



Ideals with Smital properties

Marcin Michalski¹ · Robert Rałowski¹ · Szymon Żeberski¹

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Abstract

A σ -ideal \mathcal{I} on a Polish group $(X, +)$ has the Smital Property if for every dense set D and a Borel \mathcal{I} -positive set B the algebraic sum $D + B$ is a complement of a set from \mathcal{I} . We consider several variants of this property and study their connections with the countable chain condition, maximality and how well they are preserved via Fubini products. In particular we show that there are \mathfrak{c} many maximal invariant σ -ideals with Borel bases on the Cantor space 2^ω .

Keywords Smital property · Steinhaus property · Countable chain condition · Fubini product · Maximal invariant ideal · Orthogonal ideals

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1 Introduction

We adopt the usual set-theoretical notation. We say that X is a Polish space if it is a separable and completely metrisable topological space. $\text{Bor}(X)$ denotes the family of Borel subsets of X . $\mathcal{M}(X)$ and $\mathcal{N}(X)$ are the families of meager and null subsets of

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✉ Marcin Michalski
marcin.k.michalski@pwr.edu.pl

Robert Rałowski
robert.ralowski@pwr.edu.pl

Szymon Żeberski
szymon.zeberski@pwr.edu.pl

¹ Faculty of Pure and Applied Mathematics, Wrocław University of Science and Technology, Wybrzeże Wyspiańskiego 27, 50-370 Wrocław, Poland

X respectively. Sometimes we will write briefly \mathcal{M} and \mathcal{N} if the underlying space is clear from the context.

Let \mathcal{A} be a (σ) -algebra and \mathcal{I} be a (σ) -ideal on an Abelian Polish group $(X, +)$. Throughout the paper we assume that \mathcal{I} contains all singletons and

$$(\forall I \in \mathcal{I})(\exists A \in \mathcal{A} \cap \mathcal{I})(I \subseteq A).$$

If \mathcal{A} is not explicitly stated we assume $\mathcal{A} = \text{Bor}(X)$. In such a case \mathcal{I} has a Borel base.

We say that a set A is \mathcal{I} -positive, if $A \notin \mathcal{I}$. A is called \mathcal{I} -residual if $A^c \in \mathcal{I}$, we denote this fact by $A \in \mathcal{I}^*$.

For any sets $A, B \subseteq X$ we denote the algebraic sum of these sets by $A + B$, i.e.

$$A + B = \{a + b : a \in A, b \in B\}.$$

Let us now recall the classical notion of the Steinhaus property and probably less famous notions of Smital properties which were studied in [2].

Definition 1 We say that a pair $(\mathcal{A}, \mathcal{I})$ has

- (i) the Steinhaus Property if for any $A, B \in \mathcal{A} \setminus \mathcal{I}$ the set $A - B$ has a nonempty interior;
- (ii) the Smital Property, briefly SP, if for every dense set D and every $A \in \mathcal{A} \setminus \mathcal{I}$ the set $A + D$ is \mathcal{I} -residual;
- (iii) the Weaker Smital Property, briefly WSP, if there exists a countable and dense set D such that for every $A \in \mathcal{A} \setminus \mathcal{I}$ the set $A + D$ is \mathcal{I} -residual;
- (iv) the Very Weak Smital Property, briefly VWSP, if for every $A \in \mathcal{A} \setminus \mathcal{I}$ there is a countable set D such that the set $A + D$ is \mathcal{I} -residual.

Note that \mathcal{M} and \mathcal{N} have all of these properties.

The following Proposition seems to be folklore but we could not find the proof of the second part in the literature.

Proposition 2 *The Steinhaus Property is equivalent to SP.*

Proof The Steinhaus Property implies SP. Let $A \in \mathcal{A} \setminus \mathcal{I}$ and D be countable and dense. We may assume that D is a subgroup. Suppose that $(A + D) \notin \mathcal{I}^*$. Then $(A + D)^c \in \mathcal{I}$. By the Steinhaus Property $(A + D) - (A + D)^c$ contains an open neighborhood of 0. A contradiction since $0 \notin (A + D) - (A + D)^c$.

SP implies the Steinhaus Property. Assume that $A - B$ has an empty interior for $A, B \in \mathcal{A} \setminus \mathcal{I}$. Then there is a countable dense set $D \subseteq (A - B)^c$. It follows from SP that $(B + D) \cap A \neq \emptyset$, a contradiction. □

Let $F \subseteq X \times Y$. Then for $x \in X$

$$F_x = \{y \in Y : (x, y) \in F\}$$

is the vertical section of F at x . Similarly, for $y \in Y$

$$F^y = \{x \in X : (x, y) \in F\}$$

is the horizontal section of F at y .

For $(\sigma\text{-})$ algebras $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ let $\mathcal{A} \otimes \mathcal{B} \subseteq P(X \times Y)$ denote the $(\sigma\text{-})$ algebra generated by the rectangles of the form $A \times B$ with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Definition 3 Let $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{B}, \mathcal{J})$ be pairs of $(\sigma\text{-})$ algebra - $(\sigma\text{-})$ ideal on Polish spaces X and Y respectively. Then we define the Fubini product of \mathcal{I} and \mathcal{J} as follows:

$$K \in \mathcal{I} \otimes \mathcal{J} \Leftrightarrow (\exists C \in \mathcal{A} \otimes \mathcal{B})(K \subseteq C \wedge \{x \in X : C_x \notin \mathcal{J}\} \in \mathcal{I}).$$

Notice that in the case of ideals possessing Borel bases, i.e. $\mathcal{A} = \text{Bor}(X)$ and $\mathcal{B} = \text{Bor}(Y)$ the above definition ensures the existence of a Borel base for $\mathcal{I} \otimes \mathcal{J}$.

Proposition 4 *If $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ has SP (WSP, VWSP), then $(\mathcal{A}, \mathcal{I})$ and $(\mathcal{B}, \mathcal{J})$ also have it.*

Proof Let us consider the case of WSP for \mathcal{I} . Let D be a witness that $\mathcal{I} \otimes \mathcal{J}$ has WSP. Let $A \in \mathcal{A} \setminus \mathcal{I}$. The set $R = D + (A \times Y)$ is $\mathcal{I} \otimes \mathcal{J}$ -residual, therefore

$$\tilde{R} = \{x \in X : R_x \text{ is } \mathcal{J}\text{-residual}\}$$

is \mathcal{I} -residual. Clearly, $A + \pi_X(D) = \tilde{R}$, hence we are done. □

The following definition is a variation of the ones found in [7, Definition 18.5], [9] and agrees with the notation given in [3].

Definition 5 Let X and Y be Polish spaces and let $\mathcal{F} \subseteq P(X)$, $\mathcal{G} \subseteq P(Y)$, $\mathcal{H} \subseteq P(X \times Y)$ be families of sets. Then we say that \mathcal{G} is \mathcal{H} -on- \mathcal{F} if for each set $H \in \mathcal{H}$

$$\{x \in X : H_x \in \mathcal{G}\} \in \mathcal{F}.$$

Mainly we will be interested in the case where $\mathcal{G} = \mathcal{J} \subseteq P(Y)$ is a σ -ideal, $\mathcal{F} \in \{\text{Bor}(X), \sigma(\text{Bor}(X) \cup \mathcal{I})\}$ and $\mathcal{H} \in \{\text{Bor}(X \times Y), \sigma(\text{Bor}(X \times Y) \cup \mathcal{I} \otimes \mathcal{J})\}$. Here $\sigma(\mathcal{D})$ is the σ -algebra generated by the family \mathcal{D} . We will write, for example, that \mathcal{J} is Borel-on-measurable instead of $\text{Bor}(X \times Y)$ -on- $\sigma(\text{Bor}(X) \cup \mathcal{I})$ if the context is clear. Notice that both \mathcal{M} and \mathcal{N} are Borel-on-Borel (see [7, Exercise 22.22, 22.25]).

Example 6 Measurable-on-measurable not necessarily implies Borel-on-Borel.

Proof Let $\mathcal{J} = \{\emptyset\}$ and take a Borel set B projection of which is analytic and not Borel. □

Proposition 7 *Borel-on-measurable implies measurable-on-measurable.*

Proof Assume that \mathcal{J} is Borel on measurable. Let $C \subseteq X \times Y$ be measurable with respect to $\mathcal{I} \otimes \mathcal{J}$. Then $C = (B \setminus A_1) \cup A_2$, where B is Borel and $A_1, A_2 \in \mathcal{I} \otimes \mathcal{J}$. Clearly

$$\{x \in X : A_{2,x} \notin \mathcal{J}\} \in \mathcal{I},$$

hence it is measurable. See that

$$\begin{aligned} \{x \in X : (B \setminus A_1)_x \notin \mathcal{J}\} &= \{x \in X : B_x \setminus A_{1x} \notin \mathcal{J}\} \\ &= \{x \in X : B_x \notin \mathcal{J}, A_{1x} \in \mathcal{J}\} \cup \{x \in X : B_x \notin \mathcal{J}, A_{1x} \notin \mathcal{J}, B_x \setminus A_{1x} \notin \mathcal{J}\}. \end{aligned}$$

Since $\{x \in X : B_x \notin \mathcal{J}\}$ is measurable, $\{x \in X : A_{1x} \in \mathcal{J}\} \in \mathcal{I}^*$ and

$$\{x \in X : B_x \notin \mathcal{J}, A_{1x} \notin \mathcal{J}, B_x \setminus A_{1x} \notin \mathcal{J}\} \subseteq \{x \in X : A_{1x} \notin \mathcal{J}\} \in \mathcal{I},$$

the set $\{x \in X : C_x \notin \mathcal{J}\}$ is measurable. □

2 Smital and ccc

Let \mathcal{I} be a σ -ideal in a Polish space X and assume that \mathcal{I} has a Borel base. We say that \mathcal{I} satisfies the countable chain condition (briefly: ccc) if every family of pairwise disjoint Borel \mathcal{I} -positive sets is countable. Let us also recall the following cardinal coefficient

$$\text{cov}(\mathcal{I}) = \min \left\{ |\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = X \right\}.$$

Theorem 8 *Let \mathcal{I} be a σ -ideal possessing WSP. Then \mathcal{I} satisfies ccc or $\text{cov}(\mathcal{I}) = \omega_1$.*

Proof Let \mathcal{I} be σ -ideal with WSP and let D witness it. Let $\{B_\alpha : \alpha < \omega_1\}$ be a family of pairwise disjoint Borel \mathcal{I} -positive sets. WSP implies that for each $\alpha < \omega_1$ a set $D + B_\alpha$ is \mathcal{I} -residual. If $\bigcap_{\alpha < \omega_1} (D + B_\alpha) = \emptyset$, then $\text{cov}(\mathcal{I}) = \omega_1$. On the other hand, if $\bigcap_{\alpha < \omega_1} (D + B_\alpha) \neq \emptyset$ then for $x \in \bigcap_{\alpha < \omega_1} (D + B_\alpha)$ we have

$$(\forall \alpha \in \omega_1)(\exists d \in D)(x \in d + B_\alpha).$$

D is countable, hence there exist a set $W \subseteq \omega_1$ of cardinality ω_1 and $d \in D$ such that

$$(\forall \alpha \in W)(x \in d + B_\alpha),$$

which gives $x - d \in \bigcap_{\alpha \in W} B_\alpha$, a contradiction. □

The following remark improves the result obtained in [4].

Remark 9 Let \mathcal{I} be a σ -ideal possessing WSP. Then the following statements are equivalent:

- (i) For each family of sets $\{B_\alpha : \alpha < \omega_1\} \subseteq \mathcal{B} \setminus \mathcal{I}$ there exists a set $W \subseteq \omega_1$ of cardinality ω_1 such that $\bigcap_{\alpha \in W} B_\alpha \neq \emptyset$;
- (ii) $\text{cov}(\mathcal{I}) > \omega_1$.

Proof $(ii) \Rightarrow (i)$ is a part of Theorem 8. To prove $(i) \Rightarrow (ii)$ let us suppose that $\text{cov}(\mathcal{I}) = \omega_1$. Then there is a family of sets

$$\{A_\alpha : \alpha < \omega_1\} \subseteq \mathcal{B} \cap \mathcal{I}$$

for which $\bigcup_{\alpha < \omega_1} A_\alpha = X$. Set $\tilde{A}_\alpha = \bigcup_{\beta \leq \alpha} A_\alpha$ for each $\alpha < \omega_1$. The family $\{\tilde{A}_\alpha : \alpha < \omega_1\}$ is ascending and covers X . Hence $\{\tilde{A}_\alpha^c : \alpha < \omega_1\}$ is descending family of \mathcal{I} -residual sets. Moreover, for every $W \subseteq \omega_1$ of cardinality ω_1 we have $\bigcap_{\alpha \in W} \tilde{A}_\alpha^c = \emptyset$, which contradicts (i) . \square

3 Preserving Smital properties via products

In [2] the authors present some results on various Smital properites in product spaces. Their setup is, in their words, as general as possible, concerned with algebras and ideals. It is not clear if they intended their results to hold for σ -algebras and σ -ideals or algebras and ideals only. The formulation of [2, Theorem 4.2] suggests the former since it is concerned with the Borel algebra and the families of meager and null sets. In their proof they rely implicitly on the following property.

Definition 10 Let $\mathcal{A} \subseteq P(X \times Y)$ be a (σ) -algebra and let $\mathcal{I} \subseteq P(X \times Y)$ be a (σ) -ideal. A pair $(\mathcal{A}, \mathcal{I})$ has the positive rectangle property (PRP) if for every \mathcal{I} -positive set $A \in \mathcal{A}$ there is an \mathcal{I} -positive rectangle R satisfying $R \subseteq A \cup I$ for some $I \in \mathcal{I}$.

In this section we show explicitly that PRP holds for pairs algebra-ideal. However, PRP does not hold for pairs σ -algebra - σ -ideal in general, including the relevant here pair of Borel σ -algebra and the family of null sets.

Example 11 The pair $(\text{Bor}(\mathbb{R}^2), [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega})$ does not have PRP.

Proof Let $P \subseteq \mathbb{R}$ be a perfect set such that $P \cap (P + x)$ is at most 1-point for $x \neq 0$ (see [8]). Let us set

$$B = \{(x, y) : x \in P \wedge y \in P - x\} \setminus (\mathbb{R} \times \{0\}).$$

B is Borel and $B \notin [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega}$. If $x \in B^y$, then $x \in P$ and $x \in P - y$, therefore B^y is at most 1-point.

Let us suppose that there are sets $A_1, A_2 \in \text{Bor}(\mathbb{R}^2) \setminus [\mathbb{R}]^{\leq \omega}$ and a set $K \in [\mathbb{R}]^{\leq \omega} \otimes [\mathbb{R}]^{\leq \omega}$ such that $(A_1 \times A_2) \setminus K \subseteq B$. Let $T = \{x \in \mathbb{R} : |K_x| > \omega\}$ and notice that $|T| \leq \omega$. Pick $x_0 \in A_1 \setminus T$. Then K_{x_0} is countable. By the definition of B for each $y \in A_2 \setminus K_{x_0}$ we have $K^y \supseteq A_1 \setminus (\{x_0\} \cup T)$. So $A_2 \setminus K_{x_0} \subseteq K_{x_1}$ for $x_1 \in A_1 \setminus (\{x_0\} \cup T)$, a contradiction. \square

It is clear that the pair $(\text{Bor}(\mathbb{R}^2), \mathcal{M})$ has PRP, since every nonmeager set possessing the property of Baire is nonempty and open, modulo a set of the first category. What about $(\text{Bor}(\mathbb{R}^2), \mathcal{N})$? As a warm up let us recall the following folklore result.

Proposition 12 Every set $E \subseteq [0, 1]^2$ of positive measure contains a subset of the same measure which does not contain a rectangle of positive measure.

Proof Let $E \subseteq [0, 1]^2$ have positive measure. Consider $E' = \{(x, y) \in E : x - y \in \mathbb{Q}^c\}$. To see that $\lambda(E') = \lambda(E)$ let us observe that $E'_x = E_x \cap (\mathbb{Q}^c + x)$ for every $x \in [0, 1]$ and \mathbb{Q}^c is co-null. Now, if $A \times B \subseteq E'$, A and B of positive measure, then $A - B$ should contain a nonempty open set (Steinhaus Theorem), but clearly $\mathbb{Q} \cap (A - B) = \emptyset$. A contradiction. This completes the proof. \square

This result may be improved with the following Lemma.

Lemma 13 There exists a set $F \subseteq \mathbb{R}$ such that $\lambda(F \cap U) > 0$ and $\lambda(F^c \cap U) > 0$ for every nonempty open set U .

Proof Let $(B_n : n \in \omega)$ be an enumeration of the basis of \mathbb{R} . At the step 0 let $C_0^1, C_0^2 \subseteq B_0$ be two disjoint Cantor sets of positive measure. At the step $n + 1$ let assume that we have two sequences of pairwise disjoint Cantor sets $(C_k^1 : k \leq n)$ and $(C_k^2 : k \leq n)$ which for all $i \in \{0, 1\}$ and $k \leq n$ satisfy $\lambda(C_k^i \cap B_k) > 0$. The set

$$B_{n+1} \setminus \bigcup_{k \leq n} (C_k^1 \cup C_k^2)$$

is nonempty and open, hence it contains two disjoint Cantor sets of positive measure. Denote them by C_{n+1}^1 and C_{n+1}^2 . This completes the construction and $F = \bigcup_{n \in \omega} C_n^1$ is the desired set. \square

The above Lemma will serve as a tool to prove the result from [6].

Example 14 (Erdős, Oxtoby). There is a set $E \subseteq \mathbb{R}^2$ such that $E \cap (A \times B)$ and $E^c \cap (A \times B)$ have positive measure for each $A, B \subseteq \mathbb{R}$ of positive measure.

Proof Let F be as in the formulation of Lemma 13 and set $E = \{(x, y) \in \mathbb{R}^2 : x - y \in F\}$. Let $A, B \subseteq \mathbb{R}$ have a positive measure. Then

$$\begin{aligned} \lambda(E \cap A \times B) &= \iint \chi_F(x - y) \chi_A(x) \chi_B(y) dx dy \\ &= \iint \chi_F(x) \chi_A(x + y) \chi_B(y) dx dy \\ &= \int_F \lambda((A - x) \cap B) dx. \end{aligned}$$

$\lambda((A - x) \cap B)$ is a continuous non-negative function. Furthermore, it is positive on some interval, since $\int_{\mathbb{R}} \lambda((A - x) \cap B) dx = \lambda(A \times B)$, so $\int_F \lambda((A - x) \cap B) dx > 0$. \square

Corollary 15 $(\text{Bor}(\mathbb{R}^2), \mathcal{N})$ does not have PRP.

Now we will focus on PRP for products of algebras.

Lemma 16 *Let $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ be algebras. Then $\mathcal{A} \otimes \mathcal{B} \subseteq P(X \times Y)$ consists of finite unions of rectangles.*

Proof First let us observe that complements of rectangles are finite unions of rectangles:

$$(A \times B)^c = (A \times B^c) \cup (A^c \times B) \cup (A^c \times B^c).$$

Next, see that finite intersection of a finite union of rectangles is again a finite union of rectangles. Let $T_1, T_2 \subseteq \omega$ be finite. Let A_k^n, B_k^n be rectangles from \mathcal{A} and \mathcal{B} respectively for every $n \in T_1$ and $k \in T_2$. Then

$$\bigcap_{n \in T_1} \bigcup_{k \in T_2} A_k^n \times B_k^n = \bigcup_{f \in T_2^{T_1}} \bigcap_{n \in T_1} A_{f(n)}^n \times B_{f(n)}^n.$$

$T_2^{T_1}$ is finite and finite intersections of rectangles are also rectangles, hence the proof is complete. □

For algebra \mathcal{A} and ideal \mathcal{I} let $\mathcal{A}[\mathcal{I}]$ denote the algebra generated by $\mathcal{A} \cup \mathcal{I}$.

Proposition 17 *Let $\mathcal{A} \subseteq P(X)$ and $\mathcal{B} \subseteq P(Y)$ be algebras and let \mathcal{I} be an ideal in $X \times Y$. Then $((\mathcal{A} \otimes \mathcal{B})[\mathcal{I}], \mathcal{I})$ has PRP.*

Proof Notice that $(\mathcal{A} \otimes \mathcal{B})[\mathcal{I}] = \{C \Delta I : C \in \mathcal{A} \otimes \mathcal{B}, I \in \mathcal{I}\}$. PRP follows from the previous Lemma. □

From now on let $\mathcal{A} \subseteq P(X), \mathcal{B} \subseteq P(Y)$ be σ -algebras, and $\mathcal{I} \subseteq P(X), \mathcal{J} \subseteq P(Y)$ σ -ideals.

Theorem 18 *Let \mathcal{I} and \mathcal{J} possess WSP and assume one of the following properties*

- (i) \mathcal{J} is Borel-on-Borel;
- (ii) \mathcal{J} measurable-on-measurable;
- (iii) $(\text{Bor}(X \times Y), \mathcal{I} \otimes \mathcal{J})$ has PRP.

Then $\mathcal{I} \otimes \mathcal{J}$ also has WSP.

Proof Let D_1 and D_2 witness WSP for \mathcal{I} and \mathcal{J} respectively. Let $B \in \text{Bor}(X \times Y) \setminus \mathcal{I} \otimes \mathcal{J}$. If any of the properties (i)-(iii) holds then a set $\tilde{B} = \{x \in X : B_x \notin \mathcal{J}\}$ contains a Borel, \mathcal{I} -positive set and $D_1 + \tilde{B}$ is \mathcal{I} -residual. Let us observe that

$$(D_1 \times D_2) + B \supseteq \bigcup_{d_1 \in D_1} \bigcup_{x \in \tilde{B}} (\{d_1 + x\} \times (D_2 + B_x)),$$

therefore for every $x \in D_1 + \tilde{B}$ the set $((D_1 \times D_2) + B)_x$ is \mathcal{J} -residual. Since $D_1 + \tilde{B}$ is \mathcal{I} -residual, the proof is complete. □

In [1] the authors showed that $\mathcal{M} \otimes \mathcal{N}$ and $\mathcal{N} \otimes \mathcal{M}$ have SP and thus WSP. The following corollary extends this result regarding WSP.

Corollary 19 *Let $n \in \omega$ and $\mathcal{I}_k \in \{\mathcal{M}, \mathcal{N}\}$ for any $k \leq n$. Then $\mathcal{I}_0 \otimes \mathcal{I}_1 \otimes \dots \otimes \mathcal{I}_n$ has WSP.*

Proof By [5, Lemma 3.1] the ideal $\mathcal{I}_0 \otimes \mathcal{I}_1 \otimes \dots \otimes \mathcal{I}_n$ is Borel-on-Borel for any $n \in \omega$ and $\mathcal{I}_k \in \{\mathcal{M}, \mathcal{N}\}$, $k \leq n$. \square

In [2, Theorem 4.3] the authors showed that if $\mathcal{B} = \mathcal{J} \cup \mathcal{J}^*$ and $(\mathcal{A}, \mathcal{I})$ has SP then $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ also has SP. We will generalize this result (Theorem 24). Let us start with two technical definitions.

Definition 20 We say that a pair $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ has the Tall Rectangle Hull Property (TRHP) if for every set $C \in \mathcal{A} \otimes \mathcal{B}$

$$(\exists \tilde{C} \in \mathcal{A}, I \in \mathcal{I}, J \in \mathcal{J})((\tilde{C} \setminus I) \times (Y \setminus J) \subseteq C \subseteq (\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J)).$$

If a set C fulfills the above condition we will say that it has TRHP witnessed by the triple (\tilde{C}, I, J) .

Analogously we define Wide Rectangle Hull Property (WRHP):

$$(\exists \tilde{C} \in \mathcal{B}, I \in \mathcal{I}, J \in \mathcal{J})((X \setminus I) \times (\tilde{C} \setminus J) \subseteq C \subseteq (X \times \tilde{C}) \cup (I \times Y) \cup (X \times J)).$$

Proposition 21 *If a pair $(\mathcal{A} \otimes \mathcal{B}, \mathcal{I} \otimes \mathcal{J})$ have TRHP or WRHP then it has PRP.*

Proof Let $C \in \mathcal{A} \otimes \mathcal{B}$ has TRHP witnessed by (\tilde{C}, I, J) :

$$(\tilde{C} \setminus I) \times (Y \setminus J) \subseteq C \subseteq (\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J)$$

and assume that $C \notin \mathcal{I} \otimes \mathcal{J}$. Then $\tilde{C} \times Y$ is the desired rectangle. Clearly $\tilde{C} \times Y \subseteq C$ modulo a set from $\mathcal{I} \otimes \mathcal{J}$. It is also $\mathcal{I} \otimes \mathcal{J}$ -positive, otherwise

$$(\tilde{C} \times Y) \cup (I \times Y) \cup (X \times J) \in \mathcal{I} \otimes \mathcal{J}$$

and also $C \in \mathcal{I} \otimes \mathcal{J}$.

The proof of WRHP case is almost identical. \square

Lemma 22 *The family of sets possessing TRHP is closed under countable unions and complements. The same is true for the family of sets possessing WRHP.*

Proof Proofs for both cases follow the same pattern, so without loss of generality let us focus on the case of TRHP.

Let $C = \bigcup_{n \in \omega} C_n$ and (\tilde{C}_n, I_n, J_n) witness TRHP for C_n , $n \in \omega$. Then for each $n \in \omega$

$$(\tilde{C}_n \setminus I_n) \times (Y \setminus J_n) \subseteq C_n \subseteq (\tilde{C}_n \times Y) \cup (I_n \times Y) \cup (X \times J_n).$$

Then

$$\begin{aligned} \left(\bigcup_{n \in \omega} \tilde{C}_n \setminus \bigcup_{n \in \omega} I_n\right) \times \left(Y \setminus \bigcup_{n \in \omega} J_n\right) &\subseteq \bigcup_{n \in \omega} (\tilde{C}_n \setminus I_n) \times (Y \setminus J_n) \subseteq \bigcup_{n \in \omega} C_n = C \\ &\subseteq \bigcup_{n \in \omega} \left((\tilde{C}_n \times Y) \cup (I_n \times Y) \cup (X \times J_n) \right) \\ &\subseteq \left(\left(\bigcup_{n \in \omega} C_n \right) \times Y \right) \cup \left(\left(\bigcup_{n \in \omega} I_n \right) \times Y \right) \\ &\quad \cup \left(X \times \left(\bigcup_{n \in \omega} J_n \right) \right). \end{aligned}$$

Hence, setting $\tilde{C} = \bigcup_{n \in \omega} \tilde{C}_n$, $I = \bigcup_{n \in \omega} I_n$, $J = \bigcup_{n \in \omega} J_n$ completes this part of the proof.

Now let $C = D^c$ for D witnessing TRHP with (\tilde{D}, I, J) . We have

$$(\tilde{D} \setminus I) \times (Y \setminus J) \subseteq D \subseteq (\tilde{D} \times Y) \cup (I \times Y) \cup (X \times J).$$

Through complementation

$$((\tilde{D} \times Y) \cup (I \times Y) \cup (X \times J))^c \subseteq C \subseteq ((\tilde{D} \setminus I) \times (Y \setminus J))^c.$$

Let us focus on the right-hand side

$$\begin{aligned} ((\tilde{D} \setminus I) \times (Y \setminus J))^c &= ((\tilde{D} \setminus I) \times (Y \setminus J))^c \cup ((\tilde{D} \setminus I)^c \times Y) \\ &\subseteq ((\tilde{D})^c \times Y) \cup (X \times J) \cup (I \times Y). \end{aligned}$$

Now the left-hand side. It is an intersection of the following sets

$$\begin{aligned} (\tilde{D} \times Y)^c &= (\tilde{D})^c \times Y, \\ (I \times Y)^c &= (X \setminus I) \times Y, \\ (X \times J)^c &= X \times (Y \setminus J), \end{aligned}$$

which is equal to

$$((\tilde{D})^c \setminus I) \times (Y \setminus J).$$

In summary

$$((\tilde{D})^c \setminus I) \times (Y \setminus J) \subseteq C \subseteq ((\tilde{D})^c \times Y) \cup (X \times J) \cup (I \times Y).$$

Then $((\tilde{D})^c, I, J)$ witnesses TRHP for C . The proof is complete. □

Theorem 23 *Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$. Then*

- (i) *if $\mathcal{A} = \mathcal{I} \cup \mathcal{I}^*$ then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has WRHP.*
- (ii) *if $\mathcal{B} = \mathcal{J} \cup \mathcal{J}^*$ then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has TRHP.*

Proof Let us prove (ii) (the proof of (i) is similar). Let us notice that rectangles from \mathcal{C} have TRHP. Indeed, if $C = A \times B$ then set

$$\begin{aligned} \tilde{C} &= A, \quad I = \emptyset, \quad J = B^c \text{ if } B \in \mathcal{J}^* \\ \tilde{C} &= \emptyset, \quad I = \emptyset, \quad J = B \text{ if } B \in \mathcal{J}. \end{aligned}$$

The rest of the proof relies on Lemma 22 which allows us to perform an induction over the hierarchy of sets making up the σ -algebra \mathcal{C} . □

Theorem 24 *Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ and assume that*

- (i) *$(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has TRHP and $(\mathcal{A}, \mathcal{I})$ has SP, or*
- (ii) *$(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has WRHP and $(\mathcal{B}, \mathcal{J})$ has SP.*

Then $(\mathcal{C}, \mathcal{I} \otimes \mathcal{J})$ has SP.

Proof Assume (i). Let $D \subseteq X \times Y$ be dense, set $D_1 = \pi_1(D)$ and let $B \in \mathcal{C}$ be $\mathcal{I} \otimes \mathcal{J}$ -positive. Then there are $\tilde{B} \in \mathcal{A} \setminus \mathcal{I}$ and $J \in \mathcal{J}$ such that $\tilde{B} \times (Y \setminus J) \subseteq B$. It follows that

$$D + B \supseteq D + (\tilde{B} \times (Y \setminus J)) \supseteq \bigcup_{d_1 \in D_1} \bigcup_{d_2 \in D_{d_1}} (d_1 + \tilde{B}) \times (d_2 + Y \setminus J).$$

Therefore for every $x \in D_1 + \tilde{B}$ the set $(D + B)_x$ contains a translation of $Y \setminus J$. By SP $D_1 + \tilde{B} \in \mathcal{I}^*$ thus $D + B$ is $\mathcal{I} \otimes \mathcal{J}$ -residual.

The proof assuming (ii) is analogous. □

4 Maximal invariant σ -ideals with Borel bases

There is a surprising connection between maximal invariant σ -ideals with Borel bases and Smital properties.

Proposition 25 *The following are equivalent:*

- (i) \mathcal{I} has VWSP;
- (ii) \mathcal{I} is maximal among invariant proper σ -ideals with Borel base.

Proof (i) \Rightarrow (ii) : Let us suppose that $\mathcal{J} \supsetneq \mathcal{I}$ is such an ideal. Let A be a Borel set from $\mathcal{J} \setminus \mathcal{I}$. Then there exists a countable set D such that $D + A$ is \mathcal{I} -residual, therefore \mathcal{J} -residual, hence \mathcal{J} is not proper.

(ii) \Rightarrow (i) : Let us suppose that there is a set $B \in \text{Bor} \setminus \mathcal{I}$ for which $B + D$ is not \mathcal{I} -residual for every countable set D . Consider the family

$$\mathcal{I}' = \{A \cup C : A \in \mathcal{I} \wedge (\exists D)(C \subseteq D + B \wedge |D| \leq \omega)\}.$$

It is an invariant proper σ -ideal satisfying $\mathcal{I} \subsetneq \mathcal{I}'$, which leads to a contradiction. □

Proposition 26 *Let $\{\mathcal{I}_n : n \in \omega\}$ be a countable family of pairwise distinct maximal invariant σ -ideals on X with Borel bases. Then for each $n \in \omega$ the σ -ideal \mathcal{I}_n is orthogonal to $\bigcap_{k \in \omega \setminus \{n\}} \mathcal{I}_k$.*

Proof Fix $n \in \omega$. There are sets $A_k \in (\text{Bor}(X) \cap \mathcal{I}_n) \setminus \mathcal{I}_k, k \in \omega \setminus \{n\}$. By Proposition 25 for each of them there is a countable set C_k such that $A_k + C_k \in \mathcal{I}_n \cap \mathcal{I}_k^*$. Therefore

$$\bigcup_{k \in \omega \setminus \{n\}} (A_k + C_k) \in \mathcal{I}_n \cap \bigcap_{k \in \omega \setminus \{n\}} \mathcal{I}_k^*.$$

□

Corollary 27 *Let \mathcal{I} be a maximal invariant σ -ideal with a Borel base different from \mathcal{M} and \mathcal{N} . Then there is $A \in \mathcal{I} \cap (\mathcal{M} \cap \mathcal{N})^*$.*

Let us now focus on $X = 2^\omega$. The following result incorporates techniques similar to these used in [10, Theorem 3.1].

Theorem 28 *There are \mathfrak{c} many maximal invariant σ -ideals on 2^ω .*

Proof Let $\{A_\alpha : \alpha < \mathfrak{c}\}$ be an AD family on ω , i.e. for all distinct $\alpha, \beta < \mathfrak{c}$ the set $A_\alpha \cap A_\beta$ is finite. For every $\alpha < \mathfrak{c}$ set

$$\begin{aligned} \mathcal{I}_\alpha &= \{A \subseteq 2^\omega : (\exists B \in \text{Bor}(2^\omega) A \subseteq B \wedge (\exists M \in \mathcal{M}(2^{A_\alpha})) (\forall x \in 2^{A_\alpha}) \\ &\quad (x \notin M \rightarrow B_x^\alpha \in \mathcal{N}(2^{\omega \setminus A_\alpha})))\}, \end{aligned}$$

where $B_x^\alpha = \{y \upharpoonright \omega \setminus A_\alpha : y \in B, y \upharpoonright A_\alpha = x\}$.

Let us show that $I_\alpha \neq I_\beta$ if $\alpha \neq \beta$. Set

$$\begin{aligned} M &\in \mathcal{M}(2^{A_\alpha \setminus A_\beta}) \setminus \mathcal{N}(2^{A_\alpha \setminus A_\beta}), \\ C &= \{x \in 2^\omega : x \upharpoonright (A_\alpha \setminus A_\beta) \in M\}. \end{aligned}$$

Notice that $C \in I_\alpha$. The set

$$M_\alpha = \bigcup_{t \in 2^{A_\alpha \cap A_\beta}} \left\{ y \in 2^{A_\alpha} : y \upharpoonright (A_\alpha \cap A_\beta) = t \wedge y \upharpoonright (A_\alpha \setminus A_\beta) \in M \right\}$$

is a finite union of meager sets, hence meager. For each $x \in 2^{A_\alpha} \setminus M_\alpha$ the set C_x^α is empty.

On the other hand for each $x \in 2^{A_\beta}$

$$C_x^\beta = \left\{ y \in 2^{\omega \setminus A_\beta} : y \upharpoonright (A_\alpha \setminus A_\beta) \in M \right\}.$$

The above set may be considered as a product $M \times 2^{\omega \setminus (A_\alpha \cup A_\beta)}$ of non-null set and the whole space, which is not null.

Now we will show that every \mathcal{I}_α is maximal among invariant σ -ideals on 2^ω with Borel bases. Each \mathcal{I}_α is essentially $\mathcal{M}(2^{A_\alpha}) \otimes \mathcal{N}(2^{\omega \setminus A_\alpha})$. It follows that \mathcal{I}_α has WSP, thus by Proposition 25 the proof is complete. \square

The reasoning in the above Theorem does not translate to the case of \mathbb{R} . We may ask the following question.

Question 29 *Are \mathcal{M} and \mathcal{N} the only maximal invariant σ -ideals with Borel bases in \mathbb{R} ?*

Question 30 *Is it true that for every set $G \in (\mathcal{M} \cap \mathcal{N})^*$ there is a countable set C such that $C + G = \mathbb{R}$?*

A positive answer to Question 30 would also answer positively Question 29.

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