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REVIEW ARTICLE

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Closed set of the uniqueness conditions and bifurcation criteria in generalized coupled thermoplasticity for small deformations

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Abstract This paper reports the results of a study into global and local conditions of uniqueness and the criteria excluding the possibility of bifurcation of the equilibrium state for small strains. The conditions and criteria are derived on the basis of an analysis of the problem of uniqueness of a solution involving the basic incremental boundary problem of coupled generalized thermo-elasto-plasticity. This work forms a follow-up of previous research (Śloderbach in Bifurcations criteria for equilibrium states in generalized thermoplasticity, IFTR Reports, 1980, Arch Mech 3(35):337–349, 351–367, 1983), but contains a new derivation of global and local criteria excluding a possibility of bifurcation of an equilibrium state regarding a comparison body dependent on the admissible fields of stress rate. The thermal elasto-plastic coupling effects, non-associated laws of plastic flow and influence of plastic strains on thermoplastic properties of a body were taken into account in this work. Thus, the mathematical problem considered here is not a self-conjugated problem.

Keywords Bifurcation of the equilibrium state · Conditions and criteria of uniqueness · Boundary-value problem · Generalized coupled thermo-elasto-plasticity · Comparison bodies

1 Introduction

The incremental boundary-value problem of generalized coupled thermoplasticity is formulated in this paper. This is followed by an interpretation of the uniqueness conditions for the solution of that problem. The necessary and sufficient local uniqueness conditions are deduced together with the global sufficient uniqueness condition. A similar incremental boundary-value problem of coupled generalized thermoplasticity was investigated and discussed [1,3]. In this paper, necessary and sufficient local and global conditions of uniqueness of solution of an incremental boundary-value problem of coupled generalized thermoplasticity for the case of small displacements gradients (small strains) are derived. Uniqueness conditions for the generalized coupled thermoplasticity [1–3] and suitable comparison bodies [1–7] are identified for this purpose. The derived local and global uniqueness conditions are suitable necessary and sufficient conditions excluding occurrence of the bifurcation of equilibrium state in coupled generalized thermoplasticity and for suitable comparison bodies (also in isothermal loading processes).

Early papers by Mróz [8,9] defined local conditions of uniqueness for solving an incremental boundary problem for the case of non-associated laws of plastic flow, for isothermal processes and small strains. A similar local condition was obtained by Hueckel and Maier in [10,11]. In their analysis, the stability of the material is defined by means of a condition which states that the half of the product of the stress rate tensor needs to be a positive value. The reported study was confined to the case of the isothermal theory of plasticity (with

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no thermomechanical couplings), the elastic–plastic coupling effects and non-associated laws of plastic flow being preserved. The minimum principle (a principle of stability) for incremental, isothermal elasto-plasticity with non-associated flow laws for small deformation was derived in a paper by Maier [12]. In [4,5] Mróz and Raniecki derived local necessary and sufficient conditions of uniqueness in coupled thermoplasticity for associated laws of plastic flow, without the elastic–plastic coupling effects and for small strains. The obtained local conditions were not optimal (i.e. they were not minimal). The procedure of optimization was presented in [1,3,13,14]. In Ref. [15], Raniecki and Sawczuk derived the equations of field and constitutive equations for a body of coupled thermoplasticity with associated laws of plastic flow. In Ref. [16] they applied the local conditions of uniqueness for considerations of the influence of selected thermomechanical couplings on unstable behaviour of materials and selected machine components subjected to variable thermomechanical fields. The studies into global conditions of uniqueness, stability and criteria of bifurcation for elasto-plastic bodies for the case of large strains and the associated laws of plastic flow and isothermal processes were presented by Hill, see for example [17–20]. Uniqueness conditions for the case of non-associated laws of plastic flow and large strains were derived by Raniecki and Raniecki and Bruhns [13,14]. In these papers [14,15], local and global conditions were derived with regard to a comparison body depending on kinematically admissible strain rate fields. In addition, local and global conditions of uniqueness for a comparison body dependent on statically admissible stress rate fields for a case of non-isothermal processes, large strains and non-associated laws of plastic flow were deduced in [21]. An attempt at the derivation of the local condition of uniqueness for a certain non-compressible elasto-plastic comparison body and for case of large strains was made in [22], but the study did not offer satisfactory results.

The current paper contains a summary and synthesis of this author's papers and also includes the results obtained by the earlier authors. This paper presents a closed set of uniqueness conditions and bifurcation criteria in generalized coupled thermoplasticity [1–3] for the case of small deformations. The paper analyses and includes a study regarding the influence of selected thermomechanical couplings on the uniqueness of the solution of boundary problems. A discussion is included regarding non-associated laws of plastic flow, and elastic and plastic behaviour (elastic–plastic coupling effects). Such a requirement poses a greater problem than the ones stated previously. The problem is not a mathematical self-adjoint. In comparison with the results given in [1,3], local and global conditions of uniqueness are identified with regard to a comparison body in relation to the kinematically admissible stress rate fields. Thus, a set of conditions of uniqueness and bifurcation criteria could be presented in a closed form. In contrast to the earlier papers focusing on uniqueness conditions and bifurcations of equilibrium states [4–7], this paper deals with non-associated laws of plastic flow accounting for elastic–plastic coupling effects and relations are established for a comparison body dependent on kinematically admissible stress rate fields. In addition, influence of the chosen thermomechanical couplings and influence of non-associated laws of plastic flow on some non-typical cases of behaviour of bodies under influence of thermomechanical loadings provided by the theory are specified and described.

This paper demonstrates that the local uniqueness conditions for the generalized thermo-elasto-plastic body and for the comparison bodies are the same. The methods of calculating the bifurcation state (using the global sufficient uniqueness condition) for the case of comparison bodies are less complicated than for the generalized thermo-elasto-plastic body, as there is a linear dependence between stress rate and strain rate, see [1–7] and the discussion in Sects. 4.2.2 and 4.2.4 of this paper. Thus, the use of such comparison bodies seems to be advisable.

In a generalized case, constitutive equations of coupled thermoplasticity take the form of non-associated laws of plastic flow, even if Gyarmati's [23] postulate is assumed (see [1–3]). They also include the effects of thermomechanical coupling and take into account the elastic–plastic conjugation. It means that they can be applied for the description with regard to not only metallic bodies, but also porous materials, sintered powders, rocks and soils. The paper also contains a description of special cases of the local conditions of uniqueness for more specific body models. In such less general models, constitutive functions occurring under conditions of uniqueness can take simpler form.

The obtained conditions of uniqueness seem to be important from both mathematical and practical point of view. They can offer a device useful in the estimation of critical loads. If the critical loads are exceeded, bifurcation of the equilibrium state is possible [1–7].

The incremental boundary-value problem of coupled generalized thermo-elasto-plasticity is formulated in this paper. In order to analyse the uniqueness of the boundary problem solution, it is assumed that a thermodynamic state of the body at a given instant of plastic deformation is known. It is necessary to determine rate fields of strain or stress as well as temperatures appropriately for set values of stress or strain rate and the divergence of the vector of heat flux exchanged through the surface in the elementary area (see the problems

b_1 and b_2 considered in this paper and also in [1,3]). Similar incremental boundary problems of coupled thermo-elasto-plasticity for small and large deformations were tested and discussed in the literature, see e.g. [1–9,13,14,21].

The issues of non-isothermal thermo-elasto-plasticity and isothermal elasto-plasticity with associated and non-associated laws of plastic flow and with regard to large deformations were thoroughly described in the recent 20 years, see e.g. [13,14,21,22,24–44]. The constitutive equations and plasticity conditions for isotropic and anisotropic materials were analysed in these papers, and the methods of numerical calculations were applied. In other works, e.g. [26–44], the problems regarding large thermal and elasto-plastic deformations were considered, appropriate measures to be applied for deformations, additivity of elastic and plastic deformations were established, and problems regarding actual and relative configurations were described. The topics of linear and nonlinear kinematic and isotropic hardening were analysed in papers [30,35,39]. In addition, problems regarding the stability and behaviour under thermomechanical and mechanical loading were analysed, see e.g. [24,25,27,31,33,40,44]. The majority of papers describing large deformations in thermo-elasto-plasticity apply the method of internal parameters in the frame of thermodynamics of non-reversible processes, for example see [13,14,21,22,26,28,31,33,36,43,44].

1.1 Symbols and abbreviations

c_ε and c_σ	Specific heat capacity measured at constant elastic strain and stresses in [1/(kg K)],
$\mathrm{div}\mathbf{q} = \frac{\partial q_i}{\partial x_i}$, x_i	Orthogonal Cartesian coordinates which express the initial location of the particle.
D	Dissipation of mechanical energy per unit time and volume [1,2,45,47],
E	Young's modulus,
F	Law function of plastic flow determined in the variables state space $\{T, \boldsymbol{\sigma}, K\}$ [1,2,46,47],
F_1	Generalized law function of plastic flow determined in the thermodynamic force space $\{T, \boldsymbol{\sigma}, -\Pi, K\}$ [1,2,46,47],
h	Hardening function,
M and **L**	Tensor of isothermal elastic moduli and tensor of elastic compliance, respectively, $\mathbf{M} = \left(\mathbf{L}^{-1}\right)_{\boldsymbol{\sigma}=\boldsymbol{\sigma}(Y_K^{T\varepsilon})}$, and $2(M_{ijmn}L_{mnrs})_{\boldsymbol{\sigma}=\boldsymbol{\sigma}(Y_K^{T\varepsilon})} = \delta_{is}\delta_{jr} + \delta_{ir}\delta_{js}$ and $M_{ijmn} = M_{mnij} = M_{jimn} = M_{ijnm}$,
N	Vector of pairs of tensors of the fourth and second order representing isothermal variation of the state of stress due to internal processes accompanying plastic deformation too in the state $Y_K^{T\varepsilon}$
q	Vector of the density of heat flow rate, [J/(m^{2}s)],
T	Thermodynamic temperature in [K],
T_0	Reference temperature corresponding to the TRS—it may be, for example, the ambient temperature,
∇T	Gradient T (grad$T$),
(TRS)	Abbreviation for "thermodynamic reference state", where $T = \mathrm{T}_0$, $K = 0$ (see Greek symbols) and $\boldsymbol{\varepsilon}^{\mathbf{e}} = \mathbf{0}$,
$\dot{\mathbf{x}}^{\mathrm{D}}$	Set of dissipative (mechanical) thermodynamic flows, $\dot{\mathbf{x}}^{\mathrm{D}} = \{\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}, \mathbf{q}, \dot{K}\}$,
$\mathbf{X}^{\mathrm{D}}$	Set of dissipative (mechanical) thermodynamic forces, $\mathbf{X}^{\mathrm{D}} = \{\boldsymbol{\sigma}, -\Pi, \frac{1}{T}\nabla T\}$,
Y	Yield stress in uniaxial tension,
Y_0	Initial yield stress, for $\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathbf{0}$,
Y_1	Yield stress in uniaxial tension as dependent on (π, κ, T) ;
$Y_K^{T\varepsilon} = \{T, \boldsymbol{\varepsilon}^e, K\}$ and $Y_K^{T\sigma} = \{T, \boldsymbol{\sigma}, K\}$	Variables of thermodynamic state,
$Y_\Pi^{T\varepsilon} = \{T, \boldsymbol{\varepsilon}^{\mathrm{e}}, \Pi\}$ and $Y_\Pi^{T\sigma} = \{T, \boldsymbol{\sigma}, \Pi\}$	Variables of thermodynamic state,
Z	Vector of tensor pairs composed of the fourth and second order representing the isothermal variation of elastic deformation due to the internal processes accompanying plastic deformation in state $Y_K^{T\sigma}$, then $\mathbf{Z} \Leftrightarrow \{Z_{mnkl}; Z_{mn}\}$,

Greek symbols

$\boldsymbol{\alpha}$	Symmetric tensor of thermal expansion coefficients, such that $\alpha\delta_{ij} = \text{const}$,
δ^s	Amount of entropy generated within a unit volume over a unit time and referred to a given material particle,
$\boldsymbol{\varepsilon}$	Tensor of total deformations, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathbf{e}} + \boldsymbol{\varepsilon}^{\mathbf{p}}$,
$\boldsymbol{\varepsilon}^{\mathrm{e}}$ and $\boldsymbol{\varepsilon}^{\mathrm{p}}$	Tensor of small elastic and plastic deformations,
$\mathbf{e}^{\mathrm{p}}$	Deviatoric part of plastic deformation tensors $\boldsymbol{\varepsilon}^{\mathrm{p}}$, $\mathbf{e}^{\mathrm{p}} \equiv \text{dev}\boldsymbol{\varepsilon}^{\mathrm{p}}$,
K	Pair of internal parameters $K \Leftrightarrow \{\boldsymbol{\kappa}^{(M)}; \kappa^{(N)}\}_{M=1\,m, N=1\,n}$,
$\boldsymbol{\kappa}$ and κ	Symmetric second-rank tensor and scalar internal parameter, respectively,
Λ	Plasticity multiplier,
μ and λ	Lamé elastic constants,
ν	Poisson ratio,
Π	Pair of internal thermodynamic forces associated with a pair of internal parameters K, $\Pi \Leftrightarrow \{\boldsymbol{\pi}^{(M)}, \pi^{(N)}\}_{M=1\,m, N=1\,n}$,
$\mathbf{q}$	Heat flux exchange with the neighbourhood per unit time across an unit area in $[\mathrm{J}/(\mathrm{m}^2\mathrm{s})]$,
ρ_0 and ρ	Body density in a thermodynamic reference state and in an actual one, respectively,
$\boldsymbol{\sigma}$	Cauchy stress tensor,
$\sigma_{(\mathrm{i})}$	Effective deviator of stress,
σ_0	Yield stress value obtained in the uniaxial tension test for $\boldsymbol{\varepsilon}^{\mathrm{p}} = \mathbf{0}$,

Tensors will be printed in a bold typeface. The summation convention is assumed along with the following detailed notation

$$\begin{aligned}
\mathbf{AB} &\Leftrightarrow A_{ij}B_j \quad \text{or} \quad A_{ijkl}B_{kl} \qquad (i, j, k, l, m, n, \ldots = 1, 2, 3),\\
\mathrm{tr}\mathbf{A} &\Leftrightarrow A_{kk}, \qquad \mathrm{tr}(\mathbf{AB}) \Leftrightarrow A_{ij}B_{ji},\\
\mathbf{A} : \mathbf{B} &\Leftrightarrow A_iB_i \quad \text{or} \quad A_{ij}B_{ij},\\
\mathbf{A} \otimes \mathbf{B} &\Leftrightarrow A_iB_j \quad \text{or} \quad A_{ij}B_{kl},
\end{aligned}$$

1—identity tensor, δ_{ij}—Kronecker delta, **0**—null tensor,

$$\mathrm{sym}\mathbf{A} \Rightarrow \frac{1}{2}(A_{ij} + A_{ji}), \; \mathrm{dev}\mathbf{A} = \mathbf{A} - \frac{1}{3}(\mathrm{tr}\mathbf{A})\mathbf{1} - \text{deviatoric part of tensor } \mathbf{A},$$

$$A_{i,j} = \frac{\partial A_i}{\partial x_j}, \quad \text{where} \quad x_j - \text{coordinates of a material particle},$$

$$\dot{A} = \frac{\partial A}{\partial t}, \quad \text{where} \quad (t - \text{time}), \; \frac{\partial \mathbf{A}}{\partial \mathbf{B}}\mathrm{d}\mathbf{B} \Rightarrow \frac{\partial A_{ij}}{\partial B_{kl}}dB_{kl}.$$

If $\mathbf{Z}$ denotes pairs of tensors of the fourth and the second order, then $\mathbf{Z} \Leftrightarrow \{Z_{mnkl}; Z_{mn}\}$, and if $\mathbf{M}$ is the tensor of the fourth order, then the operation $\mathbf{MZ}$ is a pair of tensors of the fourth and the second order defined as follows

$$\mathbf{MZ} \Leftrightarrow \{M_{ijmn}Z_{mnkl}; M_{ijmn}Z_{mn}\}.$$

If Π and K denote pairs of tensors of the second and the zeroth order, then the operation $\Pi \cdot K$ produces a scalar, cf. [1,2,21,45,47]

$$\Pi \cdot K = \boldsymbol{\pi} : \boldsymbol{\kappa} + \pi\kappa = \pi_{ij}\kappa_{ij} + \pi\kappa.$$

If the function F is relative to Π and K, then $F(\Pi) = F(\boldsymbol{\pi}, \pi)$ and $F(K) = F(\boldsymbol{\kappa}, \kappa)$.
The derivatives of function F with respect to a pair Π and K are defined as follows

$$\frac{\partial F}{\partial \Pi} = \left\{\frac{\partial F}{\partial \boldsymbol{\pi}}, \frac{\partial F}{\partial \pi}\right\} \quad \text{and} \quad \frac{\partial F}{\partial K} = \left\{\frac{\partial F}{\partial \boldsymbol{\kappa}}, \frac{\partial F}{\partial \kappa}\right\}.$$

The differentials of function F with respect to the pairs K and Π of tensors of the second and the zeroth order produce the following form

$$\frac{\partial F}{\partial \Pi} \cdot d\Pi = \left\{ \frac{\partial F}{\partial \pi_{ij}} d\pi_{ij} \ , \ \frac{\partial F}{\partial \pi} d\pi \right\} \quad \text{and} \quad \frac{\partial F}{\partial K} \cdot dK = \left\{ \frac{\partial F}{\partial \kappa_{ij}} \, d\kappa_{ij}, \ \frac{\partial F}{\partial \kappa} d\kappa \right\}.$$

The differential of function F with respect to the pair K of tensors of the second and the zeroth order, produces a sum, cf. [1,2,45,47], such that

$$\frac{\partial \Pi}{\partial K} dK \Leftrightarrow \left\{ \frac{\partial \pi_{ij}}{\partial \kappa_{kl}} d\kappa_{kl} + \frac{\partial \pi_{ij}}{\partial \kappa} d\kappa \ ; \ \frac{\partial \pi}{\partial \kappa_{kl}} d\kappa_{kl} + \frac{\partial \pi}{\partial \kappa} d\kappa \right\}.$$

If $\boldsymbol{\alpha}$ is a second order tensor, then the operation $\boldsymbol{\alpha} \circ (\mathbf{MZ})$ produces a pair of tensors of the second and the zeroth order

$$\boldsymbol{\alpha} \circ (\mathbf{MZ}) \Leftrightarrow \left\{ \alpha_{ij} M_{ijmn} Z_{mnkl} \ ; \ \alpha_{ij} M_{ijmn} Z_{mn} \right\}.$$

If $\mathbf{Z}$ denotes pairs of tensors of the fourth and the second order and K is a pair of tensors of the second and the zeroth order, then the operation $\mathbf{Z} * K$ produces a pair of tensors of the second order such that [1,2,45,47]

$$\mathbf{Z} * K \Leftrightarrow \left\{ Z_{ijmn} \kappa_{mn} \ ; \ Z_{ij} \kappa \right\}.$$

2 Uniqueness solution of incremental problems for homogenous processes

2.1 Fundamental assumptions and equations

A homogeneous physical body of unit mass is being considered. When the thermodynamic state of each particle of the body is the same at any moment of the process, the process is called homogeneous. In the case of such processes, the quantity div$\mathbf{q}$ occurring in the equations for temperature should be understood as the rate of global heat exchange between the physical body and the environment, and ρ_0 as a reverse of the total volume of the body.

Let us assume that the local thermodynamic state is described by the following parameters of state [1,2,7, 15,45]: $\boldsymbol{\varepsilon}^{\mathrm{e}}$—tensor of elastic strain, s—specific entropy (per unit mass),
$\boldsymbol{\kappa}^{(M)}$—the set of symmetric internal tensor parameters of second order ($M = 1, \ldots, n$),
$\boldsymbol{\kappa} = \boldsymbol{\kappa}^{\mathrm{T}}$, that is $\kappa_{ij} = \kappa_{ji}$, $\kappa^{(N)}$—the set of internal scalar parameters ($N = 1, \ldots, m$).

Now the symbol K will denote the set of internal parameters in the form of a vector of pair $K \Leftrightarrow \{\boldsymbol{\kappa}^{(M)}, \kappa^{(N)}$, $(M = 1, \ldots, n)$ and $(N = 1, \ldots, m)$, see a Greek symbols.

Differential of the internal energy depending on internal parameters of the state $\{s, \boldsymbol{\varepsilon}^{\mathrm{e}}, K\}$ has a form [1,2,7,15,45,47]

$$\mathrm{d}U(s, \boldsymbol{\varepsilon}^{e}, \ K) = T\mathrm{d}s + \frac{1}{\rho_0} \boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^{e} + \frac{1}{\rho_0} \Pi \cdot \mathrm{d}K. \tag{2.1}$$

The local approach to the principle of conservation of energy is as follows [1,2,7,15]

$$\dot{U} = \frac{1}{\rho_0} \boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \frac{1}{\rho_0} \mathrm{div}\mathbf{q}. \tag{2.2}$$

The equation of local entropy balance per unit volume of the body has the form [1,2,7,15,45]

$$\rho_0 \dot{s} = -\mathrm{div}\left(\frac{\mathbf{q}}{T}\right) + \delta^s. \tag{2.3}$$

The local formulation of the second law of thermodynamics is given by the inequality

$$\delta^s \geq 0. \tag{2.4}$$

The entropy production can be evaluated by solving a set of three Eqs. (2.1)–(2.3) for $\left(\dot{U}, \ \dot{s} \ \text{ and } \delta^s\right)$

$$T\delta^s = D - \frac{1}{T} \mathbf{q} : \nabla T, \quad \nabla T = \mathrm{grad}\, T, \tag{2.5}$$

where D expresses dissipation of mechanical energy per unit time and volume

$$D = \boldsymbol{\sigma}\colon \dot{\boldsymbol{\varepsilon}^p} - \Pi \cdot \dot{K} \geq 0 \tag{2.6}$$

and

$$\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}^e} + \dot{\boldsymbol{\varepsilon}^p}. \tag{2.7}$$

The set of forces X^D involved in (2.5), (2.6) is named as a set of dissipation forces or a set of thermodynamic impulses and a set of thermodynamic flow rates $\dot{x}^D$ joined with the set of forces X^D are the following

$$X^D = \left\{\sigma,\ -\Pi,\ \frac{1}{T}\mathrm{grad}T\right\} \quad \text{and} \quad \dot{x}^D = \left\{\dot{\boldsymbol{\varepsilon}}^p,\ \dot{K},\ \mathbf{q}\right\}. \tag{2.8}$$

2.1.1 Thermostatic identities and properties of an thermo-elasto-plastic body

The fundamental physical quantities describing the thermostatic properties of solids are defined as follows [1,2,45,47]

$$c_\varepsilon\left(Y_K^{T\varepsilon}\right) = T\frac{\partial s\left(Y_K^{T\varepsilon}\right)}{\partial T} = \frac{\partial U\left(Y_K^{T\varepsilon}\right)}{\partial T}, \quad c_\sigma = T\frac{\partial s\left(Y_K^{T\sigma}\right)}{\partial T}, \tag{2.9}$$

$$\mathbf{L}\left(Y_K^{T\sigma}\right) = \frac{\partial \boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial \boldsymbol{\sigma}}, \quad \mathbf{M}\left(Y_K^{T\varepsilon}\right) = \frac{\partial \boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}{\partial \boldsymbol{\varepsilon}^e}, \tag{2.10}$$

$$\mathbf{Z}\left(Y_K^{T\sigma}\right) = \frac{\partial \boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial K}, \quad \mathbf{R}\left(Y_\Pi^{T\sigma}\right) = \frac{\partial \boldsymbol{\varepsilon}^e\left(Y_\Pi^{T\sigma}\right)}{\partial \Pi}, \quad \mathbf{N}\left(Y_K^{T\varepsilon}\right) = \frac{\partial \boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}{\partial K}, \quad \mathbf{S}\left(Y_\Pi^{T\varepsilon}\right) = \frac{\partial \boldsymbol{\sigma}\left(Y_\Pi^{T\varepsilon}\right)}{\partial \Pi}, \tag{2.11}$$

where

$$\mathbf{N} = \left(\mathbf{R}^{-1}\right)_{\Pi=\Pi\left(Y_K^{T\varepsilon}\right)} \text{ oraz } \mathbf{Z} = \left(\mathbf{S}^{-1}\right)_{\Pi=\Pi\left(Y_K^{T\sigma}\right)}, \tag{2.12}$$

where $\mathbf{Z}$ is the vector of pairs of tensors of the fourth and second order representing the isothermal variation of elastic deformation due to the internal processes accompanying plastic deformation in state $Y_K^{T\sigma}$. Physically, it means a change of the Young's modulus caused by plastic deformations. $\mathbf{N}$ is the vector of pairs of tensors of the fourth and second order representing isothermal variation of the state of stress due to internal processes accompanying plastic deformation too in the state $Y_K^{T\varepsilon}$.

The quantities (2.9)–(2.12) are not independent. They satisfy the set of the following identities resulting from the existence of thermodynamic potentials, see e.g. [1,2,7,15,45,47])

$$\mathbf{M} = \left(\mathbf{L}^{-1}\right)_{\boldsymbol{\sigma}=\boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}, \tag{2.13}$$

$$2\left(M_{ijmn}L_{mnrs}\right)_{\boldsymbol{\sigma}=\boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)} = \delta_{is}\delta_{jr} + \delta_{ir}\delta_{js} \text{ oraz } M_{ijmn} = M_{mnij} = M_{jimn} = M_{ijnm}, \tag{2.14}$$

$$c_\sigma = \left(c_\varepsilon + \frac{T}{\rho_0}\boldsymbol{\alpha} : \mathbf{M}\boldsymbol{\alpha}\right)_{\boldsymbol{\varepsilon}=\boldsymbol{\varepsilon}\left(Y_K^{T\sigma}\right)} \quad \text{and}$$

$$\boldsymbol{\alpha}\left(Y_K^{T\sigma}\right) = \frac{\partial \boldsymbol{\varepsilon}^e}{\partial T}, \frac{1}{\rho_0}\frac{\partial \Pi\left(Y_K^{s\varepsilon}\right)}{\partial s} = \frac{\partial T\left(Y_K^{s\varepsilon}\right)}{\partial K}, \tag{2.15}$$

$$\frac{\partial s\left(Y_K^{T\varepsilon}\right)}{\partial \boldsymbol{\varepsilon}^e} = -\frac{1}{\rho_0}\frac{\partial \boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}{\partial T} = \frac{1}{\rho_0}\left(\mathbf{M}\boldsymbol{\alpha}\right)_{\boldsymbol{\sigma}=\boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}, \quad -\rho_0\frac{\partial s\left(Y_K^{T\varepsilon}\right)}{\partial K} = \frac{\partial \Pi\left(Y_K^{T\varepsilon}\right)}{\partial T}, \tag{2.16}$$

$$\frac{\partial s\left(Y_K^{T\sigma}\right)}{\partial \boldsymbol{\sigma}} = \frac{1}{\rho_0}\boldsymbol{\alpha}, \quad \alpha_{ij} = \alpha_{ji}, \quad \frac{\partial \boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial T} = \rho_0\frac{\partial s\left(Y_K^{T\sigma}\right)}{\partial \sigma} \quad \text{and}$$

$$-\rho_0\frac{\partial s\left(Y_K^{T\sigma}\right)}{\partial K} = \frac{\partial \Pi\left(Y_K^{T\sigma}\right)}{\partial T}. \tag{2.17}$$

If the thermodynamic potentials being not expressed in an additive form but in the most general form [1,2,45,47] we have the following additional identities of thermostatic couplings, which will be used in a further part of the paper

$$\begin{cases} -\mathbf{LN} = -\mathbf{L}\left(Y_K^{T\sigma}\right)\mathbf{N}\left(Y_K^{T\varepsilon}\right) = \mathbf{Z}, \\ -\mathbf{MZ} = -\mathbf{M}\left(Y_K^{T\varepsilon}\right)\mathbf{Z}\left(Y_K^{T\sigma}\right) = \mathbf{N}, \end{cases} . \tag{2.18}$$

$$\mathbf{N} = \mathbf{N}\left(Y_K^{T\varepsilon}\right) = \frac{\partial \boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}{\partial K} = \frac{\partial \Pi\left(Y_K^{T\varepsilon}\right)}{\partial \boldsymbol{\varepsilon}^e} \quad \text{and} \quad \mathbf{Z} = \mathbf{Z}\left(Y_K^{T\sigma}\right) = \frac{\partial \boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial K} = -\frac{\partial \Pi\left(Y_K^{T\sigma}\right)}{\partial \boldsymbol{\sigma}}, \tag{2.19}$$

and

$$\frac{\partial s\left(Y_K^{T\sigma}\right)}{\partial K} = \frac{\partial s\left(Y_K^{T\varepsilon}\right)}{\partial K} + \frac{1}{\rho_0}\boldsymbol{\alpha} \circ (\mathbf{MZ}), \tag{2.20}$$

$$\frac{\partial \Pi\left(Y_K^{T\sigma}\right)}{\partial T} = \frac{\partial \Pi\left(Y_K^{T\varepsilon}\right)}{\partial T} - \boldsymbol{\alpha} \circ (\mathbf{MZ}). \tag{2.21}$$

The relations (2.18), (2.20) and (2.21) are complex identities and are sometimes called thermodynamic identities of the second order. Physical interpretation is applied with regard to the identities expressing thermostatic couplings (2.19). A variation in internal forces Π due to the acting elastic strain results in the process of material hardening (softening), and variation in internal forces as a result of stress is connected with the variation in the elasticity modulus as a result of variability in internal parameters, see (2.19) and $(2.11)_3$.

The important thermostatic properties of thermo-elasto-plastic materials can be discussed by assuming consecutively that $(Y_K^{T\varepsilon},\ Y_K^{T\sigma})$ form a set of independent state parameters and evaluating the increments in the dependent parameters (cf. [1,2,45] and the formulae (2.9)–(2.21)]). Thus, we obtain the following expressions [1,2,46,47]

$$\begin{cases} T\mathrm{d}s\left(Y_K^{T\varepsilon}\right) = \gamma_1 c_\varepsilon \mathrm{d}T + \bar{\gamma}_6 \frac{T}{\rho_0}\boldsymbol{\alpha}\mathbf{M} : \mathrm{d}\boldsymbol{\varepsilon}^e - \bar{\gamma}_8 \frac{T}{\rho_0}\frac{\partial \Pi}{\partial T} \cdot \mathrm{d}K, \\ T\mathrm{d}s\left(Y_K^{T\sigma}\right) = \gamma_1 c_{\boldsymbol{\sigma}} \mathrm{d}T + \gamma_7 \frac{T}{\rho_0}\boldsymbol{\alpha} : \mathrm{d}\boldsymbol{\sigma} - \bar{\gamma}_8 \frac{T}{\rho_0}\frac{\partial \Pi}{\partial T} \cdot \mathrm{d}K, \end{cases} \tag{2.22}$$

$$\begin{cases} \mathrm{d}\boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right) = \gamma_4 \boldsymbol{\alpha}\mathrm{d}T + \gamma_2 \mathbf{L}\mathrm{d}\boldsymbol{\sigma} + \bar{\gamma}_9 \mathbf{Z}\mathrm{dK}, \\ \mathrm{d}\boldsymbol{\sigma} = \bar{\gamma}_2 \mathbf{M}\mathrm{d}\boldsymbol{\varepsilon}^{\mathrm{e}} - \gamma_5 \mathbf{M}\boldsymbol{\alpha}\mathrm{d}\dot{T} + \bar{\gamma}_{10}\mathbf{N}\dot{K}. \end{cases} \tag{2.23}$$

Eliminating $\dot{s}$ from the Eqs. (2.3), $(2.22)_1$ and making use of the Eqs. (2.9)–(2.12) and $(2.23)_1$ yields two equations for the temperature and stress rates [1,2,46,47].

$$\begin{cases} \rho_0 c_\varepsilon \dot{T} = \gamma_0 D - \bar{\gamma}_6 T\boldsymbol{\alpha} \colon \mathbf{M}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} + \bar{\gamma}_8 T \frac{\partial \Pi}{\partial T} \cdot \dot{K} + q_0, \\ \dot{\boldsymbol{\sigma}} = \bar{\gamma}_2 \mathbf{M}\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} - \gamma_5 \dot{T}\mathbf{M}\boldsymbol{\alpha} + \bar{\gamma}_{10}\mathbf{N}\dot{K}. \end{cases} \tag{2.24}$$

Eliminating $\dot{s}$ from the Eqs. (2.3), $(2.22)_2$ and making use of the Eqs. (2.9)–(2.12) and $(2.23)_2$ yields two equations for the temperature rate and of elastic strain rate [1,2,46,47].

$$\begin{cases} \rho_0 c_\sigma \dot{T} = \gamma_0 D - \gamma_7 T\boldsymbol{\alpha} \colon \dot{\boldsymbol{\sigma}} + \bar{\gamma}_8 T \frac{\partial \Pi}{\partial T} \cdot \dot{K} + q_0, \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \gamma_2 \mathbf{L}\dot{\boldsymbol{\sigma}} + \gamma_4 \boldsymbol{\alpha}\dot{T} + \bar{\gamma}_9 \mathbf{Z}\dot{K}. \end{cases} \tag{2.25}$$

Closed system of thermostatic couplings in the area of thermo-elasto-plastic interactions is presented in Fig. 1. A similar but incomplete (not closed) schemes of couplings are presented in papers [1,6,15]

The diagram shows 15 possible coupled connections between two independent parameters of state. Particular types of thermodynamic, thermomechanical and mechanical (elastic–plastic) couplings are denoted as the Greek letter γ with a suitable subscript or with subscript and dash. It was assumed that couplings with the upward (vertical or at an angle) and horizontal (to the right) arrows were expressed by the symbol $\gamma_i\,(i = 1, 2, \ldots, 15)$ (with no dash). The couplings with downward arrows (vertical or at an angle) and horizontal (to the left) arrows were denoted by the symbol with subscript and dash $\bar{\gamma}_i\,(i = 1, 2, \ldots, 15)$. Dimensionless coefficients γ_i and $\bar{\gamma}_i$ have no physical meaning. The ideas presented in, see e.g. [1–7,15,16,45–47] and others, have been introduced for simplification of interpretation of different terms of effects of couplings occurring in the equations. They can be also useful for some simplifications in general constitutive equations. They are a kind of numbers which take a value 1 (when any of the effects of couplings shown in Fig. 1 is taken into

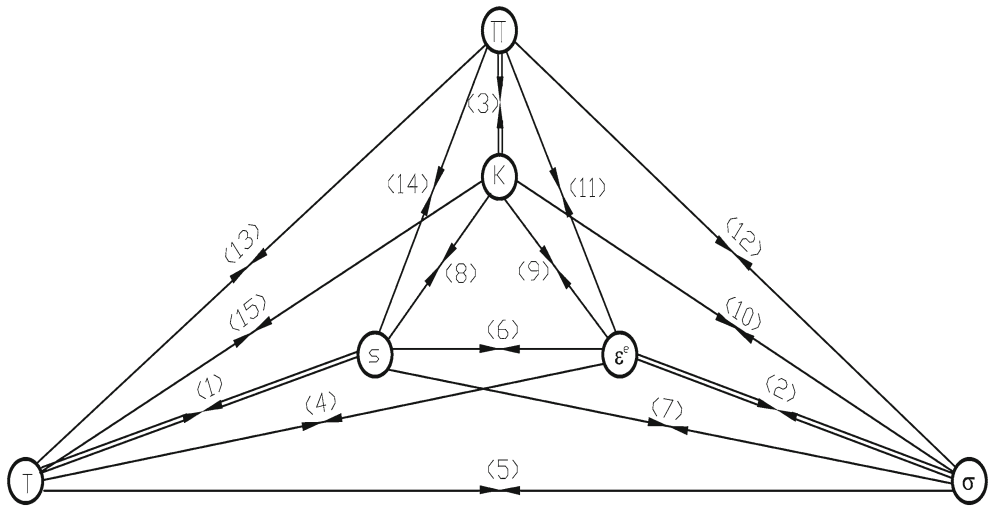

Fig. 1 Closed diagram of couplings occurring in thermomechanics of elastic–plastic bodies

account) or zero when any of effects of couplings shown in Fig. 1 is neglected. For example, when $\gamma_4 = 0$ is substituted, thermal expansion is neglected, if $\gamma_{11} = 0$, a change of internal forces caused by a change of elastic strains is neglected, when $\bar{\gamma}_6 = 0$, heat of elastic deformations is not taken into account, and in the case of $\bar{\gamma}_9 = 0$, influence of internal changes on elastic strains is omitted. If we are concentrated only on internal changes in the material caused by plastic strains and occurring as plastic hardening, i.e. the effects denoted by numbers 9–12 in Fig. 1, they can be also called the effects of elastic–plastic coupling [1,2,10,11,45–47]. Symbol γ_0 shows including dissipation heat which is not a thermostatic thermal effect, and it is not specified in the diagram in Fig. 1.

Theoretically, there are 30 physical kinds of thermodynamic, thermomechanical and mechanical (elastic–plastic) couplings. Not all of them, however, have their physical interpretation and physical explanation, and not all of them have been observed. They are a result of assumption of a general (non-additive) form for thermodynamic potentials and formal derivation of total differentials for the chosen dependent parameters of state, depending on the assumed systems of independent parameters of state [1,2,45,47].

Let us observe that the elastic strain rate involved in $(2.25)_2$ can be written as follows [1,2,46,47]

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}I} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}II}, \tag{2.26}$$

$$\begin{cases} \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}I} = \gamma_2 \mathbf{L}\dot{\boldsymbol{\sigma}} + \gamma_4 \boldsymbol{\alpha}\dot{T}, \\ \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}II} = \bar{\gamma}_9 \mathbf{Z}\dot{K}, \end{cases} \tag{2.27}$$

where $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}I}$ is termed the "uncoupled" part of the elastic strain rate, and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}II}$ is the "coupled" part of the elastic strain rate connected with the internal processes accompanying the plastic strain. Such a separation of the tensor $\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}}$ into two parts was adopted and interpreted for the case of isothermal processes in papers [10,11] and for non-isothermal processes in [1,2,45,47].

2.1.2 Rate equations: plastic flow equations

Equations of plastic flow are the following

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \Lambda \frac{\partial F_1}{\partial \boldsymbol{\sigma}}, \quad \dot{K} = \Lambda b(\boldsymbol{\sigma}, -\Pi, T, K) = \Lambda b \quad \text{or} \quad -\dot{K} = \Lambda \frac{\partial F_1}{\partial \Pi}, \tag{2.28}$$

if $F_1 = 0$ and $\Lambda \geq 0$,

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = 0, \quad \dot{K} = 0, \quad \text{if:} \quad F_1 < 0 \quad \text{or} \quad F_1 = 0 \quad \text{and} \quad \Lambda < 0, \tag{2.29}$$

where b—function describing evolution for internal parameters $\dot{K}$.

Here, $F_1 = F_1(\boldsymbol{\sigma}, -\Pi, T, K)$ is a generalized function of plastic flow determined in the space of thermodynamic forces $X^d = \{\boldsymbol{\sigma}, -\Pi\}$ such that $(F_1 = 0)$ determines the flow area in this space. The expression for an equation of evolution for the vector of pairs $\dot{K}$ in form of (2.28)$_2$ is a result of application of idea of the preparation space [1,2,45,47,48]. Using additional experimental data, in that space we can determine the equations of evolution for internal parameters K or dissipation forces Π and determine their initial value.

If Π or K is replaced by a suitable equation of state [1,2,46,47], then we will obtain the plastic flow function and suitable flow conditions F in the stress space, written as

$$F(\boldsymbol{\sigma}, K, T) = F_1(\boldsymbol{\sigma}, -\Pi, K, T)\,|_{\Pi=\Pi(\boldsymbol{\sigma},K,T)},$$
or
$$\bar{F}(\boldsymbol{\sigma}, -\Pi, T) = F_1(\boldsymbol{\sigma}, -\Pi, K, T)\,|_{K=K(\boldsymbol{\sigma},\Pi,T)}\,. \tag{2.30}$$

The factor Λ in (2.28) can be eliminated by making use of the "association condition" [1,2,7,15,46,47].

$$\dot{F}_1 = \dot{F} = 0 \text{ if } F_1 = F = 0 \quad \text{as:}$$
$$\frac{\partial F}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}\dot{T} - \Lambda h = 0 \quad \text{then:} \quad \Lambda = \left(\frac{\partial F}{\partial \boldsymbol{\sigma}}:\dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}\dot{T}\right)\frac{1}{h}, \tag{2.31}$$

where

$$h = -\frac{\partial F}{\partial K}\cdot b. \tag{2.32}$$

is what is termed the strain-hardening function.

By assuming the classical condition for plastic loading ($\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \neq 0$), if and only if $\frac{\partial F}{\partial \boldsymbol{\sigma}}: \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}\dot{T} \geq 0$, we find, by virtue of (2.31)$_2$, that $h \geq 0$. Equation (2.28)$_1$ can be expressed in the form

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \left\{\frac{1}{h}\frac{\partial F_1}{\partial \boldsymbol{\sigma}}\left(\frac{\partial F}{\partial \boldsymbol{\sigma}}: \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}\dot{T}\right) \quad \text{if} \quad F = 0 \quad \text{and} \quad \frac{\partial F}{\partial \boldsymbol{\sigma}}: \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}\dot{T} \geq 0, \right. \tag{2.33}$$

and $\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = 0$ if $F < 0$ or if $F = 0$ and $\frac{\partial F}{\partial \boldsymbol{\sigma}}: \dot{\boldsymbol{\sigma}} + \frac{\partial F}{\partial T}T < 0$,
where, by virtue of (2.30)$_1$

$$\frac{\partial F}{\partial \boldsymbol{\sigma}} = \frac{\partial F_1}{\partial \boldsymbol{\sigma}} + \frac{\partial F_1}{\partial \Pi} * \frac{\partial \Pi}{\partial \boldsymbol{\sigma}}. \tag{2.34}$$

Using the Gyarmati postulate, then $-\dot{K} = \Lambda\frac{\partial F_1}{\partial \Pi}$ and $b = \frac{\partial F_1}{\partial \Pi}$, cf. [1,2,46,47]. Making use of (2.34), (2.19)$_2$, we find

$$\Lambda\frac{\partial F}{\partial \boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \frac{\partial \boldsymbol{\varepsilon}^{\mathrm{e}}}{\partial \mathrm{K}} * \dot{K}. \tag{2.35}$$

On substituting (2.26) into (2.35), we find

$$\Lambda\frac{\partial F}{\partial \boldsymbol{\sigma}} = \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} + \dot{\boldsymbol{\varepsilon}}^{\mathrm{e}II}. \tag{2.36}$$

2.1.3 Uniqueness solution of incremental boundary-value problem

Let us assume that the thermodynamic state of the body at a certain moment t_0 of a homogenous process is known and such that the condition $F_1 = F = 0$ is satisfied. The following incremental problems can be formulated for such a type of processes. Satisfying the set of field and constitutive equations of (cf. [1,3,4,7,15,21,47]), we must find, for the time t_0, for the problems (a_1, a_2, b_1, b_2) the values

$\mathrm{a}_{1.}$ ($\dot{\boldsymbol{\varepsilon}}$ and q_0) assuming that $\dot{\boldsymbol{\sigma}}\,(t_0)$ and $\dot{T}\,(t_0)$ are prescribed,
$\mathrm{a}_{2.}$ ($\dot{\boldsymbol{\sigma}}$ and q_0) assuming that $\dot{\boldsymbol{\varepsilon}}\,(t_0)$ and $\dot{T}\,(t_0)$ are prescribed,
$\mathrm{b}_{1.}$ $(\dot{\boldsymbol{\varepsilon}}$ and $\dot{T})$ assuming that $\dot{\boldsymbol{\sigma}}\,(t_0)$ and $q_0\,(t_0)$ are prescribed,
$\mathrm{b}_{2.}$ $(\dot{\boldsymbol{\sigma}}$ and $\dot{T})$ assuming that $\dot{\boldsymbol{\varepsilon}}\,(t_0)$ and $q_0\,(t_0)$ are prescribed,

where $q_0 = -\mathrm{div}\mathbf{q}$.

It is easy to see that if a solution of the problems (a_1) and (a_2) is to be unique, it is necessary that the following respective conditions known from the isothermal theory of plasticity should be satisfied [1,3,4,7,13,14,16,47]

$$h > 0 \quad \text{and} \quad h + \mathbf{g}_p\colon \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} > 0, \tag{2.37}$$

where

$$\mathbf{g}_p = \mathbf{F}_{1,\boldsymbol{\sigma}} + \bar{\gamma}_9 \mathbf{Z} * b, \tag{2.38}$$

h is the isothermal strain-hardening function obtained in [1,2,7], $\mathbf{F}_{\boldsymbol{\sigma}} = \frac{\partial F}{\partial \boldsymbol{\sigma}}$ and $\mathbf{F}_{1,\boldsymbol{\sigma}} = \frac{\partial F_1}{\partial \boldsymbol{\sigma}}$,
b is the function describing evolution of internal parameters $\dot{K}$ [1,2,46,47].

For the problems (a_1) and (a_2), conditions (2.37) are also sufficient. However, that two solutions of the problems (b_1) and (b_2) may exist, even if the inequalities (2.37) are satisfied. The necessary uniqueness conditions for the problems (b_1) and (b_2) have the following forms [1,3,7,16,47]:

Problem $\mathbf{b_1}$

$$h_1 = h - m_\sigma F_T > 0. \tag{2.39}$$

Problem $\mathbf{b_2}$

$$H = h + \mathbf{g}_p\colon \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - \frac{1}{p}(m_\sigma + \bar{\gamma}_6 \xi \mathbf{g}_p\colon \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}})(F_T - \gamma_5 \boldsymbol{\alpha}\colon \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}}) > 0, \tag{2.40}$$

where

$$m_\sigma = \frac{1}{\rho_0 c_\sigma}\left[\gamma_0 \left(\boldsymbol{\sigma}\colon \mathbf{F}_{\boldsymbol{\sigma}} - \Pi \cdot b\right) - \gamma_3 T \left(\frac{\partial \Pi \left(Y_K^{T\sigma}\right)}{\partial \mathrm{T}} \cdot b\right)\right], \tag{2.41}$$

$$\xi = \frac{1-p}{M_\alpha^2} = \frac{T}{\rho_0 c_\sigma}, \quad p = \frac{c_\varepsilon}{c_\sigma}, \quad M_\alpha^2 = \boldsymbol{\alpha}\colon \mathbf{M}\boldsymbol{\alpha}. \tag{2.42}$$

In the case of associated laws of plastic flow ($\mathbf{F}_{1,\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}$), the quantity ($m_\sigma = m$) was analysed in Refs. [1,3,4,6,7,16,21,47] when all the elastic–plastic coupling effects being rejected ($\gamma_9 = \bar{\gamma}_9 = \gamma_{10} = \bar{\gamma}_{10} = \gamma_{11} = \bar{\gamma}_{11} = \gamma_{12} = \bar{\gamma}_{12} = 0$). By analysing the cyclic isothermal process in the space of stresses, authors of works [1,3,6,7,16,21,47] have appointed on base of account that for the majority of materials (in particular for metallic materials) m_σ is in general positive

$$m_\sigma > 0. \tag{2.43}$$

The inequalities (2.39) and (2.40) are a generalization of the uniqueness conditions appropriately derived by Mróz, Raniecki and Sawczuk, see [4,6,7,16] for the case of associated flow laws and without a elastic–plastic coupling effects. This generalization consists in the non-associated laws of plastic flow being taken into account as well as the influence of plastic deformations on the thermoelastic properties of the body. The conditions obtained in Refs. [4,6,7,16] can also be obtained from Eqs. (2.37) and (2.38) by rejecting all the effects of elastic–plastic coupling ($\gamma_9 = \bar{\gamma}_9 = \gamma_{10} = \bar{\gamma}_{10} = \gamma_{11} = \bar{\gamma}_{11} = \gamma_{12} = \bar{\gamma}_{12} = 0$) and assuming associated laws of plastic flow ($\mathbf{F}_{1,\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}$).

Condition (2.40) should be interpreted as a limitation for the functions occurring in the group of constitutive equations. If ($H = 0$), from the theory of thermo-elasto-plasticity it results that an instantaneous change of stresses and temperature is possible when the body element is not being deformed ($\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$) and it does not exchange heat with the environment ($q_0 = 0$). It means theoretically that if ($H = 0$ and $\dot{\boldsymbol{\varepsilon}} = \mathbf{0}$, $q_0 = 0$ then $\dot{\boldsymbol{\sigma}} \neq \mathbf{0}$ and $\dot{\mathrm{T}} \neq 0$). However, such phenomena do not take place in real physical bodies [1].

If ($h_1 = 0$), then in an adiabatic process ($q_0 = 0$) the body behaves similarly as the perfectly plastic body (i.e. there is no hardening). This means that momentary adiabatic plastic flow ($\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} \neq \mathbf{0}$) is possible under constant stresses ($\dot{\boldsymbol{\sigma}} = \mathbf{0}$). Such a phenomenon can occur in the range of large deformations; therefore, for small deformations we assume that constitutive functions satisfy the condition stated in (2.39).

It is worthwhile to observe that in the case of metallic materials the satisfaction of the condition (2.39) implies, in general, fulfilment of the condition (2.40) (cf. [1,3,4,6,7,16,21,47])

Let us assume that the conditions (2.39) and (2.40) are both satisfied. The solutions of the incremental problems (b_1) and (b_2) can be expressed in the following forms:

Problem $\mathbf{b}_1$

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{L}^{\mathbf{a}}\dot{\boldsymbol{\sigma}} + \frac{j}{h_1}\bar{\mathbf{K}}\dot{\boldsymbol{\sigma}} + \frac{j}{h_1} q F_T \left[\left(\mathbf{F}_{1,\sigma} + \bar{\gamma}_9 \mathbf{Z} * b\right) + \gamma_4 m_\sigma \boldsymbol{\alpha}\right] - \gamma_4 q \boldsymbol{\alpha}, \tag{2.44a}$$

$$\dot{T} = -\bar{\gamma}_7 \xi \alpha : \dot{\boldsymbol{\sigma}} + \frac{j m_\sigma}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right] + q, \tag{2.44b}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{j}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right] \mathbf{F}_{\boldsymbol{\sigma}}, \tag{2.44c}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{e}} = \mathbf{L} : \dot{\boldsymbol{\sigma}} + \gamma_4 \dot{T} \boldsymbol{\alpha} + \frac{j}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right] (\bar{\gamma}_9 \mathbf{Z} * b), \tag{2.44d}$$

where

$$\bar{\mathbf{K}} = \left[\left(\mathbf{F}_{1,\sigma} + \bar{\gamma}_9 \mathbf{Z} * b\right) + \gamma_4 m_\sigma \boldsymbol{\alpha}\right] \otimes (\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}). \tag{2.45a}$$

$$\mathbf{L}^{\mathbf{a}} = \mathbf{L} - \gamma_4 \bar{\gamma}_7 \xi \, (\boldsymbol{\alpha} \otimes \boldsymbol{\alpha}), \quad q = \frac{1}{\rho_0 c_\sigma} \operatorname{div} \mathbf{q}. \tag{2.45b}$$

$$j = \begin{cases} 1 & \text{if} \quad F = 0 \quad \text{and} \quad (\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q \geq 0, \\ 0 & \text{if} \quad F = 0 \quad \text{or} \quad F = 0 \quad \text{and} \quad (\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q < 0. \end{cases} \tag{2.45c}$$

$$\Lambda = \frac{j}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right]. \tag{2.46}$$

The symbol $\mathbf{L}^{\mathbf{a}}$ denotes the tensor of adiabatic elasticity. Let us observe that the second right-hand term of (2.44a) is not equal to the plastic strain rate, but may be considered as representing the adiabatic plastic strain rate. The tensor $\bar{\mathbf{K}}$ is asymmetric, $\bar{K}_{ijmn} \neq \bar{K}_{mnij}$. A lack of symmetry is caused by not only thermal expansion accompanying power dissipation of the plastic strain and a change of the yield point together with increase of temperature resulting from the piezoelectric effect, but also effects of elastic–plastic coupling and assumption of non-associated laws of plastic flow. It makes difficult a proof of the theorem concerning uniqueness of solution of the incremental boundary-value problem in the case of heterogeneous processes and in a consequence formulation of suitable criteria of bifurcation.

The equations for the thermodynamic flow rates can also be expressed in terms of $\dot{\boldsymbol{\sigma}}$ and q, They have the form

$$\dot{K} = \frac{j}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right] b\left(X^d, Y_K^T\right). \tag{2.47}$$

Taking into considerations the Gyarmati postulate and the resulting condition (cf. [1,3,47]), the relation (2.47) takes the form

$$-\dot{K} = \frac{j}{h_1}\left[(\mathbf{F}_{\boldsymbol{\sigma}} - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}) : \dot{\boldsymbol{\sigma}} - F_T q\right] \frac{\partial F_1\left(X^d, Y_K^T\right)}{\partial \Pi}. \tag{2.48}$$

Problem $\mathbf{b}_2$

The alternative constitutive equations, corresponding to Eqs. (2.44)–(2.46), have the forms

$$\dot{\boldsymbol{\sigma}} = \mathbf{M}^{\mathbf{a}}\dot{\boldsymbol{\varepsilon}} + \frac{\gamma_4 q}{p}\mathbf{M}\boldsymbol{\alpha} - \frac{j_1}{H}\tilde{\mathbf{K}}\dot{\boldsymbol{\varepsilon}} - \frac{j_1 q}{pH}(\gamma_5 \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - F_T)(\tilde{\varphi}\mathbf{M}\boldsymbol{\alpha} + \mathbf{B}_{\mathbf{N}}), \tag{2.49a}$$

$$\dot{T} = \frac{1}{p}(\bar{\gamma}_7 \xi \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} + q) + \frac{j_1}{pH}(m_\sigma + \bar{\gamma}_6 \xi \mathbf{g}_{\mathrm{p}} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}})\left[\mathbf{B} : \dot{\boldsymbol{\varepsilon}} + \frac{q}{p}(\gamma_5 \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - F_T)\right], \tag{2.49b}$$

$$\dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \frac{j}{H}\left[\mathbf{B} : \dot{\boldsymbol{\varepsilon}} + \frac{q}{p}(\gamma_5 \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - F_T)\right]\mathbf{F}_{1,\sigma}, \tag{2.49c}$$

where

$$j_1 = \begin{cases} 1 & \text{if} \quad F = 0 \quad \text{and} \quad \mathbf{B} : \dot{\boldsymbol{\varepsilon}} + \dfrac{q}{p}(\gamma_5 \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - F_T) \geq 0, & (2.50a) \\ 0 & \text{if} \quad F < 0 \quad \text{or} \quad F = 0 \quad \text{and} \quad \mathbf{B} : \dot{\boldsymbol{\varepsilon}} + \dfrac{q}{p}(\gamma_5 \boldsymbol{\alpha} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} - F_T) < 0. & (2.50b) \end{cases}$$

and

$$\mathbf{M}^{\mathbf{a}} = \mathbf{M} + \frac{\gamma_5\bar{\gamma}_6}{p}\xi\left(\mathbf{M}\boldsymbol{\alpha}\right)\otimes\left(\mathbf{M}\boldsymbol{\alpha}\right), \quad \mathbf{B}_{\mathbf{N}} = \mathbf{B} - \mathbf{N}_{\mathbf{Z}}, \quad -\mathbf{LN} = -\mathbf{L}\left(Y_K^{T\sigma}\right)\mathbf{N}\left(Y_K^{T\varepsilon}\right) = \mathbf{Z},$$

where

$$\mathbf{L}\left(Y_K^{T\sigma}\right) = \frac{\partial\boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial\boldsymbol{\sigma}}, \quad \mathbf{N}\left(Y_K^{T\varepsilon}\right) = \frac{\partial\boldsymbol{\sigma}\left(Y_K^{T\varepsilon}\right)}{\partial\mathrm{K}}, \quad \mathbf{Z}\left(Y_K^{T\sigma}\right) = \frac{\partial\boldsymbol{\varepsilon}^e\left(Y_K^{T\sigma}\right)}{\partial K},$$

and $\mathbf{B} = \mathbf{MF}_{\boldsymbol{\sigma}} + \bar{\gamma}_6\frac{\xi}{p}\left(\gamma_5\boldsymbol{\alpha}\colon\mathbf{MF}_{\boldsymbol{\sigma}} - F_T\right)$, $\tilde{\mathbf{K}} = \left[\tilde{\varphi}\mathbf{M}\boldsymbol{\alpha} - \mathbf{N}_{\mathbf{Z}}\right]\otimes\mathbf{B} + \mathbf{B}\otimes\mathbf{B}$.

The following additional quantities are involved in the tensor $\tilde{\mathbf{K}}$:

$$\tilde{\varphi} = \gamma_5\frac{m_\sigma}{p} + \bar{\gamma}_6\frac{\xi}{p}F_T - \bar{\gamma}_6\bar{\gamma}_{10}\frac{\xi}{p}\left\{\boldsymbol{\alpha} : \left[\mathbf{N} * \left(\gamma_3 b + \gamma_5 F_{1,\Pi}\right)\right]\right\}, \tag{2.51}$$

$$\mathbf{N}_{\mathbf{Z}} = \bar{\gamma}_{10}\mathbf{N} * b + \gamma_{12}\mathbf{N} * F_{1,\Pi} = \mathbf{N} * \left(\bar{\gamma}_{10} b + \gamma_{12} F_{1,\Pi}\right), \tag{2.52}$$

where $F_{1,\Pi} = \frac{\partial F}{\partial \Pi}$.

Symbol $\mathbf{M}^{\mathbf{a}}$ denotes the tensor of adiabatic moduli of elasticity. Similarly to the former case, the tensor interrelating the stress rate and strain rate is asymmetric because $\tilde{K}_{ijmn} \neq \tilde{K}_{mnij}$.

Let us observe that, if the conditions (2.39) and (2.40) are both satisfied, Eq. (2.49) are equivalent Eq. (2.44). They can be obtained by solving Eq. (2.44a) for $\dot{\boldsymbol{\sigma}}$ and substituting the result into Eqs. (2.44b) and (2.44c). Though Eq. (2.44) can be obtained by solving Eq. (2.49a) for $\dot{\boldsymbol{\varepsilon}}$ and substituting the result into Eqs. (2.49b) and $(2.15)_3$. Thus, conditions (2.39) and (2.40) are often called conditions of reciprocal reversibility of constitutive equations in relation to $\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\sigma}}$.

If all the thermodynamic coupling effects in the Eqs. (2.3)–(2.48) and (2.49)–(2.50) are rejected $(\gamma_1 = \gamma_3 = \bar{\gamma}_6 = \bar{\gamma}_7 = \bar{\gamma}_9 = \bar{\gamma}_{10} = \gamma_{12} = 0)$ and if $\left(\mathbf{F}_{1,\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}\right)$, those equations will constitute two equivalent set of fundamental equations of the theory of thermal stresses in an elastic–plastic body. Then $(q = \dot{T})$ or $(-\mathrm{div}\mathbf{q} = \rho_0 c_\sigma \dot{T})$ and $(p = 1)$.

3 Formulation of the incremental boundary-value problem

If the condition (2.39) is satisfied, the set of Eqs. (2.44)–(2.48) is equivalent to the fundamental set of Eqs. (2.49)–(2.50) together with the relevant evolution equations of internal parameters $\dot{K}$, when $H > 0$. The set of those equations, together with the law of heat conduction, with the equation of motion and the kinematic relations

$$\mathrm{div}\dot{\boldsymbol{\sigma}} + \rho_0\dot{\mathbf{b}}_m = \rho_0\dot{\mathbf{v}} \quad \text{and} \quad 2\varepsilon_{i,j} = v_{i,j} + v_{j,i}, \tag{3.1}$$

where $\mathbf{v}$ is the vector of velocity of particles, $\mathbf{b}_m$ is the body force, constitutes a set of fundamental field equations of generalized coupled thermo-elasto-plasticity. Together with the boundary conditions and the initial conditions, it may be used as a basis for analysis of many problems of generalized coupled thermo-elasto-plasticity, both static and dynamic [1,3,6,7,47,48].

The following static incremental boundary-value problem can be formulated [1,3,4,7,16,47].

Let the body occupy, at a time t_0, a region D in space. Let us denote by $\bar{D}$ the closure of D and by the symbol S—the boundary of $\bar{D}$. S is the closure of the sum of non-intersecting regular open surfaces S_v and S_t. Let the thermodynamic state of the body

$$\{T, \boldsymbol{\sigma}, K\} \tag{3.2}$$

and the rate of body forces $\dot{\mathbf{b}}_m$ be known, at a time t_0 and at every point $\mathbf{x}$ of the closure $\bar{D}$. It is assumed that the functions (3.2) satisfy the condition $F \leq 0$. It is also assumed that the values of the surface forces $\dot{\mathbf{t}}_0$ and the velocities of material points $\mathbf{v}^0$ are known at time t_0 over the parts S_v and S_t of the boundary, that is

$$\begin{cases} \dot{\boldsymbol{\sigma}}\mathbf{n} = \dot{\mathbf{t}}_0 & \text{for} \quad \boldsymbol{x} \in S_t, \\ \mathbf{v} = \mathbf{v}_0 & \text{for} \quad \boldsymbol{x} \in S_v, \end{cases} \tag{3.3}$$

where $\mathbf{n}$ is a unit vector normal to S, directed towards the outside of D[1,3,4,7,16,47]. Our task is to find the set of functions $(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\varepsilon}},\ \mathbf{v})$ defined in $\bar{D}$ and the function $\dot{T}$ defined in D, which satisfy, in the region D, Eqs. (2.37)–(2.40) and (3.1), expression $q = \frac{1}{\rho_0 c_\sigma}\mathrm{div}\mathbf{q}$ and the rate equations of equilibrium

$$\mathrm{div}\dot{\boldsymbol{\sigma}} + \rho_0 \dot{\mathbf{b}}_m = 0. \tag{3.4}$$

Let us observe that, knowing the functions (3.2), we can determine q at every point of the region D, directly from the $\mathbf{q} = \boldsymbol{\varphi}^q\left(\nabla T,\ Y_K^T\right)$, where $\nabla T = \mathrm{grad}T$ and $Y_K^T = \{T,\ K\}$, by differentiating T and $\mathbf{q}$ with respect to the coordinate variables x_i, where $i = 1, 2, 3$.

In the coupled generalized thermo-elasto-plasticity, the formulated incremental boundary problem plays the same role as a suitable boundary incremental problem in isothermal theory of plasticity. Namely, if its solution is ambiguous, then a solution of a general problem where the history of variation of surface forces, velocity and temperature on the surface of the considered body are given is ambiguous, too [1,4,6,7,16,47].

4 Discussion of uniqueness conditions

Tests of uniqueness of the solution of the incremental boundary problem presented in Sect. 3 belongs to the most important problems shown in this paper. Such tests and the obtained results can be the basis for formulation of two criteria allowing to estimate the critical thermodynamic state after exceeding of which bifurcation of the equilibrium state is possible. These criteria are also two sufficient conditions (local and global condition) of uniqueness of solution of the incremental boundary-value problem.

The local condition can be easily applied in practice because it is directly expressed by constitutive functions and material constants. However, it gives less accurate estimations of the critical state. The global condition gives better estimations of critical states, but its application is more difficult because it requires searching kinematically acceptable velocity fields for which the functional J (see point 4.2) reaches zero.

Derivation of both conditions uses methods presented in the previous papers [1,3–5,7,13,14,47].

4.1 Local uniqueness condition

The following theorem is proved in author's papers [1,3,47]

Theorem 1 *If the inequality*

$$h_1 = h - m_\sigma F_T > \frac{1}{2}\left[\sqrt{(\mathbf{g} : \mathbf{M^a g})\left(\bar{\mathbf{F}}_\sigma : \mathbf{M^a}\bar{\mathbf{F}}_\sigma\right)} - \mathbf{g} : \mathbf{M^a}\bar{\mathbf{F}}_\sigma\right] = h_1^*, \tag{4.1}$$

where

$$\mathbf{g} = \left(\mathbf{F}_{1,\sigma} + \bar{\gamma}_9 \mathbf{Z} * b + \gamma_4 m_\sigma \boldsymbol{\alpha}\right) \quad \text{and} \quad \bar{\mathbf{F}}_\sigma = \left(\mathbf{F}_\sigma - \bar{\gamma}_7 \xi F_T \boldsymbol{\alpha}\right), \tag{4.2}$$

is satisfied at every point of the plastic portion of the body $D_p = \{\mathbf{x}:\ F = 0\}$, there can exist only one set of functions $\left\{\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\varepsilon}},\ \dot{T}\right\}$ of class C^1 at least, which is a solution of the incremental boundary-value problem of generalized coupled thermoplasticity, which was formulated in Sect. 3.

The inequality (4.1) is the sufficient local uniqueness condition. Each thermodynamic state, for which the condition (4.1) is satisfied, is secure against bifurcation. Since in the course of a deformation process of the body the value of the strain-hardening function (the modulus) decreases, in general, therefore the value of h_1^* may be treated as an upper estimation of the unknown critical value h corresponding to the critical state.

Some particular cases of the expression (4.1) have already been mentioned in the literature. A similar condition was obtained in [10,11] in their analysis of the stability of material defined as a condition of half the product of the stress rate tensor and the strain rate tensor being positive. Their study was confined to the case of the isothermal theory of plasticity (with no thermomechanical couplings), the elastic plastic coupling effects and non-associated laws of plastic flow being preserved. An expression of this type was also obtained in [1,3–5,7,13,14,21,47] for the case of isothermal uncoupled and non-isothermal coupled theory of thermoplasticity with a non-associated law of plastic flow.

4.2 The global uniqueness condition

4.2.1 Global condition for a thermo-elasto-plastic body dependent on kinematically admissible strain rate fields

Let us assume that there exist two sets of functions $\{\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\varepsilon}},\ \dot{T},\ \mathbf{v}\}$ and $\{\dot{\boldsymbol{\sigma}}^*,\ \dot{\boldsymbol{\varepsilon}}^*,\ \dot{T}^*,\ \mathbf{v}^*\}$ which are solutions of the incremental boundary-value problem of generalized coupled thermoplasticity, which was formulated in Sect. 3. Then, the following equality must be satisfied

$$\Lambda^* = \int_D \left(\dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}^*\right):\ \left(\dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^*\right) dV = 0. \tag{4.3}$$

due to the fact that both solutions satisfy the same boundary conditions (3.3), in the case of Gauss–Ostrogradski theorem.

Let us denote by J the integrand in the expression (4.3), which depends on ($\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^*$), for an elastic–plastic body, as follows

$$J\left(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*, j_1,\ j_1^*\right) = \left[\dot{\boldsymbol{\sigma}}\left(\dot{\boldsymbol{\varepsilon}}\right) - \dot{\boldsymbol{\sigma}}\left(\dot{\boldsymbol{\varepsilon}}^*\right)\right] : \Delta\dot{\boldsymbol{\varepsilon}}, \tag{4.4}$$

where $\Delta\dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}} - \dot{\boldsymbol{\varepsilon}}^*$ and $\dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}}\left(\dot{\boldsymbol{\varepsilon}}^*\right)$,

and $j_1 = j_1\left(\dot{\boldsymbol{\varepsilon}}\right)$ and $j_1^* = j_1\left(\dot{\boldsymbol{\varepsilon}}^*\right)$ are defined by the Eqs. (2.50a) and (2.50b).

The quantities ($\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^*$) and ($\dot{\boldsymbol{\sigma}}^*$ and $\dot{\boldsymbol{\varepsilon}}^*$) are interrelated by Eq. (2.49a), which can be rewritten in a more compact form as follows

$$\dot{\boldsymbol{\sigma}} = \mathbf{M}_1\dot{\boldsymbol{\varepsilon}} - \mathbf{M}_1\mathbf{d}_1 - \frac{j_1}{H_1}\mathbf{g}^*\left[\bar{\mathbf{F}}_{\sigma}^* : \left(\dot{\boldsymbol{\varepsilon}} - \mathbf{d}_1\right) + Z_1\right], \tag{4.5}$$

where

$$\begin{cases} \mathbf{g}^* \equiv \mathbf{M}_1\mathbf{g} = \tilde{\varphi}\left(\mathbf{M}\boldsymbol{\alpha}\right) - \mathbf{N_Z} + \mathbf{B}, \quad \text{and} \quad \bar{\mathbf{F}}_{\sigma}^* \equiv \mathbf{M}_1\bar{\mathbf{F}}_{\sigma} = \mathbf{B}, \\ \mathbf{d}_1 = \gamma_5 q\boldsymbol{\alpha}, \quad \mathbf{M}_1 \equiv \mathbf{M}^{\mathbf{a}}, \quad \mathbf{Z}_1 \equiv -qF_T, H_1 \equiv H. \end{cases} \tag{4.6}$$

Since the expression (4.3) with zero at the right side provides existence of two sets of functions $\{\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\varepsilon}},\ \dot{T},\ \mathbf{v}\}$ and $\{\dot{\boldsymbol{\sigma}}^*,\ \dot{\boldsymbol{\varepsilon}}^*,\ \dot{T}^*,\ \mathbf{v}^*\}$, which are a solution of the formulated incremental boundary-value problem, so positivity of the expression (4.3), i.e. $\Lambda^* > 0$ [1,3,6,7,13,14,21,47] and (4.4) will be a condition excluding occurrence of the bifurcation state. Inequality $\Lambda^* > 0$ is a sufficient global condition of uniqueness and a global criterion excluding occurrence of the bifurcation state.

4.2.2 Global condition for a comparison body dependent on kinematically admissible strain rate fields

Let us introduce the following function J', depending on ($\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^*$)

$$J'\left(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*\right) = \Delta\dot{\boldsymbol{\varepsilon}}: \mathbf{M}_1\Delta\dot{\boldsymbol{\varepsilon}} - \frac{1}{4x^2H}\left[\left(\mathbf{g}^* + x^2\bar{\mathbf{F}}_{\sigma}^*\right) : \Delta\dot{\boldsymbol{\varepsilon}}\right]^2, \tag{4.7}$$

where x^2 is a scalar quantity.

The expression J' is a comparison body function and represents a one-parameter family of expressions of J', with respect to the parameter x^2.

The idea of reference body in coupled thermoplasticity was introduced in the papers by Mróz, Raniecki and Brunhs [4–7,13,14] and also in the author's papers [1,3,21,22,47].

The functions J and J'depend in addition to the variables ($\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^*$) on the thermodynamic state of the body (3.2).

As it results from comparison of the expression (4.7) and (4.4), it presents a certain linear dependence between ($\Delta\dot{\boldsymbol{\sigma}}$ and $\Delta\dot{\boldsymbol{\varepsilon}}$). Differentiating J' in relation to $\Delta\dot{\boldsymbol{\varepsilon}}$, we obtain a linear dependence between $\Delta\dot{\boldsymbol{\sigma}}$ and $\Delta\dot{\boldsymbol{\varepsilon}}$, which does not occur in expression (4.4), because these dependences are nonlinear.

Lemma 1 *It will be shown that if the same thermodynamic state is prescribed for J and J', then for each pair ($\dot{\boldsymbol{\varepsilon}}$ and $\dot{\boldsymbol{\varepsilon}}^*$) the following inequality holds*

$$J\left(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*, j_1, j_1^*\right) - J'\left(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*\right) \geq 0. \tag{4.8}$$

Let us introduce the following notations for the function $J\left(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*,\ j_1,\ j_1^*\right)$*:*

$$\begin{cases} J_1(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*,\ 1,\ 1) \ \text{ if } \ j_1(\dot{\boldsymbol{\varepsilon}}) = 1 \ \text{ and } \ j_1(\dot{\boldsymbol{\varepsilon}}^*) = 1, \\ J_2(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*,\ 1,\ 0) \ \text{ if } \ j_1(\dot{\boldsymbol{\varepsilon}}) = 1 \ \text{ and } \ j_1(\dot{\boldsymbol{\varepsilon}}^*) = 0, \\ J_3(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*,\ 0,\ 1) \ \text{ if } \ j_1(\dot{\boldsymbol{\varepsilon}}) = 0 \ \text{ and } \ j_1(\dot{\boldsymbol{\varepsilon}}^*) = 1, \\ J_4(\dot{\boldsymbol{\varepsilon}},\ \dot{\boldsymbol{\varepsilon}}^*,\ 0,\ 0) \ \text{ if } \ j_1(\dot{\boldsymbol{\varepsilon}}) = 0 \ \text{ and } \ j_1(\dot{\boldsymbol{\varepsilon}}^*) = 0. \end{cases} \tag{4.9}$$

Then, by evaluating the difference (4.8) for all the possible four cases (4.9), we obtain, by virtue of (4.4) and (4.5) and (4.7), that

$$\begin{cases} \left(J_1 - J'\right) H = \frac{1}{4x^2}\left(\nu_g - x^2\nu_f\right)^2 \geq 0, \\ \left(J_2 - J'\right) H = -x^2 A_\varepsilon A_\varepsilon^* + \left[x A_\varepsilon^* - \frac{1}{2x}\left(\nu_g - x^2\nu_f\right)\right]^2 \geq 0, \\ \quad \text{because} \ \ A_\varepsilon \geq 0 \ \text{ and } \ A_\varepsilon^* < 0, \\ \left(J_3 - J'\right) H = -x^2 A_\varepsilon A_\varepsilon^* + \left[x A_\varepsilon + \frac{1}{2x}\left(\nu_g - x^2\nu_f\right)\right]^2 \geq 0, \\ \quad \text{because} \ \ A_\varepsilon < 0 \ \text{ and } \ A_\varepsilon^* \geq 0, \\ \left(J_4 - J'\right) H = \left[x\nu_f + \frac{1}{2x}\left(\nu_g - x^2\nu_f\right)\right]^2 \geq 0, \end{cases} \tag{4.10}$$

where

$$\begin{cases} \nu_g = \mathbf{g}^2 : \Delta\dot{\boldsymbol{\varepsilon}}, \quad \nu_f = \bar{\mathbf{F}}_{\boldsymbol{\sigma}}^* : \Delta\dot{\boldsymbol{\varepsilon}}, \\ A_\varepsilon = \bar{\mathbf{F}}_{\boldsymbol{\sigma}}^* : (\dot{\boldsymbol{\varepsilon}} - \mathbf{d}_1) + Z_1, \quad A_\varepsilon^* = \bar{\mathbf{F}}_{\boldsymbol{\sigma}}^* : \left(\dot{\boldsymbol{\varepsilon}}^* - \mathbf{d}_1\right) + Z_1. \end{cases} \tag{4.11}$$

It follows that the inequality (4.8) is valid.

Using the expression (4.3), inequalities (4.8) and (4.10), we can formulate the following sufficient condition of uniqueness of a solution of the incremental boundary problem for the comparison body which is a stronger (safer) criterion excluding occurrence of the bifurcation state.

Theorem 2 *Let us now formulate a sufficient global uniqueness criterion (that is a criterion which excludes bifurcation). Let $H > 0$ at every point $\boldsymbol{x} \in D_{\mathrm{p}}$, in this part, where plastic deformations are occurring, i.e. where $D_{\mathrm{p}} = \{\boldsymbol{x}\colon F = 0\}$. If for every nonzero kinematically admissible and integrable velocity field* **v***, which vanishes over the part S_v of the surface, the inequality*

$$\int\limits_D J_1'(\mathbf{v})\mathrm{d}V - \int\limits_{D_{\mathrm{p}}} J_2'(\mathbf{v})\mathrm{d}V > 0, \tag{4.12}$$

is satisfied, there exists only one pair $\left\{\dot{\boldsymbol{\sigma}},\ \dot{T}\right\}$ constituting a solution of the incremental boundary-value problem in generalized coupled thermoplasticity. This criterion can easily be demonstrated.

Proof The integrands in (4.12) are

$$J_1'\left(\dot{\boldsymbol{\varepsilon}}\right) = \dot{\boldsymbol{\varepsilon}} \colon \mathbf{M}_1\dot{\boldsymbol{\varepsilon}} \quad \text{and} \quad J_2'\left(\dot{\boldsymbol{\varepsilon}}\right) = \frac{1}{4x^2 H}\left[\left(\mathbf{g}^* + x^2\bar{\mathbf{F}}_{\boldsymbol{\sigma}}^*\right) : \dot{\boldsymbol{\varepsilon}}\right]. \tag{4.13}$$

The validity of the sufficient global uniqueness criterion (4.12), being a safer criterion excluding the state of bifurcation, follows directly from the inequalities $\Lambda^* > 0$ and (4.8) and (4.10).

The integral condition (4.12) is, in particular form, of essential practical importance. If for a prescribed state $\{T, \boldsymbol{\sigma}, K\}$ it is impossible to find such a field **v** that the sum of integrals at the left-hand side of the expression (4.12) is zero, we are assured that this state is secure against bifurcation.

The idea of deriving such criterion was conceived as early as in Hill's works [17–20] for elastic–plastic bodies under large strain, for the isothermal incremental boundary-value problem. For the incremental boundary-value problem in coupled thermoplasticity in the case of associated laws of plastic flow and for small deformations such criterion has been derived by Mróz and Raniecki [4–7], for the case of non-associated laws of plastic flow by Śloderbach [1,3,47] and for large deformations by Raniecki and Bruhns [13,14] and Śloderbach and

Pajak [21]. Another sufficient global uniqueness criterion for incremental problems of isothermal plasticity of elastic–plastic bodies with non-associated laws of plastic flow has been given by Hueckel and Maier [10–12]. In paper [12] the author introduced an idea of two-point asymmetric scalar function linearly dependent on the Green function for a linearly elastic body. Application of the criterion in practice is difficult because the Green function for bodies of an arbitrary shape is usually unknown.

It is shown in [1,3,13,14,47] that the sufficient local uniqueness condition following from the requirement that the intergrand J' should be definite positive is the same as for an generalized thermo-elasto-plastic body Eqs. (2.44a) and (2.49a) or (4.5) provided that the parameter x^2 takes its optimum form

$$x_0^2 = \left(\frac{\mathbf{g}^* : \mathbf{L}_1 \mathbf{g}^*}{\bar{\mathbf{F}}_{\boldsymbol{\sigma}}^* : \mathbf{L}_1 \bar{\mathbf{F}}_{\boldsymbol{\sigma}}^*}\right)^{\frac{1}{2}}. \tag{4.14}$$

A procedure for obtaining the optimum parameter x_0^2 is also discussed in [1,3,13,14,47].

For the parameter x_0^2, the sufficient local uniqueness condition becomes the optimum (strongest) condition for the entire one-parameter family of sufficient uniqueness conditions. Now, by substituting the optimum value of the parameter (4.14) into the expressions (4.7) and (4.12) we shall obtain the optimum (strongest) integrand which will be denoted by the symbol J_0' and optimum form of the bifurcation criterion,

$$J_0' = \dot{\boldsymbol{\varepsilon}} : \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}} - \frac{1}{4H_1} \frac{\left[\left(\bar{\mathbf{F}}_{\boldsymbol{\sigma}}\colon \mathbf{M}_1 \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\right)^{\frac{1}{2}} \mathbf{g}\colon \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}} + \left(\mathbf{g}\colon \mathbf{M}_1 \mathbf{g}\right)^{\frac{1}{2}} \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\colon \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}}\right]^2}{\left(\bar{\mathbf{F}}_{\boldsymbol{\sigma}}\colon \mathbf{M}_1 \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\right)^{\frac{1}{2}} \left(\mathbf{g}\colon \mathbf{M}_1 \mathbf{g}\right)^{\frac{1}{2}}}, \tag{4.15}$$

and

$$\int\limits_D (\dot{\boldsymbol{\varepsilon}} : \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}}) \mathrm{d}V - \frac{1}{4} \int\limits_{D_\mathrm{p}} \frac{\left[\left(\bar{\mathbf{F}}_{\boldsymbol{\sigma}} : \mathbf{M}_1 \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\right)^{\frac{1}{2}} \mathbf{g}\colon \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}} + \left(\mathbf{g}\colon \mathbf{M}_1 \mathbf{g}\right)^{\frac{1}{2}} \bar{\mathbf{F}}_{\boldsymbol{\sigma}} : \mathbf{M}_1 \dot{\boldsymbol{\varepsilon}}\right]^2}{H_1 \left(\bar{\mathbf{F}}_{\boldsymbol{\sigma}} : \mathbf{M}_1 \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\right)^{\frac{1}{2}} \left(\mathbf{g}\colon \mathbf{M}_1 \mathbf{g}\right)^{\frac{1}{2}}} \mathrm{d}V > 0. \tag{4.16}$$

4.2.3 Global condition for a thermo-elasto-plastic body depending on statically admissible stress rate fields

Now, let the symbol I mean the integrand from the expression (4.3) dependent on ($\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\sigma}}^*$) for a thermo-elasto-plastic body in the following way

$$I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ j,\ j^*\right) = \left[\dot{\boldsymbol{\varepsilon}}\left(\dot{\boldsymbol{\sigma}}\right) - \dot{\boldsymbol{\varepsilon}}\left(\dot{\boldsymbol{\sigma}}^*\right)\right] : \Delta\dot{\boldsymbol{\sigma}}, \tag{4.17}$$

where $\Delta\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \dot{\boldsymbol{\sigma}}^*$, $\dot{\boldsymbol{\sigma}}^* = \dot{\boldsymbol{\sigma}}\left(\dot{\boldsymbol{\varepsilon}}^*\right)$ also $j = j\left(\hat{\boldsymbol{\sigma}}\right)$ and $j^* = j\left(\hat{\boldsymbol{\sigma}}^*\right)$.

Like in Sect. 4.2.1, functions j and j^* take the value 1—for the active plastic deformation, or 0—for the elastic loading or plastic unloading.

At present $\dot{\boldsymbol{\sigma}}^*$ and $\dot{\boldsymbol{\varepsilon}}^*$ are connected with a suitable constitutive equation, see [1,3,46] written as

$$\dot{\boldsymbol{\varepsilon}} = \mathbf{L}_1 \dot{\boldsymbol{\sigma}} + \frac{j}{h_1} \mathbf{g}\left[\bar{\mathbf{F}}_{\boldsymbol{\sigma}} \dot{\boldsymbol{\sigma}} + z_1\right] + \mathbf{d}_1, \tag{4.18}$$

where

$$\begin{cases} \mathbf{g} = \left(\mathbf{F}_{1,\sigma} + \bar{\gamma}_9 \xi \mathbf{Z} * b + \boldsymbol{\gamma}_4 m_\sigma \boldsymbol{\alpha}\right), & \mathbf{d}_1 = \boldsymbol{\gamma}_4 q \boldsymbol{\alpha}, \\ \mathbf{Z}_1 = -\mathbf{q}\mathbf{F}_\mathbf{T}, \quad \mathbf{L}_1 \equiv \mathbf{L}^\mathbf{a}, & \mathbf{M}_1 \equiv \mathbf{M}^\mathbf{a}. \end{cases} \tag{4.19}$$

Like in the case of kinematically admissible strain rate field, the expression (4.3) with the sign zero at the right side allows for existence of two sets of functions $\left\{\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\varepsilon}},\ \dot{T},\ \mathbf{v}\right\}$ and $\left\{\dot{\boldsymbol{\sigma}}^*,\ \dot{\boldsymbol{\varepsilon}}^*,\ \dot{T}^*,\ \mathbf{v}^*\right\}$, being a solution of the formulated incremental boundary problem. Thus, the positive definition of the expression (4.3) is a condition excluding occurrence of the bifurcation state, i.e. $\Lambda^* > 0$ [1,3,6,7,13,14,21,47]. The positive definition of the expression (4.17) is a result of the positive definition of (4.13), too. In this case, the inequality (4.13) is a sufficient global condition of uniqueness of solution of the incremental boundary problem for a reference body and a global criterion excluding occurrence of the bifurcation state for a case of kinematically admissible stress rate fields.

4.2.4 Global condition for a comparison body dependent on statically admissible stress rate fields

The function I' dependent on $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{\sigma}}^*$ is introduced in the following way

$$I'\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*\right) = \Delta\dot{\boldsymbol{\sigma}} : \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} - \frac{1}{4y^2 h_1}\left[\left(\mathbf{g} - y^2\bar{\mathbf{F}}_{\boldsymbol{\sigma}}\right) : \Delta\dot{\boldsymbol{\sigma}}\right]^2, \tag{4.20}$$

where y^2 is a scalar parameter.

The above expression expresses a one-parameter series of expressions I' related to the parameter y^2 . The functions I and I' depend not only on independent variables $\dot{\boldsymbol{\sigma}}$ and $\dot{\boldsymbol{\sigma}}^*$, but on the thermodynamic state as well (3.2).

In coupled generalized thermo-elasto-plasticity, an idea of the reference body dependent on statically acceptable stress rate fields was introduced in the author's papers [21,22,47]. As in the case of kinematically admissible strain rate fields, from comparison of the expression (4.20) with (4.17) it appears that it is a certain linear dependence between ($\Delta\dot{\boldsymbol{\sigma}}$ and $\Delta\dot{\boldsymbol{\varepsilon}}$). Differentiating I' in relations to $\Delta\dot{\boldsymbol{\sigma}}$, we obtain a certain linear dependence between $\Delta\dot{\boldsymbol{\varepsilon}}$ and $\Delta\dot{\boldsymbol{\sigma}}$, which does not occur in the expression (4.17), because in Eq. (4.17) those dependences are not linear.

Lemma 2 *Let us demonstrate that under a given thermodynamic state, the same for I and I', the following inequality is true for each pair $\left\{\dot{\boldsymbol{\sigma}} \text{ and } \dot{\boldsymbol{\sigma}}^*\right\}$ and each combination j and j^**

$$I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ j,\ j^*\right) - I'\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ j,\ j^*\right) \ \geq\ 0. \tag{4.21}$$

The following notations are introduced for the function $I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ j,\ j^*\right)$ as

$$\begin{cases} I_1 = I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ 1,\ 1\right) & \text{when}\ \ j\left(\dot{\boldsymbol{\sigma}}\right) = 1 \ \ \text{and}\ \ j\left(\dot{\boldsymbol{\sigma}}^*\right) = 1, \\ I_2 = I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ 1,\ 0\right) & \text{when}\ \ j\left(\dot{\boldsymbol{\sigma}}\right) = 1 \ \ \text{and}\ \ j\left(\dot{\boldsymbol{\sigma}}^*\right) = 0, \\ I_3 = I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ 0,\ 1\right) & \text{when}\ \ j\left(\dot{\boldsymbol{\sigma}}\right) = 0 \ \ \text{and}\ \ j\left(\dot{\boldsymbol{\sigma}}^*\right) = 1, \\ I_4 = I\left(\dot{\boldsymbol{\sigma}},\ \dot{\boldsymbol{\sigma}}^*,\ 0,\ 0\right) & \text{when}\ \ j\left(\dot{\boldsymbol{\sigma}}\right) = 0 \ \ \text{and}\ \ j\left(\dot{\boldsymbol{\sigma}}^*\right) = 0. \end{cases} \tag{4.22}$$

From the expressions (4.22), we obtain

$$\begin{cases} I_1 = \Delta\dot{\boldsymbol{\sigma}} : \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} + \frac{1}{h_1}\gamma_g \gamma_f, \\ I_2 = \Delta\dot{\boldsymbol{\sigma}} : \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} + \frac{1}{h_1}\gamma_g A_{\boldsymbol{\sigma}}, \\ I_3 = \Delta\dot{\boldsymbol{\sigma}} : \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}} + \frac{1}{h_1}\gamma_g A_{\boldsymbol{\sigma}}^*, \\ I_4 = \Delta\dot{\boldsymbol{\sigma}} : \mathbf{L}_1 \Delta\dot{\boldsymbol{\sigma}}. \end{cases} \tag{4.23}$$

Next, calculating the difference (4.21) for all the possible above four cases from the expressions (4.17), (4.20) and (4.22), we obtain

$$\begin{cases} \left(I_1 - I'\right) h_1 = \frac{1}{4y^2}\left(\gamma_g + y^2\gamma_f\right)^2 \geq 0, \\ \left(I_2 - I'\right) h_1 = -y^2 A_\sigma A_\sigma^* + \left[y A_\sigma^* + \frac{1}{2}\left(\gamma_g + y^2\gamma_f\right)\right]^2 \geq 0, \\ \quad \text{because}\ \ A_\sigma \geq 0 \ \ \text{and}\ \ \ A_\sigma^* < 0, \\ \left(I_3 - I\prime\right) h_1 = -y^2 A_\sigma A_\sigma^* + \left[y A_\sigma - \frac{1}{2}\left(\gamma_g + y^2\gamma_f\right)\right]^2 \geq 0, \\ \quad \text{because}\ \ A_\sigma < 0 \ \ \text{and}\ \ \ A_\sigma^* \geq 0, \\ \left(I_4 - I'\right) h_1 = \left[y\gamma_f - \frac{1}{2}\left(\gamma_g + y^2\gamma_f\right)\right]^2 \geq 0, \end{cases} \tag{4.24}$$

where $\gamma_g = \mathbf{g}\!:\, \Delta\dot{\boldsymbol{\sigma}}$ and $\gamma_f = \bar{\mathbf{F}}_{\boldsymbol{\sigma}}\!:\!\Delta\dot{\boldsymbol{\sigma}}$ and $\gamma_f = A_\sigma - A_\sigma^*$ and also

$$A_\sigma = \bar{\mathbf{F}}_{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}} + z_1 \quad \text{and} \quad A_\sigma^* = \bar{\mathbf{F}}_{\boldsymbol{\sigma}} : \dot{\boldsymbol{\sigma}}^* + z_1. \tag{4.25}$$

From the set of inequalities (4.24), it appears that the inequality (4.21) is true.

Using the inequality $\Lambda^* > 0$, see the expression (4.3), and the inequalities (4.21) and (4.24), we can formulate (like in the item 4.2.2) the following sufficient condition of uniqueness of a solution of the incremental boundary-value problem for the reference body dependent on statically admissible stress rate fields, which is a safer criterion excluding occurrence of the bifurcation state.

Theorem 3 *Let us assume $h_1 > 0$ in each point of the body $\boldsymbol{x} \in D_P$ in its part where plastic deformations take place, i.e. where $D_P = \{\boldsymbol{x}\colon F = 0\}$. If for each statically admissible stress velocity field $\dot{\boldsymbol{\sigma}}$ (or the field of stress rate difference $\Delta\dot{\boldsymbol{\sigma}}$), which disappears on a surface part (a body boundary) S_t, the following inequality is satisfied*

$$\int_D I_1'(\dot{\boldsymbol{\sigma}})\,\mathrm{d}V - \int_{D_P} I'_2(\dot{\boldsymbol{\sigma}})\,\mathrm{d}V > 0, \tag{4.26}$$

then only one pair $\{\dot{\boldsymbol{\varepsilon}},\ \dot{T}\}$ being a solution of the incremental boundary problem of coupled generalized thermoplasticity can exist. It is easy to prove that the above introduced criterion is true, like in the case of kinematically acceptable strain rate fields.

Proof Integrands occurring in the expression (4.26) have the following form

$$I_1'(\dot{\boldsymbol{\sigma}}) = \dot{\boldsymbol{\sigma}}\colon \mathbf{L}_1\dot{\boldsymbol{\sigma}} \quad \text{and} \quad I_2' = -\frac{1}{4y^2h_1}\left[\left(\mathbf{g} - y^2\bar{\mathbf{F}}_{\sigma}\right) : \dot{\boldsymbol{\sigma}}\right]^2, \tag{4.27}$$

Truth of the above condition of uniqueness for a reference body being a safer criterion excluding bifurcation results directly from satisfying the inequality $\Lambda^* > 0$ [see the expression (4.3)] and the inequalities (4.21) and (4.24).

The integral condition (4.26) presented in this form is very important from a practical point of view. Namely, if for a given thermodynamical state $\{T, \boldsymbol{\sigma}, K\}$ it is not possible to find such statically admissible stress velocity field $\dot{\boldsymbol{\sigma}}$, for which a sum of integrals occurring at the left side of the expression is equal to zero, then we must be sure that such a state is safe from the point of view of possibility of bifurcation state occurrence.

In Ref. [21] for the case of large deformations and in [47] for the case of small deformations, it is shown that the sufficient local condition of uniqueness resulting from the requirement of a positively defined integrand I' is the same as for a case of a thermo-elasto-plastic body (2.44a) and (4.18) when the parameter y^2 takes the following optimum form

$$y_0^2 = \left(\frac{\mathbf{g}\colon \mathbf{M}_1\mathbf{g}}{\bar{\mathbf{F}}_{\sigma} : \mathbf{M}_1\bar{\mathbf{F}}_{\sigma}}\right)^{\frac{1}{2}}. \tag{4.28}$$

For the parameter y_0^2, the local condition of uniqueness becomes the optimum (safest) condition from all the set of conditions. Substituting a value of (4.28) into the expressions (4.20) and (4.26), we obtain the optimum element of integration I_0' and the optimum form of the bifurcation criterion as

$$I_0' = \dot{\boldsymbol{\sigma}}\colon \mathbf{L}_1\dot{\boldsymbol{\sigma}} - \frac{1}{4h_1}\frac{\left[\left(\bar{\mathbf{F}}_{\sigma} : \mathbf{M}_1\bar{\mathbf{F}}_{\sigma}\right)^{\frac{1}{2}}\mathbf{g} : \dot{\boldsymbol{\sigma}} - \left(\mathbf{g} : \mathbf{M}_1\mathbf{g}\right)^{\frac{1}{2}}\bar{\mathbf{F}}_{\sigma} : \dot{\boldsymbol{\sigma}}\right]^2}{\left(\bar{\mathbf{F}}_{\sigma} : \mathbf{M}_1\bar{\mathbf{F}}_{\sigma}\right)^{\frac{1}{2}}\left(\mathbf{g} : \mathbf{M}_1\mathbf{g}\right)^{\frac{1}{2}}}, \tag{4.29}$$

and

$$\int_D (\dot{\boldsymbol{\sigma}} : \mathbf{L}_1\dot{\boldsymbol{\sigma}})\mathrm{d}V - \frac{1}{4}\int_{D_p}\frac{\left[\left(\bar{\mathbf{F}}_{\sigma} : \mathbf{M}_1\bar{\mathbf{F}}_{\sigma}\right)^{\frac{1}{2}}\mathbf{g} : \dot{\boldsymbol{\sigma}} - \left(\mathbf{g} : \mathbf{M}_1\mathbf{g}\right)^{\frac{1}{2}}\bar{\mathbf{F}}_{\sigma} : \dot{\boldsymbol{\sigma}}\right]^2}{h_1\left(\bar{\mathbf{F}}_{\sigma} : \mathbf{M}_1\bar{\mathbf{F}}_{\sigma}\right)^{\frac{1}{2}}\left(\mathbf{g} : \mathbf{M}_1\mathbf{g}\right)^{\frac{1}{2}}}\mathrm{d}V > 0. \tag{4.30}$$

Now we can state that in the case of the comparison body expressed by Eqs. (4.7), (4.13) or as J_1 $(4.8)_1$ dependent on kinematically admissible strain rate fields at the boundary transition $\left(\mathbf{g}^* = \bar{\mathbf{F}}_{\sigma}^*\right)$ and $(x^2 = 1)$, we obtain a body of coupled generalized thermoplasticity determined by (4.5). From the expressions (4.20) or (4.27) for the reference body dependent on statically acceptable stress velocity fields, it appears that substituting at the boundary $(\mathbf{g} = \bar{\mathbf{F}}_{\sigma})$ and the value $(y^2 = 1)$ we obtain the expression I_4 like for the thermoelastic body. Thus, reference bodies are not obtained by their mutual inversion like in the case of a thermo-elasto-plastic bodies; they are independently derived so as to satisfy the inequalities (4.8), (4.10) and (4.12) for a body dependent on kinematically acceptable strain fields and inequalities (4.21), (4.24) and (4.26) for the reference body dependent on statically acceptable stress velocity fields.

5 Local uniqueness conditions for special cases of bodies

In the constitutive equations of coupled generalized thermoplasticity (2.24) and (2.26) and in the comparison bodies as well in derived local conditions o uniqueness, see expression (4.1) the functions $\mathbf{g}$, $\bar{\mathbf{F}}_{\boldsymbol{\sigma}}$, $\mathbf{M}^{\mathbf{a}}$ and m_σ, occur. Their forms will be different in the case of less general models of bodies. In such cases, the functions will be simpler.

Case 1 The associated laws of plastic flow, when $F_{1,\Pi} = 0$, then $F_1(T, \boldsymbol{\sigma}, K) = F(T, \boldsymbol{\sigma}, K)$.

In such a case, the functions $\mathbf{g}$, $\bar{\mathbf{F}}_{\boldsymbol{\sigma}}$ and m_σ take the following forms:

$$\mathbf{g} = (\mathbf{F}_{\boldsymbol{\sigma}} + \mathbf{Z}d + m_\sigma \boldsymbol{\alpha}), \quad \bar{\mathbf{F}}_{\boldsymbol{\sigma}} = (\mathbf{F}_{\boldsymbol{\sigma}} - \xi F_T \boldsymbol{\alpha}), \tag{5.1}$$

where $m_\sigma = \frac{1}{\rho_0 c_\sigma}\left[(\boldsymbol{\sigma} : \mathbf{F}_{\boldsymbol{\sigma}} - \Pi \cdot d) - T\left(\frac{\partial \Pi(T,\boldsymbol{\sigma},K)}{\partial T} \cdot d\right)\right]$.

Function $\mathbf{M}^{(\mathrm{a})}$ is the same as in Sect. 2.

Here, the generalized function of plastic flow F_1(plastic potential) does not depend on a vector of internal pairs of dissipation forces Π, which are dependent on the stress state (see [1,3,21,45–47]). Moreover, all the effects of the thermomechanical couplings and effects of the elastic–plastic couplings are kept.

Case 2 The case not including effect of the elastic plastic coupling. Then

$$\mathbf{g} = \left(\mathbf{F}_{1,\boldsymbol{\sigma}} + m_\sigma \boldsymbol{\alpha}\right), \quad \bar{\mathbf{F}}_{\boldsymbol{\sigma}} = (\mathbf{F}_{\boldsymbol{\sigma}} - \xi F_T \boldsymbol{\alpha}), \tag{5.2}$$

where $m_\sigma = \frac{1}{\rho_0 c_\sigma}\left[(\boldsymbol{\sigma} : \mathbf{F}_{1,\boldsymbol{\sigma}} - \Pi \cdot d) - T\left(\frac{\partial \Pi(T,\boldsymbol{\sigma},K)}{\partial T} \cdot d\right)\right]$,
and the function $\mathbf{M}^{(\mathrm{a})}$ remains the same as previously.

In this case, also all the effects of thermomechanical couplings and the non-associated laws of plastic flow are still valid. A model of such a body can be useful, for example for analysis of adiabatic and non-isothermal processes of location of plastic strains and non-isothermal, adiabatic or quasi-adiabatic processes of plastic deformation and forming of metals. Such local adiabatic instabilities can occur during some machining processes, for example while milling or turning, and they cause vibrations and irregularities of the machined surface or other negative effects.

Case 3 The associated laws of plastic flow with no effect of elastic–plastic coupling. Then

$$\mathbf{F}_{1,\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}, \quad \mathbf{g} = \left(\mathbf{F}_{1,\boldsymbol{\sigma}} + m_\sigma \boldsymbol{\alpha}\right), \quad \bar{\mathbf{F}}_{\boldsymbol{\sigma}} = (\mathbf{F}_{\boldsymbol{\sigma}} - \xi F_T \boldsymbol{\alpha}), \tag{5.3}$$

where $m_\sigma = \frac{1}{\rho_0 c_\sigma}\left[(\boldsymbol{\sigma} : \mathbf{F}_{\boldsymbol{\sigma}} - \Pi \cdot d) - T\left(\frac{\partial \Pi(T,\boldsymbol{\sigma},K)}{\partial T} \cdot d\right)\right]$.

The same result described by the constitutive functions (5.3) can be also obtained according to the Gyarmati postulate, see [1,2,21,47]. In this case, we have the model of coupled thermoplasticity considered by Mróz and Raniecki [4,5], Raniecki and Sawczuk [15,16].

Case 4 The case of constitutive approximate equations.

This case of constitutive approximate functions of conjugate thermoplasticity was considered in [1,6,7,21,47], where

$$\mathbf{F}_{1,\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}, \quad \mathbf{g} = \mathbf{F}_{\boldsymbol{\sigma}}, \quad \bar{\mathbf{F}}_{\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}, \quad \mathbf{M}^{(\mathbf{a})} = \mathbf{M}, \tag{5.4}$$

and $m_\sigma = \frac{1}{\rho_0 c_\sigma}\left[(\boldsymbol{\sigma} : \mathbf{F}_{\boldsymbol{\sigma}} - \Pi \cdot d) - T\left(\frac{\partial \Pi(T,\boldsymbol{\sigma},K)}{\partial T} \cdot d\right)\right]$.

Here, from analysis of expressions (2.39), (2.40) and (4.6) it appears that $h_1 > 0$ and $H_1 = h + \mathbf{F}_{\boldsymbol{\sigma}} : \mathbf{M}\mathbf{F}_{\boldsymbol{\sigma}} > 0$.

These expressions are obtained for the same assumptions as those formulated in case 3. Moreover, the piezocaloric effect ($\xi F_T \boldsymbol{\alpha} = \mathbf{0}$) and thermal expansion caused by dissipation heat and heat of internal changes (see [1,4–7,16,47]) are neglected.

Case 5 Isothermal theory of plasticity concerning non-associated plastic flow with elastic–plastic coupling and with no thermal coupling. Then

$$\bar{\mathbf{F}}_{\boldsymbol{\sigma}} = \mathbf{F}_{\boldsymbol{\sigma}}, \ \mathbf{g} = \left(\mathbf{F}_{1,\boldsymbol{\sigma}} + \mathbf{Z}d\right), \quad \mathbf{M}^{(\mathbf{a})} = \mathbf{M}, \quad m_\sigma = 0 \quad \text{and} \quad h_1 = h. \tag{5.5}$$

Such a model of the elastic–plastic body is often applied for description of porous materials, sinters, rocks and soils [10–12]. In this model, influence of plastic strains on elastic properties of the body is included.

Case 6 Isothermal theory of plasticity concerning the non-associated law of plastic flow without the effect of elastic–plastic coupling and without the effects of thermal couplings. Then

$$\mathbf{g} = \mathbf{F_\sigma}, \quad \bar{\mathbf{F}}_\sigma = \mathbf{F_\sigma}, \mathbf{M^{(a)}} = \mathbf{M}, \quad m_\sigma = 0 \quad \text{and} \quad h_1 = h. \tag{5.6}$$

Conditions of uniqueness for this model were studied by Mróz [8,9]. In Ref. [9] Mróz also derived the sufficient local condition of uniqueness for compressible and isotropic elastic–plastic bodies.

The presented chosen cases of elastic–plastic bodies and the corresponding global and local conditions of uniqueness of a solution of the incremental boundary problem are not all possible models of bodies resulting from the generalized model of coupled thermoplasticity derived in [1–3,21,45–47]. The presented cases of body models $1 \div 6$ are more or less similar to standard models of elastic–plastic bodies, discussed previously in the literature.

6 Conclusions

1. In the paper, the necessary and sufficient conditions of uniqueness of solution of the formulated incremental boundary problem of coupled generalized thermoplasticity for small gradients of displacements (small strains) were derived. Global sufficient conditions and also local sufficient conditions (more safe for small strains) were derived. Conditions of uniqueness for the generalized thermoplastic body [1–3,21,45–47] and for suitable comparison bodies [1,3–7,13,14,21,47] were determined. The derived conditions of uniqueness (global and local) are suitable necessary and sufficient criteria excluding bifurcation of equilibrium states of coupled generalized thermoplasticity, also in isothermal loading processes. It was also shown that the local conditions of uniqueness for the generalized thermoplastic bodies and the comparison bodies have the same form. Thus, introduction of such comparison bodies seems to be proper. But the global conditions of uniqueness and the global criteria of bifurcation have different forms.
2. The set papers, see e.g. [1,3–9,13,14,16,47], are concerning problems of solution uniqueness of equilibrium states and bifurcations criteria. In this paper, however, a new global criterion was formulated for the derived comparison body dependent on statically permissible fields of stress rate. Thus, this paper is continuation of the previous author's papers [1–3,21,47]. The conditions of uniqueness and bifurcation criteria derived in the previous papers [1,3] concerned a comparison body derived for generalized coupled thermoplasticity depending on kinematically admissible strain rate fields.
3. We can assume that in the areas of the plastically deformed body where the conditions of uniqueness or the bifurcation criteria are exceeded, submicro- or microconcentrations of strains can occur. Microcracks and microlocalizations of strains are possible; then they become macrolocalizations while further developing and nucleation occurs, leading to a crack in the material. Influence of such concentrators can be especially important under variable mechanical and thermomechanical loadings, or creep, so connected with fatigue strength or material cracking [7,10,11,16,24,25,33,44,47].
4. In the case of the comparison body, the integrand J' (4.7), dependent on kinematically permissible strain rate field, while boundary transition $\left(\mathbf{g}^* = \bar{\mathbf{F}}^*_\sigma\right)$ and $(x^2 = 1)$ passes into the integrand J_1, like in the case of the elastic–plastic body in coupled thermoplasticity with associated laws of plastic flow, expressed by (4.9). From the expression for I' (4.20) for the comparison body dependent on statically admissible rate fields of stresses, it results that introducing $(\mathbf{g} = \bar{\mathbf{F}}_\sigma)$ and $(y^2 = 1)$ we obtain the expression I_4(4.23), like for the thermoelastic body. The reference bodies are not obtained by their mutual mathematical inversion (like in the case of thermo-elasto-plastic bodies), but they are independently derived so as to satisfy suitable inequalities (4.8), (4.10), (4.12) and (4.21), (4.24), (4.26).
5. In a generalized case, constitutive equations of coupled thermoplasticity are of the character of non-associated laws of plastic flow, and even in the case of assumption of Gyarmati postulate, see [1,2,46,47], they include effects of thermomechanical couplings and include a phenomenon of elastic–plastic coupling. It means that they can be applied for a description of not only plastic metallic, brittle and semi-brittle materials, but porous materials, sintered powders, rocks, soils, concretes and other materials as well [10–12,47]. Under processes of plastic deformation and plastic strain localization, unstability can occur in many processes of plastic working, both stationary and non-stationary, cold, hot and heated.

6. In the paper, only general expressions were derived for constitutive functions of coupled generalized thermoplasticity. They occur in both necessary and sufficient global and local conditions of uniqueness of solution of the formulated incremental boundary-value problem. During further investigations, the constitutive functions of coupled generalized thermoplasticity should be specified within mechanics of continuous media according to experimental results.
7. In the paper, it is assumed that gradients of displacements and strain rates are small. For simplicity purposes, it is assumed that all the mathematical operations and all the description are realized in the Cartesian coordinate system. In some last years, many papers describing some chosen kinds of thermomechanical couplings including large deformations were published, see e.g. [21–44].
8. The future tests concerning uniqueness of the solutions of incremental boundary problems can be realized also for viscoplastic materials. Hence, for the case of non-associated laws of plastic flow, the generalized function of plastic flow F_1 (plastic potential) should be replaced by the dynamic function of flow dependent on F_1, see [48–50]. In the case of the associated laws of plastic flow (then $F_1 = F$ and $\mathbf{F}_{1,\sigma} = \mathbf{F}_{\sigma}$), the function of plastic flow F should be replaced by the dynamic function dependent on F. This problem remains open to further research.

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