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Recursive Polish spaces

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Abstract

This paper is concerned with the proper way to effectivize the notion of a Polish space. A theorem is proved that shows the recursive Polish space structure is not found in the effectively open subsets of a space $\mathcal{X}$, and we explore strong evidence that the effective structure is instead captured by the effectively open subsets of the product space $\mathbb{N} \times \mathcal{X}$.

Keywords Polish spaces · Recursive presentations · Effectively open sets

Mathematics Subject Classification 03D45 · 03E15

1 Introduction

Descriptive set theory studies definable subsets of Polish spaces, i.e, separable, completely metrizable topological spaces. Although we require of a Polish space the existence of a compatible complete metric that induces the topology, a particular choice of metric is not part of the structure of a Polish space. There are many reasons for omitting a choice of metric in the definition of a Polish space, for example:

(1) The Borel hierarchy of a space only depends on the topology, not the choice of Polish metric which induces the topology;
(2) Many natural constructions of spaces have a canonical choice of topology but no canonical choice of compatible metric; for instance, in forming the product of two spaces $\mathcal{X}$ and $\mathcal{Y}$, we have a natural product topology on $\mathcal{X} \times \mathcal{Y}$, but no canonical choice of compatible metric for the product space;
(3) When studying a Polish space space, one does not hesitate to swap an initial choice of compatible metric with an alternative compatible metric which is better suited to a particular construction.[1]

[1] See, for example, section 4.F of [1] on the hyperspace of compact sets for a construction that begins by selecting a compatible metric which is bounded by 1.

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In this paper, we examine the proper "effectivization" of the structure of a Polish space, i.e., we seek to determine the "effective topology" of a recursive Polish metric space $\mathcal{X}$. As developed in Moschovakis [2], the notion of a recursive Polish metric space provides a robust setting for the effective theory of Polish metric spaces. In subsequent work (see [3]), Moschovakis defined the effective topology of a space $\mathcal{X}$ to be the collection of effectively open subsets of the product space $\mathbb{N} \times \mathcal{X}$, which is called the *frame* of the space and denoted by

$$\mathcal{R}(\mathcal{X}) = \Sigma^0_1(\mathbb{N} \times \mathcal{X}).$$

Frames are then used by Moschovakis to define the notion of a *recursive Polish space*; a recursive Polish space is a pair $(\mathcal{X}, \mathcal{R})$ such that $\mathcal{R} \subset \mathbb{N} \times \mathcal{X}$ is the frame of a recursive Polish metric space on $\mathcal{X}$.

Since the topology of a space is the collection of open sets, it is indeed tempting to define the recursive topology of a space $\mathcal{X}$ to be the collection of effectively open sets, $\Sigma^0_1(\mathcal{X})$.[2] However, this definition is not suitable for the following reason. From our intuition from classical topology, we expect that the effective topology of a product $\mathcal{X} \times \mathcal{Y}$ would be determined by the effective topologies of its factors $\mathcal{X}$ and $\mathcal{Y}$. However, the collection of effectively open sets of a space does not determine the collection of effectively open sets of its product

Main Theorem. *There are two recursive Polish metric spaces $\mathcal{X}_1, \mathcal{X}_2$ with the same underlying set $\mathcal{X}_1 = \mathcal{X}_2$ such that*

$$\Sigma^0_1(\mathcal{X}_1) = \Sigma^0_1(\mathcal{X}_2) \textit{ but } \Sigma^0_1(\mathbb{N} \times \mathcal{X}_1) \neq \Sigma^0_1(\mathbb{N} \times \mathcal{X}_2).$$

While the collection $\Sigma^0_1(\mathcal{X}_1)$ determines which subsets are effectively open, the frame $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_1)$ encodes which sequences $(U_n)_n$ of effectively open sets are recursive *uniformly in n*. Thus, the theorem essentially says that it is possible for two recursive Polish metric spaces to have the same effectively opens sets, but different sequences of uniformly effectively open sets.

We will first discuss some necessary preliminaries in Sect. 2, and then prove the Main Theorem in Sect. 3. After proving the Main Theorem—establishing that the recursive Polish space structure is not determined by $\Sigma^0_1(\mathcal{X})$—we will further discuss in Sect. 4 the strong evidence that $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$ does properly effectivize the topological structure of Polish space.

2 Background

We summarize here the basics of the theory of recursive Polish metric spaces, which is more fully developed in [3].

Fix a recursive enumeration $q_0, q_1, \ldots$ of $\mathbb{Q}$. Let $\mathcal{X} = (\mathcal{X}, d)$ be a Polish metric space. A *recursive presentation* of $\mathcal{X}$ is a function $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ such that

[2] This definition is, in fact, made in multiple places in the literature; for instance, it is made on page 110 of [2], although the definition is not used after that.

(i) the image $\mathbf{r}[\mathbb{N}] = \{\mathbf{r}_0, \mathbf{r}_1, \ldots\}$ is dense in $\mathcal{X}$; and
(ii) the relations $P^{\leq}, P^{<} \subset \mathbb{N}^3$ defined by

$$P^{\leq}(i, j, k) \iff_{\mathrm{df}} d(\mathbf{r}_i, \mathbf{r}_j) \leq q_k, \quad P^{<}(i, j, k) \iff_{\mathrm{df}} d(\mathbf{r}_i, \mathbf{r}_j) < q_k$$

are recursive.

A *recursive Polish metric space* is a triple $(\mathcal{X}, d, \mathbf{r})$, where $(\mathcal{X}, d)$ is a Polish metric space and $\mathbf{r}$ is a recursive presentation of $(\mathcal{X}, d)$. Often, when there is no risk of confusion, we will suppress the metric and presentation in our notation, simply referring to a space by its underlying set $\mathcal{X}$.

For example, $\mathbb{N}$, equipped with the metric $d(i, j) = |i - j|$ and whose presentation is the identity function $\mathrm{id} : \mathbb{N} \to \mathbb{N}$, is a recursive Polish metric space. This space plays a critical role in the effective theory, and we will refer to it simply by writing $\mathbb{N}$.

There are, of course, a wealth of examples of uncountable spaces, the most important of which are the Baire space $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ and Cantor space $\{0, 1\}^{\mathbb{N}}$. See [3] for a considerable list of interesting examples of uncountable recursive Polish metric spaces.

Our proof of the Main Theorem will make use of the following space which serves as a good source of counterexamples in the theory. Consider $\mathbb{N}$ as a metric space with the *strongly discrete* metric, dis, defined by

$$\mathrm{dis}(i, j) = 1 \text{ if } i \neq j, \text{ otherwise } \mathrm{dis}(i, j) = 0$$

It is a simple exercise to show that for the metric space $(\mathbb{N}, \mathrm{dis})$ *any arbitrary* bijection $\mathbf{r} : \mathbb{N} \to \mathbb{N}$ is a recursive presentation. For any bijection $\mathbf{r}$, we use the notation

$$\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}} = (\mathbb{N}, \mathrm{dis}, \mathbf{r}).$$

Note that, in general, $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$-recursive objects are not the same as the classical Turing computable objects on $\mathbb{N}$.

Effectively open sets Given a recursive Polish metric space $\mathcal{X}$, we associate an effectively enumerated neighborhood basis for $\mathcal{X}$ as follows. For each $s \in \mathbb{N}$, define the set[3]

$$N_s(\mathcal{X}) = \{x \in X : d(x, \mathbf{r}_{(s)_0}) < q_{(s)_1}\}.$$

We introduce the notation $\mathrm{cen}_{\mathcal{X}}(s) = \mathbf{r}_{(s)_0}$ and $\mathrm{rad}(s) = q_{(s)_1}$ so that $N_s(\mathcal{X})$ is the open ball with center $\mathrm{cen}_{\mathcal{X}}(s)$ and radius $\mathrm{rad}(s)$.

A subset $P \subset \mathcal{X}$ is *effectively open* or *semirecursive* if there is a Turing computable function $f : \mathbb{N} \to \mathbb{N}$ such that $P = \bigcup_{i \in \mathbb{N}} N_{f(i)}(\mathcal{X})$. Equivalently, $P \subset \mathcal{X}$ is effectively open if and only if there is a recursively enumerable (r.e.) relation $P_0 \subset \mathbb{N}$ that satisfies the equivalence

$$P(x) \iff (\exists s)[x \in N_s(\mathcal{X}) \wedge P_0(s)]. \tag{1}$$

3 Here, $s \mapsto ((s)_0, (s)_1)$ is some fixed recursive surjection $\mathbb{N} \to \mathbb{N}^2$.

We refer to this representation of P as a Σ^0_1-normal form. We denote by $\Sigma^0_1(\mathcal{X})$ the collection of all effectively open subsets of $\mathcal{X}$. For the space $\mathbb{N}$, $\Sigma^0_1(\mathbb{N})$ is the collection of r.e. sets from classical computability theory.

For recursive Polish metric spaces $\mathcal{X} = (\mathcal{X}, d_{\mathcal{X}}, \mathbf{r}_{\mathcal{X}})$ and $\mathcal{Y} = (\mathcal{Y}, d_{\mathcal{Y}}, \mathbf{r}_{\mathcal{Y}})$, we form the product space $\mathcal{X} \times \mathcal{Y}$ by taking as its metric

$$d_{\max}((x_1, y_1), (x_2, y_2)) = \max\{d_{\mathcal{X}}(x_1, x_2), d_{\mathcal{Y}}(y_1, y_2)\}$$

and using the recursive presentation $\mathbf{r}(i) = (\mathbf{r}_{\mathcal{X}}(i), \mathbf{r}_{\mathcal{Y}}(i))$. The product space has an alternative but—in some sense—equivalent construction, where one takes the Euclidean metric,

$$d_E((x_1, y_1), (x_2, y_2)) = \sqrt{(d_{\mathcal{X}}(x_1, x_2))^2 + (d_{\mathcal{Y}}(y_1, y_2))^2}$$

instead of the max metric.[4]

For a recursive Polish metric space $\mathcal{X}$, members of the frame $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$ have the following normal form: for any $P \in \Sigma^0_1(\mathbb{N} \times \mathcal{X})$, there exists $P_0 \in \Sigma^0_1(\mathbb{N}^2)$ such that, for every $n \in \mathbb{N}$ and $x \in \mathcal{X}$,

$$P(n, x) \iff (\exists s)[x \in N_s(\mathcal{X}) \wedge P_0(n, s)]. \tag{2}$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be recursive Polish spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is *recursive* if the relation $\{(s, x) \in \mathbb{N} \times \mathcal{X} : f(x) \in N_s(\mathcal{Y})\}$ is $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$. For the space $\mathbb{N}$, the recursive functions $f : \mathbb{N} \to \mathbb{N}$ are precisely the Turing computable functions.[5]

The following theorem summarizes the important properties of the collection of effectively open sets that we will use throughout.

Theorem 1 (Properties of Σ^0_1, see [3]) *Let $\mathcal{X}$ be a recursive Polish metric space.*

(i) *If $P, Q \subset \mathcal{X}$ are $\Sigma^0_1(\mathcal{X})$, then so too are the relations $P \wedge Q, P \vee Q \subset \mathcal{X}$ defined by*

$$(P \wedge Q)(x) \iff_{\text{df}} P(x) \wedge Q(x), \quad (P \vee Q)(x) \iff_{\text{df}} P(x) \vee Q(x).$$

(ii) *If $P \subset \mathbb{N} \times \mathcal{X}$ is $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$, then the relations obtained from bounded quantification and unbounded existential quantification,*

$$(\forall^{\leq} P)(n, x) \iff_{\text{df}} (\forall m \leq n) P(m, x),$$
$$(\exists^{\leq} P)(n, x) \iff_{\text{df}} (\exists m \leq n) P(m, x), \quad (\exists^{\mathbb{N}} P)(x) \iff_{\text{df}} (\exists n) P(n, x),$$

are also Σ^0_1.

4 Theorem 7 will address in what sense these two constructions give the same “effective topology”.

5 We will henceforth choose to consistently refer to such functions as *computable* rather than recursive.

(iii) Σ^0_1 *is closed under recursive substitutions, i.e., if* $f : \mathcal{X} \to \mathcal{Y}$ *is recursive and* $P \in \Sigma^0_1(\mathcal{Y})$*, then*

$$R(x) \iff_{\mathrm{df}} P(f(x))$$

is $\Sigma^0_1(\mathcal{X})$.

(iv) $\Sigma^0_1(\mathcal{X})$ *has a good* $\mathbb{N}$*-parameterization, i.e., there is a* Σ^0_1 *relation* $G \subset \mathbb{N} \times \mathcal{X}$ *such that the collection of the sections* $G_n = \{x \in \mathcal{X} : G(n, x)\}$ *is exactly the collection of* $\Sigma^0_1(\mathcal{X})$ *sets and, moreover, for every* $\Sigma^0_1(\mathbb{N} \times \mathcal{X})$ *relation* P *there exists a computable function* $S : \mathbb{N} \to \mathbb{N}$ *such that* $P_n = G_{S(n)}$ *for all* $n \in \mathbb{N}$.

Most of these properties are trivial and readily verified; the one possible exception is property (iv), which can be thought of as a general version of the s-m-n theorem from classical computability theory.

3 The proof of the main theorem

The spaces $\mathcal{X}_1$ and $\mathcal{X}_2$ we will use to prove the Main Theorem will both be of the form $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$, with different choices of presentation $\mathbf{r}$. If we take $\mathbf{r} = \mathrm{id}$, then $\mathcal{X}_1 = \mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}$ is easily shown to be recursively isomorphic to $\mathbb{N}$. Then, we will take $\mathcal{X}_2 = \mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$ with $\mathbf{r}$ some non-computable bijection which maps r.e. sets to r.e. sets, but not in a way that is uniform in r.e. indices. Because $\mathbf{r}$ preserves r.e. sets, $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$ will have the same effectively open sets as $\mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}$. On the other hand, the non-computability of $\mathbf{r}$ will mean that the frames disagree; e.g., the sequence of open sets $\{\mathbf{r}(0)\}, \{\mathbf{r}(1)\}, \ldots$ is uniformly recursive for the space $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$ but not the space $\mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}$.

We shall require several lemmas, the first of which characterizes the effectively open subsets of the strongly discrete space $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$.

Lemma 2 *Let* $\mathbf{r} : \mathbb{N} \to \mathbb{N}$ *be a bijection. A nonempty set* $P \subset \mathbb{N}$ *is in* $\Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$ *if and only if there is a computable function* $f : \mathbb{N} \to \mathbb{N}$ *such that*

$$P = \{\mathbf{r}_{f(i)} : i \in \mathbb{N}\} = \mathbf{r}[f[\mathbb{N}]].$$

Proof Let $P \in \Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$ be nonempty. If $P = \mathbb{N}$, the result is trivial, so suppose $P \neq \mathbb{N}$. Fix a computable $h : \mathbb{N} \to \mathbb{N}$ such that $P = \bigcup_{i \in \mathbb{N}} N_{h(i)}(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$. Since $P \neq \mathbb{N}$, the neighborhoods $N_{h(i)}(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$ enumerated by h are either empty or singletons. By computably altering h if needed, we may assume that every $N_{h(i)}(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$ is a singleton; indeed, start by enumerating some nonempty basic neighborhood contained in P (which exists since $P \neq \emptyset$), and then any time h would enumerate an empty basic neighborhood, instead return the code for the most recently enumerated code for a nonempty basic neighborhood. Now, define $f : \mathbb{N} \to \mathbb{N}$ by $f(i) = (h(i))_0$. Clearly, f is computable and

$$P = \bigcup_{i \in \mathbb{N}} N_{h(i)}(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}) = \bigcup_{i \in \mathbb{N}} \{\mathbf{r}_{(h(i))_0}\} = \{\mathbf{r}_{f(i)} : i \in \mathbb{N}\},$$

which is of the desired form.

Conversely, suppose $P = \{\mathbf{r}_{f(i)} : i \in \mathbb{N}\}$ for some computable $f : \mathbb{N} \to \mathbb{N}$. Define a computable $h : \mathbb{N} \to \mathbb{N}$ so that $\mathrm{cen}_{\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}}(h(i)) = \mathbf{r}_{f(i)}$ and $\mathrm{rad}(h(i)) = 1/2$ for all i. Then,

$$P = \left\{\mathbf{r}_{f(i)} : i \in \mathbb{N}\right\} = \bigcup_{i \in \mathbb{N}} \{\mathbf{r}_{f(i)}\} = \bigcup_{i \in \mathbb{N}} N_{h(i)}(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}),$$

so that $P \in \Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$. □

Lemma 2 has the following important consequence.

Lemma 3 *Let $\mathbf{r} : \mathbb{N} \to \mathbb{N}$ be a bijection. If for every $P \in \Sigma^0_1(\mathbb{N})$, both $\mathbf{r}[P]$ and $\mathbf{r}^{-1}[P]$ are $\Sigma^0_1(\mathbb{N})$, then $\Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}) = \Sigma^0_1(\mathbb{N})$.*

Proof Let $P \in \Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$ be nonempty. By Lemma 2, $P = \mathbf{r}[f[\mathbb{N}]]$ for some computable $f : \mathbb{N} \to \mathbb{N}$; the assumed property of $\mathbf{r}$ then immediately implies that $P \in \Sigma^0_1(\mathbb{N})$.

Now let $Q \in \Sigma^0_1(\mathbb{N})$ be nonempty. By our assumption, the set $\mathbf{r}^{-1}[Q]$ is also $\Sigma^0_1(\mathbb{N})$ and nonempty. If follows that there is a computable function $f : \mathbb{N} \to \mathbb{N}$ which enumerates $\mathbf{r}^{-1}[Q]$. Then,

$$\{\mathbf{r}_{f(i)} : i \in \mathbb{N}\} = \mathbf{r}[f[\mathbb{N}]] = \mathbf{r}[\mathbf{r}^{-1}[Q]] = Q,$$

which shows that Q is also $\Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}})$, again using Lemma 2. □

We now turn our attention to the frame of $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$. We will first prove a general criterion to establish when two recursive presentations on the same Polish metric space yield the same frame. This result is useful when working with any recursive Polish metric space.[6]

Lemma 4 *Let $(\mathcal{X}, d)$ be a Polish metric space, and let $\mathbf{r}_1, \mathbf{r}_2$ be two recursive presentations of $(\mathcal{X}, d)$. Denote the associated recursive Polish metric spaces by $\mathcal{X}_1 = (\mathcal{X}, d, \mathbf{r}_1)$ and $\mathcal{X}_2 = (\mathcal{X}, d, \mathbf{r}_2)$. The following are equivalent.*

(1) *The relation*

$$E(i, j, n) \iff_{\mathrm{df}} d(\mathbf{r}_1(i), \mathbf{r}_2(j)) < q_n$$

is $\Sigma^0_1(\mathbb{N}^3)$.

(2) $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_1) = \Sigma^0_1(\mathbb{N} \times \mathcal{X}_2)$.

(3) $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_1) \supset \Sigma^0_1(\mathbb{N} \times \mathcal{X}_2)$.

(4) *If $G^1, G^2 \subset \mathbb{N} \times \mathcal{X}$ are good $\mathbb{N}$-parametrizations of $\Sigma^0_1(\mathcal{X}_1)$ and $\Sigma^0_1(\mathcal{X}_2)$ respectively, then there is a computable $f : \mathbb{N} \to \mathbb{N}$ such that for all $e \in \mathbb{N}$, $G^2_e = G^1_{f(e)}$.*

[6] E.g., the lemma implies that on the Baire space $\mathcal{N}$, any presentation $\mathbf{r} : \mathbb{N} \to \mathcal{N}$ which is computable (in the sense of classical computability on Baire space) gives the standard recursive topological structure on $\mathcal{N}$.

Proof (1) $\Rightarrow$ (2) Assume E is $\Sigma^0_1(\mathbb{N}^3)$. By the Σ^0_1-normal form (1), to show the inclusion $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_1) \subset \Sigma^0_1(\mathbb{N} \times \mathcal{X}_2)$ it suffices to prove that the basic neighborhood relation $\{(s, x) \in \mathbb{N} \times \mathcal{X} : x \in N_s(\mathcal{X}_1)\}$ is in $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_2)$. We will do this by establishing the equivalences

$$\begin{aligned} x \in N_s(\mathcal{X}_1) &\iff (\exists t)\big[x \in N_t(\mathcal{X}_2) \wedge d(\mathrm{cen}_{\mathcal{X}_1}(s), \mathrm{cen}_{\mathcal{X}_2}(t)) + \mathrm{rad}(t) < \mathrm{rad}(s)\big] \\ &\iff (\exists t)[x \in N_t(\mathcal{X}_2) \wedge E((s)_0, (t)_0, (s)_1 - (t)_1)]. \end{aligned}$$

The only nontrivial claim here is the ($\Longrightarrow$) direction of the first equivalence. Assume $x \in N_s(\mathcal{X}_1)$ and pick a rational $\delta > 0$ such that

$$d(x, \mathrm{cen}_{\mathcal{X}_1}(s)) < \mathrm{rad}(s) - \delta.$$

By denseness of $\mathbf{r}_2[\mathbb{N}]$, choose some $\mathbf{r}_2(i)$ with $d(x, \mathbf{r}_2(i)) < \delta/2$. Choose t with $\mathrm{cen}_{\mathcal{X}_2}(t) = \mathbf{r}_2(i)$ and $\mathrm{rad}(t) = \delta/2$. Then, $x \in N_t(\mathcal{X}_2)$ and a simple triangle inequality argument shows that

$$d\big(\mathrm{cen}_{\mathcal{X}_1}(s), \mathrm{cen}_{\mathcal{X}_2}(t)\big) + \mathrm{rad}(t) < \mathrm{rad}(s).$$

To prove the converse inclusion $\mathcal{R}(\mathcal{X}_2) \subset \mathcal{R}(\mathcal{X}_1)$, we argue as above, swapping the roles of $\mathcal{X}_1$ and $\mathcal{X}_2$.

Both (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). Suppose we have a computable function $f : \mathbb{N} \to \mathbb{N}$ as in (4). Since the basic neighborhood relation $\{(s, x) \in \mathbb{N} \times \mathcal{X} : x \in N_s(\mathcal{X}_2)\}$ is $\Sigma^0_1(\mathbb{N} \times \mathcal{X}_2)$ and G^2 is a good parameterization for $\Sigma^0_1(\mathcal{X}_2)$, there is a computable function $h : \mathbb{N} \to \mathbb{N}$ such that, for all $x \in X$ and $s \in \mathbb{N}$,

$$x \in N_s(\mathcal{X}_2) \iff G^1(h(s), x).$$

Then, we have the equivalences

$$\begin{aligned} E(i, j, n) &\iff d(\mathbf{r}_1(i), \mathbf{r}_2(j)) < q_n \\ &\iff \mathbf{r}_1(i) \in N_s(\mathcal{X}_2), \quad \text{where } \mathrm{cen}_{\mathcal{X}_2}(s) = \mathbf{r}_2(j) \text{ and } \mathrm{rad}(s) = q_n \\ &\iff G^2(h(s), \mathbf{r}_1(i)) \\ &\iff G^1(f(h(s)), \mathbf{r}_1(i)). \end{aligned}$$

Using that Σ^0_1 is closed under recursive substitution, it follows that E is indeed $\Sigma^0_1(\mathbb{N}^3)$. ☐

An application of Lemma 4 gives us the following criterion to decide whether $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$ and $\mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}$ have the same frame.

Lemma 5 *Let* $\mathbf{r} : \mathbb{N} \to \mathbb{N}$ *be a bijection. Then,* $\Sigma^0_1(\mathbb{N} \times \mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}) = \Sigma^0_1(\mathbb{N} \times \mathbb{N}^{\mathrm{id}}_{\mathrm{dis}})$ *if and only if* $\mathbf{r} : \mathbb{N} \to \mathbb{N}$ *is computable.*

Proof By Lemma 4 it is enough to prove that $\mathbf{r}:\mathbb{N}\to\mathbb{N}$ is computable if and only if

$$E(i,j,n) \iff d(\mathrm{id}(i),\mathbf{r}_j) < q_n \iff d(i,\mathbf{r}_j) < q_n$$

is $\Sigma^0_1(\mathbb{N}^3)$.

Suppose $\mathbf{r}:\mathbb{N}\to\mathbb{N}$ is computable. The desired conclusion follows by noting the equivalence

$$E(i,j,n) \iff (0 < q_n \wedge \mathbf{r}_j = i) \vee (q_n > 1)$$

and that the graph of $\mathbf{r}$ is $\Sigma^0_1(\mathbb{N}^2)$.

Conversely, suppose E is $\Sigma^0_1(\mathbb{N}^3)$. Fix n_0 such that $q_{n_0} = 1/2$. Then

$$\mathbf{r}_j = i \iff d(\mathrm{id}(i),\mathbf{r}_j) < \frac{1}{2} \iff E(i,j,n_0),$$

which shows that the graph of $\mathbf{r}$ is $\Sigma^0_1(\mathbb{N}^2)$, hence $\mathbf{r}:\mathbb{N}\to\mathbb{N}$ is computable. □

The following theorem of Dekker and Myhill will give us a recursive presentation $\mathbf{r}$ of $\mathbb{N}_{\mathrm{dis}}$ which is non-computable but preserves r.e. sets.

Theorem 6 (Dekker and Myhill, see [4] 12.3) *There exists a cohesive subset of* $\mathbb{N}$, *i.e., there is an infinite set* $A\subset\mathbb{N}$ *such that for every* $P\in\Sigma^0_1(\mathbb{N})$, *either* $A\cap P$ *or* $A\setminus P$ *is finite.*

It follows that there is a non-computable bijection $\mathbf{r}:\mathbb{N}\to\mathbb{N}$ *such that for every* $P\in\Sigma^0_1(\mathbb{N})$ *both the image* $\mathbf{r}[P]$ *and the preimage* $\mathbf{r}^{-1}[P]$ *are* $\Sigma^0_1(\mathbb{N})$ *sets.*

The existence of the cohesive set is proven via a simple priority argument with infinitely many requirements and no injuries. If P_n, $n\in\mathbb{N}$, is an enumeration of the infinite r.e. subsets of $\mathbb{N}$, then we construct $A=\bigcap_{n=0}^{\infty} A_n$ as follows: $A_0=\mathbb{N}$ and, given that $A_n=\{a_{n1}<a_{n2}<\dots\}$, we set $A_{n+1}=A_n$ if $A_n\backslash P_n$ is finite, otherwise we set $A_{n+1}=\{a_{n1},\dots,a_{nn}\}\cup(A_n\backslash P_n)$. Note that we keep the first n elements of A_n in A_{n+1} to ensure that $A=\bigcap_{n=0}^{\infty} A_n$ is infinite.

To construct a bijection as in the theorem, let A be a cohesive set and define a bijection $\mathbf{r}$ so that $\mathbf{r}(n)=n$ if and only if $n\notin A$. Since A is clearly not r.e., $\mathbf{r}$ is not computable. Moreover, for every r.e. $P\subset\mathbb{N}$, $\mathbf{r}[P]$ is again r.e. since the symmetric difference $P\Delta\mathbf{r}[P]$ is finite by the cohesiveness of A. A similar argument shows that $\mathbf{r}^{-1}[P]$ is also r.e.

Now, we are ready to prove the Main Theorem.

Proof of the Main Theorem Let $\mathbf{r}:\mathbb{N}\to\mathbb{N}$ be the bijection from Theorem 6 and consider $\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}$. From two applications of Lemma 3 we have

$$\Sigma^0_1(\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}) = \Sigma^0_1(\mathbb{N}) = \Sigma^0_1(\mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}).$$

Since $\mathbf{r}$ is not computable, it follows from Lemma 5 that

$$\Sigma^0_1(\mathbb{N}\times\mathbb{N}^{\mathbf{r}}_{\mathrm{dis}}) \neq \Sigma^0_1(\mathbb{N}\times\mathbb{N}^{\mathrm{id}}_{\mathrm{dis}}),$$

which completes the proof. □

4 The frame captures the effective topology

Recall that for a recursive Polish metric space $\mathcal{X}$, the frame of $\mathcal{X}$ is $\mathcal{R}(\mathcal{X}) = \Sigma^0_1(\mathbb{N} \times \mathcal{X})$. This notion is developed in [3] and here we point out several properties that suggest that the frame captures the recursive topology of the space.

First, since a function $f : \mathcal{X} \to \mathcal{Y}$ is recursive precisely when

$$\{(s, x) \in \mathbb{N} \times \mathcal{X} : f(x) \in N_s(\mathcal{Y})\} \in \Sigma^0_1(\mathbb{N} \times \mathcal{X}),$$

it follows that whether f is recursive is determined by the frames of $\mathcal{X}$ and $\mathcal{Y}$—not the metrics.

Moreover, the frame of a space is respected by the natural, constructive operations on topological spaces:

Theorem 7 [see [3]]

(i) *For all* $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Y}$,

$$\mathcal{R}(\mathcal{X}_1) = \mathcal{R}(\mathcal{X}_2) \implies \mathcal{R}(\mathcal{Y} \times \mathcal{X}_1) = \mathcal{R}(\mathcal{Y} \times \mathcal{X}_2),$$

regardless of whether the max metric or the Euclidean metric is chosen for the product space structure.

(ii) *If* $(\mathcal{X}, d)$ *is a recursive Polish metric space and* $d_1 = \min(d, 1)$, *then* $(\mathcal{X}, d)$ *and* $(\mathcal{X}, d_1)$ *have the same frame.*

(iii) *Suppose* $\mathcal{X}_0, \mathcal{X}_1, \dots,$ $\mathcal{Y}_0, \mathcal{Y}_1, \dots$ *are sequences of spaces and that* $\mathcal{R}(\mathcal{X}_n) = \mathcal{R}(\mathcal{Y}_n)$ *uniformly in* $n \in \mathbb{N}$. *Then,*

$$\mathcal{R}(\textstyle\prod_{n\in\mathbb{N}} \mathcal{X}_n) = \mathcal{R}(\textstyle\prod_{n\in\mathbb{N}} \mathcal{Y}_n).$$

Computable spaces In addition to the properties mentioned above, the frame of a space also provides a descriptive set theoretic framework that unifies the study of recursive metric spaces with the study of notions from computable analysis in the sense of Weihrauch.

A presentation $\mathbf{r} : \mathbb{N} \to \mathcal{X}$ of a Polish metric space $\mathcal{X}$ is *computable* if the relations

$$P_W(i, j, k) \iff_{\mathrm{df}} d(r_i, r_j) < q_k, \quad Q_W(i, j, k) \iff_{\mathrm{df}} q_k < d(r_i, r_j) \tag{3}$$

are both r.e. $\mathcal{X}$ is *computable* if it admits a computable presentation.

Computable spaces were introduced by Weihrauch [5], and a great amount of research has been devoted to them. The resulting theory is similar to the theory of recursive metric spaces, but these two effective versions of metric spaces are indeed different notions.[7] Every recursive Polish metric space is computable, but the converse fails:

[7] See [6] for a detailed comparison of these two theories.

Theorem 8 (Gregoriades et al. [6]) *There is a computable Polish metric space which is not isometric with any recursive Polish metric space.*

On the level of metric space structure, we are left with these two competing effective theories; however, for the descriptive set theorist ultimately interested in the lightface definable subsets of a space, the notion of recursive Polish space eliminates the distinction between computable and recursive Polish metric spaces at the level of their effective topology.

Theorem 9 (Gregoriades et al. [6]) *If* $(\mathcal{X}, d)$ *is a computable Polish metric space, then there is a recursive Polish metric space* $(\mathcal{X}, d')$ *on the same universe* $\mathcal{X}$ *such that* $(\mathcal{X}, d)$ *and* $(\mathcal{X}, d')$ *have the same frame.*

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