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Superrosiness and dense pairs of geometric structures

Gareth J. Boxall[1]

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Abstract
Let T be a complete geometric theory and let T_P be the theory of dense pairs of models of T. We show that if T is superrosy with $U^{\text{þ}}$-rank 1 then T_P is superrosy with $U^{\text{þ}}$-rank at most ω.

Keywords Geometric structures · Dense pairs · Superrosy theories

Mathematics Subject Classification 03C45 · 03C64

1 Introduction

In [7] van den Dries studied dense pairs of o-minimal expansions of ordered groups, generalising earlier work of Robinson [13] on dense pairs of real-closed fields. The pair is formed by adding to the language of the o-minimal structure a unary predicate which is interpreted as a dense and co-dense elementary substructure. Robinson also studied pairs of algebraically closed fields. Poizat provided a generalisation of this in the stable setting in [12] and Ben-Yaacov, Pillay and Vassiliev [2] later generalised further in the simple setting. In rank one, Buechler [6] studied the case when the original structure is strongly minimal. Vassiliev [14] generalised to when it is simple with SU-rank 1, prior to the above-mentioned work with Ben-Yaacov and Pillay. Then in [3] Berenstein and Vassiliev generalised to the situation where the theory T of the original structure is geometric. So T has infinite models, eliminates $\exists^\infty$ and the algebraic closure operator satisfies the exchange principle: $a \in \text{acl}(Bc)\backslash\text{acl}(B) \Rightarrow c \in \text{acl}(Ba)$ (see [10] and [9]).

Lovely pairs of geometric structures were defined in [3] (their Definition 2.3), that terminology having previously been used in [2] for the analogous notion considered there. Definition 1.1, below, is only slightly different to Definition 2.3 in [3] and is

✉ Gareth J. Boxall
gboxall@sun.ac.za

1 Mathematics Division, Department of Mathematical Sciences, Stellenbosch University, Private Bag X1, Matieland 7602, South Africa

equivalent assuming sufficient saturation. It has the slight advantage that the important example of the real ordered field, expanded by a predicate for the subfield of all real algebraic numbers, is indeed an example (one does not have to move to an elementary extension). The term "dense pair" evokes this example and, more generally, the class of structures studied by van den Dries in [7].

Definition 1.1 Let T be a complete theory with respect to a language L. Assume T is geometric. Let P be a unary predicate such that $P \notin L$ and let $L_P = L \cup \{P\}$. A dense pair of models of T is an L_P-structure with underlying set M such that, using the language L,

(1) $P(M)$ is an elementary substructure of M,
(2) for every infinite definable $X \subseteq M$, $X \cap P(M)$ is not empty and
(3) for every infinite definable $X \subseteq M$ and every finite $B \subseteq M$, X contains an element that is not algebraic over $B \cup P(M)$.

To distinguish between the underlying L-structure and the dense pair, we refer to the former as M and the latter as $(M, P(M))$. An important part of the study of pairs has been to investigate the extent to which stability-theoretic properties transfer from the original tame structure to the slightly less tame pair. In [14], generalising earlier work of Buechler, Vassiliev studied forking in dense pairs of SU-rank 1 structures. He proved that the pair will have SU-rank at most ω and that one-types containing the formula $P(x)$ will have SU-rank at most 1. In a setting that includes both stable and o-minimal examples, it is natural to consider þ-forking instead of forking. The following definition is from [11] (see also [8] and [1]).

Definition 1.2 Let T be any complete theory with respect to a language L. Let M be a sufficiently saturated model of T, $B \subseteq M^{eq}$ a parameter set, $\varphi(x, y)$ a formula and $b \in M^{eq}$ a parameter. Then $\varphi(x, b)$ is said to þ-divide over B if there exist a parameter set $D \subseteq M^{eq}$ and a positive integer k such that $B \subseteq D$, $b \notin \operatorname{acl}(D)$ and $\{\varphi(x, b') : b' \models \operatorname{tp}(b/D)\}$ is k-inconsistent.

Let $C \subseteq M^{eq}$ be a parameter set such that $B \subseteq C$. The complete type $\operatorname{tp}(a/C)$ is said to þ-fork over B if there are a positive integer m, formulas $\varphi_1(x, y_1), \ldots, \varphi_m(x, y_m)$ and parameters $b_1, \ldots, b_m \in M^{eq}$ such that $\operatorname{tp}(a/C)$ implies $\varphi_1(x, b_1) \vee \ldots \vee \varphi_m(x, b_m)$ and, for each $i \in \{1, \ldots, m\}$, $\varphi_i(x, b_i)$ þ-divides over B.

One can hope to define an ordinal-valued $U^{þ}$-rank associated with þ-forking as follows. For each complete type $\operatorname{tp}(a/C)$ and $B \subseteq C$, $U^{þ}(a/C) \leq U^{þ}(a/B)$ and $U^{þ}(a/C) < U^{þ}(a/B)$ if and only if $\operatorname{tp}(a/C)$ þ-forks over B. When this is possible, then $U^{þ}$ is chosen to be minimal and the theory is said to be superrosy (see [8]).

It is natural to ask if the analogue of Vassiliev's result for SU-rank holds for $U^{þ}$-rank in the setting of dense pairs of geometric structures. We prove the following.

Theorem 1.3 *Let T be a geometric complete theory which is superrosy and for which every one-type has $U^{þ}$-rank at most 1. Then T_P is superrosy. Furthermore, every one-type has $U^{þ}$-rank at most ω and every one-type containing the formula $P(x)$ has $U^{þ}$-rank at most 1.*

In the case of the dense pairs of o-minimal expansions of ordered groups, studied by van den Dries [7], this result was proved by Berenstein, Ealy and Günaydın in [5]. The argument was extended to cover all dense pairs of o-minimal structures by Berenstein and Vassiliev in [3].

I communicated Theorem 1.3, which was obtained in my PhD thesis, to Berenstein and Vassiliev in time for them to refer to it in [3], and in fact they used it in the proof of their Proposition 4.8. In subsequent work [4], they proved the analogue of Theorem 1.3 for H-structures. These are expansions of geometric structures that have many properties in common with dense pairs. An example is obtained by expanding the real ordered field by a unary predicate for a dense, algebraically independent subset and then taking a sufficiently saturated elementary extension.

2 Preliminaries

For the rest of the paper, we fix a language L and a complete theory T which is assumed to be geometric. We set $L_P = L \cup \{P\}$, where P is a unary predicate and $P \notin L$. Let T_P be the L_P-theory of dense pairs of models of T, the axioms for which can be inferred from Definition 1.1. Generalising earlier work, it is shown in [3] that T_P is consistent and complete and that every formula is equivalent to a finite boolean combination of existential formulas.

We fix a sufficiently saturated model $(M, P(M)) \models T_P$. The L-reduct is denoted M and then $M \models T$. We use acl to denote the algebraic closure operator in $(M, P(M))^{eq}$, while acl_L denotes algebraic closure in M^{eq}. Similarly, $\mathrm{tp}(a/B)$ is the complete type of a over B in the sense of $(M, P(M))^{eq}$, while $\mathrm{tp}_L(a/B)$ is the complete type of a over B in the sense of M^{eq}. We follow the standard convention of abbreviating $B \cup C$ to BC.

A concept of smallness has played an important role in the study of dense pairs. A subset $X \subseteq M^n$, definable in $(M, P(M))$, is small if there is a finite set $C \subseteq M$ such that $X \subseteq \mathrm{acl}_L(CP(M))$ (see Section 2 of [5]). When we say that a set is large, we mean that it is not small. Whenever we refer to a set as a parameter set we mean to imply that its cardinality is less than κ, where κ is a cardinal such that $(M, P(M))$ is κ-saturated. A parameter set $D \subseteq M$ is said to be P-independent if, for every finite $C \subseteq D$, $\dim(C/D \cap P(M)) = \dim(C/P(M))$, where dim is defined in terms of acl_L in the usual way. Every parameter set $D \subseteq M$ is contained in a P-independent parameter set. The notion of P-independence plays a crucial role in [3]. A key result in [3] is their Lemma 2.8 which tells us that the type of a P-independent tuple is determined by the fact that it is P-independent, together with its type in the language L and its $P(x)$-type (which coordinates are in $P(M)$ and which are not).

Proposition 2.1 *Let $X \subseteq M$ be definable in $(M, P(M))$.*

(1) If $X \subseteq P(M)$ then there is a set $Y \subseteq M$ such that Y is definable in M and $X = Y \cap P(M)$.
(2) If X is large then there is a set $Y \subseteq M$ such that Y is definable in M and $(X \setminus Y) \cup (Y \setminus X)$ is small.

Proof The first statement is the $n = 1$ case of Proposition 3.4 in [3]. For the second statement, let $B \subseteq M$ be a parameter set over which X is defined. We may assume B is P-independent. For any element $a \in M$ such that $a \notin \mathrm{acl}_L(P(M)B)$, $B \cup \{a\}$ is P-independent and Lemma 2.8 in [3] tells us that $\mathrm{tp}(a/B)$ is determined by the fact that $a \notin \mathrm{acl}_L(P(M)B)$ together with $\mathrm{tp}_L(a/B)$. The result then follows by compactness. □

Proposition 2.2 *Let $D \subseteq M$ be a P-independent parameter set and let $e \in M^{eq}$. Then $e \in acl(D)$ if and only if $e \in acl_L(D)$.*

Proof The "if" part is clear. Suppose $e \in \mathrm{acl}(D)$. We may assume e is definable over D, since otherwise we could replace it with a canonical parameter (in M^{eq}) for the finite set of realisations of $\mathrm{tp}(e/D)$. Let $\bar{b} = \langle b_1, ..., b_n \rangle$ be a tuple from M that is acl_L-independent over D and such that $e \in \mathrm{acl}_L(D\bar{b})$. We may assume $\bar{b}$ has been chosen so that $D\bar{b}$ is P-independent. Now let $\bar{b}' \in M^n$ be such that $\bar{b}' \models \mathrm{tp}_L(\bar{b}/D)$ and $\bar{b}'$ is acl_L-independent over $D\bar{b}$. Suppose $e \notin \mathrm{acl}_L(D\bar{b}')$. Using acl_L-independence and Definition 1.1, we obtain $\bar{b}'' \in M^n$ such that $\bar{b}'' \models \mathrm{tp}_L(\bar{b}'/D\bar{b}e)$, $D\bar{b}''$ is P-independent and the tuples $\bar{b}$ and $\bar{b}''$ have the same $P(x)$-type. It follows by Lemma 2.8 in [3] that $\bar{b}'' \models \mathrm{tp}(\bar{b}/D)$. We then have $e \in \mathrm{acl}_L(D\bar{b}'')$. Since $\bar{b}'' \models \mathrm{tp}_L(\bar{b}'/D\bar{b}e)$, we must have $e \notin \mathrm{acl}_L(D\bar{b}'')$ and so we get a contradiction. So $e \in \mathrm{acl}_L(D\bar{b}')$. It follows that $e \in \mathrm{acl}_L(D)$. □

We conclude this section with some properties of þ-forking and $U^{\text{þ}}$-rank that we shall need, all of which are known.

Proposition 2.3 *Let B, C be parameter sets in $(M, P(M))^{eq}$ such that $B \subseteq C$. Let $a_1, a_2 \in (M, P(M))^{eq}$. If $tp(a_1/C)$ does not þ-fork over B and $tp(a_2/Ca_1)$ does not þ-fork over Ba_1 then $tp(a_1a_2/C)$ does not þ-fork over B.*

Proof Lemma 2.1.6 in [11]. □

Proposition 2.4 *Let B, C be parameter sets in $(M, P(M))^{eq}$ such that $B \subseteq C$. Let $a, a' \in (M, P(M))^{eq}$ such that $a \in acl(Ba')$. If $tp(a'/C)$ does not þ-fork over B then $tp(a/C)$ does not þ-fork over B.*

Proof Immediate from Definition 1.2. □

Proposition 2.5 *Suppose that T is superrosy, with every one-type having $U^{\text{þ}}$-rank at most 1. Let $D \subseteq M^{eq}$ be a parameter set and let $a \in M$ and $e \in M^{eq}$. If $e \in acl_L(Da) \setminus acl_L(D)$ then $a \in acl_L(De)$.*

Proof Suppose $e \in \mathrm{acl}_L(Da) \setminus \mathrm{acl}_L(D)$. Then $\mathrm{tp}(e/Da)$ þ-forks over D. By the symmetry of þ-independence (see Theorem 3.7 in [8]), it follows that $\mathrm{tp}(a/De)$ þ-forks over D. So $U^{\text{þ}}(a/De) < U^{\text{þ}}(a/D) \leq 1$. Therefore $U^{\text{þ}}(a/De) = 0$ and so $a \in \mathrm{acl}_L(De)$. □

3 Proof

We now prove Theorem 1.3. The argument is based on the proof of Theorem 3 in [5]. We begin with two lemmas.

Lemma 3.1 *Let $\varphi(x, y)$ be an L_P-formula and let $b \in (M, P(M))^{eq}$. Suppose $\varphi(x, b)$ defines an infinite set $X \subseteq P(M)$ for which b is a canonical parameter. Then $\varphi(x, b)$ does not þ-divide over the empty set.*

Proof Suppose $\varphi(x, b)$ þ-divides over the empty set. By Definition 1.2, we get a parameter set $D \subseteq (M, P(M))^{eq}$ and a positive integer k such that $b \notin \mathrm{acl}(D)$ and $\{\varphi(x, b') : b' \models \mathrm{tp}(b/D)\}$ is k-inconsistent. It is possible to choose a P-independent parameter set $D' \subseteq M$ such that each element of D is definable over D' and $b \notin \mathrm{acl}(D')$. Then $\{\varphi(x, b') : b' \models \mathrm{tp}(b/D')\}$ is k-inconsistent. Let $a \in X$ be such that $a \notin \mathrm{acl}(D'b)$. By k-inconsistency, $b \in \mathrm{acl}(D'a)$. We have

$$b \in \mathrm{acl}(D'a) \setminus \mathrm{acl}(D') \text{ and } a \notin \mathrm{acl}(D'b).$$

By Proposition 2.1, there exist an L-formula $\psi(x, z)$ and a parameter $c \in M^n$ such that the set $Y \subseteq M$ defined by $\psi(x, c)$ satisfies the equation $X = Y \cap P(M)$. Let $\sim$ be the binary relation on M^n such that $c' \sim c''$ if and only if the formula $(\psi(x, c') \wedge (\neg\psi(x, c''))) \vee ((\neg\psi(x, c')) \wedge \psi(x, c''))$ defines a finite set. Then $\sim$ is an equivalence relation on M^n and, since T is geometric, it is definable. Let $e \in M^{eq}$ be a canonical parameter for the equivalence class of c. By Definition 1.1, $e \in \mathrm{acl}(b)$. Since e determines the infinite set X to within an error of finitely many elements, it follows by k-inconsistency that $b \in \mathrm{acl}(D'e)$. We then have

$$e \in \mathrm{acl}(D'a) \setminus \mathrm{acl}(D') \text{ and } a \notin \mathrm{acl}(D'e).$$

Clearly $\mathrm{acl}_L(D'e) \subseteq \mathrm{acl}(D'e)$ and $\mathrm{acl}_L(D') \subseteq \mathrm{acl}(D')$. Since $D'a$ is P-independent, Proposition 2.2 ensures that $e \in \mathrm{acl}(D'a) \Rightarrow e \in \mathrm{acl}_L(D'a)$.

So

$$e \in \mathrm{acl}_L(D'a) \setminus \mathrm{acl}_L(D') \text{ and } a \notin \mathrm{acl}_L(D'e).$$

This contradicts Proposition 2.5 and the result follows. □

Lemma 3.2 *Let $\varphi(x, y)$ be an L_P-formula and let $b \in (M, P(M))^{eq}$. Suppose $\varphi(x, b)$ defines a large set $X \subseteq M$ for which b is a canonical parameter. Then $\varphi(x, b)$ does not þ-divide over the empty set.*

Proof Our strategy is similar to that used in the proof of Lemma 3.1. Suppose $\varphi(x, b)$ þ-divides over the empty set. We get a P-independent parameter set $D' \subseteq M$ and a positive integer k such that $b \notin \mathrm{acl}(D')$ and $\{\varphi(x, b') : b' \models \mathrm{tp}(b/D')\}$ is k-inconsistent. Let $a \in X$ be such that $a \notin \mathrm{acl}(P(M)D'b)$. This ensures that $a \notin \mathrm{acl}(D'b)$ and $D'a$ is P-independent. By k-inconsistency, $b \in \mathrm{acl}(D'a)$. We have

$$b \in \mathrm{acl}(D'a) \setminus \mathrm{acl}(D') \text{ and } a \notin \mathrm{acl}(D'b).$$

By Proposition 2.1, there exist an L-formula $\psi(x, z)$ and a parameter $c \in M^n$ such that the set $Y \subseteq M$ defined by $\psi(x, c)$ has the property that $(X \setminus Y) \cup (Y \setminus X)$ is small. Let $\sim$ be as in the previous proof and let e be a canonical parameter for the equivalence class of c. By Definition 1.1, $e \in \mathrm{acl}(b)$. Since e determines the large set X to within a small error, it follows by k-inconsistency that $b \in \mathrm{acl}(D'e)$. We again have

$$e \in \mathrm{acl}(D'a) \setminus \mathrm{acl}(D') \text{ and } a \notin \mathrm{acl}(D'e)$$

and, as before, this leads to

$$e \in \mathrm{acl}_L(D'a) \setminus \mathrm{acl}_L(D') \text{ and } a \notin \mathrm{acl}_L(D'e)$$

which contradicts Proposition 2.5. The result follows. □

Let B, C be parameter sets in $(M, P(M))^{eq}$ and suppose $B \subseteq C$. Let $a \in M$ and suppose $\mathrm{tp}(a/C)$ þ-forks over B. By Definition 1.2, there exist formulas $\varphi_1(x, y_1), ..., \varphi_m(x, y_m)$ and parameters $b_1, ..., b_m \in (M, P(M))^{eq}$ such that $\varphi_1(x, b_1) \vee ... \vee \varphi_m(x, b_m)$ is implied by $\mathrm{tp}(a/C)$ and, for each i, $\varphi_i(x, b_i)$ þ-divides over B.

Case 1: Assume $a \in P(M)$ and $a \notin \mathrm{acl}(C)$. Then, for at least one i, $\varphi_i(x, b_i) \cap P(x)$ defines an infinite set $X \subseteq P(M)$. Let $b \in (M, P(M))^{eq}$ be a canonical parameter for X and let $\varphi(x, y)$ be such that $\varphi(x, b)$ defines X. It follows that $\varphi(x, b)$ þ-divides over B and so also over the empty set. This contradicts Lemma 3.1.

Case 2: Assume $\mathrm{tp}(a/B)$ is small (some formula in $\mathrm{tp}(a/B)$ defines a small set). Let n be a positive integer and let $B' \subseteq M$ be a parameter set such that, for every realisation $\tilde{a}$ of $\mathrm{tp}(a/B)$, there is some $a' \in P(M)^n$ such that $\tilde{a} \in \mathrm{acl}(B'a')$. By Case 1, in combination with Propositions 2.3 and 2.4, if $B'_0, B'_1, B'_2, ...$ is an increasing chain of parameter sets in $(M, P(M))^{eq}$, with $B'_0 = B'$, then at most n indices j have the property that $\mathrm{tp}(a/B'_{j+1})$ þ-forks over B'_j. Suppose there is an increasing chain of parameter sets $B_0, B_1, B_2, ...$ in $(M, P(M))^{eq}$ such that $B_0 = B$ and $\mathrm{tp}(a/B_{j+1})$ þ-forks over B_j for more that n indices j. By Definition 1.2 and compactness, we could then choose $B' = B'_0 \subseteq B'_1 \subseteq ...$ in such a way as to contradict our previous assertion.

Case 3: Assume $\mathrm{tp}(a/C)$ is not small. For at least one i, $\varphi_i(x, b_i)$ defines a large set $X \subseteq M$. Let $b \in (M, P(M))^{eq}$ be a canonical parameter for X and let $\varphi(x, y)$ be such that $\varphi(x, b)$ defines X. It follows that $\varphi(x, b)$ þ-divides over B and so also over the empty set. This contradicts Lemma 3.2.

By Case 1, every one-type containing $P(x)$ has $U^{þ}$-rank at most 1. By Case 2, every small one-type has finite $U^{þ}$-rank. By Case 3, if $\mathrm{tp}(a/C)$ þ-forks over B then $\mathrm{tp}(a/C)$ must be small. It follows that every one-type has $U^{þ}$-rank at most ω. It is known (see [8]) that this implies that every type has ordinal-valued $U^{þ}$-rank and so T is superrosy. For example, one could use Propositions 2.3 and 2.4. This completes the proof of Theorem 1.3.

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