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On the rigidity of Souslin trees and their generic branches

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Abstract
We show it is consistent that there is a Souslin tree S such that after forcing with S, S is Kurepa and for all clubs $C \subset \omega_1$, $S \upharpoonright C$ is rigid. This answers the questions in Fuchs (Arch Math Logic 52(1–2):47–66, 2013). Moreover, we show it is consistent with $\diamondsuit$ that for every Souslin tree T there is a dense $X \subseteq T$ which does not contain a copy of T. This is related to a question due to Baumgartner in Baumgartner (Ordered sets (Banff, Alta., 1981), volume 83 of NATO Adv. Study Inst. Ser. C: Math. Phys. Sci., Reidel, Dordrecht-Boston, pp 239–277, 1982).

Keywords Souslin trees · Kurepa trees

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1 Introduction

Recall that an ω_1-tree is said to be Souslin if it has no uncountable chain or antichain. In [2, 3], Fuchs and Hamkins considered various notions of rigidity of Souslin trees and studied the following question: How many generic branches can Souslin trees introduce, when they satisfy certain rigidity requirements? In [2], Fuchs asks a few questions which motivate the following theorem.

Theorem 1.1 *It is consistent with* GCH *that there is a Souslin tree S such that $\Vdash_S$ "S is Kurepa and $S \upharpoonright C$ is rigid for every club $C \subset \omega_1$".*

Theorem 1.1 answers all questions in [2]. We refer the reader to [2, 3] for motivation and history.

In [1], Baumgartner proves that under $\diamondsuit^+$ there is a lexicographically ordered Souslin tree which is minimal as a tree and as an uncountable linear order. At the end of his construction he asks the following question: Does there exist a minimal

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Aronszajn line if $\diamondsuit$ holds? This question is not settled here but motivates the following proposition.

Proposition 1.2 *It is consistent with $\diamondsuit$ that if S is a Souslin tree then there is a dense $X \subset S$ which does not contain a copy of S.*

Proposition 1.2 shows it is impossible to follow the same strategy as Baumgartner's in [1], in order to show $\diamondsuit$ implies that there is a minimal Aronszajn line. More precisely, it is impossible to find a lexicographically ordered Souslin tree which is minimal as a tree and as an uncountable linear order.

This paper is organized as follows. In the next section we prove Proposition 1.2. In the third section we introduce a Souslin tree which makes itself a Kurepa tree. This tree is used in the last section, where we prove Theorem 1.1.

Let's fix some definitions, notations and conventions. Assume T, S are trees and $f : T \longrightarrow S$ is injective. Then f is said to be an *embedding* when $t <_T s \iff f(t) <_S f(s)$. T is called an ω_1-*tree* if its levels are countable and $\mathrm{ht}(T) = \omega_1$. T is said to be pruned if for all $t \in T$ and $\alpha \in \omega_1 \setminus \mathrm{ht}(t)$ there is $s \geq t$ such that $\mathrm{ht}(s) = \alpha$. If $t \in T$ and $\alpha \leq \mathrm{ht}(t)$, $t \restriction \alpha$ refers to the $\leq_T$ predecessor of t in level α. $C \subset T$ is called a *chain* if it consists of pairwise comparable elements. A chain $b \subset T$ is called a *branch* if it intersects all levels of T. An ω_1-tree U is called *minimal* if for every uncountable $X \subset U$, U embeds into X. If T is a tree and α is an ordinal, $T(\alpha) = \{t \in T : \mathrm{ht}(t) = \alpha\}$ and $T(< \alpha) = \{t \in T : \mathrm{ht}(t) < \alpha\}$. If A is a set of ordinals, $T \restriction A = \{t \in T : \mathrm{ht}(t) \in A\}$. If $t \in T$ and $U \subset T$ then $U_t = \{u \in U : t \leq_T u\}$. Assume Q is a poset and θ is a regular cardinal. We say $M \prec H_\theta$ is suitable for Q if Q and the power set of the transitive closure of Q are in M.

2 Minimality of Souslin trees and $\diamondsuit$

This section is devoted to the proof of Proposition 1.2. We will use the following terminology and notation in this section. By N we mean the set of all countable infinite successor ordinals, and $\mathbb{P}$ refers to the countable support iteration $\langle P_i, \dot{Q}_j : i \leq \omega_2, j < \omega_2 \rangle$, where $Q_j = 2^{<\omega_1}$ for each $j \in \omega_2$.

Lemma 2.1 *Assume $U = (\omega_1, <)$ is a Souslin tree, $p \in \mathbb{P}$, $\dot{X}$ is the canonical $\mathbb{P}_1$-name for the generic subset of ω_1, $p \Vdash$ "$\dot{f}$ is an embedding from U to $\dot{X}$" and for every $t \in U$ define $\varphi(p, t) = \{s \in U : \exists \bar{p} \leq p \ \bar{p} \Vdash \dot{f}(t) = s\}$. Then there is an $\alpha \in \omega_1$ such that for all $t \in U \setminus U(< \alpha)$, $\varphi(p, t)$ is not a chain.*

Proof Let $Y_p = \{y \in U : \varphi(p, y) \text{ is a chain}\}$. Y_p is downward closed and if it is countable we are done. Fix $p \in \mathbb{P}$ and assume for a contradiction that Y_p is uncountable. Let $A_p = \{t \in U : p \Vdash t \in \dot{X} \text{ or } p \Vdash t \notin \dot{X}\}$. A_p is countable. Fix $\alpha > \sup\{\mathrm{ht}(a) : a \in A_p\}$ and $y \in Y_p \setminus U(\leq \alpha)$. Since U is an Aronszajn tree and $\varphi(p, y)$ is a chain, we can choose $\beta \in \omega_1 \setminus \sup\{\mathrm{ht}(s) : s \in \varphi(p, y)\}$. For all $s \in \varphi(p, y)$, $\alpha < \mathrm{ht}(s) < \beta$ since $\emptyset \Vdash \mathrm{ht}(y) \leq \mathrm{ht}(\dot{f}(y))$. Then we can extend p to q such that $q \Vdash \dot{X} \cap (U(\leq \beta) \setminus U(< \alpha)) = \emptyset$, which contradicts $p \Vdash \dot{f}(y) \in \varphi(p, y)$. $\square$

Lemma 2.2 *Assume $U \in \mathrm{V}$ is a pruned Souslin tree and $G \subset \mathbb{P}$ is* V*-generic. Then in* $\mathrm{V}[G]$*, there is a dense $X \subset U$ which does not have a copy of U.*

Proof Let $\dot{X}$ be as in Lemma 2.1. Since U is pruned, $1_{\mathbb{P}} \Vdash \dot{X} \subset U$ is dense. We will show $1_{\mathbb{P}} \Vdash \dot{X}$ has no copy of U. Assume for a contradiction that $p \Vdash_{\mathbb{P}} \dot{f}$ is an embedding from U to $\dot{X}$. Fix a regular cardinal θ and a countable $M \prec H_\theta$ which contains $U, p, \dot{f}, 2^{\mathbb{P}}$. Also let $\langle D_n : n \in \omega \rangle$ be an enumeration of all dense open subsets of $\mathbb{P}$ in M, $\delta = M \cap \omega_1$ and $t \in U(\delta)$. For each $\sigma \in 2^{<\omega}$, find $p_\sigma \in D_{|\sigma|} \cap M$, s_σ and $t_{|\sigma|} < t$, such that:

(1) if $\sigma \sqsubset \tau$ then $p_\tau \leq p_\sigma$ and $s_\sigma \leq s_\tau$,
(2) if $\sigma \perp \tau$ then $s_\sigma \perp s_\tau$,
(3) $p_\sigma \Vdash \dot{f}(t_{|\sigma|}) = s_\sigma$.

In order to see how these sequences are constructed, let $t_0 < t$ be arbitrary and $p_\emptyset, s_\emptyset$ be such that $p_\emptyset \Vdash$ "$\dot{f}(t_0) = s_\emptyset$" and $p_\emptyset \in D_0 \cap M$. Assuming these sequences are given for all $\sigma \in 2^n$, use Lemma 2.1 to find $t_{n+1} < t$ such that $\varphi(p_\sigma, t_{n+1})$ is not a chain, for all $\sigma \in 2^n$. Let $s_{\sigma^\frown 0}, s_{\sigma^\frown 1}$ be in $\varphi(p_\sigma, t_{n+1}) \cap M$ such that $s_{\sigma^\frown 0} \perp s_{\sigma^\frown 1}$. Now find $p_{\sigma^\frown 0}, p_{\sigma^\frown 1}$ in $M \cap D_{n+1}$ which are extensions of p_σ such that $p_{\sigma^\frown i} \Vdash$ "$\dot{f}(t_{n+1}) = s_{\sigma^\frown i}$", for $i = 0, 1$.

For each $r \in 2^\omega$, let p_r be a lower bound for $\{p_\sigma : \sigma \sqsubset r\}$ and let $b_r \subset U \cap M$ be a downward closed chain such that $p_r \Vdash \dot{f}[\{s \in U : s < t\}] \subset b_r$. Note that b_r intersects all the levels of U below δ. It is obvious that p_r is an $(M, \mathbb{P})$-generic condition below p. Moreover, if r, r' are two distinct real numbers then $b_r \neq b_{r'}$. Let $r \in 2^\omega$ such that U has no element on top of b_r. Then p_r forces that $\dot{f}(t)$ is not defined, which is a contradiction. □

Now we are ready for the proof of Proposition 1.2. Let V be a model of ZFC + GCH and $G \subset \mathbb{P}$ be V-generic. Since $\mathbb{P}$ is a countable support iteration of σ-closed posets of size $\aleph_1$, it preserves all cardinals. The same argument as in Theorem 8.3 in [4] shows that $\diamondsuit$ holds in $\mathrm{V}[G]$.

Let U be a Souslin tree in $\mathrm{V}[G]$. For some $\alpha \in \omega_2$, $U \in \mathrm{V}[G \cap P_\alpha]$ since $|U| = \aleph_1$. Let $\dot{R}$ be the canonical P_α-name such that $\mathbb{P} = P_\alpha * \dot{R}$. Then $1_{P_\alpha} \Vdash \dot{R}$ is isomorphic to $\mathbb{P}$. By Lemma 2.2, there is a dense $X \subset U$ in V[G] which has no copy of U, as desired.

3 A Souslin tree with many generic branches

Definition 3.1 The poset Q is the set of all $p = (T^p, \Pi_p)$ such that:

(1) $\Delta_p \in \omega_1$ and $T^p = (\Delta_p, \leq_p)$ is a countable binary tree of height α_p such that for all $t \in T^p$ and for all $\beta \in \alpha_p \setminus \mathrm{ht}_{T^p}(t)$ there is $s \in T^p(\beta)$ with $t <_{T^p} s$.
(2) $\Pi_p = \langle \pi^p_\xi : \xi \in D_p \rangle$ where $D_p \subset \omega_2$ is countable and for each $\xi \in D_p$ there are x, y of the same height in T^p such that $\pi^p_\xi : (T^p)_x \longrightarrow (T^p)_y$ is a tree isomorphism.

We let $q \leq p$ if T^q end-extends T^p, $D_p \subset D_q$ and for all $\xi \in D_p$, $\pi^q_\xi \restriction T^p = \pi^p_\xi$.

Lemma 3.2 *Q is σ-closed. Moreover if* CH *holds, Q has the $\aleph_2$-cc.*

Proof The first part of the lemma is obvious. Assume $A \in Q^{\aleph_2}$. By thinning A out, we can assume that for all p, q in A, $T^p = T^q$, $\{D_p : p \in A\}$ is a Δ-system with root R and $|\{\langle \pi^p_\xi : \xi \in R\rangle : p \in A\}| = 1$. Now all p, q in A are compatible. □

Lemma 3.3 *If $T = \bigcup_{p \in G} T^p$ for a generic $G \subset Q$, then T is Souslin.*

Proof Obviously T is an ω_1-tree. Let τ be a Q-name and $p \Vdash_Q$ "$\tau \subset T$ is a maximal antichain". We show $p \Vdash \tau$ is countable. Let $M \prec H_\theta$ be countable, θ regular and $2^Q, \tau$ be in M. Let $\langle p_n = (T_n, \Pi_n) : n \in \omega\rangle$, be a descending (M, Q)-generic sequence with $p_0 = p$. Let $\pi^{p_n}_\xi = \pi^n_\xi$, $\delta = M \cap \omega_1$, and $R = \bigcup_{n\in\omega} T_n$. So $\mathrm{ht}(R) = \delta$ and $M \cap \omega_2 = \bigcup_{n\in\omega} D_{p_n}$. Let $\mathcal{F}$ be the set of all finite compositions of functions of the form $\bigcup_{n\in\omega} \pi^n_\xi$ with $\xi \in M \cap \omega_2$. Let $\langle f_n : n \in \omega\rangle$ be an enumeration of $\mathcal{F}$ with infinite repetition and $A = \{t \in R : \exists n \in \omega\ (p_n \Vdash t \in \tau)\}$. Observe that for all $t \in R$ there is $a \in A$ such that a, t are comparable.

Let $\langle \alpha_m : m \in \omega\rangle$ be an increasing cofinal sequence in δ. For each $t \in R$ we build an increasing sequence $\bar{t} = \langle t_m : m \in \omega\rangle$ as follows. Let $t_0 = t$. Assume t_m is given. If $R_{t_m} \cap \mathrm{dom}(f_m) = \emptyset$, choose $t_{m+1} > t_m$ with $\mathrm{ht}(t_{m+1}) > \alpha_m$. If $R_{t_m} \cap \mathrm{dom}(f_m) \neq \emptyset$, let $s \in \mathrm{dom}(f_m) \cap R_{t_m}$. Let $a \in A$ such that $a, f_m(s)$ are comparable. Let $x = \max\{f_m(s), a\}$ and $t_{m+1} > f_m^{-1}(x)$ with $\mathrm{ht}(t_{m+1}) > \alpha_m$. Let b_t be the downward closure of $\bar{t}$.

Let $B = \{f_n[b_t] : t \in R \text{ and } n \in \omega\}$. Let q be the lower bound for $\langle p_n : n \in \omega\rangle$ described as follows. $T^q = R \cup T^q(\delta)$ and for each cofinal branch $c \subset R$ there is a unique $y \in T^q(\delta)$ above c if and only if $c \in B$. For each $\xi \in M \cap \omega_2$, let $\pi^q_\xi \upharpoonright R = \bigcup_{n\in\omega} \pi^n_\xi$. Note that this determines π^q_ξ on $T^q(\delta)$ as well and $\pi^q_\xi(y)$ is defined for all $y \in T^q(\delta)$.

The condition q forces that for each $y \in T(\delta) = T^q(\delta)$ there is $a \in A$ with $a < y$. In other words q forces that $\tau = A$. Since p was arbitrary, 1_Q forces that every maximal antichain has to be countable. □

From now on T is the same tree as in Lemma 3.3. For each $\xi \in \omega_2$ let $\pi_\xi = \bigcup_{p\in G} \pi^p_\xi$, where $G \subset Q$ is generic. Observe that if $x \in \mathrm{dom}(\pi_\xi) \cap \mathrm{dom}(\pi_\eta)$ and $\xi \neq \eta$ are ordinals then there is $\alpha > \mathrm{ht}(x)$ such that for all $y \in T(\alpha) \cap T_x$, $\pi_\xi(y) \neq \pi_\eta(y)$. So forcing with T makes T Kurepa.

4 Highly rigid dense subsets of *T*

In this section we show the tree T, in the forcing extensions by $P = (2^{<\omega_1}, \supset)$, has dense subsets which are witnesses for Theorem 1.1.

Lemma 4.1 *Let $U = (\omega_1, <)$ be a pruned Souslin tree and $S \subset \omega_1$ be generic for P. Then in* V$[S]$ *the following hold.*

(1) *S is a Souslin tree when it is considered with the inherited order from U.*
(2) *$S \subset U$ is dense.*
(3) *For all clubs $C \subset \omega_1$, $S \upharpoonright C$ is rigid.*

Proof In order to see that S is Souslin, note that σ-closed posets do not add uncountable antichains to Souslin trees. Moreover by standard density arguments $S \subset U$ is dense.

Assume for a contradiction $p \Vdash_P$ "$\dot{f} : \dot{S} \restriction \dot{C} \longrightarrow \dot{S} \restriction \dot{C}$ is a nontrivial tree embedding." Let $\langle M_\xi : \xi \in \omega + 1\rangle$ be a continuous $\in$-chain of countable elementary submodels of H_θ where θ is regular and $p, \dot{f}, 2^U$ are in M_0. For each $\xi \leq \omega$, let $\delta_\xi = M_\xi \cap \omega_1$ and $t \in U(\delta_\omega)$. Let $t_n = t \restriction \delta_n$. For each $\sigma \in 2^{<\omega}$ we find $q_\sigma \in M_{|\sigma|+1} \cap P$, s_σ such that:

(1) $q_0 \leq p$, and if $\sigma \subset \tau$ then $q_\tau \leq q_\sigma$,
(2) q_σ is $(M_{|\sigma|}, P)$-generic and $q_\sigma \subset M_{|\sigma|}$,
(3) q_σ forces that $\dot{f}(t_{|\sigma|-1}) = s_\sigma$,
(4) if $\sigma \perp \tau$ then $s_\sigma \perp s_\tau$,
(5) if $\sigma \subset \tau$ then q_τ forces that $t_{|\sigma|} \in \dot{S} \restriction \dot{C}$.

Assuming q_σ and s_σ are given for all $\sigma \in 2^n$, we find $q_{\sigma\frown 0}, q_{\sigma\frown 1}, s_{\sigma\frown 0}$, and $s_{\sigma\frown 1}$. Let $\bar{q}_\sigma = q_\sigma \cup \{(t_n, 1)\}$. Obviously, $\bar{q}_\sigma \Vdash t_n \in \dot{S} \restriction \dot{C}$ and for all $\sigma \in 2^n$, $\{s \in U : \exists r \leq \bar{q}_\sigma\ r \Vdash \dot{f}(t_n) = s\}$ is uncountable. In M_{n+1}, find r_0, r_1 below $\bar{q}_\sigma$ and $s_{\sigma\frown 0}, s_{\sigma\frown 1}$ such that $s_{\sigma\frown 0} \perp s_{\sigma\frown 1}$ and $r_i \Vdash$ "$\dot{f}(t_n) = s_{\sigma\frown i}$." Let $q_{\sigma\frown i} < r_i$ be (M_{n+1}, P)-generic with $q_{\sigma\frown i} \subset M_{n+1}$, and $q_{\sigma\frown i} \in M_{n+2}$.

Let $r \in 2^\omega$ such that $\{s_\sigma : \sigma \subset r\}$ does not have an upper bound in U. Let p_r be a lower bound for $\{p_\sigma : \sigma \subset r\}$. Then p_r forces that $\dot{f}(t)$ is not defined which is a contradiction. □

Lemma 4.2 *Suppose M is suitable for Q and $\delta = M \cap \omega_1$. Let $\langle q_n : n \in \omega\rangle$ be a decreasing (M, Q)-generic sequence. Define a condition $q \in Q$ by setting $T^q = \bigcup_{n\in\omega} T^{q_n}$, $D_q = \bigcup_{n\in\omega} D_{q_n}$ and for each $\xi \in D_q$ let $\pi^q_\xi = \bigcup_{n\in\omega} \pi^{q_n}_\xi$. Also let $\Pi_q = \langle \pi^q_\xi : \xi \in D_q\rangle$. Let $\mathcal{F}$ be the set of all finite compositions of functions of the form π^q_ξ with $\xi \in D_q$. Assume $m \in \omega$ and $\langle b_i : i \in m\rangle$ are branches through T^q. Then there is an extension $q' \leq q$ such that $\alpha_{q'} \geq \delta + 1$ and for all branches $c \subset T^q$, c has an upper bound iff for some $f \in \mathcal{F}$ and $i \in m$, $f(b_i)$ is cofinal in c.*

Proof Note that $D_q = M \cap \omega_2$ and $\alpha_q = \delta$. Let $T^{q'} \restriction \delta = T^q$. Let $B = \{f(b_i) : i \in m$ and $f \in \mathcal{F}\}$. Obviously B is countable and we can fix an enumeration of B with $n \in \omega$. Let $T^{q'}(\delta + 1) = [\delta, \delta + \omega)$ and put $\delta + n$ on top of the n'th element in B. It is obvious how we should extend Π_q to $\Pi_{q'}$ with $D_q = D_{q'}$. □

Lemma 4.3 *Let $G \subset Q$ be V-generic, $p \in P$ and $\dot{S}$ be the canonical P-name for the generic subset of ω_1. Let $\dot{f}, \dot{C}$ be $P * T$-names in $\mathrm{V}[G]$ and t, x, y be pairwise incompatible in T. Suppose (p, t) forces $\dot{f}$ is an embedding from $\dot{S}_x \restriction \dot{C}$ to $\dot{S}_y \restriction \dot{C}$. For every $u \in T_x$ define $\psi(p, t, u) = \{s \in T : \exists t' > t\ \exists \bar{p} \leq p\ (\bar{p}, t') \Vdash [u \in \dot{S}_x \restriction \dot{C} \wedge \dot{f}(u) = s]\}$. Then for any $u \in T_x$ there is $u' > u$ such that $\psi(p, t, u')$ is not a chain.*

Proof Fix p, t, u as above and assume for a contradiction that for all $u' > u$ in T, $\psi(p, t, u')$ is a chain. Since T is ccc, without loss of generality we can assume that for all $q \in P$ and $\alpha \in \omega_1$, there is $\bar{q} \leq q$ such that $(\bar{q}, 1_T)$ decides the statement $\alpha \in \dot{C}$. For each $q \in P, r \in T, v \in T$ let $\alpha_{q,r,v} = \sup\{\mathrm{ht}_T(s) : s \in \psi(q, r, v)\}$. Note that if $\bar{q} \leq q$ and $\bar{r} \geq r$ then $\psi(\bar{q}, \bar{r}, v) \subseteq \psi(q, r, v)$ and $\alpha_{\bar{q},\bar{r},v} \leq \alpha_{q,r,v}$.

Let M_0, M_1 be countable elementary submodels of H_θ, θ be a regular cardinal and $\{p, t, u, x, y, \dot{f}, \dot{C}\} \in M_0 \in M_1$. Suppose $\langle p_n : n \in \omega\rangle$ is an (M_0, P)-generic sequence which is in M_1 and $p_0 \leq p$. Let $p' = \bigcup_{n\in\omega} p_n$ and $\delta_i = M_i \cap \omega_1$, for $i \in 2$. Note that $p' \Vdash \delta_0 \in \dot{C}$.

Let $\bar{p} < p'$ such that:

(1) $\bar{p} \Vdash \forall v \in T_x \cap (M_1 \setminus M_0)\, [v \in \dot{S}]$
(2) $\bar{p} \Vdash \forall v \in T_y \cap (M_1 \setminus M_0)\, [v \notin \dot{S}]$.

Let $u_0 > u$ be in $T(\delta_0)$. Since $\bar{p}$ is (M_0, Q)-generic, it forces that $\delta_0 \in \dot{C} \wedge u_0 \in \dot{S} \wedge \mathrm{ht}_{\dot{S}}(u_0) = \delta_0$. In particular, by elementarity of M_0 and basic facts on ordinal arithmetic, $\bar{p} \Vdash u_0 \in \dot{S}_x \restriction \dot{C}$.

Suppose $q < \bar{p}, r > t$ such that (q, r) decides $\dot{f}(u_0)$. Then the condition (q, r) forces that $\mathrm{ht}(\dot{f}(u_0)) \geq \delta_1$. So, $\delta_1 \leq \alpha_{\bar{p},t,u_0} \leq \alpha_{p',t,u_0} \in M_1$. But this is a contradiction. □

In the next lemma we use the following standard fact: If U is a Souslin tree and $X \subset U$ is uncountable and downward closed, then there is $x \in U$ such that $U_x \subset X$. In order to see this assume for all $v \in U$, U_v is not contained in X. Let A be the set of all minimal a outside of X. Observe that A is an uncountable antichain, contradicting the fact that U was Souslin. Lemma 4.4 finishes the proof of Theorem 1.1.

Lemma 4.4 *Assume $G * S * b$ is* V*-generic for $Q * P * \dot{T}$. Let x, y be incomparable in T. Then in* V$[G * S * b]$ *for all clubs $C \subset \omega_1$, $S_x \restriction C$ does not embed into $S_y \restriction C$.*

Proof Assume for a contradiction that (q_0, p_0, t_0) is a condition in $Q * P * \dot{T}$ which forces $\dot{f} : \dot{S}_x \restriction \dot{C} \longrightarrow \dot{S}_y \restriction \dot{C}$ is a tree embedding and x, y are incompatible in T. Note that $\dot{f}(\dot{S}_x)$ is an uncountable subset of $\dot{T}_y$ and $\dot{T}$ is a Souslin tree in V$[G][S]$. So the downward closure of $\dot{f}(\dot{S}_x)$ contains $\dot{T}_z$ for some $z > y$. Therefore, by extending $x, y, (q_0, p_0, t_0)$ if necessary, we can assume that $\dot{f}(\dot{S}_x)$ is dense in $\dot{S}_y$.

Again by extending $x, y, (q_0, p_0, t_0)$ we may assume $(q_0, p_0, t_0) \Vdash [x, y$ are in $\dot{S} \restriction \dot{C}$ and $\dot{f}(x) = y]$. Furthermore, by extending t_0 if necessary we can assume that $\mathrm{ht}(t_0) > \mathrm{ht}(y)$ and x, y, t_0 are pairwise incomparable. Since T is a ccc poset we can assume that for all $\alpha \in \omega_1$, for all u, v in T and for all $(a, b) \in P * Q$ we have $(a, b, u) \Vdash \alpha \in \dot{C} \longleftrightarrow (a, b, v) \Vdash \alpha \in \dot{C}$.

Let M be a countable elementary submodel of H_θ such that θ is regular and (q_0, p_0, t_0), $\dot{f}$ are in M. Let $\langle q_n : n \in \omega\rangle$ be a decreasing (M, Q)-generic sequence. Define $q \in Q$ as in Lemma 4.2. Let $\mathcal{F}$ be the set of all finite compositions of functions of the form π^q_ξ with $\xi \in M \cap \omega_2$. Let $\Pi_q = \langle \pi^q_\xi : \xi \in M \cap \omega_2\rangle$. Obviously, q is an (M, Q)-generic condition. Let $\langle g_n : n \in \omega\rangle$ be an enumeration of $\mathcal{F}$ with infinite repetition. Let $\langle \gamma_n : n \in \omega\rangle$ be an increasing cofinal sequence in $\delta = M \cap \omega_1$ with $\gamma_0 = 0$.

We find a decreasing sequence $\langle p_n \in P \cap M : n \in \omega\rangle$ and increasing sequences $\langle \delta_n \in \delta : n \in \omega\rangle$, $\langle t_n \in T^q : n \in \omega\rangle$, $\langle u_n \in T^q : n \in \omega\rangle$ $\langle s_n \in T^q : n \in \omega\rangle$ such that:

(1) $\delta_n \geq \gamma_n$ for all $n \in \omega$,
(2) $(q, p_n.t_n) \Vdash \min\{\mathrm{ht}_{\dot{S}}(s_n), \mathrm{ht}_{\dot{S}}(u_n), \mathrm{dom}(p_n)\} \geq \delta_n$,
(3) $\mathrm{ht}_{T^q}(t_n) \geq \mathrm{ht}_{T^q}(s_n) + 1$,
(4) $(q, p_n, 1_{T^q}) \Vdash \delta_n \in \dot{C}$,

(5) $(q, p_n, t_n) \Vdash \dot{f}(u_n) = s_n$,
(6) if $n \in \omega \setminus 1$ and $t_{n-1} \in \mathrm{dom}(g_n)$ then $g_n(t_n) \perp s_n$,
(7) if $n \in \omega \setminus 1$ and $u_{n-1} \in \mathrm{dom}(g_n)$ then $g_n(u_n) \perp s_n$.

We let $u_0 = x$, $s_0 = y$, $\delta_0 \in \omega_1$ such that (q, p_0, t_0) forces that $\min\{\mathrm{ht}_{\dot{S}}(x), \mathrm{ht}_{\dot{S}}(y), \alpha_{p_n}\} = \delta_0$. It is easy to see that this choice together with p_0, t_0 will satisfy the corresponding conditions. For given $p_n, t_n, s_n, u_n, \delta_n$ we introduce $p_{n+1}, t_{n+1}, s_{n+1}, u_{n+1}, \delta_{n+1}$.

If $t_n \notin \mathrm{dom}(g_{n+1})$ let $v = s_n$. If $t_n \in \mathrm{dom}(g_{n+1})$, let $v \geq s_n$ such that $v \perp g_{n+1}(t_n)$. Such a v exists because $\mathrm{ht}(t_n) > \mathrm{ht}(s_n)$, g_{n+1} is level preserving and the tree T^q is binary.

Claim 4.5 *There are $t'_n > t_n$, $p'_n < p_n$, $u'_n > u_n$ such that if $u_n \in \mathrm{dom}(g_{n+1})$ then (q, p'_n, t'_n) forces $[u'_n \in \mathrm{dom}(\dot{f}) \wedge v < \dot{f}(u'_n) \wedge \dot{f}(u'_n) \perp g_{n+1}(u'_n)]$.*

Proof of Claim Assume $u_n \in \mathrm{dom}(g_{n+1})$. Recall that $\dot{f}(\dot{S}_x)$ is forced to be dense in $\dot{S}_y$. Let $\bar{p}_n \leq p_n$, $\bar{t}_n \geq t_n$, $a_0 > u_n$, $v' > v$ such that $(q, \bar{p}_n, \bar{t}_n) \Vdash \dot{f}(a_0) = v'$. This is possible because q is (M, Q)-generic. Let $a > a_0$, t_n^0, t_n^1 be extensions of $\bar{t}_n$, and p_n^0, p_n^1 be extensions of $\bar{p}_n$ such that $(q, p_n^i, t_n^i) \Vdash \dot{f}(a) = s_n^i$ where $i \in 2$ and $s_n^0 \perp s_n^1$. Again, this is possible because of Lemma 4.3 and the fact that q is (M, Q)-generic. Let $a' > a$ such that $\mathrm{ht}(a') > \max\{\mathrm{ht}(s_n^0), \mathrm{ht}(s_n^1)\}$. Fix $i \in 2$ such that $g_{n+1}(a') \perp s_n^i$. Then for all $e > a'$, (q, p_n^i, t_n^i) forces that if $e \in \mathrm{dom}(\dot{f})$ then $\dot{f}(e) > s_n^i$. Moreover it forces that $g_{n+1}(e) \perp s_\sigma^i$. Therefore, $(q, p_n^i, t_n^i) \Vdash [\forall e > a'\ e \in \mathrm{dom}(\dot{f}) \longrightarrow g_{n+1}(e) \perp \dot{f}(e)]$. Let $u'_n > a'$, $p'_n < p_n^i$ and $t'_n > t_n^i$ such that $(q, p'_n, t'_n) \Vdash [u'_n \in \mathrm{dom}(\dot{f})]$. Then this condition will also force $\dot{f}(u'_n) \perp g_{n+1}(u'_n)$ and $v < \dot{f}(u'_n)$. □

Fix p'_n, t'_n, u'_n as in the claim above. By extending p'_n if necessary, we can assume that $(q, p'_n, 1_{T^q})$ decides the γ_{n+1}'st element of $\dot{C} \setminus \delta_n$ and we let δ_{n+1} be this ordinal. Let $u_{n+1} > u'_n$ such that for some $p_{n+1} < p'_n$ with $\mathrm{dom}(p_{n+1}) \geq \delta_{n+1}$, the condition $(q, p_{n+1}, 1_{T^q})$ forces that $u_{n+1} \in \dot{S} \upharpoonright \dot{C}$ and $\mathrm{ht}_{\dot{S}}(u_{n+1}) \geq \delta_{n+1}$. Let $r > t'_n$. By extending (q, p_{n+1}, r) if necessary, we can assume this condition decides $\dot{f}(u_{n+1})$. Let $s_{n+1} \in T^q$ such that $(q, p_{n+1}, r) \Vdash \dot{f}(u_{n+1}) = s_{n+1}$. Let $t_{n+1} \geq r$ such that $\mathrm{ht}(t_{n+1}) > \mathrm{ht}(s_{n+1})$. We leave it to the reader to verify that all of the conditions above hold.

Let b_0, b_1 be the downward closure of $\{u_n : n \in \omega\}$ and $\{t_n : n \in \omega\}$ respectively. By Lemma 4.2 there is $q' < q$ such that $\alpha_{q'} \geq \delta + 1$ and for all branches $c \subset T^q$, c has an upper bound in $T^{q'}$ if and only if $g_n(b_i)$ is cofinal in c for some $n \in \omega$ and $i \in 2$. Fix such a q' for the rest of the argument.

We claim that $\langle s_n : n \in \omega\rangle$ does not have an upper bound in $T^{q'}$. Suppose for a contradiction that it has an upper bound. Then for some $m \in \omega$, either

(1) $\{g_m(t_n) : n \in \omega \wedge t_n \in \mathrm{dom}(g_m)\}$ is cofinal in the downward closure of $\{s_n : n \in \omega\}$ or
(2) $\{g_m(u_n) : n \in \omega \wedge u_n \in \mathrm{dom}(g_m)\}$ is cofinal in the downward closure of $\{s_n : n \in \omega\}$.

Due to similarity of the arguments, let's assume that the first alternative happens. Since we enumerated the elements of $\mathcal{F}$ with infinite repetition, by increasing m if necessary, we can assume that $t_m \in \text{dom}(g_m)$. But then $g_m(t_m) \perp s_m$, meaning that the first alternative cannot happen, which is a contradiction. Hence $\{s_n : n \in \omega\}$ does not have an upper bound in $T^{q'}$.

Let t be the upper bound of $\langle t_n : n \in \omega \rangle$ in $T^{q'}$, and u be the upper bound for $\langle u_n : n \in \omega \rangle$ which has the lowest height δ. Let p be a lower bound for $\langle p_n : n \in \omega \rangle$ which forces that $u \in \dot{S}$. It is easy to see that $(q', p, t) \Vdash [\delta \in \dot{C} \wedge u \in \dot{S} \wedge \text{ht}_{\dot{S}}(u) = \delta]$. Also by (5), (q', p, t) forces $\dot{f}(u_n) = s_n$ for all $n \in \omega$. Hence (q', p, t) forces that $\dot{f}(u)$ is an upper bound for $\langle s_n : n \in \omega \rangle$ which is a contradiction. $\square$

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