec{\Delta}^n = (\Delta^n, \vec{P}(\Delta^n))$, where

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in [0, 1]^{n+1}\colon \sum_{i=0}^{n} t_i = 1 \right\}$$

and $\alpha = (\alpha_0, \dots, \alpha_n) \in P(\Delta^n)$ is directed if and only if the functions

$$[0, 1] \ni t \mapsto \sum_{i \geq k} \alpha_i(t) \in [0, 1]$$

are non-decreasing for $k \in \{1, \dots, n\}$.

Let us recall the definition of complete d-spaces introduced in [17] (called there "good" d-spaces).

Definition 2.2 Let $(X, \mathfrak{D})$ be a d-space. A path $\alpha\colon [0,1] \to X$ is *almost directed* if for every open subset $U \subseteq X$ and every $0 \le s < t \le 1$ such that $\alpha([s,t]) \subseteq U$, there exists a directed path $\beta \in \vec{P}(X)$ such that $\beta(0) = \alpha(s), \beta(1) = \alpha(t)$ and $\beta([0,1]) \subseteq U$.

The family of all almost directed paths $\mathfrak{D}_c$ on a d-space $(X, \mathfrak{D})$ is a d-structure on X. The d-space $(X, \mathfrak{D}_c)$ will be called *the completion* of $(X, \mathfrak{D})$; if a given d-space is equal to its completion, it is called *complete*. If $\mathfrak{S} \subseteq P(X)$ is an arbitrary family of paths, then its *completion* $\overline{\mathfrak{S}}_c$ is the smallest complete d-structure containing $\mathfrak{S}$. We add a criterion which allows to verify whether a continuous map between complete d-spaces is a d-map.

Proposition 2.3 *Let $f\colon X \to X'$ be a continuous map and let $\mathfrak{S} \subseteq P(X), \mathfrak{S}' \subseteq P(X')$ be families of paths. Assume that $f \circ \alpha \in \overline{\mathfrak{S}'}_c$ for every $\alpha \in \mathfrak{S}$. Then the map $f\colon (X, \overline{\mathfrak{S}}_c) \to (X', \overline{\mathfrak{S}'}_c)$ is a d-map.*

Proof Fix a path $\alpha \in \overline{\mathfrak{S}}_c$, $0 \le s < t \le 1$ and an open subset $U \subseteq X'$ such that $f(\alpha([s,t]) \subseteq U$. Since α is almost directed with respect to $\overline{\mathfrak{S}}$, there exists a path $\beta \in \overline{\mathfrak{S}}$ such that $\beta(0) = \alpha(s), \beta(1) = \alpha(t)$ and $\beta([0,1]) \subseteq f^{-1}(U)$. The path β is either constant, or it is a non-decreasing reparametrization of a concatenation of paths in $\mathfrak{S}$. Thus, by assumption, $f \circ \beta$ is either constant, or it is a non-decreasing reparametrization of concatenation of paths in $\mathfrak{S}'$. This implies that $f \circ \alpha$ is almost directed with respect to $\overline{\mathfrak{S}'}$; therefore $f \circ \alpha \in \overline{\mathfrak{S}}_c$. □

For every $n \ge 0$, the directed n-cube and the directed n-simplex are complete d-spaces. The d-structure on $\vec{I}^n$ is generated by the family of paths

$$\{[0,1] \ni t \mapsto (x_1, \dots, x_{k-1}, t, x_{k+1}, \dots, x_n)\colon k \in \{1, \dots, n\},\ x_i \in [0,1]\} \tag{2.1}$$

and the d-structure on $\vec{\Delta}^n$ by

$$\left\{[0,1] \ni t \mapsto (x_0, \dots, x_{k-2}, (1-t)c, tc, x_{k+1}, \dots, x_n)\colon \right. \\ \left. k \in \{1, \dots, n\},\ c, x_i \in [0,1],\ c + \sum_{i \ne k-1,k} x_i = 1 \right\}. \tag{2.2}$$

If it is clear which d-structure we consider on a given space X, we will denote it by $\vec{P}(X)$; the elements of $\vec{P}(X)$ will be called *directed paths* or *d-paths*. Given two points $x, y \in X$, let

$$\vec{P}(X)_x^y := \{\alpha \in \vec{P}(X)\colon \alpha(0) = x,\ \alpha(1) = y\} \tag{2.3}$$

be the space of directed paths from x to y. We extend the notion of directedness for paths defined on arbitrary closed intervals; a path $[a,b] \to X$ is called directed if its linear reparametrization $[0,1] \ni t \mapsto \alpha(a + t(b-a)) \in X$ is directed. The space of such paths will be denoted by $\vec{P}_{[a,b]}(X)$.

3 $\square$-Sets

In this section we introduce $\square$-sets with height function, and prove that their geometric realizations are complete d-spaces. For a more detailed discussion on $\square$-sets see also [3].

The geometric realization of a $\square$-set K is a d-space

$$|K| = \coprod_{n\geq 0} K[n] \times \vec{I}^n/(d_i^\varepsilon(c), x) \sim (c, \delta_i^\varepsilon(x)), \tag{3.1}$$

where $\delta_i^\varepsilon(s_1, \dots, s_{n-1}) = (s_1, \dots, s_{i-1}, \varepsilon, s_i, \dots, s_{n-1})$. A path $\alpha \in P(|K|)$ is directed if there exist numbers $0 = t_0 < t_1 < \cdots < t_l = 1$, cubes $c_i \in K[n_i]$ and directed paths $\beta_i\colon [t_{i-1}, t_i] \to \vec{I}^{n_i}$ such that $\alpha(t) = (c_i, \beta_i(t))$ for $t \in [t_{i-1}, t_i]$. For every $x \in |K|$ there exists a unique cube $\mathrm{supp}(x)$ of K, called *the support of* x, such that $x = (\mathrm{supp}(x), (t_1, \dots, t_n))$, and $t_i \neq 0, 1$ for $i \in \{1, \dots, n\}$. For shortness, we will further write $\vec{P}(K)$ instead of $\vec{P}(|K|)$.

Definition 3.1 *A height function* on a $\square$-set K is a function $h\colon \coprod_{n\geq 0} K[n] \to \mathbb{Z}$ such that $h(d_i^\varepsilon(c)) = h(c) + \varepsilon$ for every $n \geq 0$, $c \in K[n]$, $\varepsilon \in \{0, 1\}$ and $i \in \{1, \dots, n\}$.

Not every $\square$-set allows a height function; see the directed circle (1.5) for example. A height function h on K determines a d-map

$$h\colon |K| \ni (c, (s_1, \dots, s_n)) \mapsto h(c) + \sum_{i=1}^n s_i \in \vec{\mathbb{R}} \tag{3.2}$$

which will also be called a height function. If $\alpha \in \vec{P}(K)$ is a directed path, then $l_1(\alpha) = h(\alpha(1)) - h(\alpha(0))$, where l_1 is the L_1-arc length of Raussen [11, 5.1]. In particular, if $\alpha \in \vec{P}(K)$ is not constant on any interval, then $h \circ \alpha$ is a strictly increasing function, which implies that all directed loops on a $\square$-set with height function are constant.

Definition 3.2 Let K be a $\square$-set with height function. A d-path $\alpha\colon [a, b] \to |K|$ is *natural* if $h(\alpha(t)) = t$ for every $t \in [a, b]$. For $v, w \in K[0]$, let $\vec{N}(K)_v^w \subseteq \vec{P}_{[h(v),h(w)]}(|K|)_v^w$ be the space of natural paths from v to w.

The space $\vec{N}(K)_v^w$ can be regarded as a subspace $\vec{P}(K)_v^w$ via the reparametrizing inclusion

$$\vec{N}(K)_v^w \ni \alpha \mapsto \alpha \circ (t \mapsto a + (b - a)t) \in \vec{P}(K)_v^w. \tag{3.3}$$

On the other hand, by [9, 2.15] there exists *a naturalization map*

$$nat\colon \vec{P}(K)_v^w \to \vec{N}(K)_v^w, \tag{3.4}$$

which is a left inverse of (3.3). As shown in [9], these maps are mutually inverse homotopy equivalences.

We conclude this section with the following

Proposition 3.3 *If K is a $\square$-set with height function, then its geometric realization $|K|$ is a complete d-space.*

Proof Let $\preceq$ be the minimal reflexive and transitive relation on $\coprod_{n\geq 0} K[n]$ such that $d_i^0(c) \preceq c$ and $c \preceq d_i^1(c)$ for $n \geq 0, c \in K[n], i \in \{1, \ldots, n\}$. The relation $\preceq$ is a partial order since, for $c \neq c'$, $c \preceq c'$ implies that either $h(c) < h(c')$, or $h(c) = h(c')$ and $\dim(c) < \dim(c')$. Moreover, it follows immediately from the definition that $\mathrm{supp}(\alpha(0)) \preceq \mathrm{supp}(\alpha(1))$ for every directed path α on $|K|$.

Let α be an almost directed path in $|K|$ and let, for an arbitrary cube c of K, $J_c \subseteq [0, 1]$ be the closure of the set $\{s \in [0, 1]\colon \mathrm{supp}(\alpha(s)) = c\}$. For $s < s'$ there exists a directed path from $\alpha(s)$ to $\alpha(s')$; therefore $\mathrm{supp}(\alpha(s)) \preceq \mathrm{supp}(\alpha(s'))$. This implies that all the sets J_c are either closed intervals or empty. Thus α is a concatenation of almost directed paths each of which lies in a single cube. Every such path is directed, hence so is α. $\square$

4 d-simplicial complexes

In this section, we recall the definition of a d-simplicial complex and prove that every proper $\square$-set with height function has a d-simplicial triangulation.

Definition 4.1 [16] *A d-simplicial complex* M is a triple $(V_M, S_M, \leq_M)$, where (V_M, S_M) is a simplicial complex and $\leq_M$ is a binary relation of V_M such that

- for every simplex $A \in S_M$, the restriction $(\leq_M)|_A$ is a total order on A,
- if $v \leq_M v'$, then $\{v, v'\} \in S_M$.

A d-simplicial map $f\colon M \to M'$ is a simplicial map such that $v \leq_M w$ implies $f(v) \leq_{M'} f(w)$ for $v, w \in V_M$. *The geometric realization* of a d-simplicial complex M is the space

$$|M| = \left\{ \sum_{v\in V_M} t_v v\colon t_v \geq 0,\ \sum_{v \in V_M} t_v = 1,\ \{v\colon t_v > 0\} \in S_M \right\}$$

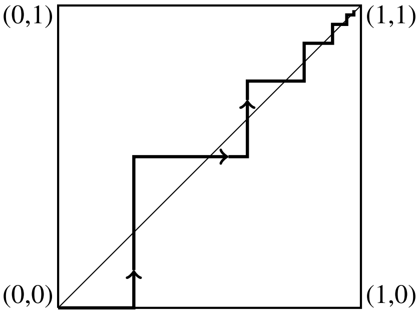
with the maximal topology and the minimal complete d-structure such that the inclusions of simplices

$$\vec{\Delta}^n \ni (t_0, \ldots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i \in |M| \tag{4.1}$$

are d-maps for all $\{v_0 < \cdots < v_n\} \in S_M$.

Remark The completion of d-structure is necessary to obtain proper d-structures on the triangulations of $\square$-sets. For example, the directed square $\vec{\square}^2$ admits a triangulation that is a d-simplicial complex with vertices $(i, j), i, j \in \{0, 1\}$, maximal simplices

$((0,0) < (0,1) < (1,1))$ and $((0,0) < (1,0) < (1,1))$, and $(i,j) \leq (i',j')$ if and only if $i \leq i'$, $j \leq j'$. However, the infinite staircase path on the picture below



(4.2)

is not directed with respect to the (non-completed) d-structure induced by the inclusions of simplices, though it is directed with respect to the d-structure of the square.

Example 4.2 If P is a poset, then the nerve of P, denoted by $\mathcal{N}P$, is defined by $V_{\mathcal{N}P} = P$, $\leq_{\mathcal{N}P} = \leq_P$ and $A \subseteq P$ is a simplex of $\mathcal{N}P$ if and only if A is a totally ordered subset.

Remark d-simplicial complexes can be regarded as special cases of simplicial sets. For a d-simplicial complex M one can define a simplicial set whose n-simplices are functions $\sigma\colon \{0,\dots,n\} \to V_M$ such that $\sigma(i) \leq_M \sigma(j)$ for every $i \leq j$ and the image of σ is a simplex of M.

For the rest of the section we assume that K is a proper $\square$-set. We will construct a triangulation of K, i.e., a d-simplicial complex Tr_K such that $|K|$ and Tr_K are d-homeomorphic. With notation as in (1.1), let

$$\begin{aligned} S_{\mathrm{Tr}(K)} := \{&\{d_{f_0}(c),\dots,d_{f_k}(c)\} \subseteq K[0]\colon \\ &c \in K[n],\ f_0 < f_1 < \dots < f_k,\ f_i\colon \{1,\dots,n\} \to \{0,1\}\}, \end{aligned} \tag{4.3}$$

where $f_j < f_k$ means that $f_j \neq f_k$ and $f_j(i) \leq f_k(i)$ for all i. Introduce a binary relation $\leq_{\mathrm{Tr}(K)}$ on $K[0]$ by

$$v \leq_{\mathrm{Tr}(K)} v' \Leftrightarrow \exists_{c\in K[n]} \exists_{f \leq f'\colon \{1,\dots,n\}\to\{0,1\}}\, d_f(c) = v,\ d_{f'}(c) = v'. \tag{4.4}$$

Every element $A = \{d_{f_0}(c) < \dots < \dots, d_{f_k}(c)\}$, $c \in K[n]$ of $S_{\mathrm{Tr}(K)}$ can be written as

$$A = \{d_{f_0\circ e}(c') < \dots < \dots, d_{f_k\circ e}(c')\},$$

where $c' = d_{f_k^{-1}(0), f_0^{-1}(1)}(c)$ and

$$e\colon \{1,\dots,n - |f_k^{-1}(0)| - |f_0^{-1}(1)|\} \to \{1,\dots,n\} \setminus (f_k^{-1}(0) \cup f_0^{-1}(1))$$

is an increasing bijection. Thus, A has a unique presentation such that $f_0 \equiv 0$ and $f_k \equiv 1$. By the properness of K, the cube c is determined by its extreme vertices $d^0(c) = d_{f_0}(c)$ and $d^1(c) = d_{f_k}(c)$ and, for every vertex v of c, there exists a unique f such that $v = d_f(c)$. Such a presentation of a simplex of $\mathrm{Tr}(K)$ will be called *canonical*.

Proposition 4.3 *If K is a proper $\square$-set, then*

$$\mathrm{Tr}(K) = (V_{\mathrm{Tr}(K)} := K[0], S_{\mathrm{Tr}(K)}, \leq_{\mathrm{Tr}(K)})$$

is a d-simplicial complex.

Proof It follows immediately from the definition that $\mathrm{Tr}(K)$ is a simplicial complex, and that $v \leq_{\mathrm{Tr}(K)} v'$ implies $\{v, v'\} \in S_{\mathrm{Tr}(K)}$. It remains to prove that, for every $A \in S_{\mathrm{Tr}(K)}$, the restriction $\leq_{\mathrm{Tr}(K)} |_A$ is a total order. We have

$$A = \{v_0 = d_{f_0}c \leq_{\mathrm{Tr}(K)} v_1 = d_{f_1}c \leq_{\mathrm{Tr}(K)} \cdots \leq_{\mathrm{Tr}(K)} v_k = d_{f_k}c\}$$

for some $n \geq 0, c \in K[n]$ and $f_0 < \cdots < f_k$. Since K is proper, the vertices $v_0, \ldots, v_k$ are all different. Clearly $\leq_{\mathrm{Tr}(K)} |_A$ contains a total order. Assume that $v_j \leq_{\mathrm{Tr}(K)} v_i$ for $i < j$. Then there exists $c' \in K[n']$ and $f_0', f_1'\colon \{1, \ldots, n'\} \to \{0, 1\}$, $f_0' < f_1'$, such that $d_{f_0'}c' = v_j$ and $d_{f_1'}c' = v_i$. We have

$$d^0(d_{f_j^{-1}(0), f_i^{-1}(1)}c) = v_i \neq v_j = d^1(d_{f_j^{-1}(0), f_i^{-1}(1)}c),$$
$$d^0(d_{(f_1')^{-1}(0), (f_0')^{-1}(1)}c') = v_j \neq v_i = d^1(d_{(f_1')^{-1}(0), (f_0')^{-1}(1)}c'),$$

then $d_{f_j^{-1}(0), f_i^{-1}(1)}c$ and $d_{(f_1')^{-1}(0), (f_0')^{-1}(1)}c'$ are two different cubes with the same sets of extreme vertices, which contradicts the properness of K. □

The d-simplicial complex $\mathrm{Tr}(K)$ will be called *the triangulation* of K.

Our next goal is to construct a d-homeomorphism between the geometric realization of K and the geometric realization of its triangulation. Let A be a simplex of $\mathrm{Tr}(K)$ and let $(d_{f_0}(c), \ldots, d_{f_k}(c))$, $c \in K[n]$ be its unique presentation such that $f_0 \equiv 0$, $f_k \equiv 1$. Define a continuous map

$$F_A\colon |A| \ni \sum_{i=0}^{k} t_i d_{f_i}(c) \mapsto \left(c, \sum_{i=0}^{k} t_i f_i\right) \in |K|, \tag{4.5}$$

where functions $f\colon \{1, \ldots, n\} \to [0, 1]$ are regarded as points $(f(1), \ldots, f(n)) \in \vec{I}^n$.

Proposition 4.4 *If $x \in |A| \cap |A'|$, then $F_A(x) = F_{A'}(x)$.*

Proof It is sufficient to check this for

$$A' = (d_{f_0}(c), \dots, d_{f_k}(c)),$$
$$A = (d_{f_0}(c), \dots, d_{f_{j-1}}(c), d_{f_{j+1}}(c), \dots, d_{f_k}(c)),$$

$f_0 \equiv 0$, $f_k \equiv 1$. If $j \neq 0, k$, then the preceding presentation of A is canonical and the equation $F_A = F_{A'}|_{|A|}$ is clearly satisfied. For $j = 0$ the canonical presentation of A is

$$A = \left(d_{f_1 \circ e}(d^1_{f_0^{-1}(1)}(c)), \dots, d_{f_k \circ e}(d^1_{f_0^{-1}(1)}(c))\right),$$

where $e\colon \{1, \dots, n - |f_1^{-1}(1)|\} \to \{1, \dots, n\} \setminus f_1^{-1}(1)$ is the increasing bijection. We have

$$\begin{aligned} f_A\left(\sum_{i=1}^k t_i d_{f_i}(c)\right) &= f_A\left(\sum_{i=1}^k t_i d_{f_i \circ e}(d^1_{f_0^{-1}(1)}(c))\right) = \left(d^1_{f_0^{-1}(1)}(c), \sum_{i=1}^k t_i (f_i \circ e)\right) \\ &= \left(c, \delta^1_{f_0^{-1}(1)}\left(\sum_{i=1}^k t_i (f_i \circ e)\right)\right) \\ &= \left(c, \sum_{i=1}^k t_i f_i\right) = f_{A'}\left(\sum_{i=1}^k t_i d_{f_i}(c)\right). \end{aligned}$$

The case $j = k$ is similar. □

As a consequence, the maps F_A glue to a continuous map $F_K\colon |\mathrm{Tr}(K)| \to |K|$.

Proposition 4.5 *F_K is a d-homeomorphism.*

Proof For $c \in K[n]$ and a permutation σ of $\{1, \dots, n\}$ define a set

$$S_{c,\sigma} := \{(c, (t_1, \dots, t_n)) \in |K|\colon t_{\sigma(1)} \le t_{\sigma(2)} \le \dots \le t_{\sigma(n)}\}$$

and a sequence of functions $f_0^\sigma, \dots, f_n^\sigma\colon \{1, \dots, n\} \to \{0, 1\}$ by

$$f_i^\sigma(j) = \begin{cases} 0 & \text{for } \sigma(j) > i \\ 1 & \text{for } \sigma(j) \le i. \end{cases}$$

Clearly the sets $S_{c,\sigma}$ cover $|K|$. Similarly to the proof of Proposition 4.4 we can show that the maps

$$G_{c,\sigma}\colon S_{c,\sigma} \ni (c, (t_1, \dots, t_n)) \mapsto \sum_{i=0}^n (t_{\sigma(i+1)} - t_{\sigma(i)}) d_{f_i^\sigma}(c) \in |\mathrm{Tr}(K)|,$$

where by convention $t_{\sigma(0)} = 0$ and $t_{\sigma(n+1)} = 1$, glue to a map $G_K\colon |K| \to |\operatorname{Tr}(K)|$, which is the inverse of F. Thus F_K is a homeomorphism. Next, we need to prove that both F_K and G_K are d-maps. By definition, $\vec{P}(|\operatorname{Tr}(K)|)$ is a complete d-structure generated by paths having the form (cf. 2.2, 4.1)

$$\omega\colon s \mapsto b(1-s)d_{f_k}(c) + bsd_{f_{k+1}}(c) + \sum_{i\neq k,k+1} t_i d_{f_i}(c)$$

for $n \geq 0$, $c \in K[n]$, $0 \equiv f_0 < \cdots < f_n \equiv 1$, $k \in \{0,\ldots,n-1\}$, $b + \sum_{i\neq k,k+1} t_i = 1$. The image

$$F_K(\omega(s)) = F_c(\omega(s)) = b(1-s)f_k + bsf_{k+1} + \sum_{i\neq k,k+1} t_i f_i$$

is a directed path in $|c| \subseteq |K|$ since $f_k < f_{k+1}$. Since $|K|$ is a complete d-space (by Proposition 3.3) from Proposition 2.3 follows that F_K is a d-map. A similar argument applies for G_K. □

We conclude with some obvious properties of the triangulation.

Proposition 4.6 *Let K be a finite proper $\square$-set with height function.*

(1) F_K maps bijectively vertices of $|\operatorname{Tr}(K)|$ into vertices of $|K|$.
(2) For each simplex $A \in S_{\operatorname{Tr}(K)}$ there exists a cube of K such that $F_K(|A|) \subseteq |c|$.
(3) Tr is a functor from the category of proper $\square$-sets into the category of d-simplicial complexes. Moreover, the maps F_K define a natural equivalence of functors $K \mapsto |K|$ and $K \mapsto |\operatorname{Tr}(K)|$. □

5 Tame paths

Recall [16] that a d-simplicial complex M has no loops if for every sequence of vertices $v_0 \leq \cdots \leq v_n \leq v_0$ in M we have $v_0 = \cdots = v_n$. We say that a $\square$-set K has no loops if, for every vertex $v \in K[0]$, every cube chain from v to v is empty. If K admits a height function, then, clearly, it has no loops. Immediately from the construction follows that the triangulation of a proper $\square$-set having no loops also has no loops. In this section we prove that the space of directed paths between two arbitrary vertices of a proper $\square$-set having no loops is homotopy equivalent to its subspace containing only tame paths, i.e. paths that cross from one cube to another *at vertices only*. It is a consequence of the results from [16] for d-simplicial complexes.

Definition 5.1 Let M be a d-simplicial complex. A directed path $\alpha \in \vec{P}(M)$ is *tame* if, for each $0 \leq s < t \leq 1$, there exists a simplex $A \in S_M$ such that $\alpha|_{[s,t]} \subseteq |A|$, or a vertex $v \in V_M$ such that $v \in \alpha([s,t])$. Let $\vec{P}_t(M)_v^w \subseteq \vec{P}(M)_v^w$ be the subspace of tame paths from v to w, $v, w \in M[0]$.

Proposition 5.2 [16, Section 6] *Let M be a finite d-simplicial complex having no loops. There exists a d-map $R_M\colon |M| \to |M|$ such that*

(1) $R_M(v) = v$ *for every vertex* $v \in V_M$*,*
(2) for every $x \in |M|$*, there exists a simplex* A *such that* $x, R_M(x) \in |A|$*,*
(3) for $v, w \in V_M$ *and* $\alpha \in P(\vec{M})_v^w$ *the path* $R_M \circ \alpha$ *is tame.*

The tameness of paths on $\square$-sets is defined quite similarly:

Definition 5.3 Let K be a $\square$-set. A path $\alpha \in \vec{P}(|K|)$ is *tame* if, for each $0 \le s < t \le 1$, there exists a cube $c \in K[n]$ such that $\alpha|_{[s,t]} \subseteq |c|$, or a vertex $v \in K[0]$ such that $v \in \alpha([s,t])$.

For the remaining part of this section we assume that K is a proper $\square$-set with height function.

Proposition 5.4 *If* $\alpha \in \vec{P}(\mathrm{Tr}(K))$ *is tame (in the simplicial sense), then* $F_K \circ \alpha \in \vec{P}(K)$ *is also tame (in the cubical sense).*

Proof This follows immediately from Proposition 4.6(1) and (2). □

Proposition 5.5 *Let* X *be a space and* $f_1, f_2 \colon X \to \vec{P}(K)_v^w$ *continuous maps. Assume that, for every* $t \in [0,1]$*, there exists a cube* c *such that* $f_1(\alpha(t)), f_2(\alpha(t)) \in |c|$*. Then* f_1 *and* f_2 *are homotopic.*

Proof For $c \in K[n]$ the map $|i_c| \colon \vec{I}^n \ni x \mapsto (c,x) \in |c|$ is a d-homeomorphism, since K is proper. The homotopy between f_1 and f_2 is given by

$$H_s(x)(t) = |i_c|((1-s)|i_c|^{-1}(f_1(\alpha(t)) + s|i_c|^{-1}(f_2(\alpha(t))), \quad f_1(x), f_2(x) \in |c|.$$

It is clear that this is well-defined (since convex combinations of d-paths in $\vec{I}^n$ are d-paths), continuous and does not depend on the choices of cubes c for respective points. □

Remark If K has height function, then convex combinations of natural paths are natural, so the analogue of Proposition 5.5 holds also for maps into $\vec{N}(K)_v^w$.

The main result of this section is the following

Theorem 5.6 *Assume that* K *is a finite proper* $\square$*-set with height function. For* $v, w \in K[0]$*, the inclusion*

$$\vec{P}_t(K)_v^w \subseteq \vec{P}(K)_v^w$$

is a homotopy equivalence.

Proof The triangulation $\mathrm{Tr}(K)$ is finite and has no loops. Let $R_{\mathrm{Tr}(K)} \colon |\mathrm{Tr}(K)| \to |\mathrm{Tr}(K)|$ be a map satisfying the conditions of Proposition 5.2, and let $F_K \colon |\mathrm{Tr}(K)| \to |K|$ denote the homeomorphism from Proposition 4.5. For $\alpha \in \vec{P}(K)_v^w$ the path $F_K \circ R_{\mathrm{Tr}(K)} \circ (F_K)^{-1} \circ \alpha$ is a d-path with endpoints v, w [Proposition 5.2(1)] and is tame [Propositions 5.2(3), 5.4]. Therefore the map

$$\vec{P}(K)_v^w \ni \alpha \mapsto F_K \circ R_{\mathrm{Tr}(K)} \circ (F_K)^{-1} \circ \alpha \in \vec{P}_t(\mathrm{Tr}(K))_v^w$$

is well-defined and is a homotopy inverse of the inclusion $\vec{P}_t(K)_v^w \subseteq \vec{P}(K)_v^w$. The latter statement follows from Propositions 5.2(2), 5.5 and 4.6(2). □

6 A cover by cube chains

Let K be a finite proper $\square$-set with height function and let $v, w \in K[0]$. Let

$$\vec{N}_t(K)_v^w = (nat)^{-1}(\vec{P}_t(K)_v^w),$$

(cf. 3.4) denote the space of natural tame paths; thanks to Theorem 5.6 and [9, 2.16] the spaces $\vec{N}_t(K)_v^w$, $\vec{N}(K)_v^w$, $\vec{P}_t(K)_v^w$ and $\vec{P}(K)_v^w$ are all homotopy equivalent via normalization maps and inclusions. In this section we construct a good closed cover of $\vec{N}_t(K)_v^w$ indexed with the poset of cube chains from v to w.

Recall that a cube chain from v to w is a sequence of cubes $\mathbf{c} = (c_1, \dots, c_l)$ such that $d^0(c_1) = v$, $d^1(c_l) = w$ and $d^1(c_i) = d^0(c_{i-1})$ for $1 \leq i < l$. Recall also that

$$d_{k,A}(\mathbf{c}) = \left(c_1, \dots, c_{k-1}, d^0_{\bar{A}}(c_k), d^1_A(c_k), c_{k+1}, \dots, c_l\right) \in \mathrm{Ch}(K)_v^w. \qquad (6.1)$$

for $k \in \{1, \dots, l\}$, $A \cap \bar{A} = \emptyset$, $A \cup \bar{A} = \{1, \dots, \dim(c_k)\}$ and that $\leq$ is the transitive-reflexive closure of relations $d_{k,A}(\mathbf{c}) \leq \mathbf{c}$ on $\mathrm{Ch}(K)_v^w$. For a cube chain $\mathbf{c} = (c_1, \dots, c_l)$ we will write $l^{\mathbf{c}} = l$, $n_i^{\mathbf{c}} = \dim(c_i)$, $v_i^{\mathbf{c}} = d^0(c_{i+1}) = d^1(c_i)$ and $b_k^{\mathbf{c}} = n_1^{\mathbf{c}} + \dots + n_k^{\mathbf{c}}$, $k \in \{1 \dots l\}$. The upper index $\mathbf{c}$ will be omitted if it does not lead to confusion.

Definition 6.1 We say that a natural path $\alpha \in \vec{N}_t(K)_v^w$ *lies in a cube chain* $\mathbf{c} \in \mathrm{Ch}(K)_v^w$ if

$$\alpha\left(\left[b_{i-1}^{\mathbf{c}}, b_i^{\mathbf{c}}\right]\right) \subseteq |c_i|$$

for $i \in \{1, \dots, l\}$. Let $\vec{N}(K, \mathbf{c}) \subseteq \vec{N}_t(K)$ be the subspace of tame paths lying in $\mathbf{c}$.

Note that $\alpha(b_i^{\mathbf{c}}) = v_i^{\mathbf{c}}$ for $\alpha \in \vec{N}(K, \mathbf{c})$, $i \in \{0, \dots, l^{\mathbf{c}}\}$.

For $v, w, z \in K[0]$ and cube chains $\mathbf{c} = (c_1, \dots, c_l) \in \mathrm{Ch}(K)_v^w$, $\mathbf{c}' = (c'_1, \dots, c'_{l'}) \in \mathrm{Ch}(K)_w^z$ we define *the concatenation* $\mathbf{c} * \mathbf{c}' \in \mathrm{Ch}(K)_v^z$ of $\mathbf{c}$ and $\mathbf{c}'$ by

$$\mathbf{c} * \mathbf{c}' = (c_1, \dots, c_l, c'_1, \dots, c'_{l'}). \qquad (6.2)$$

The concatenation of paths induces a homeomorphism

$$\vec{N}(K, \mathbf{c}) \times \vec{N}(K, \mathbf{c}') \subseteq \vec{N}(K, \mathbf{c} * \mathbf{c}'). \qquad (6.3)$$

Proposition 6.2 *Let* $v, w \in K[0]$, $\mathbf{c}, \mathbf{c}' \in \mathrm{Ch}(K)_v^w$.

(1) $\vec{N}(K, \mathbf{c})$ *is a contractible closed subspace of* $\vec{N}_t(K)_v^w$.
(2) The intersection $\vec{N}(K, \mathbf{c}) \cap \vec{N}(K, \mathbf{c}')$ *is either empty or there exists a cube chain* $\mathbf{c} \cap \mathbf{c}' \in \mathrm{Ch}(K)_v^w$ *such that* $\vec{N}(K, \mathbf{c} \cap \mathbf{c}') = \vec{N}(K, \mathbf{c}) \cap \vec{N}(K, \mathbf{c}')$.
(3) $\vec{N}(K, \mathbf{c}) \subseteq \vec{N}(K, \mathbf{c}')$ *if and only if* $\mathbf{c} \leq \mathbf{c}'$.
(4) $\bigcup_{\mathbf{c} \in \mathrm{Ch}(K)} \vec{N}(K, \mathbf{c}) = \vec{N}_t(K)$.

Proof Assume that $h(v) = 0$, $h(w) = n$, $\mathbf{c} = (c_1, \dots, c_l)$, $\mathbf{c}' = (c'_1, \dots, c'_{l'})$.

(1) Let $\gamma \in \vec{N}(K, \mathbf{c})$ be the diagonal of $\mathbf{c}$, i.e. a path given by

$$\gamma(t) = \left(c_i, \left(\tfrac{t-b^{\mathbf{c}}_{i-1}}{n^{\mathbf{c}}_i}, \dots, \tfrac{t-b^{\mathbf{c}}_{i-1}}{n^{\mathbf{c}}_i}\right)\right)$$

for $t \in [b^{\mathbf{c}}_{i-1}, b^{\mathbf{c}}_i]$. The identity map of $\vec{N}(K, \mathbf{c})$ and the constant map with value γ satisfy the assumptions of Proposition 5.5; therefore the space $\vec{N}(K, \mathbf{c})$ is contractible. Closedness of $\vec{N}(K, \mathbf{c})$ is clear.

(2) Assume that $\alpha \in \vec{N}(K, \mathbf{c}) \cap \vec{N}(K, \mathbf{c}')$. The proof is by induction with respect to n. If $n = 1$, then $\mathbf{c} = (c_1) = (c'_1) = \mathbf{c}'$ since c_1 and c'_1 have the same extreme vertices. Assume that $n > 1$ and $n^{\mathbf{c}}_1 < n^{\mathbf{c}'}_1$. Since $\alpha(n^{\mathbf{c}}_1) = v^{\mathbf{c}}_1$ and $\alpha([0, n^{\mathbf{c}'}_1]) \subseteq |c'_1|$ then $v^{\mathbf{c}}_1$ is a vertex of the cube c'_1, say $v^{\mathbf{c}}_1 = d_{A,\bar{A}}(c'_1)$ for $A \cap \bar{A} = \emptyset$, $A \cup \bar{A} = \{1, \dots, n^{\mathbf{c}'}_1\}$. We have $d^0(d^0_A(c'_1)) = v = d^0(c_1)$ and $d^1(d^0_A(c'_1)) = v^{\mathbf{c}}_1 = d^1(c_1)$; by the properness of K this implies $c_1 = d^0_A(c'_1)$. Furthermore, $\alpha([n^{\mathbf{c}}_1, n^{\mathbf{c}'}_1]) \subseteq |d^1_{\bar{A}}(c'_1)|$, and then $\alpha|_{[n^{\mathbf{c}}_1, n]}$ is contained in the cube chain $(d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_{l'})$. We have

$$\begin{aligned}
\vec{N}(K, \mathbf{c}) \cap \vec{N}(K, \mathbf{c}') &= \vec{N}(K, \mathbf{c}) \cap \vec{N}\left(K, \left(c_1, d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n\right)\right) \\
&= \vec{N}(K, (c_1)) \times \left(\vec{N}(K, (c_2, \dots, c_n)) \cap \vec{N}\left(K, \left(d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n\right)\right)\right) \\
&= \vec{N}(K, c_1 * ((c_2, \dots, c_n) \cap (d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n)))
\end{aligned}$$

since $(c_2, \dots, c_n) \cap (d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n)$ is a cube chain in $\mathrm{Ch}(K)^w_{v^{\mathbf{c}}_1}$ by the inductive assumption. The case $n^{\mathbf{c}}_1 = n^{\mathbf{c}'}_1$ is similar, only the term $d^1_{\bar{A}}(c'_1)$ is dropped (since it is a vertex).

(3) Immediately from the definition, we have $\vec{N}(K, d_{k,A}(\mathbf{c})) \subseteq \vec{N}(K, \mathbf{c})$. Assume that $\vec{N}(K, \mathbf{c}) \subseteq \vec{N}(K, \mathbf{c}')$; this is equivalent to the equation $\mathbf{c} = \mathbf{c} \cap \mathbf{c}'$. Assume that the statement is true for all pairs of chains having length less than n. Using the argument from the previous point we obtain

$$\mathbf{c} \cap \mathbf{c}' = c_1 * \left((c_2, \dots, c_n) \cap \left(d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n\right)\right)$$

By the inductive assumption, $(c_2, \dots, c_n) \leq (d^1_{\bar{A}}(c'_1), c'_2, \dots, c'_n)$ and therefore $\mathbf{c} \leq \mathbf{c}'$.

(4) Let $\alpha\colon [0, n] \to |K|$ be a natural tame path from v to w. If $\alpha(t)$ is a vertex then t is an integer. Thus, there is a finite sequence $0 = k_0 < \dots < k_l = n$ of integers such that $\alpha(k_i) = v_i$ is a vertex, and restrictions $\alpha|_{(k_{i-1}, k_i)}$ do not contain vertices in its images. By the tameness of α, for each $i \in \{1, \dots, l\}$ there exists a cube c_i such that $\alpha([k_{i-1}, k_i]) \subseteq |c_i|$. Obviously v_{i-1} and v_i are vertices of c_i and then there exist unique (by properness) subsets $A, B \subseteq \{1, \dots, \dim(c_i)\}$ such that $d_{A,\bar{A}}(c) = v_{i-1}$, $d_{B,\bar{B}}(c) = v_i$ (where $\bar{A}$, $\bar{B}$ stand for suitable complements). The segment $\alpha|_{[k_{i-1}, k_i]}$ is contained in $c'_i := d_{B,\bar{A}}(c_i)$ and $v_{i-1} = d^0(c'_i)$, $v_i = d^1(c'_i)$. As a consequence, α is contained in the cube chain $(c'_1, \dots, c'_l)$. □

This allows us to prove Theorem 1.2 in the following special case. Let $\mathbf{fph\Box Set}_*^*$ be the category of finite proper $\Box$-sets with height function and $\Box$-maps that preserve height functions. For $(K, v, w) \in \mathbf{fph\Box Set}_*^*$ let $\varepsilon_{(K,v,w)}$ be the composition

$$\begin{aligned}\varepsilon_{(K,v,w)}\colon \vec{P}(K)_v^w \supseteq \vec{P}_t(K)_v^w \xrightarrow{nat} \vec{N}_t(K)_v^w \\ \xleftarrow{\simeq} \operatorname{hocolim}_{\mathbf{c}\in \mathrm{Ch}(K,\mathbf{c})} \vec{N}(K,\mathbf{c}) \xrightarrow{\simeq} |\,\mathrm{Ch}(K)_v^w\,|.\end{aligned} \tag{6.4}$$

All the maps in the sequence and homotopy equivalences. This follows from Theorem 5.6 for the left-most inclusion, [9, 2.16] for the *nat* map, and from [14, 4.1] and Proposition 6.2 for the remaining two maps. This implies that $\varepsilon_{(K,v,w)}$ is well-defined up to homotopy, and it is a homotopy equivalence. Furthermore, all the maps are functorial with respect to (K, v, w). As a consequence, we obtain the following.

Proposition 6.3 *The maps $\varepsilon_{(K,v,w)}$ define a natural equivalence of functors*

$$\vec{P} \simeq |\,\mathrm{Ch}\,|\colon \mathbf{fph\Box Set}_*^* \to \mathbf{hTop}.\square$$

The proof of Theorem 1.2 in full generality is postponed to Sect. 9.

7 Permutahedra

In this section we study the posets of faces of products of permutohedra (see [15, p.18]), which play an important role in the description of the posets of cube chains on $\Box$-sets. There are (at least) three ways to describe the face poset of the $(n-1)$-dimensional permutahedron:

- As ordered partitions of the set $\{1, \ldots, n\}$, i.e., as sequences

$$a_1^1 a_2^1 \ldots a_{n_1}^1 \,\Big|\, a_1^2 a_2^2 \ldots a_{n_2}^1 \,\Big|\, \ldots \,\Big|\, a_1^l a_2^l \ldots a_{n_l}^l \tag{*}$$

  such that $n = n_1 + \cdots + n_l$ and $\{1, \ldots, n_l\} = \{a_i^k\}_{i=1,\ldots,n_k}^{k=1,\ldots,l}$. The partitions are ordered by the relation of (ordered) refinement.
- As weak strict orderings of $\{1, \ldots, n\}$, ordered by inclusion. *A weak strict ordering* is a reflexive and transitive relation such that any two elements are comparable, though, not necessarily anti-symmetric. The ordered partition (*) corresponds to the weak strict ordering $\sqsubseteq$ defined by

$$a_i^k \sqsubseteq a_{i'}^{k'} \Leftrightarrow k \leq k'.$$

- As surjective functions $f\colon \{1, \ldots, n\} \to \{1, \ldots, k\}$ ordered by refinement (see 7.1). Such a function determines a weak strict ordering $\sqsubseteq_f$ such that $i \sqsubseteq_f i'$ if and only if $f(i) \leq f(i')$. On the other hand, every weak strict ordering $\sqsubseteq$ on $\{1, \ldots, n\}$ determines the unique surjective function

$$f_{\sqsubseteq}\colon \{1, \ldots, n\} \to \{1, \ldots, k\}, \tag{7.1}$$

called *the characteristic function* of $\sqsubseteq$, characterized by $i \sqsubseteq j$ if and only if $f_{\sqsubseteq}(i) \leq f_{\sqsubseteq}(j)$.

The first description is most common and, probably, more intuitive than the others. We will pass freely between these three approaches; however, we will mainly use surjective functions since they allow more rigorous arguments.

Definition 7.1 Let O_n be the poset whose elements are surjective functions

$$f\colon \{1, \dots, n\} \to \{1, \dots, k\},$$

where $k = k(f)$ is a positive integer, and $f \preceq g$ if and only if f is *a refinement* of g, i.e., there exists a non-decreasing function h such that $g = h \circ f$.

Clearly, the function f is determined uniquely. The poset O_n has a greatest element: the constant function with value 1, which will be denoted by f_n. Let

$$\partial O_n = O_n \setminus \{f_n\}. \tag{7.2}$$

Proposition 7.2 *The pairs* $(|O_n|, |\partial O_n|)$ *and* (D^{n-1}, S^{n-2}) *are homeomorphic.*

Proof As mentioned before, the poset O_n is isomorphic to the face lattice of the $(n-1)$-dimensional permutohedron P^{n-1} (cf. [15, p.18]), with f_n corresponding to its body (i.e. the single $(n-1)$-dimensional cell), and ∂O_n to its boundary. Hence the pair $(|O_n|, |\partial O_n|)$ is homeomorphic to $(P^{n-1}, \partial P^{n-1})$ and hence to (D^{n-1}, S^{n-2}). □

Next, we will identify the product $O_{n_1} \times \dots \times O_{n_l}$ with the sub-poset of O_n, $n = \sum_k n_k$, consisting of those weak strict orderings $\sqsubseteq$ of $\{1, \dots, n\}$ which satisfy

$$1, \dots, n_1 \sqsubset n_1 + 1, \dots, n_1 + n_2 \sqsubset \dots \sqsubset n - n_l + 1, \dots, n,$$

where $i \sqsubset i'$ means that $i \sqsubseteq i'$ and not $i' \sqsubseteq i$.

Let $\mathrm{Seq}(n)$ be the set of all sequences of positive integers $\mathbf{n} = (n_1, \dots, n_{l(\mathbf{n})})$ such that $n = n_1 + \dots + n_{l(\mathbf{n})}$. For a given partition $\mathbf{n} \in \mathrm{Seq}(n)$, let

- $b_k(\mathbf{n}) = \sum_{j=1}^{k} n_j$, for $k \in \{0, \dots, l(\mathbf{n})\}$,
- $f_{\mathbf{n}}\colon \{1, \dots, n\} \to \{1, \dots, l(\mathbf{n})\}$ be the unique non-decreasing surjective function that takes value k for exactly n_k integers, i.e., $f_{\mathbf{n}}(i) = k$ if and only if $b_{k-1}(\mathbf{n}) < i \leq b_k(\mathbf{n})$,
- $O_{\mathbf{n}} := (O_n)_{\preceq f_{\mathbf{n}}} = \{f \in O_n \colon f \preceq f_{\mathbf{n}}\}$.

Notice that $O_{(n)} = O_n$. Whenever it does not lead to confusion, we will write l or b_k instead of $l(\mathbf{n})$ or $b_k(\mathbf{n})$.

Fix $\mathbf{n} \in \mathrm{Seq}(n)$. For every $f \in O_{\mathbf{n}}$, there exists a non-decreasing function $h\colon \{1, \dots, k(f)\} \to \{1, \dots, l\}$ such that $h \circ f = f_{\mathbf{n}}$. Thus, for $k \in \{1, \dots, l\}$, f determines a function

$$f^k\colon \{1, \dots, n_k\} \xrightarrow{i \mapsto i + b_{k-1}} f_{\mathbf{n}}^{-1}(k) \xrightarrow{f_{\mathbf{n}}} h^{-1}(k) \xrightarrow{j \mapsto j + 1 - \min(h^{-1}(k))} \{1, \dots, r_k\}, \tag{7.3}$$

which is an element of O_{n_k}. On the other hand, a sequence

$$(f^k\colon \{1,\dots,n_k\} \to \{1,\dots,r_k\}) \in O_{n_k},$$

for $k \in \{1,\dots,l\}$, determines an element $f \in O_{\mathbf{n}}$ such that

$$f(i) = f^k(i - b_{k-1}) + \sum_{j=1}^{k-1} r_j. \tag{7.4}$$

These constructions give an isomorphism of posets $O_{\mathbf{n}} \cong O_{n_1} \times \cdots \times O_{n_l}$. Thus, $O_{\mathbf{n}}$ is isomorphic to the face lattice of the product of permutahedra of dimensions $n_1 - 1, \dots, n_k - 1$ respectively; as a consequence, there is a homeomorphism

$$(|O_{\mathbf{n}}|, |\partial O_{\mathbf{n}}|) \simeq (D^{n-l}, S^{n-l-1}), \tag{7.5}$$

where $\partial O_{\mathbf{n}} := O_{\mathbf{n}} \setminus \{f_{\mathbf{n}}\}$, which generalizes Proposition 7.2.

The main goal of this section is to construct a fundamental class of the pair (7.5) in simplicial homology [7, p. 104]; to achieve this, we need to introduce a notation for the simplices of the nerve $\mathcal{N}O_n$ of O_n. Before passing to formal definitions, let us present a general idea of this notation.

Example 7.3 Here follows an example of a 2-simplex in $\mathcal{N}O_n$, $n = 6$:

$$S = (V_0 = (3|5|6|14|2), \quad V_1 = (35|614|2), \quad V_2 = (35|6142)).$$

To this simplex we can assign a permutation $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 4 & 2 \end{pmatrix}$ on $n = 6$ letters that determines in what order integers appear in every sequence; we can choose compatible orders for all vertices of S. Then we assign a function

$$\tau\colon \{1,\dots,5\} \to \{0,\dots,3\}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 1 & 0 & 2 \end{pmatrix};$$

$\tau(i) = j$ means that the vertical line between the i-th and the $(i+1)$-th entry appears in $V_0, \dots, V_{i-1}$ and does not appear in $V_i, \dots, V_2$. The pair (σ, τ) determines S; on the other hand, S determines uniquely τ but not σ.

Let

- Σ_n be the set of permutations of $\{1,\dots,n\}$,
- T_n^d be the set of functions $\tau\colon \{1,\dots,n-1\} \to \{0,\dots,d+1\}$ such that $\tau^{-1}(j) \neq \emptyset$ for $0 < j \leq d$.

For $\mathbf{n} \in \mathrm{Seq}(n)$ define $\tau_{\mathbf{n}} \in T_n^0$ by $\tau_{\mathbf{n}}(i) = f_{\mathbf{n}}(i+1) - f_{\mathbf{n}}(i)$, and for $\tau \in T_n^0$ define $f^\tau \in O_n$ by

$$f^\tau(i) = 1 + \sum_{i=1}^{k-1} \tau(i). \tag{7.6}$$

Clearly, $f^{\tau_{\mathbf{n}}} = f_{\mathbf{n}}$, and this defines a bijection between T_n^0 and $\mathrm{Seq}(n)$. For $\tau \in T_n^d$ let

$$\Sigma_\tau = \{\sigma \in \Sigma_n \colon f^\tau \sigma = f^\tau\}. \tag{7.7}$$

Let us enlist some properties of the functions f^τ:

Proposition 7.4 *(a) If $\tau, \psi \in T_n^0$, then $f^\tau \preceq f^\psi$ if and only if $\tau(i) \geq \psi(i)$ for all $i \in \{1, \dots, n-1\}$.*
(b) $f^\tau \sigma \preceq f^\psi \varphi$ if and only if $f^\tau \preceq f^\psi$ and $\sigma\varphi^{-1} \in \Sigma_\psi$.
(c) Every element of O_n has the form $f^\tau \sigma$ for $\sigma \in \Sigma_n$, $\tau \in T_n^0$.

Proof To prove (a), assume that $\tau(i) \geq \psi(i)$ for $\tau, \psi \in T_n^0, i \in \{1, \dots, n-1\}$. Clearly $\tau(i) = 0$ implies that $\psi(i) = 0$; thus, if $\tau(i) = \tau(j)$, then $\psi(i) = \psi(j)$. As a consequence, the formula $h(k) = (f^\psi \circ (f^\tau)^{-1})(k)$ defines a non-decreasing surjection such that $f^\psi = h \circ f^\tau$. On the other hand, if $\tau(i) = 0$ and $\psi(i) = 1$, then $f^\psi(i) \neq f^\psi(i+1)$ and $f^\tau(i) = f^\tau(i+1)$; therefore, f^ψ cannot be written as a composition $h \circ f^\tau$.

If $f^\tau \preceq f^\psi$ and $\sigma\phi^{-1} \in \Sigma_\psi$, then

$$f^\psi \varphi = f^\psi \sigma \varphi^{-1} \varphi = f^\psi \sigma = h f^\tau \sigma,$$

where h is a function such that $f^\psi = h f^\tau$. If $f^\psi \varphi = h f^\tau \sigma$, then $f^\psi \varphi \sigma^{-1} = h f^\tau$ is a non-decreasing function. Hence $\varphi\sigma^{-1} \in \Sigma_\psi$ and $f^\psi = h f^\tau$, which implies (b). The point (c) is clear. □

Let us introduce the following maps between integers:

$$\mathsf{d}_k(i) = \begin{cases} i & \text{for } i < k \\ i+1 & \text{for } i \geq k \end{cases} \qquad \mathsf{s}_k(i) = \begin{cases} i & \text{for } i \leq k \\ i-1 & \text{for } i > k \end{cases}$$

$$\mathsf{p}_k(i) = \begin{cases} 0 & \text{for } i \leq k \\ 1 & \text{for } i > k \end{cases} \qquad \mathsf{t}_k(i) = \begin{cases} k & \text{for } i = k+1 \\ k+1 & \text{for } i = k \\ i & \text{for } i \neq k, k+1. \end{cases}$$

furthermore, for $r \leq s$ denote $\mathsf{t}_r^s = \mathsf{t}_r \mathsf{t}_{r+1} \dots \mathsf{t}_{s-1}$; it is easy to check that $\mathsf{t}_r^s(i) = i$ for $i < r$ or $i > s$, $\mathsf{t}_r^s(s) = r$, and $\mathsf{t}_r^s(i) = i+1$ for $r \leq i < s$.

Every element $\tau \in T_n^d$ determines a sequence $(\mathsf{p}_0 \circ \tau, \dots, \mathsf{p}_d \circ \tau)$ of $d+1$ elements of T_n^0. Thus, for $\sigma \in \Sigma_n$ and $\tau \in T_n^d$, we can define a d-simplex in O_n:

$$a[\sigma, \tau] := (f^{\mathsf{p}_0\tau}\sigma, f^{\mathsf{p}_1\tau}\sigma, \dots, f^{\mathsf{p}_d\tau}\sigma). \tag{7.8}$$

Notice that Proposition 7.4(b) implies that $f^{\mathsf{p}_0\tau}\sigma \preceq f^{\mathsf{p}_1\tau}\sigma \preceq \dots \preceq f^{\mathsf{p}_d\tau}\sigma$. This definition coincides with Example 7.3

Proposition 7.5 *For $\sigma \in \Sigma_n$, $\tau \in T_n^d$ we have*

(a) $\partial_i(a[\sigma,\tau]) = a[\sigma, \mathsf{s}_i\tau]$, *where* ∂_i *denotes the simplex with* i*-th vertex skipped,* $i \in \{0, \dots, d\}$,
(b) $a[\sigma,\tau] = a[\varphi,\psi]$ *if and only if* $\tau = \psi$ *and* $\sigma\varphi^{-1} \in \Sigma_\tau$.

Proof This follows immediately from Proposition 7.4 and the definition. □

For $\mathbf{n} \in \mathrm{Seq}(n)$ define

$$\Sigma_{\mathbf{n}} := \{\sigma \in \Sigma_n \colon f_{\mathbf{n}} \circ \sigma = f_{\mathbf{n}}\}, \tag{7.9}$$

$$T^d_{\mathbf{n}} := \{\tau \in T^d_n \colon \forall_{k=1,\dots,l(\mathbf{n})-1}\ \tau(b_k(\mathbf{n})) = d+1\}. \tag{7.10}$$

Notice that $T^0_{\mathbf{n}} = \{\tau \in T^0_n \colon f^\tau \preceq f_{\mathbf{n}}\}$, and $T^d_{\mathbf{n}} = \{\tau \in T^d_n \colon \mathsf{p}_d\tau \in T^0_{\mathbf{n}}\}$.

Proposition 7.6 *Let* $\mathbf{n} \in \mathrm{Seq}(n)$, $\sigma \in \Sigma_n$, $\tau \in T^d_n$. *Then* $a[\sigma,\tau]$ *is a simplex in* $\mathcal{N}O_{\mathbf{n}}$ *if and only if* $\sigma \in \Sigma_{\mathbf{n}}$ *and* $\tau \in T^d_{\mathbf{n}}$.

Proof Clearly $a[\sigma,\tau]$ is a simplex of $\mathcal{N}O_{\mathbf{n}}$ if and only if $f^{\mathsf{p}_d\tau}\sigma \preceq f_{\mathbf{n}} = f^{\tau_{\mathbf{n}}}$. By Proposition 7.4 this holds if and only if $\sigma \in \Sigma_{\mathbf{n}}$ and $\mathsf{p}_d\tau \preceq \tau_{\mathbf{n}}$. □

The next step is to construct a generator of the top homology class of the pair $(\mathcal{N}O_{\mathbf{n}}, \mathcal{N}\partial O_{\mathbf{n}})$. It is clear that this a combination of all simplices of $\mathcal{N}O_{\mathbf{n}}$ having the maximal dimension; the difficult part is to choose ± 1 coefficients at these simplices. For $\sigma \in \Sigma_n$ let $\mathrm{sgn}(\sigma) \in \{\pm 1\}$ be the sign of the permutation σ. For $\mathbf{n} \in \mathrm{Seq}(n)$ and $\tau \in T^{n-l}_{\mathbf{n}}$, the image of τ contains $\{1,\dots,n-l\}$, and $\tau(b_j) = n-l$ for $j = 1,\dots,l-1$. Then the restriction

$$\tau \colon \{1,\dots,n-1\} \setminus \{b_1,\dots,b_{l-1}\} \to \{1,\dots,n-l\}$$

is a bijection. Let $\varrho_{\mathbf{n}} \colon \{1,\dots,n-l\} \to \{1,\dots,n-1\} \setminus \{b_1,\dots,b_{l-1}\}$ be the (unique) increasing bijection. The composition $\tau\varrho_{\mathbf{n}}$ is a permutation on $n-l$ letters, and we define *the sign* of $\tau \in T^{n-l}_{\mathbf{n}}$ as

$$\mathrm{sgn}_{\mathbf{n}}(\tau) = \mathrm{sgn}(\tau\varrho_{\mathbf{n}}). \tag{7.11}$$

Define a chain $g_{\mathbf{n}} \in C_{n-l}(\mathcal{N}O_{\mathbf{n}})$ by

$$g_{\mathbf{n}} = \sum_{\sigma \in \Sigma_{\mathbf{n}}} \sum_{\tau \in T^{n-l}_{\mathbf{n}}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}_{\mathbf{n}}(\tau) a[\sigma,\tau]. \tag{7.12}$$

We will show that $g_{\mathbf{n}}$ represents a fundamental class in $H_{n-l}(|O_{\mathbf{n}}|, |\partial O_{\mathbf{n}}|)$ (cf. 7.5).

Proposition 7.7 *Let* $\partial \colon C_{n-l}(\mathcal{N}O_{\mathbf{n}}) \to C_{n-l-1}(\mathcal{N}O_{\mathbf{n}})$ *be the differential of the simplicial homological chain complex. Then*

$$\partial(g_{\mathbf{n}}) = (-1)^{n-l} \sum_{\sigma \in \Sigma_{\mathbf{n}}} \sum_{\tau \in T^{n-l(\mathbf{n})}_{\mathbf{n}}} \mathrm{sgn}(\sigma)\,\mathrm{sgn}_{\mathbf{n}}(\tau) a[\sigma, \mathsf{s}_{n-l}\tau].$$

As a consequence, $g_{\mathbf{n}}$ *represents a generator of* $H^{n-l(\mathbf{n})}(\mathcal{N}(O_{\mathbf{n}}), \mathcal{N}(\partial O_{\mathbf{n}}))$.

Proof We have

$$\begin{aligned}\partial(g_{\mathbf{n}}) &= \sum_{\sigma\in\Sigma_{\mathbf{n}}}\sum_{\tau\in T_{\mathbf{n}}^{n-l}} \operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)\sum_{j=0}^{n-l}(-1)^j\partial_j(a[\sigma,\tau])\\
&= \sum_{\sigma,\tau}\sum_{j=0}^{n-l}(-1)^j\operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_j\tau] = \sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_0\tau]\\
&\quad+\sum_{j=1}^{n-l-1}(-1)^j\sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_j\tau]\\
&\quad+(-1)^{n-l}\sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_{n-l}\tau].\end{aligned}$$

For the first summand, we have $a[\sigma,\mathsf{s}_0\tau] = a[\mathsf{t}_{\tau^{-1}(1)}\sigma,\mathsf{s}_0\tau]$, since $\mathsf{t}_{\tau^{-1}(1)}\sigma\sigma^{-1} = \mathsf{t}_{\tau^{-1}(1)} \in \Sigma_{\mathsf{s}_0\tau}$, therefore the first summand equals

$$\sum_{\tau}\operatorname{sgn}_{\mathbf{n}}(\tau)\sum_{\sigma\colon\sigma(\tau^{-1}(1))<\sigma(\tau^{-1}(1)+1)}\operatorname{sgn}(\sigma)a[\sigma,\mathsf{s}_0\tau]+\operatorname{sgn}(\mathsf{t}_{\tau^{-1}(1)}\sigma)a[\mathsf{t}_{(\tau^{-1}(1))}\sigma,\mathsf{s}_0\tau]\\
= 0.$$

For the second summand, for $j \in \{1,\dots,n-l-1\}$, we have $\mathsf{s}_j\tau = \mathsf{s}_j\mathsf{t}_j\tau$, therefore

$$\begin{aligned}&\sum_{\sigma,\tau}\operatorname{sgn}(\sigma)\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_j\tau]\\
&\quad=\sum_{\sigma}\operatorname{sgn}(\sigma)\sum_{\tau\colon\tau^{-1}(j)<\tau^{-1}(j+1)}\operatorname{sgn}_{\mathbf{n}}(\tau)a[\sigma,\mathsf{s}_j\tau]+\operatorname{sgn}_{\mathbf{n}}(\mathsf{t}_j\tau)a[\sigma,\mathsf{s}_j\mathsf{t}_j\tau]=0.\end{aligned}$$

Hence only the third summand remains. The maximal element of $O_{\mathbf{n}}$, namely $f_{\mathbf{n}}$, does not appear as a vertex of any $a[\sigma,\mathsf{s}_{n-l}\tau]$. Thus, $\partial(g_{\mathbf{n}}) \in C_{n-l-1}(\mathcal{N}(\partial O_{\mathbf{n}}))$ and $g_{\mathbf{n}}$ is a cycle, regarded as an element of $C_{n-l}(\mathcal{N}(O_{\mathbf{n}}),\mathcal{N}(\partial O_{\mathbf{n}}))$. Since the coefficients of $g_{\mathbf{n}}$ on all simplices are ±1, it represents the generator in the homology group. □

Let $\mathbf{n} \in \operatorname{Seq}(n)$. For $k \in \{1,\dots,l\}$, $r \in \{1,\dots,n_k-1\}$ denote

$$\mathbf{n}[k,r] := (n_1,\dots,n_{k-1},r,n_k-r,n_{k+1},\dots,n_l) \in \operatorname{Seq}(n). \tag{7.13}$$

Lemma 7.8 *For each $\tau \in T_{\mathbf{n}}^{n-l}$, there exists a unique pair of integers $k \in \{1,\dots,l\}$, $r \in \{1,\dots,n_k-1\}$ such that $\tau(b_{k-1}+r) = n-l$, i.e., there is a bijection*

$$T_{\mathbf{n}}^{n-l} \ni \tau \mapsto \mathsf{s}_{n-l}\tau \in \coprod_{k=1}^{l}\coprod_{r=1}^{n_k-1} T_{\mathbf{n}[k,r]}^{n-l-1}.$$

Furthermore, if $\mathsf{s}_{n-l}\tau \in T^{n-l-1}_{\mathbf{n}[k,r]}$, *then*

$$\operatorname{sgn}_{\mathbf{n}[k,r]}(\mathsf{s}_{n-l}\tau) = (-1)^{n-l-b_{k-1}-r+k-1}\operatorname{sgn}_{\mathbf{n}}(\tau).$$

Proof The existence of k and r, as well as the fact that $\mathsf{s}_{n-l}\tau \in T^{n-l-1}_{\mathbf{n}[k,r]}$, follows immediately from the definitions. It remains to prove that the signs of permutations $\mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]} \in \Sigma_{n-l-1}$ and $\tau\varrho_{\mathbf{n}} \in \Sigma_{n-l}$ (cf. 7.11) differ by $(-1)^{n-l-b_{k-1}-r+k-1}$. Denote $v = b_{k-1} - (k-1) + r$; clearly

$$\varrho_{\mathbf{n}}(j) = \begin{cases} \varrho_{\mathbf{n}[k,r]}(j) & \text{for } j \in \{1,\dots,v-1\}, \\ b_{k-1}+r & \text{for } j = v, \\ \varrho_{\mathbf{n}[k,r]}(j-1) & \text{for } j \in \{v+1,\dots,n-l\}. \end{cases}$$

since $\operatorname{im}(\tau\varrho_{\mathbf{n}[k,r]}) \subseteq \{1,\dots,n-l-1\}$, we obtain

$$\tau\varrho_{\mathbf{n}}(j) = \begin{cases} \mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}(j) & \text{for } j \in \{1,\dots,v-1\} \\ \tau(b_{k-1}+r) = n-l & \text{for } j = v \\ \mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}(j-1) & \text{for } j \in \{v+1,\dots,n-l\}. \end{cases}$$

Then

$$\tau\varrho_{\mathbf{n}}(j) = \begin{cases} \mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}\mathsf{t}^{n-l}_{v}(j) & \text{for } j \in \{1,\dots,n-l-1\} \\ n-l & \text{for } j = n-l, \end{cases}$$

The permutations $\tau\varrho_{\mathbf{n}} \in \Sigma_{n-l}$ and $\mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}\mathsf{t}^{n-l}_{v} \in \Sigma_{n-l-1}$ have equal signs. As a consequence,

$$\begin{aligned} \operatorname{sgn}_{\mathbf{n}}(\tau) &= \operatorname{sgn}(\tau\varrho_{\mathbf{n}}) = \operatorname{sgn}(\mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}\mathsf{t}^{n-l}_{v}) = (-1)^{n-l-v}\operatorname{sgn}(\mathsf{s}_{n-l}\tau\varrho_{\mathbf{n}[k,r]}) \\ &= (-1)^{n-l-v}\operatorname{sgn}_{\mathbf{n}[k,r]}(\mathsf{s}_{n-l}\tau) = (-1)^{n-l-b_{k-1}-r+k-1}\operatorname{sgn}_{\mathbf{n}[k,r]}(\mathsf{s}_{n-l}\tau). \end{aligned}$$

□

Fix $\mathbf{n} \in \operatorname{Seq}(n)$. For $k \in \{1,\dots,l\}$, $r \in \{1,\dots,n_k-1\}$, let

$$S_{\mathbf{n}}(k,r) := \{A \subseteq f_{\mathbf{n}}^{-1}(k) = \{b_{k-1}+1,\dots,b_k\} \colon |A| = r\}. \tag{7.14}$$

Remark Notice that elements of $\coprod_k \coprod_r S_{\mathbf{n}}(k,r)$ are in 1–1 correspondence with maximal elements of $\partial O_{\mathbf{n}}$ or, more geometrically, facets of the product of permutahedra $P^{n_1} \times \cdots \times P^{n_l}$.

For $A \in S_{\mathbf{n}}(k,r)$, let $\xi_{k,A} \in \Sigma_{\mathbf{n}}$ be the permutation determined by the following conditions:

- $\xi_{k,A}(i) = i$ for $f_{\mathbf{n}}(i) \neq k$,
- $\xi_{k,A}|_A \colon A \to \{b_{k-1}+1,\dots,b_{k-1}+r\}$ be strictly increasing,

- $\xi_{k,A}|_{f_{\mathbf{n}}^{-1}(k)\setminus A}\colon f_{\mathbf{n}}^{-1}(k)\setminus A \to \{b_{k-1}+r+1,\ldots,b_k\}$ be strictly increasing.

Proposition 7.9 *Assume that* $\mathbf{n} \in \mathrm{Seq}(n), k \in \{1,\ldots,l\}, r \in \{1,\ldots,b_k-1\}$ *and* $A \in S_{\mathbf{n}}(k,r)$. *Then* $\mathrm{sgn}(\xi_{k,A}) = \mathrm{sgn}(A-b_{k-1})$, *where* $A-b_{k-1} := \{i-b_{k-1}\colon i \in A\}$.

Proof Assume that $A = (a_1 < \cdots < a_r)$. We have

$$\xi_{k,A} = \mathsf{t}^{a_r}_{b_{k-1}+r} \ldots \mathsf{t}^{a_2}_{b_{k-1}+2}\mathsf{t}^{a_1}_{b_{k-1}+1},$$

hence $\mathrm{sgn}(\xi_{k,A})$ equals -1 to the power

$$\sum_{j=1}^{r} a_j - \sum_{j=1}^{r}(b_{k-1}+j) = \sum_{j=1}^{r}(a_j-b_{k-1}) - \sum_{j=1}^{r} j,$$

which coincides with the definition given in Theorem 1.4 in the introduction. □

Lemma 7.10 *For every* $\mathbf{n} \in \mathrm{Seq}(n), k \in \{1,\ldots,l\}, r \in \{1,\ldots,n_k-1\}$ *the function*

$$S_{\mathbf{n}}(k,r) \times \Sigma_{\mathbf{n}[k,r]} \ni (A,\omega) \mapsto \omega\xi_{k,A} \in \Sigma_{\mathbf{n}} \tag{*}$$

is a bijection.

Proof For arbitrary $\sigma \in \Sigma_{\mathbf{n}}$, let $A = \sigma^{-1}(\{b_{k-1}+1,\ldots,b_{k-1}+r\})$. Then $\sigma\xi_{k,A}^{-1} \in \Sigma_{\mathbf{n}[k,r]}$, and the inverse to the function (*) is given by $\sigma \mapsto (A, \sigma\xi_{k,A}^{-1})$. □

Every permutation $\xi \in \Sigma_{\mathbf{n}}$ induces an automorphism $\sigma^*\colon O_{\mathbf{n}} \to O_{\mathbf{n}}$ by the formula $\sigma^*(f) = f\sigma$; note that $\xi_* a[\sigma,\tau] = a[\sigma\xi,\tau]$. Define an inclusion $\iota_{k,A}$ as the composition $O_{\mathbf{n}[k,r]} \subseteq O_{\mathbf{n}} \xrightarrow{(\xi_{k,A})^*} O_{\mathbf{n}}$.

Proposition 7.11 *For* $\mathbf{n} \in \mathrm{Seq}(n)$ *we have*

$$\partial g_{\mathbf{n}} = \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)} (-1)^{k+r+b_{k-1}+1}\,\mathrm{sgn}(A-b_{k-1})(\iota_{k,A})_*(g_{\mathbf{n}[k,r]}).$$

Proof By 7.7 we have

$$\partial g_{\mathbf{n}} = (-1)^{n-l}\sum_{\tau\in T_{\mathbf{n}}^{n-l}} \mathrm{sgn}_{\mathbf{n}}(\tau)\sum_{\sigma\in\Sigma_{\mathbf{n}}} \mathrm{sgn}(\sigma)a[\sigma, \mathsf{s}_{n-l}\tau].$$

Then, by Lemma 7.8

$$\partial g_{\mathbf{n}} = \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{\psi\in T_{\mathbf{n}[k,r]}^{n-l-1}} (-1)^{k+r+b_{k-1}+1}\,\mathrm{sgn}_{\mathbf{n}[k,r]}(\psi)\sum_{\sigma\in\Sigma_{\mathbf{n}}} \mathrm{sgn}(\sigma)a[\sigma,\psi].$$

Finally, using Propositions 7.9 and 7.10, we obtain

$$\begin{aligned}
\partial g_{\mathbf{n}} &= \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{\psi\in T^{n-l-1}_{\mathbf{n}[k,r]}}(-1)^{k+r+b_{k-1}+1}\operatorname{sgn}_{\mathbf{n}[k,r]}(\psi)\\
&\qquad\sum_{\omega\in\Sigma_{\mathbf{n}[k,r]}}\sum_{A\in S_{\mathbf{n}}(k,r)}\operatorname{sgn}(\omega\xi_{k,A})a[\omega\xi_{k,A},\psi]\\
&= \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)}(-1)^{k+r+b_{k-1}+1}\operatorname{sgn}(\xi_{k,A})\\
&\qquad\sum_{\psi\in T^{n-l-1}_{\mathbf{n}[k,r]}}\sum_{\omega\in\Sigma_{\mathbf{n}[k,r]}}\operatorname{sgn}_{\mathbf{n}[k,r]}(\psi)\operatorname{sgn}(\omega)(\iota_{k,A})_*a[\omega,\psi]\\
&= \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)}(-1)^{k+r+b_{k-1}+1}\operatorname{sgn}(A-b_{k-1})(\iota_{k,A})_*(g_{\mathbf{n}[k,r]}).
\end{aligned}$$

□

8 CW-decomposition

Throughout the whole section, K is a finite proper $\square$-set with height function h, and $v, w \in K[0]$ are two vertices such that $h(v) = 0$, $h(w) = n$. For $s \in \mathbb{Z}$, define a function

$$\mathsf{u}_s\colon \mathbb{Z} \ni i \mapsto \begin{cases} 0 & \text{for } i > s\\ 1 & \text{for } i < s\\ * & \text{for } i = s.\end{cases}$$

Proposition 8.1 *For every $c \in K[n]$, the map*

$$I_c\colon O_n \ni f \mapsto (d_{\mathsf{u}_1 f}(c), d_{\mathsf{u}_2 f}(c), \ldots, d_{\mathsf{u}_{k(f)} f}(c)) \in \mathrm{Ch}_{\leq(c)}(K),$$

is an isomorphism of posets.

Proof We will construct an inverse of I_c. For an arbitrary $\mathbf{b} = (b_1, \ldots, b_l) \in \mathrm{Ch}_{\leq(c)}(K)$ and $k \in \{1, \ldots, l\}$, there exists a unique, since K is proper, presentation $b_k = d_{A_k,B_k}(c)$ where A_k and B_k are disjoint subsets of $\{1, \ldots, n\}$. For $X \subseteq \{1, \ldots, n\}$ denote $\bar{X} = \{1, \ldots, n\} \setminus X$. We have $B_1 = \emptyset$ (since $d^0(b_1) = d^0(c)$), $A_l = \emptyset$ (since $d^1(b_l) = d^1(c)$), $B_{k+1} = \bar{A}_k$ (since $d^0(b_{k+1}) = d^1(b_k)$), and $A_k \cup B_k \neq \{1, \ldots, n\}$ (since $\dim(b_k) > 0$). Therefore, we have a strictly increasing sequence

$$\emptyset = B_1 \subsetneq \bar{A}_1 = B_2 \subsetneq \cdots \subsetneq \bar{A}_{l-1} = B_l \subsetneq \bar{A}_l = \{1, \ldots, n\},$$

which defines a unique function $J_c(\mathbf{b})\colon \{1,\dots,n\} \to \{1,\dots,l\}$ such that $i \in \bar{A}_{f(i)} \setminus B_{f(i)}$. Clearly, J_c is inverse to I_c. □

For $v, w \in K[0]$ and $\mathbf{c} = (c_1,\dots,c_l) \in \mathrm{Ch}(K)_v^w$, define a map

$$I_{\mathbf{c}}\colon O_{\mathbf{n}} \ni f \mapsto I_{c_1}(f^1) * \cdots * I_{c_l}(f^l) \in \mathrm{Ch}_{\leq \mathbf{c}}(K), \tag{8.1}$$

where f^i are defined as in (7.3).

Proposition 8.2 *For every $\mathbf{c} \in \mathrm{Ch}(K)_v^w$, $I_{\mathbf{c}}$ is an isomorphism of posets.*

Proof Every $\mathbf{b} \in \mathrm{Ch}_{\leq \mathbf{c}}(K)$ has a unique presentation as $\mathbf{b}^1 * \cdots * \mathbf{b}^l$, where $\mathbf{b}^k \in \mathrm{Ch}_{\leq (c_k)}(K)$. Define

$$J_{\mathbf{c}}(\mathbf{b}) = (J_{c_1}(\mathbf{b}^1),\dots,J_{c_l}(\mathbf{b}^l)) \in O_{n_1} \times \cdots \times O_{n_l} \simeq O_{\mathbf{n}},$$

where J_{c_k} is the map from the proof of Proposition 8.1; $J_{\mathbf{c}}$ is the inverse of $I_{\mathbf{c}}$. □

Proposition 8.3 *Theorem 1.3 holds for finite proper bi-pointed □-sets with height function.*

Proof By Propositions 8.2 and 7.5, for every $\mathbf{c} \in \mathrm{Ch}(K)$ the space $|\mathrm{Ch}_{<\mathbf{c}}(K)_v^w|$ is homeomorphic to $S^{\dim(\mathbf{c})-1}$. Therefore, $\mathrm{Ch}(K)_v^w$, augmented with a minimal element $\emptyset$, is a CW poset in the sense of [1]. Thus, by [1, 3.1], $|\mathrm{Ch}(K)_v^w|$ is a regular CW-complex with cells

$$A_{\mathbf{c}} := |\mathrm{Ch}_{\leq \mathbf{c}}(K)_v^w| \setminus |\mathrm{Ch}_{<\mathbf{c}}(K)_v^w|$$

for $\mathbf{c} \in \mathrm{Ch}(K)_v^w$. Every □-map $f\colon K \to K'$ induces a morphism of posets $\mathrm{Ch}(K)_v^w \to \mathrm{Ch}(K')_{f(v)}^{f(w)}$, which maps $\mathrm{Ch}_{\leq \mathbf{c}}(K)_v^w$ into $\mathrm{Ch}_{\leq f(\mathbf{c})}(K)_{f(v)}^{f(w)}$; therefore, $f_*\colon |\mathrm{Ch}(K)_v^w| \to |\mathrm{Ch}(K')_{f(v)}^{f(w)}|$ is cellular. □

Let $\mathrm{Ch}^{\leq n}(K)_v^w \subseteq \mathrm{Ch}(K)_v^w$ be the subset of cube chains having dimension less of equal n; clearly $|\mathrm{Ch}^{\leq n}(K)_v^w|$ is the n-skeleton of $|\mathrm{Ch}(K)_v^w|$. For a chain $\mathbf{c} \in \mathrm{Ch}(K)_v^w$ of type $\mathbf{n}$ and dimension n let

$$g_{\mathbf{c}} := (I_{\mathbf{c}})_*(g_{\mathbf{n}}) \in C_n(|\mathrm{Ch}^{\leq n}(K)_v^w|, |\mathrm{Ch}^{\leq n-1}(K)_v^w|). \tag{8.2}$$

The group $H_n(|\mathrm{Ch}^{\leq n}(K)_v^w|, |\mathrm{Ch}^{\leq n-1}(K)_v^w|)$ is a free $\mathbb{Z}$-module generated by (simplicial, homological) chains $g_{\mathbf{c}}$ for $\mathbf{c} \in \mathrm{Ch}^{=n}(K)_v^w$.

Proposition 8.4 *Assume that $\mathbf{n} \in \mathrm{Seq}(n)$, $k \in \{1,\dots,l = l(\mathbf{n})\}$, $r \in \{1,\dots,n_k - 1\}$, and $A \in S_{\mathbf{n}}(k,r)$. Then the diagram*

$$\begin{tikzcd} O_{\mathbf{n}[k,r]} \arrow[r, "I_{d_{k,(A-b_{k-1})}(\mathbf{c})}"] \arrow[d, "\iota_{k,A}"'] & \mathrm{Ch}(K) \arrow[d, "="] \\ O_{\mathbf{n}} \arrow[r, "I_{\mathbf{c}}"] & \mathrm{Ch}(K) \end{tikzcd}$$

commutes.

Proof It is sufficient to prove this in the case when $\mathbf{n} = (n)$, $\mathbf{c} = (c)$ and $k = 1$. Let $f\colon \{1,\dots,n\} \to \{1,\dots,m\}$ be an element of $O_{\mathbf{n}[k,r]} = O_{(r,n-r)}$. There exists a presentation $m = m_1 + m_2$ such that $f(i) \le m_1$ for $i \in \{1,\dots,r\}$ and $f(i) > m_1$ for $i \in \{r+1,\dots,n\}$. Furthermore,

$$f^1\colon \{1,\dots,r\} \to \{1,\dots,m_1\}$$

is the restriction of f, and f_2 is the composition

$$\begin{aligned}&\{1,\dots,n-r\} \xrightarrow{i\mapsto i+r} \{r+1,\dots,n\}\\ &\quad\xrightarrow{i\mapsto f(i)} \{m_1+1,\dots,m_2\} \xrightarrow{i\mapsto i-m_1} \{1,\dots,m_2\}.\end{aligned}$$

We will write ξ instead of $\xi_{1,A}$. We have

$$I_{(c)}(\iota_{1,A}(f)) = I_c(f\xi) = (d_{\mathsf{u}_1 f\xi}(c),\dots,d_{\mathsf{u}_m f\xi}(c))$$

and

$$\begin{aligned}I_{d_{1,A}((c))}(f) &= I_{(d^0_{\bar{A}}(c),d^1_A(c))}(f) = I_{d^0_{\bar{A}}(c)}(f^1) * I_{d^1_A(c)}(f^2)\\ &= \left(d_{\mathsf{u}_1 f^1} d^0_{\bar{A}}(c),\dots,d_{\mathsf{u}_{m_1} f^1} d^0_{\bar{A}}(c), d_{\mathsf{u}_1 f^2} d^1_A(c),\dots,d_{\mathsf{u}_{m_2} f^2} d^1_A(c)\right)\end{aligned}$$

Since $\xi|_A$ is an increasing bijection $A \to \{1,\dots,r\}$, for $s \in \{1,\dots,r\}$ we have (cf. 1.2) $d_{\mathsf{u}_s f^1} d^0_{\bar{A}}(c) = d_g$, where $g|_{\bar{A}} = 0 = \mathsf{u}_s f\xi$, and $g|_A = f^1\xi = f\xi$. Hence, $g = \mathsf{u}_s f\xi$. A similar argument applies for $s \in \{r+1,\dots,n\}$. □

Proposition 8.5 *Let* $\mathbf{c} \in \mathrm{Ch}(K)^w_v$ *be a cube chain of type* $\mathbf{n}$. *Then*

$$\partial(g_{\mathbf{c}}) = \sum_{k=1}^{l(\mathbf{n})} \sum_{r=1}^{n_k-1} \sum_{A\in S_{\mathbf{n}}(k,r)} (-1)^{k+r+b_{k-1}+1} \operatorname{sgn}(A-b_{k-1}) g_{d_{k,A-b_{k-1}}(\mathbf{c})}$$

Proof We will write $A' = A - b_{k-1}$. By 8.4 and 7.11, we have

$$\begin{aligned}\partial(g_{\mathbf{c}}) &= \partial(I_{\mathbf{c}}(g_{\mathbf{n}})) = I_{\mathbf{c}}(\partial g_{\mathbf{n}})\\ &\overset{(7.25)}{=} I_{\mathbf{c}}\left(\sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)} (-1)^{k+r+b_{k-1}+1}\operatorname{sgn}(A')\iota_{k,A}(g_{\mathbf{n}[k,r]})\right)\\ &= \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)} (-1)^{k+r+b_{k-1}+1}\operatorname{sgn}(A') I_{d_{k,A'}(\mathbf{c})}(g_{\mathbf{n}[k,r]})\\ &= \sum_{k=1}^{l}\sum_{r=1}^{n_k-1}\sum_{A\in S_{\mathbf{n}}(k,r)} (-1)^{k+r+b_{k-1}+1}\operatorname{sgn}(A') g_{d_{k,A'}(\mathbf{c})}.\end{aligned}$$

□

Proposition 8.6 *Theorem 1.4 holds for finite proper bi-pointed $\square$-sets with height function.*

Proof By Theorem 1.3, $H_n(|\mathrm{Ch}^{\leq m}(K)_v^w|, |\mathrm{Ch}^{\leq m-1}(K)_v^w|)$ is a free $\mathbb{Z}$-module generated by (simplicial, homological) chains $g_{\mathbf{c}}$ for $\mathbf{c} \in \mathrm{Ch}^{=m}(K)_v^w$, and the differentials are calculated in Proposition 8.5. $\square$

9 Non-looping length covering

In this section we generalize the results obtained for finite proper $\square$-sets with height function to a larger class of $\square$-sets that have proper non-looping length coverings. Recall that $\mathbf{pc}\square\mathbf{Set}_*^*$ is the category of finite bi-pointed $\square$-sets having proper non-looping coverings.

For $K \in \mathbf{pc}\square\mathbf{Set}_*^*$ and $n \geq 0$ define a $\square$-subset $\tilde{K}_n \subseteq \tilde{K}$ by $\tilde{K}_n[d] = K[d] \times \{0, \dots, n-d\}$ (cf. 1.4). For every n, the formula $(K, v, w) \mapsto (\tilde{K}_n, (v, 0), (w, n))$ defines a functor from $\mathbf{pc}\square\mathbf{Set}_*^*$ to $\mathbf{fph}\square\mathbf{Set}_*^*$.

Let $p\colon \tilde{K} \ni (c, k) \mapsto c \in K$ denote the obvious projection, and let $p_n = p|_{\tilde{K}_n}$.

Proposition 9.1 *Assume that $(K, v, w) \in \mathbf{pc}\square\mathbf{Set}_*^*$. All maps in the sequence*

$$\coprod_{n\geq 0} \vec{N}(\tilde{K}_n)_{(v,0)}^{(w,n)} \subseteq \coprod_{n\geq 0} \vec{P}(\tilde{K}_n)_{(v,0)}^{(w,n)} \subseteq \coprod_{n\geq 0} \vec{P}(\tilde{K})_{(v,0)}^{(w,n)} \xrightarrow{\coprod p_*} \vec{P}(K)_v^w$$

are homotopy equivalences, and they are functorial with respect to (K, v, w).

Proof For the right-hand map this follows [11, Proposition 5.3] and for the left-hand one from [9, 2.5]. Every directed path in $\tilde{K}$ with endpoints in $\tilde{K}_n$ lies in $\tilde{K}_n$ so the middle inclusion is a homeomorphism. $\square$

This criterion shows that, when proving Theorem 1.2, we can restrict to the case when K if a finite proper $\square$-set with height function.

Proposition 9.2 *For an arbitrary $\square$-set K and $v, w \in K[0]$ the sequence of $\square$-maps $\tilde{K}_n \subseteq \tilde{K} \xrightarrow{pr} K$ induces morphism of posets*

$$\coprod_{n\geq 0} \mathrm{Ch}(\tilde{K}_n)_{(v,0)}^{(w,n)} \xrightarrow{\cong} \coprod_{n\geq 0} \mathrm{Ch}(\tilde{K})_{(v,0)}^{(w,n)} \xrightarrow{\simeq} \mathrm{Ch}(K)_v^w,$$

which are isomorphisms.

Proof The formula

$$\mathrm{Ch}(K)_v^w \ni (c_i)_{i=1}^l \mapsto \left(c_i, \sum_{j=1}^{i-1} \dim(c_j) \right) \in \coprod_{n\geq 0} \mathrm{Ch}(\tilde{K}_n)_{(v,0)}^{(w,n)}$$

defines the inverse function and all the maps involved commute with $d_{K,A}$. $\square$

Proof of Theorem 1.2 If K is a $\square$-set with the proper non-looping covering, then, for $v, w \in K[0]$, there is a sequence of homotopy equivalences

$$\vec{P}(K)_v^w \xleftarrow{\simeq} \coprod_{n\geq 0} \vec{P}(\tilde{K}_n)_{(v,0)}^{(w,n)} \xrightarrow{\coprod \varepsilon_{\tilde{K}_n}} \coprod_{n\geq 0} |\operatorname{Ch}(\tilde{K}_n)_{(v,0)}^{(w,n)}| \xrightarrow{\simeq} |\operatorname{Ch}(K)_v^w|,$$

according to Propositions 9.1, 6.3 and 9.2 respectively. □

Proof of Theorems 1.3 and 1.4 By Proposition 9.2, these follow from Propositions 8.3 and 8.6, respectively. □

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