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Horizontal product differentiation in Varian's model of sales

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Abstract
We consider the explicit introduction of firms' choice of location into Varian's model of sales. In our model, firms compete for both uninformed and informed consumers in a two-stage spatial competition model in which firms choose price and location. We obtain a result where both prices and locations are randomized in the subgame perfect equilibrium. The difference between each firm's choice of location in the subgame perfect equilibrium is neither purely maximized at both ends of a line nor purely minimized at the center. Also, the expected profits in a subgame perfect equilibrium are equal to the maximum profit from an uninformed market in the absence of informed consumers. Thus, even when product differentiation is explicitly introduced into a Varian-type model, Varian's implication can be retained; the opportunity for profit in an informed market is lost due to competition.

Keywords Varian's model of sales · Hotelling game · Spatial competition · Price and location dispersion

1 Introduction

Varian (1980) proposed a model in which temporary price discounts, popularly known as "sales," can occur in equilibrium as a mixed strategy. In a market with complete information, all agents usually have the same price information about products. A fundamental assumption in Varian's model is that some consumers are informed of both products' presence and their prices, while others are only informed of one product's presence and its price. Varian developed the earliest successful oligopoly pricing model, which analyzed equilibrium price dispersion in this market structure.

In Varian's model, every product is homogeneous. Moreover, the characteristic of products is exogenously assigned. Few products are strictly homogeneous. Firms

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usually develop their products to differentiate themselves to compete with their opponents. As a more realistic setting, we consider a competition where a firm differentiates its product from its opponent's. In addition, Varian's model is not suitable for empirical research about price competition when product differentiation is endogenous. Hence, we analyze a model in which product characteristics are determined endogenously.

In this study, we analyze horizontal product differentiation in Varian's model of sales as a location-then-price competition on Hotelling's linear city model (Hotelling 1929). We formulate our model as a two-stage game, like a spatial competition of D'Aspremont et al. (1979). In the first period, firms choose their location on Hotelling's linear city. In the second period, firms compete in prices given their locations. Our motivation has the following two features.

First, in our model, consumers are located on three points in the [0, 1] interval: both opposite endpoints and the middle point. Consumers at the middle point correspond to Varian's informed consumers. They know both products' presence and their prices. In contrast, the consumers at both opposite endpoints correspond to Varian's uninformed consumers. The consumers at the endpoint 0 only know firm 1's product and its price, and those at the endpoint 1 only know firm 2's product and its price. One interpretation is that those consumers are oblivious of the other product's price because their search costs are prohibitively high. So they rationally choose to know only one price, and all consumers in our model make an entirely rational purchasing decision based on their information.

Second, in our model, firms control two parameters: location, z, and price, p. Firm 1 chooses its location z_1 within [0,1/2], and firm 2 chooses its location z_2 within [1/2,1]; one reason is that both firms do not have an incentive to choose a location point beyond the center which is closer to consumers who are uninformed of their products. Consequently, this model is symmetric for the center where informed consumers are located. Thus, each firm is a monopolist to the consumer group at each end, that is, at 0 and 1. By contrast, both firms compete for consumers at the middle point.

We obtained two main results. First, the equilibrium behaviors in our model are randomized in both the location and price stages. Consequently, the difference between each firm's choice of location in a subgame perfect equilibrium is neither purely maximized at both ends of a line nor purely minimized at the center. In our model, these two typical results stochastically occur in a subgame perfect equilibrium.

Second, the expected profits in a subgame perfect equilibrium are equal to the maximum profit from an uninformed market in the absence of informed consumers. Hence, the result of firms' profits in the subgame perfect equilibrium in our model is the same as Varian's result for price competition. Moreover, each firm's equilibrium profit is equal to each firm's uninformed consumers' reservation value. This result indicates that Bertrand's law concerning the loss of opportunity to gain excess profit in a competitive market still holds even when product differentiation is explicitly considered.

In the literature of Hotelling's model, we briefly review models in which location or price dispersion occurs in equilibrium. First of all, in the models mentioned here, all the consumers on the Hotelling's line are completely informed. In contrast, in our model, some consumers are uninformed, as in Varian's model. Osborne and Pitchik (1986) analyzed a mixed strategy equilibrium in Hotelling's location game without price choice. Osborne and Pitchik (1987) study a subgame perfect equilibrium in the spatial competition model in which the firms use mixed strategies in the second stage. Their equilibrium location is not dispersed. Gal-or (1982) is an early result of a price equilibrium with mixed strategies in Hotelling's model. She did not analyze an equilibrium location. Bester et al. (1996), Matsumura and Matsushima (2009), and Eaton and Tweedle (2012) also analyzed a mixed strategy equilibrium in Hotelling's location game.

In the literature mentioned here, unit demand is an important assumption. Our model considers two types of consumers, but each consumer purchases only one good following traditional literature. Some studies consider the consumer preference for variety in models with single-peaked preferences and unit demand. Kim and Serfes (2006) considered that each consumer purchases one or two products. They obtain a similar equilibrium where some consumers remain loyal to one brand while others consume both brands. However, location and price dispersion did not occur in their equilibrium.

Our analysis proceeds as follows. We construct a two-stage game: firms choose a location in the first stage and a price in the second stage. We solve this game using backward induction. We analyze price competition in a subgame given a location pair and obtain an equilibrium price dispersion for any location. Then, we analyze the subgame perfect location equilibrium in the first stage. Finally, we discuss the results.

2 The model

There are two firms ($i = 1, 2$) in this model. Both firms sell a single product. These two firms compete for three types of consumers: consumers who are informed about both firms' products and consumers who are informed about either one firms' product. Now, we assume that all types of consumers are distributed over the interval [0, 1] to introduce a firm's choice of location into Varian's model. C_1 is located at 0 and C_2 at 1. C_3 is located at an equal distance from both C_1 and C_2, that is, at 1/2. We assume that both C_1 and C_2 consumers have equal size and normalize the size to 1. We set the size of C_3 consumers to x and assume that $x \geq 1$.

Our model is a two-stage spatial competition. In the first stage, both firms choose their own location in the interval [0, 1]. In the second stage, given their location pair, each firm chooses a price for the product. z_i denotes the location of Firm i. We assume $z_1 \in [0, 1/2]$ and $z_2 \in [1/2, 1]$. p_i denotes the price of Firm i's product.

We define each consumer's utility when they purchase a product with location z_i at price p_i as follows: We assume that the reservation value of C_1 for Firm 1's product and the reservation value of C_2 for Firm 2's product are equal, and we normalize these reservation values to 1. We assume that C_3 has the same reservation value for both firms' products, and this reservation value is defined by y.

$$\begin{cases} u^{C_1} = 1 - \{p_1 + z_1^2\}, \\ u^{C_2} = 1 - \{p_2 + (1 - z_2)^2\}, \\ u^{C_3} = y - \left\{ p_i + \left(\frac{1}{2} - z_i\right)^2 \right\}. \end{cases} \tag{1}$$

Each consumer C_k $(k = 1, 2, 3)$ purchases one unit of the product from either one of the two firms. The distance between each consumer's location and the location of a firm measures the consumer's disutility. In our model, this disutility is measured using the quadratic cost function.[1]

Next, we consider each firm's profit in a price subgame given the location pair (z_1, z_2), taking each consumer's choice into account. Firm i's profit is defined as the sum of the profit gained from C_i and C_3. Here, we assume that the production cost is 0.

We consider Firm 1's profit. C_2 does not purchase the product from Firm 1. Let $\pi_1(p_1, p_2)$ denote Firm 1's profit. $\pi_1(p_1, p_2)$ is the total sum of $\pi_1^{C_1}(p_1)$, which denotes Firm 1's profit gained from C_1, and $\pi_1^{C_3}(p_1, p_2)$, which denotes Firm 1's profit gained from C_3:

$$\pi_1(p_1, p_2) = \pi_1^{C_1}(p_1) + \pi_1^{C_3}(p_1, p_2). \tag{2}$$

First, we define $\pi_1^{C_1}(p_1)$. From (1), C_1 purchases Firm 1's product when $1 - (p_1 + z_1^2) \geq 0$ is satisfied. Thus, we obtain:

$$\pi_1^{C_1}(p_1) = \begin{cases} p_1, & \text{if } p_1 \leq 1 - z_1^2, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Next, we consider $\pi_1^{C_3}(p_1, p_2)$. We calculate C_3's utility, u^{C_3}, as follows:

$$u^{C_3} = \begin{cases} y - \{p_1 + (1/2 - z_1)^2\}, & \text{if buying from 1,} \\ y - \{p_2 + (1/2 - z_2)^2\}, & \text{if buying from 2,} \\ 0, & \text{otherwise.} \end{cases}$$

Here, we find that C_3 purchases only Firm 1's product when the following two equations are satisfied: $p_1 + (1/2 - z_1)^2 < p_2 + (1/2 - z_2)^2$ and $p_1 + (1/2 - z_1)^2 \leq y$. It follows that

$$p_1 < p_2 + (1/2 - z_2)^2 - (1/2 - z_1)^2 \text{and} p_1 \leq y - (1/2 - z_1)^2.$$

We assume that C_3 is divided equally between Firm 1 and Firm 2 when C_3 is indifferent between choosing either Firm 1's or Firm 2's product, $p_1 + (1/2 - z_1)^2 = p_2 + (1/2 - z_2)^2 \leq y$.

Now, we define $\pi_1^{C_3}$, which denotes a firm's profit obtained from C_3, as follows:

[1] In what follows, we sometimes call this the 'transportation cost.'

$$\pi_1^{C_3}(p_1, p_2) = \begin{cases} p_1 x, & \text{if } p_1 < p_2 + (1/2 - z_2)^2 - (1/2 - z_1)^2 \text{ and } p_1 \le y - (1/2 - z_1)^2, \\ p_1 \frac{x}{2}, & \text{if } p_1 = p_2 + (1/2 - z_2)^2 - (1/2 - z_1)^2 \text{ and } p_1 \le y - (1/2 - z_1)^2, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

Substituting (3) and (4) with (2), we obtain Firm 1's profit in a price subgame given a location pair. We also define Firm 2's profit in the same way because of symmetry. Since Firm 2's profit π_2 is the total sum of the profit gained from C_2 and C_3, we define π_2 as follows:

$$\pi_2(p_1, p_2) = \pi_2^{C_2}(p_2) + \pi_2^{C_3}(p_1, p_2). \tag{5}$$

3 Price game

In this section, we characterize an equilibrium profit in a price subgame for all given pairs of (z_1, z_2). Because our model is symmetric at 1/2, we focus on the case of $z_1 + z_2 \le 1$. In other words, we primarily analyze Firm 2 located closer to the center than Firm 1. This case can be further divided into the following two cases by the threshold value for z_1: $z_1 \in [0, \bar{z}_1]$ or $z_1 \in (\bar{z}_1, 1/2]$. These cases are classified according to whether or not C_3 is attractive to Firm 1. $\bar{z}_1$ is given by evaluating $1 - z_1^2 = \{y - (1/2 - z_1)^2\}(1 + x)$ for z_1.

$$\bar{z}_1 = \frac{1}{2} - \frac{\sqrt{(x-1)^2 + 4x(y - 1/4)(1+x)} - 1}{2x}.$$

The necessary and sufficient condition for $\bar{z}_1 \in [0, 1/2]$ is:

$$\frac{3}{4} \le y(1+x) \le \frac{5+x}{4}. \tag{6}$$

The right inequality of (6) is obtained from the condition that Firm i is indifferent between C_i and C_3 if they are located at the end point of the line. For all z_1, $\{y - (1/2 - z_1)^2\}(1 + x) > 1 - z_1^2$ if and only if $(y - 1/4)(1 + x) > 1$. Therefore, $\bar{z}_1$ does not exist if $(y - 1/4)(1 + x) > 1$. From $\bar{z}_1 \in [0, 1/2]$, $(y - 1/4)(1 + x) \le 1$ must hold. Thus, we obtain $y(1 + x) \le \frac{5+x}{4}$, which is given by solving $(y - 1/4)(1 + x) \le 1$. The left inequality of (6) is obtained from the condition that Firm i is indifferent between C_i and C_3 if they are located at the center of the line, because Firm i can obtain the profit $\pi_i^{C_i}(p_i)$ when it charges $p_i = 3/4$.

The right inequality of (6) also implies that

$$yx < \frac{3}{4}(1+x). \tag{7}$$

To understand (7), suppose that Firm i is a monopolist at the center. Then, the left-hand side and the right-hand side of (7) are its profits when it charges y and then obtains all the informed consumers, and when it charges 3/4 and then obtains both informed and uninformed consumers, respectively. If (7) fails, the firm wants to sell only to informed consumers. Thus, (7) requires that C_3 is small enough to prevent such a scenario.

For all $z_1 \in (\bar{z}_1, 1/2]$, $1 - z_1^2 < \{y - (1/2 - z_1)^2\}(1 + x)$ is satisfied. If $z_1 \leq \bar{z}_1$, a pure strategy equilibrium exists in a price subgame. On the contrary, if $\bar{z}_1 < z_1$, there does not exist a pure strategy equilibrium. We also define $\bar{z}_2$ in the same way, that is, $\bar{z}_2 = 1 - \bar{z}_1$.

First, we analyze the case $z_1 \in [0, \bar{z}_1]$. This is the case of $1 - z_1^2 \geq \{y - (1/2 - z_1)^2\}(1 + x)$ where the informed market at C_3 is not attractive to Firm 1 because Firm 1 gains a higher profit when it sells a product to C_1 only than when it sells to both C_1 and C_3. When C_3 is attractive to Firm 1, the direction of the inequality sign is inverted.

In this case, we obtain a pure strategy equilibrium in a price subgame. We classify the following two cases according to z_2. First, $z_2 \in [\bar{z}_2, 1]$. Thus, z_2 satisfies $1 - (1 - z_2)^2 \geq \{y - (1/2 - z_2)^2\}(1 + x)$ (Proposition 1). Second, $z_2 \in [1/2, \bar{z}_2)$. Therefore, z_2 satisfies $1 - (1 - z_2)^2 < \{y - (1/2 - z_2)^2\}(1 + x)$ (Proposition 2). In each case, we show that the equilibrium profit in a subgame is unique.

Proposition 1 *If*

$$1 - z_1^2 \geq \{y - (1/2 - z_1)^2\}(1 + x), \tag{8}$$

$$1 - (1 - z_2)^2 \geq \{y - (1/2 - z_2)^2\}(1 + x) \tag{9}$$

hold, then the equilibrium profit vector is $(1 - z_1^2, 1 - (1 - z_2)^2)$, *and is unique.*

Proof See Appendix B.1. □

Proposition 2 *If*

$$1 - z_1^2 \geq \{y - (1/2 - z_1)^2\}(1 + x), \tag{10}$$

$$1 - (1 - z_2)^2 < \{y - (1/2 - z_2)^2\}(1 + x) \tag{11}$$

hold, then the equilibrium profit vector is $(1 - z_1^2, \{y - (1/2 - z_2)^2\}(1 + x))$ *and is unique.*

Proof See Appendix B.2. □

Remark 1 Proposition 1 shows that an equilibrium profit is unique in a price subgame. However, there may be more than one equilibrium; that is, an equilibrium strategy that achieves an equilibrium profit is not always unique. For example, when

$$1 - z_1^2 = \{y - (1/2 - z_1)^2\}(1 + x), \tag{12}$$

$$1-(1-z_2)^2 > \{y-(1/2-z_2)^2\}(1+x) \tag{13}$$

hold. It follows from (13) that a price vector $1-(1-z_2)^2$ is a dominant strategy of Firm 2. On the contrary, if Firm 2 chooses $1-(1-z_2)^2$, it follows from (12) that Firm 1 is indifferent to both $1-z_1^2$ and $y-(1/2-z_1)^2$. Thus, we find that all strategy pairs are equilibria, where Firm 1 uses a mixed strategy that combines $1-z_1^2$ with $y-(1/2-z_1)^2$ at any ratio it likes when Firm 2 chooses $1-(1-z_2)^2$.

Second, we analyze the case $z_1 \in (\bar{z}_1, 1/2]$ that corresponds to the case where C_3 is attractive to Firm 1. In this case, there does not exist a pure strategy equilibrium. In the following, we focus on a mixed-strategy equilibrium.

Proposition 3 shows the equilibrium profit in a mixed-strategy equilibrium. Furthermore, neither firm can obtain this profit in a pure strategy equilibrium. Thus, we calculate a mixed-strategy equilibrium. Propositions 4 and 5 show a mixed strategy equilibrium in which both firms obtain the equilibrium profit shown in Proposition 3. Propositions 4 and 5 are classified according to the upper bound of the price support that Firm 2 can charge. These include the equilibria of Varian's model of price dispersion in which product differentiation is explicitly considered.

First, we characterize the equilibrium profit. When the informed market at C_3 is attractive for both firms z_1 and z_2, the following equations are satisfied:

$$\begin{cases} 1-z_1^2 < \{y-(1/2-z_1)^2\}(1+x), \\ 1-(1-z_2)^2 < \{y-(1/2-z_2)^2\}(1+x). \end{cases} \tag{14}$$

In a price subgame in which $z_1+z_2 \le 1$ holds, Firm 2 is located close to C_3, while Firm 1 is located close to its monopoly market. Firm 1 gains at most its monopoly profit because it is located further away from C_3 than Firm 2.

Let (π_1^*, π_2^*) be an equilibrium profit vector, which is fixed in this price subgame. Let $\bar{p}_i^*$ be the upper bound of Firm i's strategy in this equilibrium and $\underline{p}_i^*$ be the lower bound of Firm i's strategy in this equilibrium.

We show that $\pi_1^* = 1-z_1^2$. Initially, Lemma 1 finds that Firm 1 charges $p_1 \le 1-z_1^2$ because it loses its own uninformed consumers. Moreover, Lemma 3 shows that Firm 1 sets $\underline{p}_1^* \ge (1-z_1^2)/(1+x)$. Finally, Lemma 4 determines that the equilibrium profit of Firm 1 is $\pi_1^* = 1-z_1^2$. Once we determine the lower bound of the support of Firm 1's strategy and its equilibrium profit, we also determine Firm 2's lower bound $\underline{p}_2^*$. From Lemma 5, we obtain π_2^*. See Appendix A.1, A.2, A.3, A.4 and A.5 for more details on these lemmas. Here, we obtain the following proposition.

Proposition 3 *If* (14) *holds, then the equilibrium profit is unique and the equilibrium profit vector is*

$$(\pi_1^*, \pi_2^*) = (1-z_1^2, 1-z_1^2 + [(1/2-z_1)^2 - (1/2-z_2)^2](1+x)).$$

Proof From Lemmas 4–5 in Appendix A.4 and A.5, we obtain this result. □

Now, we obtain that the equilibrium profit of each firm is determined uniquely. In the following, we show a mixed strategy equilibrium for firms to obtain this equilibrium profit. In these equilibria, each firm's equilibrium profit is the same as that of Proposition 3.

Proposition 4 *If $y-(1/2-z_2)^2 < 1-(1-z_2)^2$ holds, then there exists an equilibrium such that the equilibrium strategy of each firm is*

$$F^{1*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-z_1^2}{1+x}, \\ 1 - \frac{\pi_2^*-(p+T_1)}{(p+T_1)x}, & \text{if } p \in \left[\frac{1-z_1^2}{1+x}, y-(1/2-z_1)^2\right], \\ F^{1*}(y-(1/2-z_1)^2), & \text{if } p \in [y-(1/2-z_1)^2, 1-z_1^2), \\ 1, & \text{if } p \ge 1-z_1^2. \end{cases} \tag{15}$$

$$F^{2*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-z_1^2}{1+x} + T_1, \\ 1 - \frac{\pi_1^*-(p-T_1)}{(p-T_1)x}, & \text{if } p \in \left[\frac{1-z_1^2}{1+x} + T_1, y-(1/2-z_2)^2\right), \\ 1, & \text{if } p \ge y-(1/2-z_2)^2, \end{cases} \tag{16}$$

where $T_1 = (1/2-z_1)^2 - (1/2-z_2)^2 \ge 0$. *This equilibrium profit* (π_1^*, π_2^*) *is* $(1-z_1^2, 1-z_1^2 + [(1/2-z_1)^2 - (1/2-z_2)^2](1+x))$.

Proof See Appendix B.3. □

Proposition 5 *If $1-(1-z_2)^2 \le y-(1/2-z_2)^2$ holds, then there exists an equilibrium such that the equilibrium strategy of each firm is*

$$F^{1*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-z_1^2}{1+x}, \\ 1 - \frac{\pi_2^*-(p+T_1)}{(p+T_1)x}, & \text{if } p \in \left[\frac{1-z_1^2}{1+x}, 1-(1-z_2)^2 - T_1\right], \\ F(1-(1-z_2)^2 - T_1), & \text{if } p \in [1-(1-z_2)^2 - T_1, 1-z_1^2), \\ 1, & \text{if } p \ge 1-z_1^2. \end{cases}$$
$$F^{2*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-z_1^2}{1+x} + T_1, \\ 1 - \frac{\pi_1^*-(p-T_1)}{(p-T_1)x}, & \text{if } p \in \left[\frac{1-z_1^2}{1+x} + T_1, 1-(1-z_2)^2\right), \\ 1, & \text{if } p_2 \ge 1-(1-z_2)^2, \end{cases}$$

where $T_1 = (1/2-z_1)^2 - (1/2-z_2)^2 \ge 0$. *This equilibrium profit* (π_1^*, π_2^*) *is* $(1-z_1^2, 1-z_1^2 + [(1/2-z_1)^2 - (1/2-z_2)^2](1+x))$.

Proof See Appendix B.4. □

In equilibrium, Firm 2 gains a strictly greater profit than Firm 1 because Firm 2 is strictly located closer to C_3 than Firm 1. However, we also find that Firm 2's profit fails to achieve the maximum profit that it gains in its present location if it acts as a monopolist in this location (see Lemma 6 in Appendix A.6 for more details). This shows that the competitive effect of Firm 1's behavior affects the rent that Firm 2 enjoys.

Remark 2 The equilibrium strategies involve symmetric location pairs. For symmetric location pairs, a symmetric equilibrium strategy profile exists where the equilibrium profit is equal to the equilibrium profit in an asymmetric equilibrium. For example, an equilibrium shown by Proposition 4 exists such that

$$F^{1*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-z_1^2}{1+x}, \\ 1 - \frac{\pi_2^* - p}{px}, & \text{if } p \in \left[\frac{1-z_1^2}{1+x}, y - (1/2 - z_1)^2\right], \\ F^{1*}(y - (1/2 - z_1)^2), & \text{if } p \in [y - (1/2 - z_1)^2, 1 - z_1^2), \\ 1, & \text{if } p \ge 1 - z_1^2. \end{cases}$$

$$F^{2*}(p) = \begin{cases} 0, & \text{if } p \le \frac{1-(1-z_2)^2}{1+x}, \\ 1 - \frac{\pi_1^* - p}{px}, & \text{if } p \in \left[\frac{1-(1-z_2)^2}{1+x}, y - (1/2 - z_2)^2\right), \\ F^{2*}(y - (1/2 - z_2)^2), & \text{if } p \in [y - (1/2 - z_2)^2, 1 - (1 - z_2)^2), \\ 1, & \text{if } p \ge 1 - (1 - z_2)^2. \end{cases}$$

4 Location game

Let $\Pi_i(z_1, z_2)$ denote Firm i's profit in a location game:

$$\Pi_1(z_1, z_2) = \begin{cases} 1 - z_1^2, & \text{if } z_1 \in [0, \bar{z}_1], \\ (y - (1/2 - z_1)^2)(1 + x), & \text{if } z_1 \in [\bar{z}_1, 1/2], z_2 \in [\bar{z}_2, 1], \\ 1 - z_1^2, & \text{if } z_1 \in [\bar{z}_1, 1/2], z_2 \in [1/2, \bar{z}_2], z_1 + z_2 \le 1, \\ 1 - (1 - z_2)^2 + T_2(1 + x), & \text{if } z_1 \in [\bar{z}_1, 1/2], z_2 \in [1/2, \bar{z}_2], z_1 + z_2 > 1, \end{cases} \tag{17}$$

$$\Pi_2(z_1, z_2) = \begin{cases} 1 - (1 - z_2)^2, & \text{if } z_2 \in [\bar{z}_2, 1], \\ (y - (1/2 - z_2)^2)(1 + x), & \text{if } z_1 \in [0, \bar{z}_1], z_2 \in [1/2, \bar{z}_2], \\ 1 - (1 - z_2)^2, & \text{if } z_1 \in [\bar{z}_1, 1/2], z_2 \in [1/2, \bar{z}_2], z_1 + z_2 \ge 1, \\ 1 - z_1^2 + T_1(1 + x), & \text{if } z_1 \in [\bar{z}_1, 1/2], z_2 \in [1/2, \bar{z}_2], z_1 + z_2 < 1, \end{cases} \tag{18}$$

where $T_j = (1/2 - z_j)^2 - (1/2 - z_i)^2, (i = 1, 2, j \ne i)$. $\bar{z}_i, i = 1, 2$ are defined in the previous section.

In this section, we consider symmetric location equilibrium strategies. In the following, we consider the case of $y(1 + x) \ge 1$. We show that a symmetric mixed-strategy equilibrium exists in this case. First, we show that the equilibrium pay-offs of all equilibrium location pairs are identical.

Proposition 6 *The equilibrium pay-offs of all equilibrium location pairs are equal to* 1.

Proof Firm 1 gains profit $\Pi_1 = 1$ with a probability of 1 if it is located at $z_1 = 0$. It follows that the equilibrium profit is not strictly less than 1. Suppose that Firm 1 gains a profit strictly greater than 1. If Firm 1 is located at $\tilde{z}_1$, which is the left edge of its location strategy, it follows from the discussion above that its profit is $1 - \tilde{z}_1^2$. However, this profit is less than 1. Because of the continuity in the profit functions at all location points, Firm 1 cannot gain an equilibrium profit greater than 1, which is approximately $\tilde{z}_1$. Thus, Firm 1 cannot gain a profit that is strictly greater than 1 in equilibrium. □

Mixed Strategy Equilibrium 1 We consider equilibrium location pairs when x, y is such that $(y - 1/4)(1 + x) > 1$; in other words, informed consumers at 1/2 are attractive for both firms for all location pairs; therefore, $\bar{z}_i, i = 1, 2$ does not exist in the range [0, 1/2] or [1/2, 1]. We obtain the profit functions as follows:

$$\Pi_1(z_1, z_2) = \begin{cases} 1 - z_1^2, & \text{if } z_1 + z_2 \leq 1, \\ 1 - (1 - z_2)^2 + T_2(1 + x), & \text{if } z_1 + z_2 > 1. \end{cases} \tag{19}$$

$$\Pi_2(z_1, z_2) = \begin{cases} 1 - (1 - z_2)^2, & \text{if } z_1 + z_2 \geq 1, \\ 1 - z_1^2 + T_1(1 + x), & \text{if } z_1 + z_2 < 1. \end{cases} \tag{20}$$

The closer a firm is located to 1/2, the more profit it gains when it is located asymmetrically with respect to the other firm. However, both firms obtain at most their own monopoly profits in symmetric location pairs.

Given Firm 2's strategy $G_2(z_2)$, the expected profit of Firm 1 when located at z_1 is given by the following:

$$\begin{aligned} E[\Pi_1] = &\int_{1/2}^{1-z_1} (1 - z_1^2) g_2(z_2) dz_2 \\ &+ \int_{1-z_1}^{1} \left[1 - (1 - z_2)^2 + \{(1/2 - z_2)^2 - (1/2 - z_1)^2\}(1 + x)\right] g_2(z_2) dz_2, \end{aligned} \tag{21}$$

where $g_2(z_2)$ denotes the density function of a distribution function G_2.

Proposition 7 *Let* x, y *satisfy* $(y - 1/4)(1 + x) > 1$. *Then, a mixed strategy equilibrium exists such that the equilibrium strategy of each firm is*

$$G_1^*(z_1) = \frac{2z_1}{x(1 - 2z_1) + 1}, \qquad 0 \leq z_1 \leq 1/2,$$

$$G_2^*(z_2) = \frac{(2z_2 - 1)(1 + x)}{x(2z_2 - 1) + 1}, \qquad 1/2 \leq z_2 \leq 1,$$

where $G_2^*(z_2)$ *is a distribution function because* $(G_2^*)' > 0$, $G_2^*(1/2) = 0$, *and* $G_2^*(1) = 1$. $G_1^*(z_1)$ *is a distribution function for similar reasons.*

Proof See Appendix B.5. □

Mixed Strategy Equilibrium 2 Next, we consider the case in which x, y lies in the region in which both $y(1+x) \geq 1$ and $(y-1/4)(1+x) \leq 1$ hold. The profit functions in this region are characterized by $\bar{z}$. Thus, (17) and (18) denote the profits for Firms 1 and 2, respectively. Profit functions are more complicated than in the previous case. Heuristically, we construct an equilibrium in this case using G^* obtained from Proposition 7. Now, we consider $\hat{z}_2$, which satisfies

$$(1-(1-\hat{z}_2)^2)G_2^*(\hat{z}_2) + \{1-G_2^*(\hat{z}_2)\}\left\{y-(\hat{z}_2-1/2)^2\right\}(1+x) = 1. \tag{22}$$

Solving (22), we obtain the following:

$$\hat{z}_2 = \frac{1}{2} + 2\left(y - \frac{1}{1+x}\right).$$

Here, we check the properties of $\hat{z}_2$. First, from $y(1+x) \geq 1$, it follows that $y - \frac{1}{1+x} \geq 0$. Then, it follows that $1/2 \leq \hat{z}_2$.

Next, we show that $\hat{z}_2 \leq \bar{z}_2$. From the definition of $\bar{z}_2$, we obtain $(y-(1/2-\bar{z}_2)^2)(1+x) = 1-(1-\bar{z}_2)^2$. The left-hand side of this equation monotonically decreases with respect to $\bar{z}_2$, while the right-hand side monotonically increases. Therefore, in the following, we show that $(y-(1/2-\hat{z}_2)^2)(1+x) \geq 1-(1-\hat{z}_2)^2$. From $(y-(\hat{z}_2-1/2)^2)(1+x) = 1 + \frac{1}{2}(1-\hat{z}_2)(2\hat{z}_2-1)(1+x)$, we determine that $\frac{1}{2}(1-\hat{z}_2)(2\hat{z}_2-1)(1+x) > -(1-\hat{z}_2)^2$. Hence, $\hat{z}_2 \leq \bar{z}_2$.

In the same way, owing to symmetry, we obtain that $\hat{z}_1 = \frac{1}{2} - 2\left(y - \frac{1}{1+x}\right)$.

Using $\hat{z}_2$, we show that a symmetric equilibrium exists such that Firm 2 plays a mixed strategy, which is the same as G_2^* on the support $[1/2, \hat{z}_2]$, while the remainder of the probability, $1-G_2^*$, is attached to being located at $z_2 = 1$.

Given this strategy of Firm 2, $E[\Pi_1]$ denotes the expected profit of Firm 1 for all points z_1 on the interval $[1-\hat{z}_2, 1/2]$. We conclude that

$$\begin{aligned} E[\Pi_1] = & \int_{1/2}^{1-z_1} (1-z_1^2) g_2(z_2) dz_2 \\ & + \int_{1-z_1}^{\hat{z}_2} [1-(1-z_2)^2 + \left\{(1/2-z_2)^2 - (1/2-z_1)^2\right\}(1+x)] g_2(z_2) dz_2 \\ & + \left\{1-G_2(\hat{z}_2)\right\}\left\{y-(1/2-z_1)^2\right\}(1+x), \end{aligned} \tag{23}$$

where $g_2(z_2)$ denotes the density function of the distribution function G_2.

Proposition 8 *Let x, y satisfy both $y(1+x) \geq 1$ and $(y-1/4)(1+x) \leq 1$. Then, there exists an equilibrium such that each firm plays.*

$$G_1^{**}(z_1) = \begin{cases} 0, & \text{if } z_1 < 0, \\ \dfrac{2\hat{z}_1}{x(1-2\hat{z}_1)+1}, & \text{if } 0 \le z_1 \le \hat{z}_1, \\ \dfrac{2z_1}{x(1-2z_1)+1}, & \text{if } \hat{z}_1 \le z_1 \le 1/2, \end{cases}$$

$$G_2^{**}(z_2) = \begin{cases} \dfrac{(2z_2-1)(1+x)}{x(2z_2-1)+1}, & \text{if } 1/2 \le z_2 \le \hat{z}_2, \\ \dfrac{(2\hat{z}_2-1)(1+x)}{x(2\hat{z}_2-1)+1}, & \text{if } \hat{z}_2 \le z_2 < 1, \\ 1, & \text{if } 1 \le z_2. \end{cases}$$

Proof See Appendix B.6. □

Pure Strategy Equilibrium When y is relatively small, C_3 consumers do not want to purchase products because they are not sufficiently attracted to them. For this reason, firms sell products that have distinctive characteristics recognized by uninformed consumers. This situation can occur when C_3 is a large market, but the reservation value y is sufficiently small. Thus, it appears that C_3 does not exist. Finally, we characterize this type of subgame as a perfect equilibrium.

Proposition 9 *When* x, y *satisfies* $1 > y(1+x)$, *we have a subgame perfect equilibrium such that*

$$z_1^* = 0, z_2^* = 1.$$

Proof Firm 1 gains profit $\max\{1 - z_1^2, (y - (1/2 - z_1)^2)(1+x)\}$ when it is located at z_1. Given $z_1^* = 0, z_2^* = 1$, suppose that Firm 1 deviates to $z_1 \in (0, 1/2]$. Because $1 > y(1+x)$, Firm 1 gains a profit strictly less than 1. Therefore, Firm 1 never deviates. Because of the symmetry, we show that this is the same for Firm 2. Thus, $z_1^* = 0, z_2^* = 1$ is a subgame perfect location pair. □

Remark 3 When $y(1+x) \ge 1$ is satisfied, an asymmetric pure strategy equilibrium exists such that $(z_1^*, z_2^*) = \left(0, \frac{1}{2}\right) \text{or} \left(\frac{1}{2}, 1\right)$. This corresponds to the regions mentioned in Propositions 7 and 8. If $(y - 1/4)(1+x) \le 1$ is satisfied, the firms behave in the same manner as they would in a monopoly. That is, a firm located at 0 or 1 maximizes its own monopoly profit, while a firm located at 1/2 maximizes its profit from its own uninformed and informed consumers. Thus, firms do not deviate. If $(y - 1/4)(1+x) > 1$ is satisfied, a firm located at either 0 or 1 cannot gain a strictly higher profit than the monopoly profit, because it is always far away from 1/2, regardless of how it moves. Thus, the firm does not deviate. In contrast, given the strategy of a firm that specializes in its own uninformed consumers, a firm located at 1/2 decreases its profit when it moves from 1/2 to another location. Thus, this firm does not deviate.

5 Conclusions and remarks

In our model, firms compete for both uninformed and informed consumers in a two-stage spatial competition model in which firms choose price and location. We show a subgame perfect equilibrium in which the firms' strategies are symmetric at the middle point of the line.

In the equilibrium, each firm randomly chooses its location. Consequently, various equilibrium location patterns were realized. For example, this mixed strategy equilibrium includes two typical positions. Maximum differentiation occurs with a positive probability that each firm is located towards either end of the interval. Minimal differentiation also occurs with a positive probability that both firms towards the center. In other words, the equilibrium location is dispersed in a subgame perfect equilibrium.

The main result of this study depends on the behavior of uninformed consumers. When many uninformed consumers exist, location dispersion does not occur (Proposition 9). Moreover, this equilibrium dispersion changes continuously according to the change in the size of informed consumers, x. When the size of informed consumers converges to 0, both dispersion types disappear. Conversely, when few uninformed consumers exist, both location and price dispersion in equilibrium always occur. In other words, product differentiation between products in Varian's model depends on the assumption that at least some consumers are uninformed.

Our model can be regarded as explicitly introducing product differentiation choice into Varian's model of sales. In our model, there are two types of consumers. There seems to be an opportunity to earn excess profit from informed consumers as well as uninformed consumers. However, from Proposition 6, the firm's expected profit in a subgame perfect equilibrium is set as the maximum monopoly profit (equal to one). This result is the same as Varian's equilibrium profit. Thus, even when product differentiation is explicitly introduced into a Varian-type model, Varian's implication can be retained; the opportunity for profit in an informed market is lost due to the competition.

A Lemmas

A.1 Lemma 1

Lemma 1 *When (7) holds, in any equilibrium of a price subgame corresponding to a pair of location points, Firm 1 chooses a price such that* $p_1 \leq 1 - z_1^2$, *and Firm 2 chooses a price such that* $p_2 \leq 1 - (1 - z_2)^2$.

Proof Given z_1, z_2, we fix Firm 2's strategy in a price subgame. We must show that the total sum of the profit gained from C_1 and C_3 when it charges price $1 - z_1^2$ is greater than the profit gained from C_3 when it charges price $p_1 > 1 - z_1^2$.

Let ρ denote the probability of C_3 purchasing Firm 1's product when Firm 1 charges p_1. Let ρ^* denote the probability of C_3 purchasing Firm 1's product when Firm 1 charges $1-z_1^2$.

Since $p_1 > 1-z_1^2$, we obtain $\rho \le \rho^*$. Here, we show that $p_1\rho x < (1-z_1^2)(1+\rho^* x)$. When $\rho = 0$, this equation is obvious. Suppose that $\rho > 0$. Then, $p_1 \le y$ must hold. Now, from (7), we obtain

$$\begin{aligned}(1-z_1^2)(1+\rho^* x) - p_1\rho x &\ge \rho(1-z_1^2) + [\rho^*(1-z_1^2) - \rho p_1]x\\ &\ge \rho[(1-z_1^2)(1+x) - p_1 x] \ge \rho[(1-z_1^2)(1+x) - yx] > 0.\end{aligned}$$

We have shown this result for Firm 1. Because of the symmetry, we can show that Firm 2 charges $p_2 \le 1-(1-z_2)^2$ in an equilibrium in a price subgame. □

A.2 Lemma 2

Lemma 2 $\pi_1^* \ge 1-z_1^2$.

Proof Firm 1 obtains profit $1-z_1^2$ whenever it chooses $p_1 = 1-z_1^2$ for any of Firm 2's strategies. If $\pi_1^* < 1-z_1^2$, Firm 1 increases its profit to $1-z_1^2$, which is contradictory to an equilibrium. □

A.3 Lemma 3

Lemma 3 *If* $\pi_1^* > 1-z_1^2$, *then* $\underline{p}_1^* > \frac{1-z_1^2}{1+x}$.

Proof If $\pi_1^* > 1-z_1^2$ and $\underline{p}_1^* \le \frac{1-z_1^2}{1+x}$, Firm 1 is at most $1-z_1^2+(1+x)\varepsilon$ when it chooses $\underline{p}_1^* + \varepsilon, (\varepsilon > 0)$. This profit is strictly less than π_1^* when $\varepsilon \to 0$. Therefore, this is contradictory to the definition of $\underline{p}_1^*$, which is the lower bound of the support of an equilibrium strategy. □

A.4 Lemma 4

Lemma 4

$$\pi_1^* = 1-z_1^2.$$

Proof If $\pi_1^* > 1-z_1^2$ holds, then it follows from (14) and Lemma 3 that a positive number ε exists such that

$$\underline{p}_1^* - \varepsilon > \frac{1-z_1^2}{1+x}, \tag{24}$$

$$\frac{1-(1-z_2)^2}{1+x} + \varepsilon < \min\{1-(1-z_2)^2, y-(1/2-z_2)^2\} \tag{25}$$

holds.

Now, we consider Firm 2's profit when it chooses

$$\tilde{p}_2 = \frac{1-(1-z_2)^2}{1+x} + \varepsilon$$

for Firm 1's strategy. From (24),

$$\begin{aligned} &\underline{p}_1^* + (1/2 - z_1)^2 - [\tilde{p}_2 + (1/2 - z_2)^2] \\ &\quad = \underline{p}_1^* - \varepsilon - \frac{1-(1-z_2)^2}{1+x} + (1/2 - z_1)^2 - (1/2 - z_2)^2 \\ &\quad > \frac{1-z_1^2}{1+x} - \frac{1-(1-z_2)^2}{1+x} + (1/2 - z_1)^2 - (1/2 - z_2)^2 \geq 0 \end{aligned} \tag{26}$$

are satisfied. Thus, we have $\underline{p}_1^* + (1/2 - z_1)^2 > \tilde{p}_2 + (1/2 - z_2)^2$. Additionally, from (25), we find that Firm 2 gains profit $\tilde{p}_2(1+x) = 1-(1-z_2)^2 + \varepsilon(1+x)$ from both C_2 and C_3. Thus, we have $\pi_2^* \geq \tilde{p}_2(1+x) > 1-(1-z_2)^2$. Therefore, we obtain $\pi_2^* > 1-(1-z_2)^2$, if $\pi_1^* > 1-z_1^2$.

Now, from Lemma 1, we have

$$\bar{p}_1^* \leq 1 - z_1^2.$$

Therefore, it suffices to show that $\bar{p}_2^* \geq \bar{p}_1^* + (1/2 - z_1)^2 - (1/2 - z_2)^2$. If $\bar{p}_2^* < \bar{p}_1^* + (1/2 - z_1)^2 - (1/2 - z_2)^2$, Firm 1 obtains at most $1 - z_1^2$ when it chooses $\bar{p}_1^*$ because it never obtains C_3, which contradicts $\pi_1^* > 1 - z_1^2$.

Here, we find that $\pi_2^* > 1-(1-z_2)^2$ is inconsistent with $\bar{p}_2^* > \bar{p}_1^* + (1/2 - z_1)^2 - (1/2 - z_2)^2$. The reason is that Firm 2 obtains at most $1-(1-z_2)^2$ when it chooses $\bar{p}_2^*$. Thus, we obtain $\bar{p}_2^* = \bar{p}_1^* + (1/2 - z_1)^2 - (1/2 - z_2)^2$.

Assume that Firm 1 does not choose $\bar{p}_1^*$ with a positive probability, Firm 2 obtains at most $1-(1-z_2)^2$ when it chooses $\bar{p}_2^*$, which contradicts $\pi_2^* > 1-(1-z_2)^2$. In the same way, if Firm 2 does not choose $\bar{p}_2^*$ with a positive probability, Firm 1 obtains at most $1 - z_1^2$ when it chooses $\bar{p}_1^*$. This is contradictory to $\pi_1^* > 1 - z_1^2$. Thus, both firms choose $\bar{p}_i^*$ with a positive probability.

Let $F^i(p)$ be a probability distribution function when Firm i chooses a price less than p. We define Firm 1's profit as follows when it chooses $\bar{p}_1^*$:

$$\bar{p}_1^* \left[1 + \left\{ 1 - \lim_{\varepsilon \to 0} F^2(\bar{p}_2^* - \varepsilon) \right\} \frac{x}{2} \right]. \tag{27}$$

If Firm 1 chooses $\bar{p}_1^* - \varepsilon$, it obtains at least

$$(\bar{p}_1^* - \varepsilon) \left[1 + \left\{ 1 - \lim_{\varepsilon \to 0} F^2(\bar{p}_2^* - \varepsilon) \right\} x \right]. \tag{28}$$

If $\varepsilon \to 0$, then (27) < (28). This contradicts the optimality that Firm 1 chooses $\bar{p}_1^*$ with a positive probability. Thus, we obtain $\pi_1^* = 1 - z_1^2$. □

A.5 Lemma 5

Lemma 5

$$\pi_2^* = 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x).$$

Proof From Lemma 4, $\pi_1^* = 1 - z_1^2$. We obtain

$$\underline{p}_1^* \geq \frac{1 - z_1^2}{1 + x}.$$

Assuming that $\pi_2^* > 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x)$, we obtain $\underline{p}_2^* > \frac{1-z_1^2}{1+x} + (1/2 - z_1)^2 - (1/2 - z_2)^2$. However, given this strategy of Firm 2, Firm 1 always obtains profit $\pi_1 > 1 - z_1^2 = \pi_1^*$. This is because Firm 1 can always charge a price of $p_1 \in (\frac{1-z_1^2}{1+x}, \underline{p}_2^* + (1/2 - z_2)^2 - (1/2 - z_1)^2)$ to C_3 consumers, which is strictly higher than $\frac{1-z_1^2}{1+x}$ and lower than the lower bound of Firm 2's price $\underline{p}_2^* + (1/2 - z_2)^2$, which includes transportation costs. Thus, Firm 1 increases its profit when it deviates to price p_1 in this open interval, which is contradictory to π_1^* being the maximum profit.

Thus, we determine that if $\pi_1^* = 1 - z_1^2$, then $\pi_2^* \leq 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x)$.

Next, we show that it is not the case that $\pi_2^* < 1 - z_1^2 + [(1/2 - z_1)^2(1 + x)$. Assuming that $\pi_2^* < 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x)$, if Firm 2 chooses

$$\dot{p}_2 = \frac{1 - z_1^2}{1 + x} + (1/2 - z_1)^2 - (1/2 - z_2)^2 - \varepsilon,$$

it obtains C_3 because $\dot{p}_2$ satisfies $\underline{p}_1^* + (1/2 - z_1)^2 > \dot{p}_2 + (1/2 - z_2)^2$ and $y - (\dot{p}_2 + (1/2 - z_2)^2) = y - \frac{1-z_1^2}{1+x} - (1/2 - z_1)^2 + \varepsilon > 0$ is satisfied. Here, from (14), $\varepsilon > 0$ is chosen as being satisfied by $y - \frac{1-z_1^2}{1+x} - (1/2 - z_1)^2 + \varepsilon > 0$.

Now, from $x \geq 1$, we obtain $\frac{1-z_1^2}{1+x} + (1/2 - z_1)^2 \leq 1 - (1 - z_2)^2 + (1/2 - z_2)^2$. Because the left-hand side of the equation is at most 3/4, the minimum of the right-hand side is 3/4. Thus, C_2 purchases Firm 2's product because $1 - (1 - z_2)^2 - \dot{p}_2 \geq \varepsilon > 0$.

Therefore, we obtain Firm 2's profit as follows:

$$\dot{\pi}_2 = \dot{p}_2(1 + x) = 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x) - \varepsilon(1 + x).$$

However, this contradicts the definition that π_2^* is the maximum profit because an ε exists such that $\pi_2^* < \dot{\pi}_2$ when we take a sufficiently small ε.

Thus, we obtain $\pi_2^* = 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x)$ if $\pi_1^* = 1 - z_1^2$. □

A.6 Lemma 6

Lemma 6 *If* $1 - z_1^2 < \{y - (1/2 - z_1)^2\}(1 + x)$ *is satisfied, then*

$$\{y - (1/2 - z_2)^2\}(1 + x) > 1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x).$$

Proof Given that condition $1 - z_1^2 < \{y - (1/2 - z_1)^2\}(1 + x)$ holds, we determine that $1 - z_1^2 + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x) < \{y - (1/2 - z_1)^2\}(1 + x) + [(1/2 - z_1)^2 - (1/2 - z_2)^2](1 + x) = \{y - (1/2 - z_2)^2\}(1 + x)$. □

B Proofs

B.1 Proof of Proposition 1

Proof Firm 1's maximized profit gained from C_1 is higher than the maximum profit gained from both C_1 and C_3. Thus, we obtain $\max_{p_1,p_2} \pi_1(p_1, p_2) = 1 - z_1^2$. Therefore, in an equilibrium, Firm 1 never obtains a strictly higher profit than $1 - z_1^2$.

Next, Firm 1 obtains $1 - z_1^2$ whenever it charges $p_1 = 1 - z_1^2$ for any p_2. Thus, in an equilibrium, Firm 1 never obtains strictly less than $1 - z_1^2$. Similarly, Firm 2 never obtains a profit other than $1 - (1 - z_2)^2$ in equilibrium. Here, the price profile $(1 - z_1^2, 1 - (1 - z_2)^2)$ enables each firm to maximize its profit. Thus, this profile is a unique equilibrium. □

B.2 Proof of Proposition 2

Proof We show Firm 1's profit. From (10), we obtain $\max_{p_1,p_2} \pi_1(p_1, p_2) = 1 - z_1^2 \geq \{y - (1/2 - z_1)^2\}(1 + x)$. Thus, in an equilibrium, Firm 1 never obtains a strictly higher profit than $1 - z_1^2$. Moreover, Firm 1 obtains $1 - z_1^2$ whenever it charges $p_1 = 1 - z_1^2$ for any p_2. Thus, in an equilibrium, Firm 1 never obtains strictly less than $1 - z_1^2$. In an equilibrium, Firm 1 never chooses a price other than $1 - z_1^2$ and $y - (1/2 - z_1)^2$.

Next, we show that Firm 1 never chooses $y - (1/2 - z_1)^2$ with any positive probability in an equilibrium. If Firm 1 plays a mixed strategy profile in which it chooses either $y - (1/2 - z_1)^2$ at a positive probability $\eta > 0$ or $1 - z_1^2$ at a positive probability $1 - \eta > 0$, then Firm 2 does not have the best response to the strategy of Firm 1. The reason is that from (11), if Firm 2 chooses price $y - (1/2 - z_2)^2 - \varepsilon, (\varepsilon > 0)$ because of Remark ??, both C_2 and C_3 and gains profit $\{y - (1/2 - z_2)^2 - \varepsilon\}(1 + x)$. However, if Firm 2 chooses $y - (1/2 - z_2)^2$, it obtains at most $\{y - (1/2 - z_2)^2\}(1 + x/2)$ because the prices of both firms, including the transportation costs, are equal for C_3.

Thus, in an equilibrium, Firm 1 chooses a price of $1 - z_1^2$ at probability 1. Given this strategy of Firm 1, from (11), Firm 2's best response is uniquely determined.

Therefore, a unique equilibrium $(1 - z_1^2, y - (1/2 - z_2)^2)$ exists. Thus, a unique equilibrium profit $(1 - z_1^2, \{y - (1/2 - z_2)^2\}(1 + x))$ is determined. □

B.3 Proof of Proposition 4

Proof We show that for any of Firm 2's mixed strategies, Firm 1 gains a constant expected profit when it adopts a strategy that belongs to the support of its strategy. Given the $F^{2*}(p)$ of Firm 2, Firm 1 charges $p_1 \in [\frac{1-z_1^2}{1+x}, y - (1/2 - z_1)^2)$ and gains the expected profit:

$$p_1 + p_1 x(1 - F^{2*}(p_1 + T_1)) = p_1 + p_1 x\left(\frac{\pi_1^* - p_1}{p_1 x}\right) = \pi_1^*$$

because Firm 1 obtains C_3 consumers if Firm 2 charges p such that $p > p_1 + T_1$ holds. Similarly, Firm 2's expected profit is π_2^* when it charges $p_2 \in [\frac{1-z_1^2}{1+x} + T_1, y - (1/2 - z_2)^2]$. because Firm 2 obtains C_3 consumers if Firm 1 charges p such that $p > p_2 - T_1$ holds.

Now, we show that (15)–(16) are equilibrium strategy profiles. From $\pi_1^* = 1 - z_1^2$, Firm 1 does not deviate to a price strictly lower than $\frac{1-z_1^2}{1+x}$. $\frac{1-z_1^2}{1+x}$ is the lower bound of Firm 1's support of its strategy. Thus, Firm 2 never charges a price strictly lower than $\frac{1-z_1^2}{1+x} + T_1$, which is the lower bound of Firm 2's strategy support.

Next, we show that Firm 1 never deviates to price $\dot{p}_1$ such that $y - (1/2 - z_1)^2 < \dot{p}_1 < 1 - z_1^2$ holds. If Firm 1 charges $\dot{p}_1$, then it obtains only C_1 consumers and cannot obtain C_3 consumers because $\dot{p}_1$ has exceeded the reservation price of C_3 consumers. Thus, Firm 1 obtains a strictly lower profit than $1 - z_1^2$ if it charges $\dot{p}_1$ Therefore, Firm 1 does not deviate.

Next, we show that Firm 1 never deviates to price $\ddot{p}_1 = y - (1/2 - z_1)^2$. If Firm 1 charges $\ddot{p}_1$, it does not obtain C_3 consumers, except when Firm 2 charges $p_2 = y - (1/2 - z_2)^2$. In this case, the C_3 market is divided equally between both firms. Thus, we obtain Firm 1's expected profit as follows:

$$\ddot{p}_1\left(1 + \frac{x}{2}\left(\frac{\pi_1^* - \ddot{p}_1}{\ddot{p}_1 x}\right)\right) = \frac{1}{2}\left(\ddot{p}_1 + \pi_1^*\right).$$

However, from $z_1 + z_2 \leq 1$ and $y - (1/2 - z_2)^2 < 1 - (1 - z_2)^2$, we obtain $\frac{1}{2}\left(\ddot{p}_1 + \pi_1^*\right) - \pi_1^* = \frac{1}{2}\left(\ddot{p}_1 - \pi_1^*\right) < 0$. Thus, Firm 1 never deviates to price $\ddot{p}_1 = y - (1/2 - z_1)^2$.

In the same way, we show that Firm 2 does not deviate to a price other than $p_2 \in [\frac{1-z_1^2}{1+x} + T_1, y - (1/2 - z_2)^2]$. □

B.4 Proof of Proposition 5

Proof In an equilibrium, $\pi_i^*, (i = 1, 2)$ is constant regardless of the selected strategy. This is the same as we have shown for Proposition 4. We show that the strategy profiles are in equilibrium. From $\pi_1^* = 1 - z_1^2$, Firm 1 does not deviate to a price strictly lower than $\frac{1-z_1^2}{1+x}$. Because Firm 1's lower bound of support for its equilibrium strategy is $\frac{1-z_1^2}{1+x}$, it does not follow that Firm 2 charges a price strictly lower than $\frac{1-z_1^2}{1+x} + T_1$, which is the lower bound of its strategy support.

Next, we show that Firm 1 does not deviate, choosing price $\dot{p}_1$ such that $1 - (1 - z_1)^2 - T_1 < \dot{p}_1 < 1 - z_1^2$. Suppose that $F^{2*}(p)$ is given, it follows with a probability of 1 that Firm 2's price charged to C_3, including transport costs, is strictly lower than Firm 1's price. Thus, only C_1 purchases Firm 1's product. By contrast, in this case, Firm 1's profit is strictly lower than π_1^*. Thus, Firm 1 does not deviate. □

B.5 Proof of Proposition 7

Proof We show that for any given $G_2^*(z_2)$, Firm 1 gains an expected profit equal to 1 when it is located at $z_1 \in [0, 1/2]$. Substituting G_2^* into (21), owing to the partial integration of the second term of this equation, we obtain

$$
\begin{aligned}
E[\Pi_1^*] &= (1 - z_1^2)[G_2^*(z_2)]_{1/2}^{1-z_1} \\
&\quad + \left[[1 - (1 - z_2)^2 + \{(1/2 - z_2)^2 - (1/2 - z_1)^2\}(1 + x)]G_2^*(z_2)\right]_{1-z_1}^{1} \\
&\quad - \int_{1-z_1}^{1} [1 - (1 - z_2)^2 + \{(1/2 - z_2)^2 - (1/2 - z_1)^2\}(1 + x)]' G_2^*(z_2) dz_2 \\
&= 1 + (1/4 - (1/2 - z_1)^2)(1 + x) - (1 + x) \int_{1-z_1}^{1} (2z_2 - 1) dz_2 \\
&= 1 + (1/4 - (1/2 - z_1)^2)(1 + x) - (1 + x) \left[\left(z_2 - \frac{1}{2}\right)^2\right]_{1-z_1}^{1} = 1.
\end{aligned}
$$

Because of the symmetry, we obtain the same result for Firm 2 in the same way. □

B.6 Proof of Proposition 8

Proof It is sufficient to show Firm 1's case because of symmetry. Given $G_2^{**}(z_2)$, Firm 1 can gain an expected profit equal to 1 on either $z_1 = 0$ or all the $z_1 \in [1 - \hat{z}_2, 1/2]$ intervals.

Firm 1 can gain a profit equal to 1 at $z_1 = 0$. Thus, it is sufficient to show that Firm 1 gains an expected profit equal to 1 when it is located in any $z_1 \in [1 - \hat{z}_2, 1/2]$ interval. Substituting G_2^{**} with (23), using partial integration, we have

$$\begin{aligned}
E[\Pi_1^*] = &(1 - z_1^2)[G_2^{**}(z_2)]_{1/2}^{1-z_1} \\
&+ \left[[1 - (1 - z_2)^2 + \{(1/2 - z_2)^2 - (1/2 - z_1)^2\}(1 + x)]G_2^{**}(z_2)\right]_{1-z_1}^{\hat{z}_2} \\
&- \int_{1-z_1}^{\hat{z}_2} [1 - (1 - z_2)^2 + \{(1/2 - z_2)^2 - (1/2 - z_1)^2\}(1 + x)]' G_2^{**}(z_2) dz_2 \\
&+ \left\{1 - G_2^{**}(\hat{z}_2)\right\}\left\{y - (1/2 - z_1)^2\right\}(1 + x) \\
= &[1 - (1 - \hat{z}_2)^2 + \{(1/2 - \hat{z}_2)^2 - (1/2 - z_1)^2\}(1 + x)]G_2^{**}(\hat{z}_2) \\
&- \int_{1-z_1}^{\hat{z}_2} \{(2z_2 - 1)x + 1\}G_2^{**}(z_2) dz_2 \\
&+ \left\{1 - G_2^{**}(\hat{z}_2)\right\}\left\{y - (1/2 - z_1)^2\right\}(1 + x) \\
= &[1 - (1 - \hat{z}_2)^2]G_2^{**}(\hat{z}_2) \\
&+ \left\{1 - G_2^{**}(\hat{z}_2)\right\}\left\{y - (1/2 - z_1)^2\right\}(1 + x) \\
&+ [(1/2 - \hat{z}_2)^2 - (1/2 - z_1)^2](1 + x)G_2^{**}(\hat{z}_2) \\
&- (1 + x)[(z_2 - 1/2)^2]_{1-z_1}^{\hat{z}_2} \\
= &[1 - (1 - \hat{z}_2)^2]G_2^{**}(\hat{z}_2) \\
&+ \left\{1 - G_2^{**}(\hat{z}_2)\right\}\left\{y - (1/2 - z_1)^2\right\}(1 + x) \\
&- \left\{1 - G_2^{**}(\hat{z}_2)\right\}[(1/2 - \hat{z}_2)^2 - (1/2 - z_1)^2](1 + x) \\
= &[1 - (1 - \hat{z}_2)^2]G_2^{**}(\hat{z}_2) + \{1 - G_2^{**}(\hat{z}_2)\}\left\{y - (\hat{z}_2 - 1/2)^2\right\}(1 + x) = 1.
\end{aligned}$$

Thus, the expected profit is constant in support of an equilibrium.

Finally, we show that for any given strategy of Firm 2, Firm 1 does not choose $z_1 \in (0, \hat{z}_1)$. Given $G_2^{**}(\hat{z}_2)$, Firm 1 does not deviate from $z_1 \in (0, \bar{z}_1]$. This is because Firm 1 gains $1 - \dot{z}_1^2 < 1$ when it is located on $\dot{z}_1 \in (0, \bar{z}_1]$.

Given $G_2^{**}(\hat{z}_2)$, Firm 1 does not choose $z_1 \in (\bar{z}_1, \hat{z}_1)$ when it uses a pure strategy. When $1 - \hat{z}_2 = \hat{z}_1$ holds, we have

$$(1 - \hat{z}_1^2)G_2^{**}(\hat{z}_2) + \{1 - G_2^{**}(\hat{z}_2)\}\left\{y - (\hat{z}_1 - 1/2)^2\right\}(1 + x) = 1.$$

Let $f(z_1)$ denote Firm 1's profit when it chooses $z_1 \in (\bar{z}_1, \hat{z}_1)$. We have

$$f(z_1) = (1 - z_1^2)G_2^{**}(\hat{z}_2) + \{1 - G_2^{**}(\hat{z}_2)\}\left\{y - (z_1 - 1/2)^2\right\}(1 + x). \tag{29}$$

When the derivatives of (29) are taken with respect to z_1, we have

$$f'(z_1) = -2z_1 G_2^{**}(\hat{z}_2) + \{1 - G_2^{**}(\hat{z}_2)\}(1 - 2z_1)(1 + x).$$

We determine that $f'' = -2 - 2(1 - G)x < 0$, and we have

$$f'(\hat{z}_1) = -2\hat{z}_1 G_2^{**}(\hat{z}_2) + \{1 - G_2^{**}(\hat{z}_2)\}(1 - 2\hat{z}_1)(1 + x) = 0.$$

Thus, we determine that both $f' \geq 0$ and $f'' < 0$ hold for all $z_1 \in [\bar{z}_1, \hat{z}_1]$. (29) is monotonically increasing. Therefore, Firm 1 deviates from the left-hand side of $\hat{z}_1$. □

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