Metrika (2008) 68:323–331
DOI 10.1007/s00184-007-0160-5

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Strong laws for weighted sums of NA random variables

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Received: 12 December 2005 / Published online: 4 December 2007

Abstract Strong laws are established for linear statistics that are weighted sums of an negatively associated (NA) random sample. The results obtained not only generalize the results of Sung (Stat. Probab. Lett. 52:413–419, 2001) to NA random variables, but also extend and sharpen them.

Keywords Almost sure convergence · Weighted sums · NA

1 Introduction

For negatively associated (NA) random variables: Joag and Proschan (1983) gives the following definition

Definition (Joag and Proschan 1983) A finite family of random variables $\{X_i, 1 \le i \le n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets, T_1 and T_2, of $\{1, 2, \ldots, n\}$, we have

$$\mathrm{Cov}(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \le 0,$$

whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Let $\{X, X_i, i \ge 1\}$ be a sequence of i.i.d. random variables and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be a triangular array of constants. The almost sure (a.s.) limiting behavior

Research supported by Youth Talent Foundation of Zhejiang Gongshang University (Q07-07).

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of weighted sums $\sum_{i=1}^{n} a_{ni} X_i$ has been studied by many authors (see, Sung 2001; Bai and Cheng 2000; Choi and Sung 1987; Cuzick 1995; Wu 1999). Recently Sung (2001) proved the following strong laws of large numbers:

Theorem A *Let* $\{X, X_i, i \geq 1\}$ *be a sequence of i.i.d. random variables satisfying* $EX = 0$ *and for any* $h, \gamma > 0$, $E\exp(h|X|^{\gamma}) < \infty$. *And let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a triangular array of constants satisfying* $A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty$, $A_{\alpha,n} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}/n$ *for some* $1 < \alpha \leq 2$. *Then for* $0 < \gamma < 1$ *and* $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}}$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \ a.s.,$$

moreover, for $\gamma > 1$ *and* $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}+\delta}$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \ a.s.$$

where $\delta = 1 - \frac{1}{\gamma} - \frac{\gamma-1}{1+\alpha\gamma-\alpha}$.

Theorem B *Let* $\{X, X_i, i \geq 1\}$ *be a sequence of i.i.d. random variables satisfying* $EX = 0$ *and for some* $h, \gamma > 0$, $E\exp(h|X|^{\gamma}) < \infty$. *And let* $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ *be a triangular array of constants satisfying* $A_\alpha = \limsup_{n\to\infty} A_{\alpha,n} < \infty$, $A_{\alpha,n} = \sum_{i=1}^{n} |a_{ni}|^{\alpha}/n$ *for some* $1 < \alpha \leq 2$. *Then for* $0 < \gamma \leq 1$ *and* $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}+\beta}$ *for* $\beta > 0$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \ a.s.,$$

moreover, for $\gamma > 1$ *and* $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}+\delta+\beta}$ *for* $\beta > 0$

$$\sum_{i=1}^{n} a_{ni} X_i / b_n \to 0 \ a.s.$$

where $\delta = 1 - \frac{1}{\gamma} - \frac{\gamma-1}{1+\alpha\gamma-\alpha}$.

The main purpose of this paper is to establish the Marcinkiewicz-Zygmund strong laws for linear statistics of NA sequences of random variables. The results obtained not only generalize the results of Sung (2001) to NA random variables, but also extend and sharpen them.

2 The Marcinkiewicz–Zygmund strong laws

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, $a_n = O(b_n)$ will represent $a_n \leq Cb_n$, and $a_n \ll b_n$ will represent $a_n = O(b_n)$.

In order to prove our results, we need the following lemma and the concept of complete convergence. As for complete convergence, let $\{X, X_n, n \geq 1\}$ be a sequence of independent indentically distribution random variables (i.i.d) random variables and denote $S_n = \sum_{i=1}^n X_i$. The Hsu–Robbins–Erdös law of large numbers (Hsu and Robbins 1947; Erdös 1949) states that

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a foundamental theorm in probability theory and has been intensively investigated by many authors in the past decades. We can see in Petrov (1995), Chow and Teicher (1997) and Stout (1974). There have been many extensions in various directions for Hsu-Robbins-Erdös law of large numbers.

In order to prove our results, we need the following lemma.

Lemma 2.1 (Su et al. 1996; Shao 2000 or Yang 2000) *Let $\{X_i, i \geq 1\}$ be a sequence of NA random variables, $EX_i = 0$, $E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that*

$$E \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} X_i \right|^p \leq C \left\{ \sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2 \right)^{p/2} \right\}.$$

The following is the main result in this paper.

Theorem 2.1 *Let $\{X, X_i, i \geq 1\}$ be a sequence of NA random variables with indentical distributions. And let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a triangular array of constants satisfying $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$ for $0 < \alpha \leq 2$. Let $T_n = \sum_{i=1}^{n} a_{ni} X_i, n \geq 1$, $b_n = n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}}$. $EX = 0$ when $1 < \alpha \leq 2$. We assume that for some $h, \gamma > 0$, $E \exp(h|X|^\gamma) < \infty$. Then*

$$\forall \varepsilon > 0, \quad \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leq j \leq n} |T_j| > \varepsilon b_n \right) < \infty. \tag{2.1}$$

Proof Without loss of generality, we can assume that $a_{ni} \geq 0$, for all $1 \leq i \leq n, n \geq 1$. $\forall i \geq 1$, define $X_i^{(n)} = X_i I(|X_i| \leq b_n) + b_n I(X_i > b_n) - b_n I(X_i < -b_n)$, $T_j^{(n)} = \sum_{i=1}^{j} (a_{ni} X_i^{(n)} - E a_{ni} X_i^{(n)})$, then $\forall \varepsilon > 0$,

$$P\left(\max_{1\le j\le n}|T_j| > \varepsilon b_n\right) \le P\left(\max_{1\le j\le n}|X_j|>b_n\right)+P\left(\max_{1\le j\le n}|T_j^{(n)}|>\varepsilon b_n-\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right|\right). \tag{2.2}$$

First we show that

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right| \to 0, as\ n\to\infty. \tag{2.3}$$

By $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ and the Hölder inequality, $\forall 1\le k<\alpha$,

$$\sum_{i=1}^{n}|a_{ni}|^k \le \left(\sum_{i=1}^{n}|a_{ni}|^{k\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}}\left(\sum_{i=1}^{n}1\right)^{\frac{\alpha-k}{\alpha}} \le Cn. \tag{2.4}$$

When $1<\alpha\le 2$, using $EX=0$, (2.4), the Markov inequality and $E\exp(h|X|^{\gamma})<\infty$, as $n\to\infty$,

$$\begin{aligned}
&b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right| \\
&\le b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|X_i|>b_n)+\sum_{i=1}^{n}|a_{ni}|P(|X_i|>b_n) \\
&\ll b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|>b_n)+nP(|X|>b_n) \\
&\le Cb_n^{-1}nE|X|I(|X|>b_n)+n\frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})} \\
&\ll Cb_n^{-1}n\sum_{k=n}^{\infty}E|X|I(b_k<|X|\le b_{k+1})+n\exp(-hb_n^{\gamma}) \\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}P(|X|>b_k)+ne^{-hn^{\gamma/\alpha}\log n} \\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}\frac{E\exp(h|X|^{\gamma})}{\exp(hb_k^{\gamma})}+n^{1-hn^{\gamma/\alpha}} \\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}(k+1)^{\frac{1}{\alpha}}(\log(k+1))^{\frac{1}{\gamma}}k^{-hk^{\gamma/\alpha}}+n^{1-hn^{\gamma/\alpha}} \\
&\le Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}nn^{-1}+n^{1-hn^{\gamma/\alpha}} \\
&= Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}+n^{1-hn^{\gamma/\alpha}}\to 0.
\end{aligned} \tag{2.5}$$

By $\sum_{i=1}^{n}|a_{ni}|^{\alpha}=O(n)$ and the Hölder inequality, $\forall k\geq\alpha$, then

$$\sum_{i=1}^{n}|a_{ni}|^{k}=\sum_{i=1}^{n}|a_{ni}|^{\alpha}|a_{ni}|^{k-\alpha}\leq Cnn^{\frac{k-\alpha}{\alpha}}=Cn^{\frac{k}{\alpha}}. \tag{2.6}$$

When $0<\alpha\leq 1$, using (2.6), the Markov inequality and $E\exp(h|X|^{\gamma})<\infty$, as $n\to\infty$,

$$\begin{aligned}
&b_n^{-1}\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}Ea_{ni}X_i^{(n)}\right|\\
&\leq b_n^{-1}\sum_{i=1}^{n}E|a_{ni}X_i|I(|X_i|\leq b_n)+\sum_{i=1}^{n}|a_{ni}|P(|X_i|>b_n)\\
&\ll b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I(|X|\leq b_n)+n^{1/\alpha}P(|X|>b_n)\\
&\leq Cb_n^{-1}n^{\frac{1}{\alpha}}E|X|I(|X|\leq b_n)+n^{1/\alpha}\frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})}\\
&=Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}E|X|I(b_{k-1}<|X|\leq b_k)+n^{1/\alpha}\exp(-hb_n^{\gamma})\\
&\leq Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}b_kP(|X|>b_{k-1})+n^{1/\alpha}e^{-hn^{\gamma/\alpha}\log n}\\
&\leq Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}b_k\frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})}+n^{\frac{1}{\alpha}-hn^{\gamma/\alpha}}\\
&\leq Cb_n^{-1}n^{\frac{1}{\alpha}}\sum_{k=2}^{n}k^{\frac{1}{\alpha}}(\log k)^{\frac{1}{\gamma}}(k-1)^{-h(k-1)^{\gamma/\alpha}}+n^{\frac{1}{\alpha}-hn^{\gamma/\alpha}}\\
&\leq Cn^{-\frac{1}{\alpha}}(\log n)^{-\frac{1}{\gamma}}n^{\frac{1}{\alpha}}+n^{\frac{1}{\alpha}-hn^{\gamma/\alpha}}\\
&=C(\log n)^{-\frac{1}{\gamma}}+n^{\frac{1}{\alpha}-hn^{\gamma/\alpha}}\to 0.
\end{aligned} \tag{2.7}$$

(2.3) follows from (2.5) and (2.7).

From (2.2) and (2.3), it follows that for large enough n

$$P\left(\max_{1\leq j\leq n}|T_j|>\varepsilon b_n\right)\leq\sum_{j=1}^{n}P(|X_j|>b_n)+P\left(\max_{1\leq j\leq n}|T_j^{(n)}|>\frac{\varepsilon}{2}b_n\right).$$

Hence we need only to prove that

$$I =: \sum_{n=1}^{\infty} n^{-1} \sum_{j=1}^{n} P(|X_j| > b_n) < \infty,$$
$$II =: \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n} |T_j^{(n)}| > \frac{\varepsilon}{2} b_n\right) < \infty. \tag{2.8}$$

From the fact that $E\exp(h|X|^{\gamma}) < \infty$, it follows easily that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n^{-1} n P(|X| > b_n) \\ &= \sum_{n=1}^{\infty} P(|X| > b_n) \\ &\le \sum_{n=1}^{\infty} \frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})} \\ &\le C \sum_{n=1}^{\infty} \frac{1}{n^{hn^{\gamma/\alpha}}} < \infty. \end{aligned} \tag{2.9}$$

By Lemma 2.1, it follows that

$$\begin{aligned} II &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} E \max_{1\le j\le n} |T_j^{(n)}|^q \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \left\{ \sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^q + \left(\sum_{j=1}^{n} E|a_{nj} X_j^{(n)}|^2 \right)^{q/2} \right\} \\ &=: II_1 + II_2. \end{aligned} \tag{2.10}$$

Let $\max(2, \alpha, \gamma + 1) \le q$, using (2.6), we have

$$\begin{aligned} II_1 &= C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \left\{ \sum_{i=1}^{n} |a_{ni}|^q E|X|^q I(|X| \le b_n) + \sum_{i=1}^{n} |a_{ni}|^q P(|X| > b_n) \right\} \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} \left\{ n^{\frac{q}{\alpha}} E|X|^q I(|X| \le b_n) + n^{\frac{q}{\alpha}} P(|X| > b_n) \right\} \\ &\le C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} n^{\frac{q}{\alpha}} \sum_{k=2}^{n} E|X|^q I(b_{k-1} < |X| \le b_k) \\ &\quad + C \sum_{n=2}^{\infty} n^{-1} b_n^{-q} n^{\frac{q}{\alpha}} \frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})} \end{aligned}$$

$$\leq C\sum_{k=2}^{\infty}\sum_{n=k}^{\infty} n^{-1+\frac{q}{\alpha}} n^{-q/\alpha}(\log n)^{-q/\gamma} b_k^q P(|X|>b_{k-1})$$
$$+C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}$$
$$\leq C\sum_{k=2}^{\infty} b_k^q \frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})} + C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}$$
$$\leq C\sum_{k=2}^{\infty} k^{\frac{q}{\alpha}}(\log k)^{\frac{q}{\gamma}}(k-1)^{-h(k-1)^{\gamma/\alpha}}$$
$$+C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}} < \infty. \tag{2.11}$$

By $0<\alpha\leq 2$, (2.6) and $q\geq \max\{2, \gamma+1\}$, we have

$$II_2 = C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\left\{\left(\sum_{i=1}^{n}|a_{ni}|^2\right)^{q/2}(E|X|^2 I(|X|\leq b_n))^{q/2}\right.$$
$$\left.+\left(\sum_{i=1}^{n}|a_{ni}|^2 P(|X|>b_n)\right)^{q/2}\right\}$$
$$\leq C\sum_{n=2}^{\infty} n^{-1} b_n^{-q}\left\{(n^{\frac{2}{\alpha}})^{q/2}(E|X|^2 I(|X|\leq b_n))^{q/2} + (n^{\frac{2}{\alpha}})^{q/2} P(|X|>b_n)\right\}$$
$$= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n} E|X|^2 I(b_{k-1}<|X|\leq b_k)\right)^{q/2}$$
$$+C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} P(|X|>b_n)$$
$$\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n} b_k^2 P(|X|>b_{k-1})\right)^{q/2}$$
$$+C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\frac{E\exp(h|X|^{\gamma})}{\exp(hb_n^{\gamma})}$$
$$\leq C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n} b_k^2\frac{E\exp(h|X|^{\gamma})}{\exp(hb_{k-1}^{\gamma})}\right)^{q/2}$$
$$+C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}$$

$$
\begin{aligned}
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n}\frac{k^{\frac{2}{\alpha}}(\log k)^{\frac{2}{\gamma}}}{\exp(h(k-1)^{\gamma/\alpha}\log(k-1))}\right)^{q/2}\\
&\quad +C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}\\
&= C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n} k^{\frac{2}{\alpha}}(\log k)^{\frac{2}{\gamma}}(k-1)^{-h(k-1)^{\gamma/\alpha}}\right)^{q/2}\\
&\quad +C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}\\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma}\left(\sum_{k=2}^{n} k^{-2}\right)^{q/2} + C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}}\\
&\le C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} + C\sum_{n=2}^{\infty} n^{-1}(\log n)^{-q/\gamma} n^{-hn^{\gamma/\alpha}} < \infty. \qquad (2.12)
\end{aligned}
$$

Putting (2.11) and (2.12) into (2.10) yields $II < \infty$. Now we complete the proof of Theorem 2.1. □

Corollary 2.1 *Under the conditions of Theorem 2.1,*

$$
\lim_{n\to\infty}\frac{|T_n|}{b_n} = 0 \; a.s.
$$

Proof By (2.1), we have

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\le j\le n}|T_j| > \varepsilon b_n\right)\\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} n^{-1} P\left(\max_{1\le j\le n}|T_j| > \varepsilon n^{\frac{1}{\alpha}}(\log n)^{\frac{1}{\gamma}}\right)\\
&\ge \frac{1}{2}\sum_{i=1}^{\infty} P\left(\max_{1\le j\le 2^i}|T_j| > \varepsilon 2^{\frac{i+1}{\alpha}}(\log 2^{i+1})^{\frac{1}{\gamma}}\right).
\end{aligned}
$$

By the Borel–Cantelli Lemma, we have

$$
P\left(\max_{1\le j\le 2^i}|T_j| > \varepsilon 2^{\frac{i+1}{\alpha}}\left(\log 2^{i+1}\right)^{\frac{1}{\gamma}}, i.o.\right) = 0.
$$

Hence

$$\lim_{i\to\infty} \frac{\max\limits_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{\alpha}} (\log 2^{i+1})^{\frac{1}{\gamma}}} = 0 \ a.s.$$

and using

$$\max_{2^{i-1}\le n<2^i} \frac{|T_n|}{b_n} \le 2^{\frac{2}{\alpha}} \frac{\max\limits_{1\le j\le 2^i} |T_j|}{2^{\frac{i+1}{\alpha}} (\log 2^{i+1})^{\frac{1}{\gamma}}} \left(\frac{i+1}{i-1}\right)^{\frac{1}{\gamma}},$$

we have

$$\lim_{n\to\infty} \frac{|T_n|}{b_n} = 0 \ a.s.$$

□

Remark Corollary 2.1 not only generalizes the results of Sung (2001) to NA random variables, but also extends and sharpens them.

Acknowledgments The author would like to thank the anonymous referee for his/her valuable comments.

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