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On convergence of solutions to variational–hemivariational inequalities

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Abstract. In this paper we investigate the convergence behavior of the solutions to the time-dependent variational–hemivariational inequalities with respect to the data. First, we give an existence and uniqueness result for the problem, and then, deliver a continuous dependence result when all the data are subjected to perturbations. A semipermeability problem is given to illustrate our main results.

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1. Introduction

Variational–hemivariational inequalities represent a special class of inequalities which involve both convex and nonconvex functions. Elliptic hemivariational and variational–hemivariational inequalities were introduced by Panagiotopoulos in the 1980s and studied in many contributions, see [15,17] and the references therein. Various classes of such inequalities have been recently investigated, for instance, in [7,9,10,12,20,22]. They play an important role in describing many mechanical problems arising in solid and fluid mechanics.

In this paper we study the following time-dependent variational–hemivariational inequality: find $u\colon \mathbb{R}_+ = [0, +\infty) \to X$ such that, for all $t \in \mathbb{R}_+$, $u(t) \in K$ and

$$\begin{aligned} &\langle Au(t) - f(t), v - u(t)\rangle_X + \varphi(u(t), v) - \varphi(u(t), u(t)) \\ &\quad + j^0(u(t); v - u(t)) \geq 0 \quad \text{for all} \quad v \in K, \end{aligned} \tag{1}$$

where K is a nonempty, closed and convex subset of a reflexive Banach space X, $A\colon X \to X^*$ and $\varphi\colon K \times K \to \mathbb{R}$ are given maps to be specified later, $j\colon X \to \mathbb{R}$ is a locally Lipschitz function, and $f\colon \mathbb{R}_+ \to X^*$ is fixed. The notation $j^0(u; v)$ stands for the generalized directional derivative of j at point $u \in X$ in the direction $v \in X$. The goal of the paper is to study the convergence of solution of the variational–hemivariational inequality (1) when the data A, f, φ, j and K are subjected to perturbations.

The dependence of solutions to elliptic variational–hemivariational inequalities on the data has been studied only recently. For such inequalities the dependence with respect to functions φ and j was investigated in [13], where A and K were not subjected to perturbations. A result on the dependence of solutions to elliptic variational inequalities with respect to perturbations of the set K of a special form was studied

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in [19]. There, the data A, φ and f were independent of perturbations. For a class of elliptic history-dependent variational–hemivariational inequalities studied in [21], the convergence result was obtained in a case when φ depends on a history-dependent operator, and A does not depend on perturbations. A result on the convergence with respect to the set of constraints K were studied for elliptic quasivariational inequalities in [1]. In all aforementioned papers the convergence results were applied to various mathematical models of deformable bodies in contact mechanics. Note that a result on the dependence of solutions to evolution second order hemivariational inequalities with respect to perturbations of the operators can be found in [8]. Furthermore, it is well known that the continuous dependence results are of importance in optimal control and identification problems, see, e.g., [2,9,23].

The aim of the paper is twofold. First, we consider the class of abstract time-dependent variational–hemivariational inequalities of the form (1) for which we study the dependence of the solution with respect to the data A, f, φ, j and K. Our hypotheses on φ and j are different than the one used in the aforementioned papers. Moreover, the set of constraints is of a more general form.

Second, we illustrate the applicability of the convergence results in the study of a semipermeability problem. Semipermeability problems were first considered in [5] for convex potentials (which lead to monotone relations) and, later, in [11,16,17] for nonconvex superpotentials (leading to nonmonotone relations). They concern the treatment of semipermeable membranes either in the interior or on the boundary of the body and arise, for instance, in flow problems through porous media and heat conduction problems. In the current paper we study a semipermeability problem involving simultaneously both monotone and nonmonotone relations. Its weak formulation is a variational–hemivariational inequality. Note that the convergence results for semipermeability problems are provided here for the first time. Finally, we underline that our convergence results of Sect. 3 are also applicable to various problems in contact mechanics like a nonlinear elastic contact problem with normal compliance condition with unilateral constraint, and a contact problem with the Coulomb friction law in which the friction bound is supposed to depend on the normal displacement, studied in, e.g., [1,6,13,19].

The rest of this paper is organized as follows. In Sect. 2, we will introduce some necessary preliminary materials. Section 3 is devoted to the proofs of convergence results for the elliptic variational–hemivariational inequality and its time-dependent counterpart. In Sect. 4, we apply the results to a semipermeability problem.

2. Preliminaries

In this section we recall notation, basic definitions and a result on unique solvability of a variational–hemivariational inequality.

Let $(X, \|\cdot\|_X)$ be a Banach space. We denote by X^* its dual space and by $\langle\cdot,\cdot\rangle_X$ the duality pairing between X^* and X. The strong and weak convergences in X are denoted by "$\to$" and "$\rightharpoonup$," respectively.

Let $C(\mathbb{R}_+;X)$ be the space of continuous functions defined on interval $\mathbb{R}_+ = [0,+\infty)$ with values in X. For a subset $K \subset X$ the symbol $C(\mathbb{R}_+;K)$ denotes the set of continuous functions on $\mathbb{R}_+$ with values in K. We also recall that the convergence of a sequence $\{x_n\}_{n\geq 1}$ to the element x, in the space $C(\mathbb{R}_+;X)$, can be described as follows

$$\begin{cases} x_n \to x \quad \text{in } C(\mathbb{R}_+;X), \quad \text{as } n\to\infty \quad \text{if and only if} \\ \max\limits_{t\in[0,k]} \|x_n(t)-x(t)\|_X \to 0, \quad \text{as } n \to \infty, \quad \text{for all} \quad k \in \mathbb{N}. \end{cases} \tag{2}$$

We recall the definitions of the convex subdifferential, the (Clarke) generalized gradient and the pseudomonotone single-valued operators, see [3,4].

Definition 1. A function $f\colon X \to \mathbb{R}$ is said to be lower semicontinuous (l.s.c.) at u, if for any sequence $\{u_n\}_{n\geq 1} \subset X$ with $u_n \to u$, we have $f(u) \leq \liminf f(u_n)$. A function f is said to be l.s.c. on X, if f is l.s.c. at every $u \in X$.

Definition 2. Let $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and l.s.c. function. The mapping $\partial\varphi_c\colon X \to 2^{X^*}$ defined by

$$\partial\varphi_c(u) = \{\, u^* \in X^* \mid \langle u^*, v-u\rangle_X \leq \varphi(v) - \varphi(u) \quad \text{for all} \quad v \in X \,\}$$

for $u \in X$, is called the subdifferential of φ. An element $u^* \in \partial_c\varphi(u)$ is called a subgradient of φ in u.

Definition 3. Given a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$, we denote by $\varphi^0(u; v)$ the (Clarke) generalized directional derivative of φ at the point $u \in X$ in the direction $v \in X$ defined by

$$\varphi^0(u; v) = \limsup_{\lambda\to 0^+,\, \zeta\to u} \frac{\varphi(\zeta+\lambda v) - \varphi(\zeta)}{\lambda}.$$

The generalized gradient of φ at $u \in X$, denoted by $\partial\varphi(u)$, is a subset of X^* given by

$$\partial\varphi(u) = \{\, u^* \in X^* \mid \varphi^0(u; v) \geq \langle u^*, v\rangle_X \quad \text{for all} \quad v \in X \,\}.$$

Furthermore, a locally Lipschitz function $\varphi\colon X \to \mathbb{R}$ is said to be regular (in the sense of Clarke) at $u \in X$, if for all $v \in X$ the directional derivative $\varphi'(u; v)$ exists, and for all $v \in X$, we have $\varphi'(u; v) = \varphi^0(u; v)$. The function is regular (in the sense of Clarke) on X if it is regular at every point in X.

Definition 4. A single-valued operator $F\colon X \to X^*$ is said to be pseudomonotone, if it is bounded (sends bounded sets into bounded sets) and satisfies the inequality

$$\langle Fu, u-v\rangle \leq \liminf \langle Fu_n, u_n - v\rangle_X \quad \text{for all} \quad v \in X,$$

where $u_n \rightharpoonup u$ in X with $\limsup \langle Fu_n, u_n - u\rangle_X \leq 0$.

The following result provides a useful characterization of a pseudomonotone operator.

Lemma 5. (see [12, Proposition 1.3.66]) *Let X be a reflexive Banach space and $F\colon X \to X^*$ be a single-valued operator. The operator F is pseudomonotone if and only if F is bounded and satisfies the following condition: if $u_n \rightharpoonup u$ in X and $\limsup\langle Fu_n, u_n - u\rangle_X \leq 0$, then $Fu_n \rightharpoonup Fu$ in X^* and $\lim\langle Fu_n, u_n - u\rangle_X = 0$.*

The following notion of the Mosco convergence of sets will be useful in the next sections. For the definitions, properties and other modes of set convergence, we refer to [4, Chapter 4.7] and [14].

Definition 6. Let $(X, \|\cdot\|)$ be a normed space and $\{K_\rho\}_{\rho>0} \subset 2^X \setminus \{\emptyset\}$. We say that K_ρ converge to K in the Mosco sense, $\rho \to 0$, denoted by $K_\rho \xrightarrow{M} K$ if and only if the two conditions hold

(m1) for each $x \in K$, there exists $\{x_\rho\}_{\rho>0}$ such that $x_\rho \in K_\rho$ and $x_\rho \to x$ in X,

(m2) for each subsequence $\{x_\rho\}_{\rho>0}$ such that $x_\rho \in K_\rho$ and $x_\rho \rightharpoonup x$ in X, we have $x \in K$.

For the following properties of the Mosco convergence, we refer to [14, p. 520].

Remark 7. Let $K_\rho \xrightarrow{M} K$. Then, $K \neq \emptyset$ implies $K_\rho \neq \emptyset$ and the opposite is not true. Also, if K_ρ is a closed and convex set for all $\rho > 0$, then K is also closed and convex.

Finally, we recall a result on existence and uniqueness of solution to the following variational–hemivariational inequality.

Problem 8. *Find $u \in K$ such that*

$$\langle Au - f, v-u\rangle_X + \varphi(u, v) - \varphi(u, u) + j^0(u; v-u) \geq 0 \quad \text{for all} \quad v \in K, \tag{3}$$

Problem 8 was studied in [13] where results on its unique solvability, continuous dependence on the data and a penalty method were provided. We need the following hypotheses on the data of Problem 8.

$$K \text{ is nonempty, closed and convex subset of } X. \tag{4}$$

$$f \in X^*. \tag{5}$$

$$\begin{cases} A\colon X \to X^* \text{ is an operator such that} \\ \text{(a) } A \text{ is pseudomonotone.} \\ \text{(b) there exists } m_A > 0 \text{ such that} \\ \qquad \langle Au_1 - Au_2, u_1 - u_2\rangle_X \geq m_A \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \\ \text{(c) there exist } \alpha_A > 0, \alpha_1, \alpha_2 \in \mathbb{R}, u_0 \in K \text{ such that} \\ \qquad \langle Au, u - u_0\rangle \geq \alpha_A \|u\|_X^2 - \alpha_1 \|u\|_X - \alpha_2 \\ \quad \text{for all } u \in X. \end{cases} \tag{6}$$

$$\begin{cases} \varphi\colon K \times K \to \mathbb{R} \text{ is a function such that} \\ \text{(a) } \varphi(u, \cdot)\colon K \to \mathbb{R} \text{ is convex and l.s.c. on } K, \text{ for all } u \in K. \\ \text{(b) there exists } \alpha_\varphi > 0 \text{ such that} \\ \qquad \varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \\ \qquad \leq \alpha_\varphi \|u_1 - u_2\|_X \|v_1 - v_2\|_X \\ \quad \text{for all } u_1, u_2, v_1, v_2 \in K. \end{cases} \tag{7}$$

$$\begin{cases} j\colon X \to \mathbb{R} \text{ is a function such that} \\ \text{(a) } j \text{ is locally Lipschitz.} \\ \text{(b) there exist } c_0, c_1 \geq 0 \text{ such that} \\ \qquad \|\partial j(u)\|_{X^*} \leq c_0 + c_1 \|u\|_X \text{ for all } u \in X. \\ \text{(c) there exists } \alpha_j \geq 0 \text{ such that} \\ \qquad j^0(u_1; u_2 - u_1) + j^0(u_2; u_1 - u_2) \leq \alpha_j \|u_1 - u_2\|_X^2 \\ \quad \text{for all } u_1, u_2 \in X. \end{cases} \tag{8}$$

The following existence and uniqueness result was established in Theorem 18 of [13].

Theorem 9. *Assume that (4)–(8) hold and the following smallness conditions are satisfied*

$$\alpha_\varphi + \alpha_j < m_A \quad \textit{and} \quad \alpha_j < \alpha_A. \tag{9}$$

Then Problem 8 has a unique solution $u \in K$.

3. Convergence of solutions

In this section we study the dependence of the solution to Problem 8 with respect to the operator A, functions f, φ and j, and the constraint set K.

Continuous dependence for Problem 8 has been investigated earlier in some particular cases. For example, it was studied in Theorem 23 in [13], where A and K are independent of $\rho > 0$ and the hypotheses on the behavior of φ_ρ and j_ρ are different than ours. Furthermore, the dependence of solution to an elliptic variational inequality with respect to perturbations of the set K_ρ was studied in [19] under the hypotheses $j \equiv 0$, A, φ and f are independent of ρ, φ satisfies additional assumptions, and the constraint sets K_ρ satisfy the following hypothesis

$$\left\{\begin{array}{l} K_\rho = c(\rho)K + d(\rho)\theta \text{ is such that} \\ \quad \text{(a)}\ \ K \text{ is a nonempty, closed and convex subset of } X. \\ \quad \text{(b)}\ \ 0_X \in K_\rho \text{ and } \theta \text{ is a given element of } X. \\ \quad \text{(c)}\ \ c\colon (0,+\infty) \to \mathbb{R} \text{ is such that } c(\rho) \to 1, \text{ as } \rho \to 0. \\ \quad \text{(d)}\ \ d\colon (0,+\infty) \to \mathbb{R} \text{ is such that } d(\rho) \to 0, \text{ as } \rho \to 0. \end{array}\right. \tag{10}$$

We make the following observation.

Remark 10. Note that if K_ρ, for $\rho > 0$, is defined by (10), then $K_\rho \xrightarrow{M} K$, as $\rho \to 0$. Indeed, for each $x \in K$, we define $x_\rho \in K$ by $x_\rho = c(\rho)x + d(\rho)\theta \in K_\rho$. From (10)(c) and (d), it follows that $x_\rho \to x$ in X. Hence, the condition (m1) in Definition 6 holds. Moreover, for each subsequence $\{x_\rho\}_{\rho>0}$ such that $x_\rho \in K_\rho$ and $x_\rho \rightharpoonup x$ in X, there exists $x'_\rho \in K$ such that $x_\rho = c(\rho)x'_\rho + d(\rho)\theta$. Again, from (10)(c) and (d), we infer that $x'_\rho \rightharpoonup x$ in X. Since K is closed and convex, it is weakly closed. Hence, $x \in K$ which implies that the condition (m2) in Definition 6 is satisfied.

Consider the following perturbed version of Problem 8.

Problem 11. *Find $u_\rho \in K_\rho$ such that for all $v_\rho \in K_\rho$, we have*

$$\langle A_\rho u_\rho - f_\rho, v_\rho - u_\rho\rangle_X + \varphi_\rho(u_\rho, v_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; v_\rho - u_\rho) \geq 0. \tag{11}$$

We formulate the hypotheses needed for the continuous dependence result. Let $\rho > 0$.

$$\left\{\begin{array}{l} K,\ K_\rho \text{ are sets such that} \\ \quad \text{(a) } K,\ K_\rho \text{ satisfy (4).} \\ \quad \text{(b) } K_\rho \xrightarrow{M} K, \text{ as } \rho \to 0. \end{array}\right. \tag{12}$$

$$\left\{\begin{array}{l} f,\ f_\rho \text{ are functions such that} \\ \quad \text{(a) } f,\ f_\rho \text{ satisfy (5).} \\ \quad \text{(b) } f_\rho \to f \ \text{ in } \ X^*, \text{ as } \rho \to 0. \end{array}\right. \tag{13}$$

$$\left\{\begin{array}{l} A,\ A_\rho\colon X \to X^* \text{ are operators such that} \\ \quad \text{(a) } A, A_\rho \text{ satisfy (6) with } m_A > 0, \alpha_A > 0, \alpha_1, \alpha_2 \in \mathbb{R},\ u_0 \in K, \\ \qquad \text{and } m_{A_\rho} > 0, \alpha_{A_\rho} > 0, \alpha_{1\rho}, \alpha_{2\rho} \in \mathbb{R},\ u_{0\rho} \in K_\rho, \text{respectively.} \\ \quad \text{(b) there exist } c_A > 0 \text{ and } \alpha_\rho > 0 \text{ with } \alpha_\rho \to 0, \text{ as } \rho \to 0 \text{ such} \\ \qquad \text{that } \|A_\rho u - Av\|_{X^*} \leq c_A\left(\alpha_\rho + \|u - v\|_X\right) \ \text{ for all } \ u, v \in X \\ \qquad \text{with } \ \|u\|_X, \|v\|_X \leq M, \text{ where } M > 0 \text{ is independent of } \rho. \end{array}\right. \tag{14}$$

$$\begin{cases} \varphi\colon K\times K\to\mathbb{R},\ \varphi_\rho\colon K_\rho\times K_\rho\to\mathbb{R} \text{ are functions such that} \\ \quad \text{(a) } \varphi,\ \varphi_\rho \text{ satisfy (7) with } \alpha_\varphi>0 \text{ and } \alpha_{\varphi_\rho}>0, \text{ respectively.} \\ \quad \text{(b) for all } u_\rho, v_\rho \text{ such that } u_\rho, v_\rho\in K_\rho \text{ for each } \rho>0 \text{ with} \\ \qquad u_\rho\rightharpoonup u \text{ in } X \text{ and } v_\rho\to v \text{ in } X, \text{we have} \\ \qquad \limsup\,(\varphi_\rho(u_\rho,v_\rho)-\varphi_\rho(u_\rho,u_\rho))\le\varphi(u,v)-\varphi(u,u). \\ \quad \text{(c) there exists a nondecreasing function } c_\varphi\colon\mathbb{R}_+\to\mathbb{R}_+ \text{ such} \\ \qquad \text{that for all } u,v_1,v_2\in K_\rho, \text{ we have} \\ \qquad \varphi_\rho(u,v_1)-\varphi_\rho(u,v_2)\le c_\varphi(\|u\|_X)\|v_1-v_2\|_X. \end{cases} \tag{15}$$

$$\begin{cases} j,\ j_\rho\colon X\to\mathbb{R} \text{ are functions such that} \\ \quad \text{(a) } j,\ j_\rho \text{ satisfy (8) with } \alpha_j\ge 0, c_0, c_1\ge 0 \\ \qquad \text{and } \alpha_{j_\rho}\ge 0, c_{0\rho}, c_{1\rho}\ge 0, \text{ respectively.} \\ \quad \text{(b) for all } u_\rho, v_\rho \text{ such that } u_\rho, v_\rho\in K_\rho \text{ for each } \rho>0 \text{ with} \\ \qquad u_\rho\rightharpoonup u \text{ in } X \text{ and } v_\rho\to v \text{ in } X, \text{we have} \\ \qquad \limsup\, j_\rho^0(u_\rho;v_\rho-u_\rho)\le j^0(u;v-u). \end{cases} \tag{16}$$

$$\begin{cases} \text{(a) there exist } m_0, m_1, m_2>0 \text{ such that for } \rho>0 \text{ sufficiently small} \\ \qquad \alpha_{\varphi_\rho}+\alpha_{j_\rho}\le m_0<m_{A_\rho} \quad\text{and}\quad \alpha_{\varphi_\rho}+\alpha_{j_\rho}\le m_1<m_2\le\alpha_{A_\rho}. \\ \text{(b) there exists } M_0>0 \text{ such that for all } \rho>0 \text{ sufficiently small} \\ \qquad \max\{\alpha_{1\rho},\alpha_{2\rho},c_{0\rho},c_{1\rho},\|u_{0\rho}\|\}\le M_0. \end{cases} \tag{17}$$

The following result ensures the existence, uniqueness and convergence of Problem 11.

Theorem 12. *Assume that hypotheses* (12)(a), (13)(a), (14)(a), (15)(a), (16)(a) *and* (17)(a) *are satisfied. Then,*

(i) *for each* $\rho>0$*, Problem* 11 *has a unique solution* $u_\rho\in K_\rho$*,*

(ii) *if, in addition,* (9), (12)(b), (13)(b), (14)(b), (15)(b)(c), (16)(b), (17)(b) *hold, then the sequence* $\{u_\rho\}$ *converges in* X*, as* $\rho\to 0$*, to the solution* u *of Problem 8.*

Proof. (i) The existence and uniqueness result for Problem 11 follows from Theorem 9.

(ii) Let $\rho>0$ and $u_\rho\in K_\rho$ be the unique solution to Problem 11. First, we will show that there exists a constant $c>0$ independent of ρ such that for all $\rho>0$ sufficiently small

$$\|u_\rho\|_X\le c. \tag{18}$$

From conditions (8) and (16)(*a*), we have

$$\begin{aligned} j_\rho^0(u_\rho;u_{0\rho}-u_\rho) &= j_\rho^0(u_\rho;u_{0\rho}-u_\rho)+j_\rho^0(u_{0\rho};u_\rho-u_{0\rho})-j_\rho^0(u_{0\rho};u_\rho-u_{0\rho}) \\ &\le \alpha_{j_\rho}\|u_\rho-u_{0\rho}\|_X^2+|\max\{\langle\zeta_\rho,u_\rho-u_{0\rho}\rangle\mid\zeta_\rho\in\partial j_\rho(u_{0\rho})\}| \\ &\le \alpha_{j_\rho}\|u_\rho-u_{0\rho}\|_X^2+(c_{0\rho}+c_{1\rho}\|u_{0\rho}\|_X)\|u_\rho-u_{0\rho}\|_X. \end{aligned}$$

Taking $v_\rho = u_{0\rho} \in K_\rho$ in inequality (11), we obtain

$$\begin{aligned}
&\alpha_{A_\rho}\|u_\rho\|_X^2 - \alpha_{1\rho}\|u_\rho\|_X - \alpha_{2\rho} \le \langle A_\rho u_\rho, u_\rho - u_{0\rho}\rangle_X \\
&\quad\le \varphi_\rho(u_\rho, u_{0\rho}) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; u_{0\rho} - u_\rho) + \langle f_\rho, u_\rho - u_{0\rho}\rangle_X \\
&\quad\le \big(\varphi_\rho(u_\rho, u_{0\rho}) - \varphi_\rho(u_\rho, u_\rho) + \varphi_\rho(u_{0\rho}, u_\rho) - \varphi_\rho(u_{0\rho}, u_{0\rho})\big) \\
&\qquad+ \big(\varphi_\rho(u_{0\rho}, u_{0\rho}) - \varphi_\rho(u_{0\rho}, u_\rho)\big) + j_\rho^0(u_\rho; u_{0\rho} - u_\rho) + \langle f_\rho, u_\rho - u_{0\rho}\rangle_X \\
&\quad\le \alpha_{\varphi_\rho}\|u_\rho - u_{0\rho}\|_X^2 + \alpha_{j_\rho}\|u_\rho - u_{0\rho}\|_X^2 \\
&\qquad+ (c_\varphi(\|u_{0\rho}\|_X) + c_{0\rho} + c_{1\rho}\|u_{0\rho}\|_X + \|f_\rho\|_{X^*})\|u_\rho - u_{0\rho}\|_X.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&(\alpha_{A_\rho} - \alpha_{\varphi_\rho} - \alpha_{j_\rho})\|u_\rho\|_X^2 \\
&\quad\le \Big((2\alpha_{\varphi_\rho} + 2\alpha_{j_\rho} + c_{1\rho})\|u_{0\rho}\|_X + c_\varphi(\|u_{0\rho}\|_X) + \alpha_{1\rho} + c_{0\rho} + \|f_\rho\|_{X^*}\Big)\|u_\rho\|_X \\
&\qquad+ (\alpha_{\varphi_\rho} + \alpha_{j_\rho} + c_{1\rho})\|u_{0\rho}\|_X^2 + (c_\varphi(\|u_{0\rho}\|_X) + c_{0\rho} + \|f_\rho\|_{X^*})\|u_{0\rho}\|_X + \alpha_{2\rho}.
\end{aligned}$$

Hence, by hypothesis (17), we can find a constant $c > 0$ independent of ρ such that, for all $\rho > 0$ sufficiently small, condition (18) holds.

Exploiting (18) and the reflexivity of X, by passing to a subsequence if necessary, we may suppose that the sequence $\{u_\rho\}$, $u_\rho \in K_\rho$ for each $\rho > 0$, converges weakly to some $u \in X$, as $\rho \to 0$. By the condition $(m2)$ of Definition 6, we deduce that $u \in K$.

We will show that $u \in K$ is a solution of Problem 8. From (m1) in Definition 6, we can find a sequence $\{u'_\rho\}$ such that $u'_\rho \in K_\rho$ and $u'_\rho \to u$ in X. Taking $v_\rho = u'_\rho$ in (11) we have

$$\langle A_\rho u_\rho - f_\rho, u'_\rho - u_\rho\rangle + \varphi_\rho(u_\rho, u'_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; u'_\rho - u_\rho) \ge 0.$$

Then, from (13)(b), (14)(b), (15)(b) and (16)(b), using the fact that A_ρ is a bounded operator, we have

$$\begin{aligned}
&\limsup \langle Au_\rho, u_\rho - u\rangle \\
&\quad\le \limsup \langle Au_\rho - A_\rho u_\rho, u_\rho - u\rangle + \limsup \langle A_\rho u_\rho, u_\rho - u\rangle \\
&\quad\le \limsup c_A\, \alpha_\rho\, \|u_\rho - u\| + \limsup \langle A_\rho u_\rho, u_\rho - u\rangle \\
&\quad\le \limsup \langle A_\rho u_\rho, u_\rho - u'_\rho\rangle + \limsup \langle A_\rho u_\rho, u'_\rho - u\rangle \\
&\quad\le \limsup \big(\langle f_\rho, u_\rho - u'_\rho\rangle + \varphi_\rho(u_\rho, u'_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; u'_\rho - u_\rho)\big) \le 0.
\end{aligned}$$

Since A is pseudomonotone, by Lemma 5, we infer

$$Au_\rho \rightharpoonup Au \ \text{ in } \ X^* \tag{19}$$

$$\lim\langle Au_\rho, u_\rho - u\rangle = 0. \tag{20}$$

Let $w \in K$. From hypothesis (12) and $(m1)$ in Definition 6, we find a sequence $\{w_\rho\}$ such that $w_\rho \in K_\rho$ for each $\rho > 0$ and $w_\rho \to w$ in X, as $\rho \to 0$. We set $v_\rho = w_\rho$ in inequality (11), and obtain

$$\langle A_\rho u_\rho - f_\rho, w_\rho - u_\rho\rangle + \varphi_\rho(u_\rho, w_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; w_\rho - u_\rho) \ge 0.$$

Since $u_\rho \rightharpoonup u$ in X, $w_\rho \to w$ in X, from (19) and (20), we have

$$\lim\langle Au_\rho, u_\rho - w_\rho\rangle = \lim\langle Au_\rho, u_\rho - u\rangle + \lim\langle Au_\rho, u - w_\rho\rangle = \langle Au, u - w\rangle.$$

Using the latter, from (13)(b), (14)(b), (15)(b) and (16)(b) again, we deduce that

$$\begin{aligned}\langle Au, u-w\rangle &= \limsup \langle Au_\rho, u_\rho - w_\rho\rangle \\ &\le \limsup \langle Au_\rho - A_\rho u_\rho, u_\rho - w_\rho\rangle + \limsup \langle A_\rho u_\rho, u_\rho - w_\rho\rangle \\ &\le \limsup c_A\, \alpha_\rho\, \|u_\rho - w_\rho\| + \limsup \langle A_\rho u_\rho, u_\rho - w_\rho\rangle \\ &\le \limsup \big(\langle f_\rho, u_\rho - w_\rho\rangle + \varphi_\rho(u_\rho, w_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; w_\rho - u_\rho)\big) \\ &\le \langle f, u-w\rangle + \varphi(u,w) - \varphi(u,u) + j^0(u; w-u).\end{aligned}$$

Since $w \in K$ is arbitrary, we have shown that

$$\langle Au - f, w-u\rangle + \varphi(u,w) - \varphi(u,u) + j^0(u; w-u) \ge 0 \ \text{ for all } \ w \in K,$$

which implies that $u \in K$ solves Problem 8. Since every subsequence of $\{u_\rho\}$ converges weakly to the same limit ($u \in K$ is the unique solution to Problem 8), the whole sequence $\{u_\rho\}$ converges weakly to $u \in K$.

Finally, we show the strong convergence $u_\rho \to u$ in X, as $\rho \to 0$. Since $K_\rho \xrightarrow{M} K$, as $\rho \to 0$, by condition $(m1)$ of Definition 6, we can find a sequence $\{\tilde{u}_\rho\}$ such that $\tilde{u}_\rho \in K_\rho$ for each $\rho > 0$ and $\tilde{u}_\rho \to u$, as $\rho \to 0$. Choosing $v_\rho = \tilde{u}_\rho$ in (11), we have

$$\begin{aligned} m_{A_\rho}\|u_\rho - \tilde{u}_\rho\|_X^2 &\le \langle A_\rho u_\rho - A_\rho \tilde{u}_\rho, u_\rho - \tilde{u}_\rho\rangle_X \\ &= \langle A_\rho u_\rho, u_\rho - \tilde{u}_\rho\rangle_X + \langle A_\rho \tilde{u}_\rho, \tilde{u}_\rho - u_\rho\rangle_X \\ &\le \varphi_\rho(u_\rho, \tilde{u}_\rho) - \varphi_\rho(u_\rho, u_\rho) + j_\rho^0(u_\rho; \tilde{u}_\rho - u_\rho) + \langle f_\rho - A_\rho \tilde{u}_\rho, u_\rho - \tilde{u}_\rho\rangle_X.\end{aligned}$$

It follows from (14)(b) that

$$\begin{aligned} &\limsup \langle -A_\rho \tilde{u}_\rho, u_\rho - \tilde{u}_\rho\rangle_X \\ &\quad = \limsup \langle Au - A_\rho \tilde{u}_\rho, u_\rho - \tilde{u}_\rho\rangle_X + \limsup \langle -Au, u_\rho - \tilde{u}_\rho\rangle_X \\ &\quad \le \limsup c_A\, (\alpha_\rho + \|\tilde{u}_\rho - u\|)\|u_\rho - \tilde{u}_\rho\| + \limsup \langle -Au, u_\rho - \tilde{u}_\rho\rangle_X \\ &\quad = 0.\end{aligned}$$

Passing to the upper limit, as $\rho \to 0$, and exploiting (13)(b), (15)(b) and (16)(b), we deduce that $\limsup \|u_\rho - \tilde{u}_\rho\|_X^2 \le 0$. Hence, we obtain $\|u_\rho - \tilde{u}_\rho\|_X \to 0$. Finally, we have

$$0 \le \lim \|u_\rho - u\|_X \le \lim \|u_\rho - \tilde{u}_\rho\|_X + \lim \|\tilde{u}_\rho - u\|_X = 0,$$

which implies that $u_\rho \to u$ in X, as $\rho \to 0$. This completes the proof. □

We now consider the following time-dependent variational–hemivariational inequality.

Problem 13. *Find a function $u\colon \mathbb{R}_+ \to X$ such that, for all $t \in \mathbb{R}_+$, $u(t) \in K$ and*

$$\begin{aligned}&\langle Au(t) - f(t), v - u(t)\rangle_X + \varphi(u(t), v) - \varphi(u(t), u(t)) \\ &\quad + j^0(u(t); v - u(t)) \ge 0 \ \text{ for all } \ v \in K.\end{aligned} \tag{21}$$

We have the following existence and uniqueness result.

Theorem 14. *Assume that the hypotheses of Theorem 9 hold and $f \in C(\mathbb{R}_+; X^*)$. Then, Problem 13 has a unique solution $u \in C(\mathbb{R}_+; K)$.*

Proof. We apply Theorem 9 for any $t \in \mathbb{R}_+$. We deduce that Problem 13 has a unique solution $u(t) \in K$. The fact $u \in C(\mathbb{R}_+; K)$ can be proved from the proof of Theorem 5 in [21]. □

The perturbed problem corresponding to Problem 13 reads as follows.

Problem 15. *Find a function* $u_\rho\colon \mathbb{R}_+ \to X$ *such that, for all* $t \in \mathbb{R}_+$, $u_\rho(t) \in K_\rho$ *and*

$$\langle A_\rho u_\rho(t) - f_\rho(t), v_\rho - u_\rho(t)\rangle_X + \varphi_\rho(u_\rho(t), v_\rho) - \varphi_\rho(u_\rho(t), u_\rho(t)) + j^0_\rho(u_\rho(t); v_\rho - u_\rho(t)) \geq 0 \quad \text{for all} \quad v_\rho \in K_\rho. \tag{22}$$

The following result concerns the pointwise convergence of solutions to Problem 15.

Theorem 16. *Assume that the hypotheses of Theorem 12 are satisfied. Suppose that for all* $\rho > 0$, $f_\rho \in C(\mathbb{R}_+; X^*)$ *and* $f_\rho(t) \to f(t)$ *in* X^* *for all* $t \in \mathbb{R}_+$, *as* $\rho \to 0$. *Then,*

(i) *for each* $\rho > 0$, *Problem 15 has the unique solution* $u_\rho \in C(\mathbb{R}_+; K_\rho)$*;*

(ii) *for each* $t \in \mathbb{R}_+$, *there is a subsequence* $\{u_\rho\}$ *such that* $u_\rho(t) \to u(t)$ *in* X, *as* $\rho \to 0$, *where* $u \in C(\mathbb{R}_+; K)$ *is the unique solution to Problem 13.*

Proof. By applying Theorems 12 and 14, we know that Problem 15 has a unique solution $u_\rho \in C(\mathbb{R}_+; K_\rho)$ for all $\rho > 0$. Moreover, for each $t \in \mathbb{R}_+$, there is a subsequence $\{u_\rho\}$ such that $u_\rho(t) \to u(t)$ in X, as $\rho \to 0$, where $u \in C(\mathbb{R}_+; K)$ is the unique solution to Problem 13. □

Finally, we conjecture that under additional hypotheses the convergence result of Theorem 16(ii) can be strengthen to the uniform convergence of $u_\rho \to u$ in $C(\mathbb{R}_+; X)$, as $\rho \to 0$, which will be studied in the future. We note that a convergence result for Problem 15 with $j \equiv 0$, A, f and φ independent of ρ, and K_ρ of the form (10) was provided in [1] under assumption that A is Lipschitz continuous and φ depends on a history-dependent operator.

4. Semipermeability problem

In this section we consider a semipermeability problem to which our main results of Sect. 3 can be applied. First, we state the classical formulation of the problem, then we provide its variational formulation, and finally we obtain results on its weak solvability and convergence of solutions.

The motivation comes from semipermeability problems studied in [5, Chapter I] for monotone relations, and in [15, Chapter 5.5.3] and [16] for nonmonotone relations which lead to variational and hemivariational inequalities, respectively. We consider the stationary heat conduction problem with constraints and both the interior and the boundary semipermeability relations. Nevertheless, similar problems can be formulated in electrostatics and in flow problems through porous media, where the semipermeability relations are realized by natural and artificial membranes of various types, see [5,11,15–17]. We will analyze a very general situation which leads to a variational–hemivariational inequality problem and provide examples which satisfy our hypotheses.

Let Ω be a bounded domain of $\mathbb{R}^d$ with Lipschitz continuous boundary $\partial\Omega = \Gamma$ which consists of two disjoint measurable parts Γ_1 and Γ_2 such that $m(\Gamma_1) > 0$. The classical model for the heat conduction problem is described by the following boundary value problem.

Problem 17. *Find a temperature* $u\colon \Omega \times \mathbb{R}_+ \to \mathbb{R}$ *such that*

$$-\operatorname{div} a(\boldsymbol{x}, \nabla u) = \tilde{f}(t, u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{23}$$

$$\tilde{f}(t, u) = f_1(t) + f_2(u), \quad -f_2(u) \in \partial h(\boldsymbol{x}, u) \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{24}$$

$$u(t) \in U \quad \text{for} \quad t \in \mathbb{R}_+, \tag{25}$$

$$u = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{26}$$

$$-\frac{\partial u}{\partial \nu_a} \in k(u)\partial g_c(\boldsymbol{x}, u) \quad \text{on} \quad \Gamma_2 \times \mathbb{R}_+. \tag{27}$$

Now, we describe the problem (23)–(27). Equation (23) is the stationary heat equation related to the nonlinear operator in divergence form, with the time-dependent heat source $\tilde{f} = \tilde{f}(t,u)$ where time plays a role of a parameter. The function $\tilde{f}$ in (24) admits an additive decomposition on $f_1 = f_1(t)$ which is prescribed and independent of the temperature u, and $f_2 = f_2(u)$ which is a multivalued function of u in the Clarke subgradient term. Here $h = h(\boldsymbol{x}, r)$ is a function which is assumed to be locally Lipschitz in the second argument. Condition (25) introduces an additional constraint for the temperature (or the pressure of the fluid). The temperature u is constrained to belong to a convex, closed set U. For example, the set U can represent a bilateral obstacle which means that we look for the temperature within prescribed bounds in the domain Ω, see Example 24. The homogeneous (for simplicity) Dirichlet boundary condition is supposed in (26). In the boundary condition (27) the expression $\frac{\partial u}{\partial \nu_a} = a(\boldsymbol{x}, \nabla u) \cdot \boldsymbol{\nu}$ represents the heat flux through the part Γ_2, where $\boldsymbol{\nu}$ denotes the outward unit normal on Γ. Here, $g = g(\boldsymbol{x}, r)$ is a prescribed function, convex in its second argument, $\partial_c g$ stands for its convex subdifferential, and a given function k is positive. Note that in (27) we deal with the nonlinearity which is determined by a law of the form $k\partial_c g$. In such a case we cannot deal with a variational inequality since there is not, in general, a function g_1 with $\partial_c g_1 = k\partial_c g$.

We introduce the following spaces

$$V = \{\, v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \,\}, \quad H = L^2(\Omega). \tag{28}$$

Since $m(\Gamma_1) > 0$, on V we can consider the norm $\|v\|_V = \|\nabla v\|_{L^2(\Omega)^d}$ for $v \in V$ which is equivalent on V to the $H^1(\Omega)$ norm. By $\gamma \colon V \to L^2(\Gamma)$ we denote the trace operator which is known to be linear, bounded and compact. Moreover, by γv we denote the trace of an element $v \in H^1(\Omega)$.

In order to study the variational formulation of Problem 17, we need the following hypotheses.

$$\left\{\begin{array}{l} a\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d \text{ is such that} \\ \quad \text{(a) } a(\cdot, \boldsymbol{\xi}) \text{ is measurable on } \Omega \text{ for all } \boldsymbol{\xi} \in \mathbb{R}^d, \\ \quad\quad \text{and } a(\boldsymbol{x}, 0) = 0 \text{ for a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(b) } a(\boldsymbol{x}, \cdot) \text{ is continuous on } \mathbb{R}^d \text{ for a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(c) } \|a(\boldsymbol{x}, \boldsymbol{\xi})\| \le m_a\,(1 + \|\boldsymbol{\xi}\|) \;\text{ for all }\; \boldsymbol{\xi} \in \mathbb{R}^d,\; \text{a.e. } \boldsymbol{x} \in \Omega \\ \quad\quad \text{with } m_a > 0. \\ \quad \text{(d) } (a(\boldsymbol{x}, \boldsymbol{\xi}_1) - a(\boldsymbol{x}, \boldsymbol{\xi}_2)) \cdot (\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2) \ge \alpha_a \,\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|^2 \\ \quad\quad \text{for all }\; \boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d,\; \text{a.e. } \boldsymbol{x} \in \Omega \;\text{ with } \alpha_a > 0. \end{array}\right. \tag{29}$$

$$\left\{\begin{array}{l} h\colon \Omega \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } h(\cdot, r) \text{ is measurable on } \Omega \text{ for all } r \in \mathbb{R} \text{ and there} \\ \quad\quad\quad \text{exists } \overline{e} \in L^2(\Omega) \text{ such that } h(\cdot, \overline{e}(\cdot)) \in L^1(\Omega). \\ \quad \text{(b) } h(\boldsymbol{x}, \cdot) \text{ is locally Lipschitz on } \mathbb{R},\; \text{a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(c) there exist } \overline{c}_0,\, \overline{c}_1 \ge 0 \text{ such that} \\ \quad\quad\quad |\partial h(\boldsymbol{x}, r)| \le \overline{c}_0 + \overline{c}_1 |r| \text{ for all } r \in \mathbb{R},\; \text{a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(d) there exists } \alpha_h \ge 0 \text{ such that} \\ \quad\quad\quad h^0(\boldsymbol{x}, r_1; r_2 - r_1) + h^0(\boldsymbol{x}, r_2; r_1 - r_2) \le \alpha_h |r_1 - r_2|^2 \\ \quad\quad \text{for all } r_1, r_2 \in \mathbb{R},\; \text{a.e. } \boldsymbol{x} \in \Omega. \end{array}\right. \tag{30}$$

$$\begin{cases} g\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R} \text{ is such that} \\ \quad \text{(a) } g(\cdot, r) \text{ is measurable on } \Gamma_2 \text{ for all } r \in \mathbb{R}. \\ \quad \text{(b) } g(\boldsymbol{x}, \cdot) \text{ is convex on } \Gamma_2, \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \quad \text{(c) there exists } L_g > 0 \text{ such that} \\ \qquad\qquad |g(\boldsymbol{x}, r_1) - g(\boldsymbol{x}, r_2)| \le L_g |r_1 - r_2| \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_2. \end{cases} \tag{31}$$

$$\begin{cases} k\colon \Gamma_2 \times \mathbb{R} \to \mathbb{R}_+ \text{ is such that} \\ \quad \text{(a) } k(\cdot, r) \text{ is measurable on } \Gamma_2 \text{ for all } r \in \mathbb{R}. \\ \quad \text{(b) there exists } L_k > 0 \text{ such that} \\ \qquad\qquad |k(\boldsymbol{x}, r_1) - k(\boldsymbol{x}, r_2)| \le L_k |r_1 - r_2| \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } \boldsymbol{x} \in \Gamma_2. \\ \quad \text{(c) } k(\boldsymbol{x}, 0) = 0 \text{ for a.e. } \boldsymbol{x} \in \Omega. \end{cases} \tag{32}$$

$$U \text{ is a closed, convex subset of } V, \quad f_1 \in C(\mathbb{R}_+; H). \tag{33}$$

Below we provide examples of functions a and h.

Example 18. We provide an example of a function $a\colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ which satisfies hypothesis (29). Let $a(\boldsymbol{x}, \boldsymbol{\xi}) = \phi(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}$ for all $\boldsymbol{\xi} \in \mathbb{R}^d$, a.e. $\boldsymbol{x} \in \Omega$, where

$$\begin{cases} \phi\colon \Omega \to \mathbb{R} \text{ is measurable and there are constants } d_1, d_2 > 0 \\ \text{such that for a.e. } \boldsymbol{x} \in \Omega, \text{ we have } d_1 \le \phi(\boldsymbol{x}) \le d_2 \end{cases} \tag{34}$$

and

$$\begin{cases} \psi\colon \mathbb{R}_+ \to \mathbb{R} \text{ is piecewise continuously differentiable,} \\ \text{and there are constants } d_3, d_4, d_5 > 0 \text{ such that for} \\ \text{all } r \ge 0, \text{ we have } |\psi(r)| \le d_3, \ d_4 \le \psi(r) + 2\psi'(r) r \le d_5. \end{cases} \tag{35}$$

It is evident that (29)(a), (b) and (c) hold with $m_a = d_2 d_3$. We will verify the strong monotonicity condition (29)(d). Let $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2 \in \mathbb{R}^d$ and $\boldsymbol{x} \in \Omega$. For $t \in [0, 1]$, we put $\overline{\boldsymbol{\xi}}(t) = \boldsymbol{\xi}_2 + t(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2)$. We have

$$\begin{aligned} a(\boldsymbol{x}, \boldsymbol{\xi}_1) - a(\boldsymbol{x}, \boldsymbol{\xi}_2) &= \phi(\boldsymbol{x})\psi(\|\boldsymbol{\xi}_1\|^2)\boldsymbol{\xi}_1 - \phi(x)\psi(\|\boldsymbol{\xi}_2\|^2)\boldsymbol{\xi}_2 \\ &= \phi(\boldsymbol{x}) \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}(\psi(\|\overline{\boldsymbol{\xi}}(t)\|^2)\overline{\boldsymbol{\xi}}(t))\, \mathrm{d}t \\ &= \phi(\boldsymbol{x}) \int_0^1 (2\psi'(\|\overline{\boldsymbol{\xi}}(t)\|^2)\|\overline{\boldsymbol{\xi}}(t)\|\|\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2\|\overline{\boldsymbol{\xi}}(t) + \psi(\|\overline{\boldsymbol{\xi}}(t)\|^2)(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2))\, \mathrm{d}t. \end{aligned}$$

Then

$$\begin{aligned}
&(a(\boldsymbol{x},\boldsymbol{\xi}_1)-a(\boldsymbol{x},\boldsymbol{\xi}_2))\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\\
&=\phi(\boldsymbol{x})\int_0^1\big(2\psi'(\|\overline{\boldsymbol{\xi}}(t)\|^2)\|\overline{\boldsymbol{\xi}}(t)\|\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|\,\overline{\boldsymbol{\xi}}(t)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\\
&\quad+\psi(\|\overline{\boldsymbol{\xi}}(t)\|^2)(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\cdot(\boldsymbol{\xi}_1-\boldsymbol{\xi}_2)\big)\,\mathrm{d}t\\
&=\phi(\boldsymbol{x})\int_0^1\big(2\psi'(\|\overline{\boldsymbol{\xi}}(t)\|^2)\|\overline{\boldsymbol{\xi}}(t)\|^2+\psi(\|\overline{\boldsymbol{\xi}}(t)\|^2)\big)\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2\,\mathrm{d}t\\
&\ge d_1d_4\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|^2.
\end{aligned}$$

Hence, condition (29)(d) follows with $\alpha_a=d_1d_4$. We also observe that if $\psi\equiv 1$, then $a(\boldsymbol{x},\boldsymbol{\xi})=\phi(\boldsymbol{x})\boldsymbol{\xi}$ for all $\boldsymbol{\xi}\in\mathbb{R}^d$, a.e. $\boldsymbol{x}\in\Omega$ which leads to the linear operator A in the divergence form.

Example 19. Consider the following example. Let $h\colon\mathbb{R}\to\mathbb{R}$ be defined by

$$h(r)=\begin{cases}0 & \text{if } r<0,\\ \frac{r^2}{2} & \text{if } r\in[0,1),\\ -\frac{r^2}{2}+3r-\frac{3}{2} & \text{if } r\in[1,3),\\ \frac{r^2}{2}-3r+\frac{15}{2} & \text{if } r\ge 3.\end{cases}$$

Then, its subdifferential is given by

$$\partial h(r)=\begin{cases}0 & \text{if } r<0,\\ r & \text{if } r\in[0,1),\\ [1,2] & \text{if } r=1,\\ -r+3 & \text{if } r\in[1,3),\\ r-3 & \text{if } r\ge 3.\end{cases}$$

It can be proved that the function h satisfies condition (30) with $\bar{c}_0=2$, $\bar{c}_1=1$ and $\alpha_h=3$. For more examples of functions which satisfy this condition, we refer to Examples 16 and 17 in [13].

We turn to the variational formulation of Problem 17. Let $v\in U$ and $t\in\mathbb{R}_+$. We multiply (23) by $v-u$, use Green's formula, decompose the surface integral on two parts on Γ_1 and Γ_2 and take into account that $v-u=0$ on Γ_1.

$$\begin{aligned}
&\int_\Omega a(\boldsymbol{x},\nabla u)\cdot\nabla(v-u)\,\mathrm{d}x-\int_{\Gamma_2}(\frac{\partial u}{\partial\nu_a})(v-u)\,\mathrm{d}\Gamma\\
&\quad=\int_\Omega f_1(t)(v-u)\,\mathrm{d}x+\int_\Omega f_2(u)(v-u)\,\mathrm{d}x.
\end{aligned}\tag{36}$$

From (23), (24) and definitions of subgradients, we have

$$\begin{aligned}
-f_2(u)\,r&\le h^0(\boldsymbol{x},u;r)\ \text{ in }\ \Omega,\\
-\frac{\partial u}{\partial\nu_a}(r-u)&\le k(u)(g(\boldsymbol{x},r)-g(\boldsymbol{x},u))\ \text{ on }\ \Gamma_2
\end{aligned}$$

for all $r\in\mathbb{R}$. Using these inequalities in (36), we obtain the following variational–hemivariational inequality.

Problem 20. *Find* $u\colon \mathbb{R}_+ \to U$ *such that for all* $t \in \mathbb{R}_+$

$$\int_\Omega a(\boldsymbol{x}, \nabla u(t)) \cdot \nabla(v - u(t))\, dx + \int_{\Gamma_2} \big(k(u(t))g(\boldsymbol{x}, v) - k(u(t))g(\boldsymbol{x}, u(t))\big)\, d\Gamma$$
$$+ \int_\Omega h^0(\boldsymbol{x}, u(t); v - u(t))\, dx \geq \int_\Omega f_1(t)(v - u(t))\, dx$$

for all $v \in U$.

The following result concerns the well posedness of Problem 20.

Theorem 21. *Assume that* (29)–(33) *hold and the following smallness condition is satisfied*

$$L_k L_g \|\gamma\|^2 + \alpha_h < \alpha_a. \tag{37}$$

Then, Problem 20 has a unique solution $u \in C(\mathbb{R}_+; U)$.

Proof. We apply Theorem 14 in the following functional framework: $X = V$, $K = U$, $f(t) = f_1(t)$ for all $t \in \mathbb{R}_+$ and

$$A\colon V \to V^*, \quad \langle Au, v\rangle_V = \int_\Omega a(\boldsymbol{x}, \nabla u) \cdot \nabla v \, \mathrm{d}x \text{ for } u, v \in V, \tag{38}$$

$$\varphi\colon V \times V \to \mathbb{R}, \quad \varphi(u, v) = \int_{\Gamma_2} k(u) g(v) \, \mathrm{d}\Gamma \text{ for } u, v \in V, \tag{39}$$

$$j\colon V \to \mathbb{R}, \quad j(v) = \int_\Omega h(v) \, \mathrm{d}x \text{ for } v \in V. \tag{40}$$

With this notation, we can see that Problem 20 is equivalent to Problem 13. We now check the hypotheses of Theorem 14.

First, since V is a closed linear subspace of the Sobolev space $H^1(\Omega)$, containing $H_0^1(\Omega)$, it is straightforward to prove that under hypotheses (29), the operator A is bounded and pseudomonotone, for details see, e.g., [18, Theorem 4.6] or [24, Proposition 26.12]. It is clear that condition (29)(d) implies that operator A is strongly monotone with constant $m_A = \alpha_a$. Using the strong monotonicity condition, for $u_0 \in K$ and $u \in V$, we have

$$\langle Au, u - u_0\rangle = \langle Au - Au_0, u - u_0\rangle + \langle Au_0, u - u_0\rangle$$
$$\geq m_A \|u - u_0\|_X^2 - \|Au_0\|_{X^*} \|u - u_0\|_X.$$

From the following elementary inequalities

$$\big|\|u\|_X - \|u_0\|_X\big| \leq \|u - u_0\|_X,$$
$$\|Au_0\|_{X^*} \|u - u_0\|_X \leq \|Au_0\|_{X^*} \|u\|_X + \|Au_0\|_{X^*} \|u_0\|_X,$$

we obtain

$$\langle Au, u - u_0\rangle \geq m_A(\|u\|_X - \|u_0\|_X)^2 - \|Au_0\|_{X^*}\|u\|_X - \|Au_0\|_{X^*}\|u_0\|_X$$
$$= m_A \|u\|_X^2 - (2m_A\|u_0\|_X + \|Au_0\|_{X^*})\|u\|_X + m_A\|u_0\|_X^2 - \|Au_0\|_{X^*}\|u_0\|_X,$$

which proves condition (6)(c) with $\alpha_A = m_A$ and α_1, $\alpha_2 \in \mathbb{R}$.

Next, hypothesis (8) is a consequence of (30), which holds with $\alpha_j = \alpha_h$, $c_0 = \bar{c}_0$ and $c_1 = \bar{c}_1$.

Next, we will verify (7). Condition (31) implies (7)(a). To prove (7)(b), let u, $v \in V$. We have

$$\begin{aligned}
&\varphi(u_1, v_2) - \varphi(u_1, v_1) + \varphi(u_2, v_1) - \varphi(u_2, v_2) \\
&= \int_{\Gamma_2} (k(u_1) - k(u_2))(g(v_1) - g(v_2))\, \mathrm{d}\Gamma \\
&\leq \int_{\Gamma_2} L_k |u_1(x) - u_2(x)| L_g |v_1(x) - v_2(x)| \mathrm{d}s \\
&\leq L_k L_g \|\gamma\|^2 \|u_1 - u_2\|_V \|v_1 - v_2\|_V .
\end{aligned}$$

Hence, it follows that (7)(b) is satisfied with $\alpha_\varphi = L_k L_g \|\gamma\|^2$. The smallness conditions (9) are a consequence of assumption (37).

Therefore, we deduce that all hypotheses of Theorem 14 are satisfied. By applying Theorem 14, we conclude that Problem 20 has a unique solution $u \in C(\mathbb{R}_+; U)$. □

We now turn to the dependence of solution to Problem 20 on the perturbation of the mapping a, functions k, g, h and f_1, and the set U. We consider the following perturbation of Problem 20.

Problem 22. *Find $u_\rho \colon \mathbb{R}_+ \to U_\rho$ such that for all $t \in \mathbb{R}_+$*

$$\begin{aligned}
&\int_\Omega a_\rho(\boldsymbol{x}, \nabla u_\rho(t)) \cdot \nabla(v_\rho - u_\rho(t))\, dx + \int_{\Gamma_2} (k_\rho(u_\rho(t)) g_\rho(\boldsymbol{x}, v_\rho) - k_\rho(u_\rho(t)) g_\rho(\boldsymbol{x}, u_\rho(t)))\, d\Gamma \\
&\quad + \int_\Omega h_\rho^0(\boldsymbol{x}, u_\rho(t); v_\rho - u_\rho(t))\, dx \geq \int_\Omega f_{1\rho}(t)(v_\rho - u_\rho(t))\, dx
\end{aligned}$$

for all $v_\rho \in U_\rho$.

For the data of Problem 22, we introduce the following hypotheses.

$$U,\, U_\rho \ \text{ are closed convex sets in } V, \ \text{ and } \ U_\rho \xrightarrow{M} U, \ \text{as } \rho \to 0. \tag{41}$$

$$f_1,\, f_{1\rho} \in C(\mathbb{R}_+; H) \ \text{ and } \ f_{1\rho}(t) \to f_1(t) \text{ in } H \text{ for all } t \in \mathbb{R}_+, \ \text{as } \rho \to 0. \tag{42}$$

$$\begin{cases}
a, a_\rho \colon \Omega \times \mathbb{R}^d \to \mathbb{R}^d \text{ are functions such that} \\
\quad \text{(a) } a, a_\rho \text{ satisfy (29) with constants } m_a, \alpha_a > 0, \\
\qquad \text{and } m_{a_\rho}, \alpha_{a_\rho} > 0, \text{respectively.} \\
\quad \text{(b) there exist } c_a > 0 \text{ and } \beta_\rho > 0 \text{ with } \beta_\rho \to 0, \text{ as } \rho \to 0 \text{ such that} \\
\qquad \|a_\rho(\boldsymbol{x}, \boldsymbol{\xi}) - a(\boldsymbol{x}, \boldsymbol{\eta})\| \leq c_a(\beta_\rho + \|\boldsymbol{\xi} - \boldsymbol{\eta}\|) \text{ for all } \boldsymbol{\xi}, \boldsymbol{\eta} \in \mathbb{R}^d \text{ with} \\
\qquad \|\boldsymbol{\xi}\|, \|\boldsymbol{\eta}\| \leq M_1, \ \text{a.e. } \boldsymbol{x} \in \Omega, \ \text{where } M_1 \text{ is independent of } \rho.
\end{cases} \tag{43}$$

$$\begin{cases} h,\, h_\rho\colon \Omega\times\mathbb{R}\to\mathbb{R} \text{ are functions such that}\\ \quad \text{(a) } h,\, h_\rho \text{ satisfy (30) with } \alpha_h>0,\, \overline{c}_0,\, \overline{c}_1\ge 0\\ \qquad \text{and } \alpha_{h_\rho}>0,\, \overline{c}_{0\rho},\, \overline{c}_{1\rho}\ge 0, \text{ respectively.}\\ \quad \text{(b) for all } \{r_\rho\},\{s_\rho\}\subset\mathbb{R} \text{ with } r_\rho\to r \text{ and } s_\rho\to s, \text{ we have}\\ \qquad \limsup h^0_\rho(\boldsymbol{x},r_\rho;s_\rho)\le h^0(\boldsymbol{x},r;s), \text{ a.e. } \boldsymbol{x}\in\Omega.\\ \quad \text{(c) either } h(\boldsymbol{x},\cdot) \text{ or } -h(\boldsymbol{x},\cdot) \text{ is regular in the sense of Clarke}\\ \qquad \text{for a.e. } \boldsymbol{x}\in\Omega. \end{cases} \tag{44}$$

$$\begin{cases} g,\, g_\rho\colon \Gamma_2\times\mathbb{R}\to\mathbb{R} \text{ are functions such that}\\ \quad \text{(a) } g,\, g_\rho \text{ satisfy (31) with } L_g>0 \text{ and } L_{g_\rho}>0, \text{ respectively.}\\ \quad \text{(b) for all } \{r_\rho\},\{s_\rho\}\subset\mathbb{R} \text{ with } r_\rho\to r \text{ and } s_\rho\to s, \text{ we have}\\ \qquad \lim\,(g_\rho(\boldsymbol{x},r_\rho)-g_\rho(\boldsymbol{x},s_\rho))=g(\boldsymbol{x},r)-g(\boldsymbol{x},s), \text{ a.e. } \boldsymbol{x}\in\Gamma_2. \end{cases} \tag{45}$$

$$\begin{cases} k,\, k_\rho\colon \Gamma_2\times\mathbb{R}\to\mathbb{R} \text{ are functions such that}\\ \quad \text{(a) } k,\, k_\rho \text{ satisfy (32) with } L_k>0 \text{ and } L_{k_\rho}>0, \text{ respectively.}\\ \quad \text{(b) for all } \{r_\rho\}\subset\mathbb{R} \text{ with } r_\rho\to r\in\mathbb{R},\\ \qquad \text{we have } \lim\, k_\rho(\boldsymbol{x},r_\rho)=k(\boldsymbol{x},r), \text{ a.e. } \boldsymbol{x}\in\Gamma_2. \end{cases} \tag{46}$$

$$\begin{cases} \quad \text{(a) there exist } \overline{m}_0,\overline{m}_1>0 \text{ such that for all } \rho>0 \text{ sufficiently}\\ \qquad \text{small, we have } L_kL_g\|\gamma\|^2+\alpha_h\le\overline{m}_0<\overline{m}_1\le m_a.\\ \quad \text{(b) there exists } M_1>0 \text{ such that for all } \rho>0 \text{ sufficiently}\\ \qquad \text{small, we have } \max\{L_{k_\rho},L_{g_\rho},\overline{c}_{0\rho},\overline{c}_{1\rho}\}\le M_1. \end{cases} \tag{47}$$

We have the following convergence result for Problem 22.

Theorem 23. *Assume that (37), (41), (42), (43), (44), (45), (46) and (47) are satisfied. Then for each $\rho>0$, Problem 22 has a unique solution $u_\rho\in C(\mathbb{R}_+;U_\rho)$ and for each $t\in\mathbb{R}_+$, there is a subsequence of $\{u_\rho\}$, denoted by $\{u_\rho\}$ again, such that $u_\rho(t)\to u(t)$ in V, as $\rho\to 0$, where $u\in C(\mathbb{R}_+;U)$ is the unique solution to Problem 20.*

Proof. First, we note that, by Theorem 21, for every $\rho>0$, Problem 22 has a unique solution $u_\rho\in C(\mathbb{R}_+;U_\rho)$. Then, we shall apply Theorem 16 in the following framework: $X=V$, $K_\rho=U_\rho$, $f_\rho(t)=f_{1\rho}(t)$ for all $t\in\mathbb{R}_+$ and

$$A_\rho\colon V\to V^*,\quad \langle A_\rho u,v\rangle=\int_\Omega a_\rho(\boldsymbol{x},\nabla u(x))\cdot\nabla v(x)\,\mathrm{d}x \text{ for } u,v\in V, \tag{48}$$

$$\varphi_\rho\colon V\times V\to\mathbb{R},\quad \varphi_\rho(u,v)=\int_{\Gamma_2} k_\rho(u(x))g_\rho(v(x))\,\mathrm{d}\Gamma \text{ for } u,v\in V, \tag{49}$$

$$j_\rho\colon V\to\mathbb{R},\quad j_\rho(v)=\int_\Omega h_\rho(v(x))\,\mathrm{d}x \text{ for } v\in V. \tag{50}$$

We now check the hypotheses of Theorem 16. It is clear that (13), (14)(a), (16)(a) and (17) follow from (41), (43), (44)(a) and (47), respectively.

We now show that (14)(b) and (16)(b) are satisfied. For all u, v, $w \in V$, from hypothesis (43)(b), we have

$$\begin{aligned}\langle A_\rho u - Av, w\rangle &= \int_\Omega (a_\rho(\boldsymbol{x}, \nabla u(x)) - a(\boldsymbol{x}, \nabla v(x))) \cdot \nabla w(x)\,\mathrm{d}x \\ &\le c_a \int_\Omega (\beta_\rho + \|\nabla u(x) - \nabla v(x)\|)\|\nabla w(x)\|\,\mathrm{d}x \\ &\le c_a(\sqrt{2}\beta_\rho|\Omega| + \|u-v\|_V)\|w\|_V.\end{aligned}$$

Then

$$\|A_\rho u - Av\|_{V^*} \le c_a(\sqrt{2}m(\Omega)\beta_\rho + \|u-v\|_V).$$

Hence, we deduce that (14)(b) holds.

Next, we prove condition (16)(b). Let $\{u_\rho\}$, $\{v_\rho\}$ be such that u_ρ, $v_\rho \in U_\rho$ for each $\rho > 0$ with $u_\rho \rightharpoonup u$ in V and $v_\rho \to v$ in V. Since the embedding $V \subset H$ is compact, we get the strong convergences $u_\rho \to u$ in H and $v_\rho \to v$ in H. Then, by passing to a subsequence, if necessary, we have $u_\rho(\boldsymbol{x}) \to u(\boldsymbol{x})$ and $v_\rho(\boldsymbol{x}) \to v(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$. On the other hand, we recall that for j_ρ and j, by [12, Theorem 3.47], we have the following inequality

$$j^0(v; w) \le \int_\Omega h^0(\boldsymbol{x}, v(x); w(x))\,\mathrm{d}x \quad \text{for all} \quad v, w \in V \tag{51}$$

and if, in addition, (44)(c) is assumed, then (51) holds with equality. From (44)(b) and (51), by Fatou's lemma, we have

$$\begin{aligned}&\limsup\, j_\rho^0(u_\rho; v_\rho - u_\rho) \le \limsup \int_\Omega h_\rho^0(u_\rho(x); v_\rho(x) - u_\rho(x))\,\mathrm{d}x \\ &\le \int_\Omega \limsup h_\rho^0(u_\rho(x); v_\rho(x) - u_\rho(x))\,\mathrm{d}x \le \int_\Omega h^0(u(x); v(x) - u(x))\,\mathrm{d}x \\ &= j^0(u; v-u).\end{aligned}$$

Hence, condition (16)(b) is verified.

Next, we verify (15). Condition (15)(a) is obvious. Assume that $\{u_\rho\}$, $\{v_\rho\}$ are such that u_ρ, $v_\rho \in U_\rho$ for each $\rho > 0$ with $u_\rho \rightharpoonup u$ in V and $v_\rho \to v$ in V. From the compactness of the trace operator, it follows that $\gamma u_\rho \to \gamma u$ in $L^2(\Gamma)$ and $\gamma v_\rho \to \gamma v$ in $L^2(\Gamma)$. Then, at least for a subsequence, we have

$\gamma u_\rho(\boldsymbol{x}) \to \gamma u(\boldsymbol{x})$ and $\gamma v_\rho(\boldsymbol{x}) \to \gamma v(\boldsymbol{x})$. Using (45)(b) and (46)(b), by Fatou's lemma, we have

$$\begin{aligned}
&\limsup(\varphi_\rho(u_\rho, v_\rho) - \varphi_\rho(u_\rho, u_\rho))\\
&\quad = \limsup \int_{\Gamma_2} k_\rho(\gamma u_\rho(x))(g_\rho(\gamma v_\rho(x)) - g(\gamma u_\rho(x)))\,\mathrm{d}\Gamma\\
&\quad \le \int_{\Gamma_2} \limsup k_\rho(\gamma u_\rho(x))(g_\rho(\gamma v_\rho(x)) - g_\rho(\gamma u_\rho(x)))\,\mathrm{d}\Gamma\\
&\quad = \int_{\Gamma_2} k(\gamma u(x))(g(\gamma v(x)) - g(\gamma u(x)))\,\mathrm{d}\Gamma\\
&\quad = \varphi(u, v) - \varphi(u, u).
\end{aligned}$$

Hence, condition (15)(b) holds.

Finally, we will prove that (15)(c) is satisfied. In fact, from (45)(a) and (46)(a), for all $u, v_1, v_2 \in U_\rho$, we have

$$\begin{aligned}
&\varphi_\rho(u, v_1) - \varphi_\rho(u, v_2)\\
&\quad = \int_{\Gamma_2} k_\rho(\gamma u(x))(g_\rho(\gamma v_1(x)) - g_\rho(\gamma v_2(x)))\,\mathrm{d}\Gamma\\
&\quad = \int_{\Gamma_2} (k_\rho(\gamma u(x)) - k_\rho(0) + k_\rho(0))(g_\rho(\gamma v_1(x)) - g_\rho(\gamma v_2(x)))\,\mathrm{d}\Gamma\\
&\quad \le \int_{\Gamma_2} L_{k_\rho}|\gamma u(x)| L_{g_\rho}|\gamma v_1(x) - \gamma v_2(x)|\mathrm{d}s\\
&\quad \le L_{k_\rho} L_{g_\rho} \|\gamma\|^2 \|u\|_V \|v_1 - v_2\|_V\\
&\quad \le M_1^2 \|\gamma\|^2 \|u\|_V \|v_1 - v_2\|_V.
\end{aligned}$$

Hence, it follows that condition (15)(c) holds with the function $c_\varphi(r) = M_1^2\|\gamma\|^2 r$ for $r \in \mathbb{R}_+$. From Theorem 16, we deduce that $u_\rho(t) \to u(t)$ in V for all $t \in \mathbb{R}_+$, as $\rho \to 0$. This completes the proof. □

We conclude this section with the following examples.

Example 24. Hypothesis (41) is satisfied for the following constraint sets for a bilateral obstacle problem.

$$U = U(\psi_1, \psi_2) = \{\, v \in V \mid \psi_1 \le v \le \psi_2 \ \text{ a.e. in } \ \Omega \,\},$$
$$U_\rho = U(\psi_{1\rho}, \psi_{2\rho}) = \{\, v \in V \mid \psi_{1\rho} \le v \le \psi_{2\rho} \ \text{ a.e. in } \ \Omega \,\},$$

where ψ_1, $\psi_{1\rho}$, ψ_2, $\psi_{2\rho} \in V \cap H^2(\Omega)$. It is clear that U and U_ρ are closed convex subsets of V. We will show that if $(\psi_{1\rho}, \psi_{2\rho}) \rightharpoonup (\psi_1, \psi_2)$ in $(V \cap H^2(\Omega)) \times (V \cap H^2(\Omega))$, as $\rho \to 0$, then

$$U_\rho \xrightarrow{M} U, \text{ as } \rho \to 0. \tag{52}$$

In fact, let $v_\rho \in U_\rho$ be such that $v_\rho \rightharpoonup v$ in V, as $\rho \to 0$. Since

$$U_\rho = \{\, z \in V \mid z \ge \psi_{1\rho} \ \text{ a.e. in } \ \Omega \,\} \cap \{\, z \in V \mid z \le \psi_{2\rho} \ \text{ a.e. in } \ \Omega \,\},$$

we obtain $v_\rho - \psi_{1\rho} \in \{\, z \in V \mid z \ge 0 \ \text{ a.e. in } \ \Omega \,\}$ and $v_\rho - \psi_{2\rho} \in \{\, z \in V \mid z \le 0 \ \text{ a.e. in } \ \Omega \,\}$. Moreover, since the sets $\{\, z \in V \mid z \ge 0 \ \text{ a.e. in } \ \Omega \,\}$ and $\{\, z \in V \mid z \le 0 \ \text{ a.e. in } \ \Omega \,\}$ are weakly closed by Mazur's theorem, we deduce that $v-\psi_1 \in \{\, z\in V \mid z\geq 0 \ \text{ a.e. in } \ \Omega\,\}$ and $v-\psi_2 \in \{\, z\in V \mid z\leq 0 \ \text{ a.e. in } \ \Omega\,\}$, and hence, $v\in U$.

On the other hand, for any $v\in U$, there exist $v_1 \in \{\, z\in V \mid z\geq 0 \ \text{ a.e. in } \ \Omega\,\}$ and $v_2 \in \{\, z\in V \mid z\leq 0 \ \text{ a.e. in } \ \Omega\,\}$ such that $v=v_1+\psi_1=v_2+\psi_2$.

Using the compactness embedding theorem, it is clear that $(\psi_{1\rho},\psi_{2\rho})\to(\psi_1,\psi_2)$ in $V\times V$. Put $v_\rho=v_1+\psi_{1\rho}$. Then, for ρ small enough, we get $v_\rho\in U_\rho$. Hence, $v_\rho=v_1+\psi_{1\rho}\to v_1+\psi_1=v_2+\psi_2=v$ in V. Therefore, the convergence (52) holds.

Example 25. (i) Let a, $a_\rho\colon \Omega\times\mathbb{R}^d\to\mathbb{R}^d$ be functions defined by

$$a(\boldsymbol{x},\boldsymbol{\xi})=\phi(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi},\quad a_\rho(\boldsymbol{x},\boldsymbol{\xi})=\phi_\rho(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}$$

for all $\boldsymbol{\xi}\in\mathbb{R}^d$, a.e. $\boldsymbol{x}\in\Omega$ with $\rho>0$. Assume that ϕ and ϕ_ρ satisfy condition (34) with constants d_1, $d_2>0$ uniformly with respect to ρ, ψ satisfies condition (35) and the following condition is satisfied

$$\begin{cases}\text{there exist } w\in L^\infty(\Omega) \text{ and } \tilde\beta_\rho>0 \text{ with } \tilde\beta_\rho\to 0, \text{ as } \rho\to 0\\ \text{such that } |\phi_\rho(\boldsymbol{x})-\phi(\boldsymbol{x})|\leq\tilde\beta_\rho\, w(\boldsymbol{x}) \ \text{ for a.e. } \boldsymbol{x}\in\Omega.\end{cases}\tag{53}$$

From Example 18 it is clear that (43)(a) is satisfied. We show that condition (43)(b) holds. To this end, let $M_1>0$, $\boldsymbol{\xi}$, $\boldsymbol{\eta}\in\mathbb{R}^d$ with $\|\boldsymbol{\xi}\|\leq M_1$ and $\|\boldsymbol{\eta}\|\leq M_1$, and $t\in[0,1]$. We put $\boldsymbol{\zeta}(t)=\boldsymbol{\xi}+t(\boldsymbol{\xi}-\boldsymbol{\eta})$. Then, for a.e. $\boldsymbol{x}\in\Omega$, we have

$$\begin{aligned}&\|a_\rho(\boldsymbol{x},\boldsymbol{\xi})-a(\boldsymbol{x},\boldsymbol{\eta})\|=\|\phi_\rho(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}-\phi(\boldsymbol{x})\psi(\|\boldsymbol{\eta}\|^2)\boldsymbol{\eta}\|\\
&\quad\leq\|\phi_\rho(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}-\phi(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}\|+\|\phi(\boldsymbol{x})\psi(\|\boldsymbol{\xi}\|^2)\boldsymbol{\xi}-\phi(\boldsymbol{x})\psi(\|\boldsymbol{\eta}\|^2)\boldsymbol{\eta}\|\\
&\quad\leq|\phi_\rho(\boldsymbol{x})-\phi(\boldsymbol{x})||\psi(\|\boldsymbol{\xi}\|^2)|\|\boldsymbol{\xi}\|+|\phi(\boldsymbol{x})|\|\int_0^1\frac{\mathrm{d}}{\mathrm{d}t}(\psi(\|\boldsymbol{\zeta}(t)\|^2)\boldsymbol{\zeta}(t))\,\mathrm{d}t\|\\
&\quad=|\phi_\rho(\boldsymbol{x})-\phi(\boldsymbol{x})||\psi(\|\boldsymbol{\xi}\|^2)|\|\boldsymbol{\xi}\|\\
&\qquad+|\phi(\boldsymbol{x})|\|\int_0^1(2\psi'(\|\boldsymbol{\zeta}(t)\|^2)(\boldsymbol{\zeta}(t)\cdot(\boldsymbol{\xi}-\boldsymbol{\eta}))\boldsymbol{\zeta}(t)+\psi(\|\boldsymbol{\zeta}(t)\|^2)(\boldsymbol{\xi}-\boldsymbol{\eta}))\,\mathrm{d}t\|\\
&\quad\leq|\phi_\rho(\boldsymbol{x})-\phi(\boldsymbol{x})||\psi(\|\boldsymbol{\xi}\|^2)|\|\boldsymbol{\xi}\|\\
&\qquad+|\phi(\boldsymbol{x})|\int_0^1(2\psi'(\|\boldsymbol{\zeta}(t)\|^2)\|\boldsymbol{\zeta}(t)\|^2+\psi(\|\boldsymbol{\zeta}(t)\|^2))\|\boldsymbol{\xi}-\boldsymbol{\eta}\|\,\mathrm{d}t\\
&\quad\leq d_3M_1|\phi_\rho(\boldsymbol{x})-\phi(\boldsymbol{x})|+d_2d_5\|\boldsymbol{\xi}-\boldsymbol{\eta}\|.\end{aligned}$$

Here we have used the conditions $d_1\leq\phi(\boldsymbol{x})\leq d_2$ for a.e. $\boldsymbol{x}\in\Omega$ and $|\psi(r)|\leq d_3$, $d_4\leq\psi(r)+2\psi'(r)r\leq d_5$ for all $r\geq 0$. From hypothesis (53), for a.e. $\boldsymbol{x}\in\Omega$, we deduce

$$\|a_\rho(\boldsymbol{x},\boldsymbol{\xi})-a(\boldsymbol{x},\boldsymbol{\eta})\|\leq c_a\,(\tilde\beta_\rho+\|\boldsymbol{\xi}-\boldsymbol{\eta}\|)$$

for a.e. $\boldsymbol{x}\in\Omega$ with $c_a>0$. Therefore, the condition (43)(b) is satisfied.

(ii) Let h, $h_\rho\colon \Omega \times \mathbb{R} \to \mathbb{R}$ be functions defined by

$$h(\boldsymbol{x}, r) = \alpha(\boldsymbol{x})\overline{h}(r), \quad h_\rho(\boldsymbol{x}, r) = \alpha_\rho(\boldsymbol{x})\overline{h}(r)$$

for all $r \in \mathbb{R}$, a.e. $\boldsymbol{x} \in \Omega$. Suppose that the function $\overline{h}\colon \mathbb{R} \to \mathbb{R}$ is such that $\overline{h}$ is locally Lipschitz, $|\partial\overline{h}(r)| \le \overline{c}_{0h} + \overline{c}_{1h}|r|$ for all $r \in \mathbb{R}$ with $\overline{c}_{0h}$, $\overline{c}_{1h} \ge 0$, $\overline{h}(r_1; r_2 - r_1) + \overline{h}(r_2; r_1 - r_2) \le \alpha_{0h}|r_1 - r_2|^2$ for all r_1, $r_2 \in \mathbb{R}$ with $\alpha_{0h} \ge 0$, and either $\overline{h}$ of $-\overline{h}$ is regular in the sense of Clarke. Moreover, let

$$0 < \alpha_0 \le \alpha(\boldsymbol{x}),\ \alpha_\rho(\boldsymbol{x}) \le \alpha_1 \quad \text{for a.e. } \ \boldsymbol{x} \in \Omega.$$

If $\alpha_\rho(\boldsymbol{x}) \to \alpha(\boldsymbol{x})$ for a.e. $\boldsymbol{x} \in \Omega$, as $\rho \to 0$, then condition (44) is satisfied.

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