

PERIODICITY FOR SUBQUOTIENTS OF THE MODULAR CATEGORY \mathcal{O}

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Abstract. In this paper we study the category \mathcal{O} over the hyperalgebra of a reductive algebraic group in positive characteristics. For any locally closed subset K of weights, we define a subquotient $\mathcal{O}_{[K]}$ of \mathcal{O} . It has the property that its simple objects are parametrized by elements in K . We then show that $\mathcal{O}_{[K]}$ is equivalent to $\mathcal{O}_{[K+p^l\gamma]}$ for any dominant weight γ if $l > 0$ is an integer such that $K \cap (K - p^l\eta) = \emptyset$ for all weights $\eta > 0$. Hence it is enough to understand the subquotients inside the dominant (or the antidominant) chamber.

Introduction

One of the cornerstones of the rational representation theory of a reductive algebraic group G over a field k of characteristic $p > 0$ is Steinberg’s tensor product theorem. For a dominant weight μ we denote by $L(\mu)$ the irreducible representation of G with highest weight μ . For a dominant weight λ there is a p -adic extension $\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \dots + p^l\lambda_l$ with uniquely defined restricted weights $\lambda_0, \dots, \lambda_l$, and Steinberg’s tensor product theorem states that

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \dots \otimes L(\lambda_l)^{[l]},$$

where $(\cdot)^{[n]}$ denotes the n -fold Frobenius twist. In order to prove the theorem, it is, by induction, sufficient to prove the following. If λ_0 is restricted and γ is dominant, then $L(\lambda_0) \otimes L(\gamma)^{[l]}$ is a simple G -module (it must then be isomorphic to $L(\lambda_0 + p^l\gamma)$ for weight reasons). The theorem has numerous generalizations (for nondominant weights, quantum groups, or for representations of the hyperalgebra of G , see, for example, [A]).

The above serves as a motivation for us to consider the functor $(\cdot) \otimes L(\gamma)^{[l]}$ on the category of modules of the hyperalgebra (or algebra of distributions) U associated with G . This algebra coincides with the universal enveloping algebra of the Lie algebra of G in the case that the ground field is of characteristic 0, but this is not the case in positive characteristics. The category of finite dimensional rational

representations of G can be identified with the category of finite dimensional U -modules. As U admits a triangular decomposition one can, as in the classical characteristic 0 case, drop the finite dimensionality condition and instead consider the *highest weight category* \mathcal{O}^1 . The category \mathcal{O} for U shares many familiar properties with its characteristic 0 relative, which was studied intensely over multiple decades (for an overview on the most essential results, see [H2]). But in the modular case it has an additional property that does not occur in characteristic 0, and which is closely connected to Steinberg’s theorem: it inhibits a periodicity structure on subquotients.

In this article we consider subquotient categories $\mathcal{O}_{[K]}$ of \mathcal{O} associated to locally closed subsets of the weight lattice X (a locally closed subset of X is a union of intervals $[\lambda, \mu]$ with respect to the usual partial order). We then prove the following.

Theorem. *Suppose that K is a locally closed quasi-bounded subset of X . Suppose that $l > 0$ is such that $K \cap (K - p^l\eta) = \emptyset$ for all weights $\eta > 0$. Then the functor $(\cdot) \otimes L(\gamma)^{[l]}$ induces an equivalence $\mathcal{O}_{[K]} \xrightarrow{\sim} \mathcal{O}_{[K - p^l\gamma]}$ for all dominant weights $\gamma \in X$.*

Apart from Steinberg’s tensor product theorem, the periodicity of (p, Δ) -filtrations (cf. [A, Cor.4.3]) could also find a conceptual foundation in the above theorem.

Note that if K is finite, then an integer l as required in the theorem always exists. Once such an l is fixed, we still can choose an arbitrary dominant weight γ for the statement to hold. This shows that we can transfer the subquotient $\mathcal{O}_{[K]}$ deep inside the dominant chamber, or, by reversing the statement, deep inside the antidominant chamber. In both cases, the representation theory has some valuable additional features. By [A], projective covers exist in \mathcal{O} for the simple objects $L(\lambda)$ for antidominant weights λ , while the tilting modules $T(\lambda)$ exist in \mathcal{O} for all dominant λ .

We use this opportunity to also state and prove some basic results on the modular category \mathcal{O} for future reference, such as Krull–Remak–Schmidt decompositions, the existence of projectives in truncated subcategories, BGGH-reciprocity, and some additional results on modules admitting a Verma flag. Most of these results are well known in the characteristic 0 case and the proofs can be found in [H2]. If the proof in the modular case does not need adjustment, we will simply refer to the appropriate result in [H2]. We give a more detailed proof if some adjustments are required.

The modular category \mathcal{O}

Let \mathfrak{g} be a semisimple complex Lie algebra with root system R . For $\alpha \in R$ we denote by α^\vee the associated coroot, and by X the weight lattice. We fix a basis $\Pi \subset R$ and denote by $R^+ \subset R$ the corresponding system of positive roots. Then we denote by \leq the induced partial order on X , i.e., $\mu \leq \lambda$ if and only if $\lambda - \mu$ can be written as a sum of elements of R^+ . We let $U_{\mathbb{C}} = U(\mathfrak{g})$ be the universal enveloping algebra of \mathfrak{g} .

¹for simplicity we only consider in this article modules with integral weights.

1. Kostant’s integral form

Let $\{e_\alpha, f_\alpha, h_\beta \mid \alpha \in R^+, \beta \in \Pi\}$ be a Chevalley basis of \mathfrak{g} . By $U_{\mathbb{Z}} \subset U_{\mathbb{C}}$ we denote Kostant’s integral form (cf. [K]). Recall that $U_{\mathbb{Z}}$ is the unital subring of $U_{\mathbb{C}}$ that is generated by the elements $e_\alpha^{(n)} := e_\alpha^n/n!, f_\alpha^{(n)} := f_\alpha^n/n!$ and $\binom{h_\beta}{n} := h_\beta(h_\beta - 1) \cdots (h_\beta - n + 1)/n!$ with $n > 0, \alpha \in R^+$ and $\beta \in \Pi$. We denote by $U_{\mathbb{Z}}^+$ the subring generated by the $e_\alpha^{(n)}$ ’s, by $U_{\mathbb{Z}}^-$ the subring generated by the $f_\alpha^{(n)}$ ’s, and by $U_{\mathbb{Z}}^0$ the subring generated by the $\binom{h_\beta}{n}$ ’s. The following integral version of the PBW-theorem is [K, Thm. 1].

Theorem 1.

- (1) The algebra $U_{\mathbb{Z}}^+$ is free over \mathbb{Z} , and the elements $\prod_{\alpha \in R^+} e_\alpha^{(n_\alpha)}$ with $n_\alpha \geq 0$ form a basis.
- (2) The algebra $U_{\mathbb{Z}}^-$ is free over \mathbb{Z} , and the elements $\prod_{\alpha \in R^+} f_\alpha^{(n_\alpha)}$ with $n_\alpha \geq 0$ form a basis.
- (3) The algebra $U_{\mathbb{Z}}^0$ is free over \mathbb{Z} , and the elements $\prod_{\beta \in \Pi} \binom{h_\beta}{n_\beta}$ with $n_\beta \geq 0$ form a basis.
- (4) The multiplication defines an isomorphism

$$U_{\mathbb{Z}}^- \otimes_{\mathbb{Z}} U_{\mathbb{Z}}^0 \otimes_{\mathbb{Z}} U_{\mathbb{Z}}^+ \xrightarrow{\sim} U_{\mathbb{Z}}$$

of \mathbb{Z} -modules.

The products in parts (1) and (2) above should be taken with respect to a fixed but arbitrary order on R^+ . The algebra $U_{\mathbb{Z}}^0$ is commutative, so no order is needed.

Note that $U_{\mathbb{Z}}$ can be considered as the algebra of distributions of the semisimple and simply connected \mathbb{Z} -group scheme $G_{\mathbb{Z}}$ associated with R (cf. [J, Sect. II.1.12]). It inherits a Hopf algebra structure from $U_{\mathbb{C}}$.

2. The modular category \mathcal{O}

From now on we fix a field k of characteristic $p > 0$. We set $U := U_{\mathbb{Z}} \otimes_{\mathbb{Z}} k$ and define U^+, U^-, U^0 likewise. As before, we can consider U as the algebra of distributions of the group scheme G_k . It inherits a Hopf algebra structure from $U(\mathfrak{g})$ via $U_{\mathbb{Z}}$. The category of U -modules hence obtains a tensor product structure $(\cdot) \otimes (\cdot)$.

To any $\lambda \in X$ we can associate a character $\chi_\lambda: U^0 \rightarrow k$ that maps $\binom{h_\beta}{n}$ to the image of $\binom{\langle \lambda, \beta^\vee \rangle}{n}$ in k . For a U^0 -module M and $\lambda \in X$ we define

$$M_\lambda := \{m \in M \mid h.m = \chi_\lambda(h)m \text{ for all } h \in U^0\}$$

and call this the λ -weight space of M . We say that M is a weight module if $M = \bigoplus_{\lambda \in X} M_\lambda$, and we say that λ is a weight of M if $M_\lambda \neq 0$. We say that a U^+ -module M is U^+ -locally finite if M is the union of its finite dimensional U^+ -submodules.

Definition 1. The category \mathcal{O} is the full subcategory of the category of U -modules that contains all objects that are weight modules and U^+ -locally finite.

Note that this is a slightly more general definition than the original definition in [BGG] or [H2], where objects in \mathcal{O} are also supposed to be finitely generated. We do not impose this condition, as already in the affine Kac–Moody case, this

would not lead to an abelian category (i.e., submodules of Verma modules need not be finitely generated).

If M is an object in \mathcal{O} , then every submodule and every quotient of M is also contained in \mathcal{O} , and \mathcal{O} is an abelian category. It is easy to check that \mathcal{O} is stable under the tensor product on U -modules.

3. Verma modules and simple quotients

The algebra U^0 normalizes U^+ , so $U^{\geq 0} := U^0U^+ \subset U$ is a subalgebra. For any $\lambda \in X$ we denote by k_λ the one-dimensional $U^{\geq 0}$ -module on which U^0 acts via the character χ_λ , and U^+ acts via the augmentation $U^+ \rightarrow k$ (that sends all generators $e_\alpha^{(n)}$ to 0 for $n > 0$). The induced U -module

$$\Delta(\lambda) := U \otimes_{U^{\geq 0}} k_\lambda$$

is called the *Verma module with highest weight λ* . We denote by $v_\lambda = 1 \otimes 1$ the obvious generator of $\Delta(\lambda)$. The following proposition collects the basic facts about Verma modules. The arguments for the proofs are standard and do not depend on the characteristic of k . The characteristic 0 version can be found in [H2, Sects. 1.2 & 1.3].

Proposition 2. *Let $\lambda \in X$.*

- (1) *The weight space $\Delta(\lambda)_\lambda$ is one dimensional, and $\Delta(\lambda)_\mu \neq 0$ implies $\mu \leq \lambda$. Moreover, each weight space of $\Delta(\lambda)$ is finite dimensional.*
- (2) *The U -module $\Delta(\lambda)$ is contained in \mathcal{O} .*
- (3) *For any object M in \mathcal{O} , the map $\text{Hom}_{\mathcal{O}}(\Delta(\lambda), M) \rightarrow M_\lambda, f \mapsto f(v_\lambda)$ is injective with image $\{m \in M_\lambda \mid e_\alpha^{(n)}.m = 0 \text{ for all } \alpha \in R^+, n > 0\}$.*
- (4) *There exists a unique simple quotient $L(\lambda)$ of $\Delta(\lambda)$ in \mathcal{O} .*

Moreover, the set $\{L(\lambda)\}_{\lambda \in X}$ is a full set of representatives for the simple isomorphism classes of \mathcal{O} .

4. Finite dimensional simple modules

Denote by $X^+ = \{\lambda \in X \mid \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Pi\}$ the set of *dominant weights*. For a dominant weight λ we denote by $V(\lambda)$ the *Weyl module* with highest weight λ . It is constructed as follows. We denote by $L_{\mathbb{C}}(\lambda)$ the simple, finite dimensional $U_{\mathbb{C}}$ -module with highest weight λ and choose a nonzero vector $v \in L_{\mathbb{C}}(\lambda)$. The Weyl module for $G_{\mathbb{Z}}$ is then defined by $V_{\mathbb{Z}}(\lambda) = U_{\mathbb{Z}}.v \subset L_{\mathbb{C}}(\lambda)$. It turns out that this is a free abelian group of finite rank inside $L_{\mathbb{C}}(\lambda)$ and $V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{C} = L_{\mathbb{C}}(\lambda)$. The Weyl module in the modular case is then $V(\lambda) = V_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} k$. It is a finite dimensional U -module with highest weight λ and it is contained in \mathcal{O} .

Lemma 3. *The object $L(\lambda)$ is finite dimensional if and only if λ is a dominant weight.*

Proof. If λ is a dominant weight, then the Weyl module $V(\lambda)$ is a finite dimensional U -module with highest weight λ and it contains $L(\lambda)$ as a quotient. Hence $L(\lambda)$ is finite dimensional. If λ is not dominant, then there is a simple root α such that $\langle \lambda, \alpha^\vee \rangle < 0$. We consider the α -string in $L(\lambda)$ through the highest weight: $\bigoplus_{n \geq 0} L(\lambda)_{\lambda - n\alpha}$. This is a highest weight module for the subalgebra U^α of U

generated by the $e_\alpha^{(n)}, f_\alpha^{(n)}, \binom{h_\alpha}{n}$ and $n \geq 0$. The algebra U^α is the algebra of distributions associated to the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the field k . The structure of the highest weight modules can be worked out explicitly (for example, see [A]). The fact that $\langle \lambda, \alpha^\vee \rangle < 0$ implies that $\bigoplus_{n \geq 0} L(\lambda)_{\lambda - n\alpha}$ must be infinite dimensional, and hence so is $L(\lambda)$. \square

5. Characters and multiplicities

Our definition of category \mathcal{O} doesn't assume that the weight spaces of objects are finite dimensional, so we cannot assign a character to each object in the usual way. So let us define $\widehat{\mathcal{O}}$ as the category of all U -modules M that satisfy the following:

- M is a weight module and M_μ is a finite dimensional k -vector space for all $\mu \in X$.
- There exist $\gamma_1, \dots, \gamma_l \in X$ such that $M_\mu \neq 0$ implies $\mu \leq \gamma_i$ for at least one $i \in \{1, \dots, l\}$.

Then $\widehat{\mathcal{O}}$ is a subcategory of \mathcal{O} . It is analogous to the category \mathcal{O} defined in [DGK, Sect. 3]. Clearly, the Verma modules $\Delta(\lambda)$ and their simple quotients $L(\lambda)$ are contained in $\widehat{\mathcal{O}}$.

Let $\widehat{\mathbb{Z}[X]}$ be the set of maps $f: X \rightarrow \mathbb{Z}$ that have the property that there exist $\gamma_1, \dots, \gamma_l \in X$ such that $f(\mu) \neq 0$ implies $\mu \leq \gamma_i$ for at least one $i \in \{1, \dots, l\}$. Then for any object M in $\widehat{\mathcal{O}}$, the map $\text{ch } M: X \rightarrow \mathbb{Z}$, $\text{ch } M(\mu) = \dim_k M_\mu$ is an element in $\widehat{\mathbb{Z}[X]}$ that we call the character of M .

Note the following analogue of Proposition 3.2 in [DGK] holds verbatim. The proof is the same.

Proposition 4. *Let M be an object in $\widehat{\mathcal{O}}$ and $\lambda \in X$. Then there exists an increasing filtration $0 = M_0 \subset M_1 \subset \dots \subset M_t = M$ of objects in $\widehat{\mathcal{O}}$ and a subset J of $\{1, \dots, t\}$ such that the following holds:*

- (i) *For $j \in J$, the quotient M_j/M_{j-1} is isomorphic to $L(\lambda_j)$ for some $\lambda_j \geq \lambda$.*
- (ii) *For $j \notin J$ and any $\mu \geq \lambda$ we have $(M_j/M_{j-1})_\mu = 0$.*

A filtration as above is called a *local composition series* for M at λ . Let $M, \lambda, 0 = M_0 \subset M_1 \subset \dots \subset M_t = M$ and J be as above. It is easy to check (using the character of M in $\widehat{\mathbb{Z}[X]}$) that for all $\mu \geq \lambda$ the number $[M : L(\mu)] := |\{j \in J \mid M_j/M_{j-1} \cong L(\mu)\}|$ is independent of choices (as long as $\mu \geq \lambda$). This number is called the *multiplicity of $L(\mu)$ in M* . Moreover, one also checks that $\sum_{\mu \in X} [M : L(\mu)] \cdot \text{ch } L(\mu)$ is a well-defined element in $\widehat{\mathbb{Z}[X]}$ that equals $\text{ch } M$. For more details, see Section 3 in [DGK].

6. A duality on \mathcal{O}

Recall that there is an antiautomorphism σ on $U_{\mathbb{C}}$ that maps e_α to f_α and h_α to h_α . It hence leaves $U_{\mathbb{Z}}$ stable, so we obtain an antiautomorphism on U that we denote by the same symbol. For a U -module M we denote by M^σ its twist by σ . Then $(M^\sigma)_\lambda = M_\lambda$ as a vector space, so the σ -twist preserves weight modules. For a U -module M we denote by $M^* = \bigoplus_{\lambda \in X} \text{Hom}_k(M_\lambda, k)$ the restricted linear dual. It acquires the structure of a U -module by setting $(x \cdot \phi)(m) = \phi(\sigma(x) \cdot m)$ for all $m \in M, \phi \in M^*$ and $x \in U$.

Lemma 5. *Suppose that M is an object in $\widehat{\mathcal{O}}$. Then M^* is an object in \mathcal{O} as well and we have a functorial identification $(M^*)^* = M$.*

Proof. As observed above, M^* is a weight module. As the set of weights of M^* coincides with the set of weights of M , it is bounded from above, hence M^* must be U^+ -locally finite. So it is contained in \mathcal{O} as well. It again satisfies the assumptions of the above lemma, and it is immediate that $(M^*)^* = M$ canonically. \square

Clearly, L^* is a simple object in \mathcal{O} if L is. From the fact that the characters agree we deduce that $L(\lambda)^* \cong L(\lambda)$ for all $\lambda \in X$.

We denote by $\nabla(\lambda) = \Delta(\lambda)^*$ the dual of the Verma module with highest weight λ .

Lemma 6. *We have $\dim_k \text{Hom}_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) = \delta_{\lambda, \mu}$ and $\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), \nabla(\mu)) = 0$ for all $\lambda, \mu \in X$.*

(Here, $\text{Ext}_{\mathcal{O}}^1$ means the Yoneda-extension group.)

Proof. For all $\lambda, \mu \in X$, we have

$$\text{Hom}_{\mathcal{O}}(\Delta(\lambda), \nabla(\mu)) \cong \{v \in \nabla(\mu)_{\lambda} \mid e_{\alpha}^{(n)}v = 0 \text{ for all } \alpha \in R^+, n > 0\}$$

by the universal property of Verma modules. This is a one-dimensional space if $\lambda = \mu$. If $\lambda \not\leq \mu$, then $\nabla(\mu)_{\lambda} = 0$ and hence there are no nontrivial homomorphisms from $\Delta(\lambda)$ to $\nabla(\mu)$. From any nonzero homomorphism $f: \Delta(\lambda) \rightarrow \nabla(\mu)$ we obtain by dualizing a nonzero homomorphism $f^*: \Delta(\mu) \rightarrow \nabla(\lambda)$. Hence the existence of a nonzero homomorphism implies $\lambda \leq \mu$ and $\mu \leq \lambda$, hence $\lambda = \mu$. We have proven the first statement.

Let $0 \rightarrow \nabla(\mu) \rightarrow M \rightarrow \Delta(\lambda) \rightarrow 0$ be an exact sequence. If $\lambda \not\leq \mu$, then λ is a maximal weight of M , so the universal property of $\Delta(\lambda)$ implies that this sequence splits. By dualizing we obtain another exact sequence $0 \rightarrow \nabla(\lambda) \rightarrow M^* \rightarrow \Delta(\mu) \rightarrow 0$, which splits if $\mu \not\leq \lambda$. But the conditions $\lambda < \mu$ and $\mu < \lambda$ cannot both hold. Hence the second statement is also true. \square

7. The Frobenius twist

The group scheme G_k is induced from an integral version $G_{\mathbb{Z}}$ via base change (cf. [C]). Hence there exists a Frobenius endomorphism $G_k \rightarrow G_k$ (an endomorphism of k -group schemes). This endomorphism induces an endomorphism on its k -algebra of distributions. We hence obtain an endomorphism $\text{Fr}: U \rightarrow U$ that is given by the following formulas:

$$\begin{aligned} e_{\alpha}^{(n)} &\mapsto \begin{cases} e_{\alpha}^{(n/p)}, & \text{if } p \text{ divides } n, \\ 0, & \text{if } p \text{ does not divide } n, \end{cases} \\ f_{\alpha}^{(n)} &\mapsto \begin{cases} f_{\alpha}^{(n/p)}, & \text{if } p \text{ divides } n, \\ 0, & \text{if } p \text{ does not divide } n, \end{cases} \\ \binom{h_{\alpha}}{n} &\mapsto \begin{cases} \binom{h_{\alpha}}{n/p}, & \text{if } p \text{ divides } n, \\ 0, & \text{if } p \text{ does not divide } n. \end{cases} \end{aligned}$$

These formulas follow from explicit calculations for the additive and multiplicative groups G_a and G_m ; see [J, Sect. 3.1].

If M is a U -module, then we denote by $M^{[1]}$ the U -module that we obtain from M by pulling back the action homomorphism along Fr. If M is a weight module, then so is $M^{[1]}$ and we have $M_\lambda^{[1]} = 0$ unless $\lambda \in pX$, in which case $M_\lambda^{[1]} = M_{\lambda/p}$. If M is U^+ -locally finite, then so is $M^{[1]}$ and we obtain a functor $(\cdot)^{[1]}: \mathcal{O} \rightarrow \mathcal{O}$. This is called the *Frobenius twist*. We denote by $(\cdot)^{[l]}$ the functor obtained by applying $(\cdot)^{[1]}$ l -times. As Fr is surjective, $L^{[1]}$ is simple for all simple objects L in \mathcal{O} . A quick check on the highest weights yields the following result.

Lemma 7. *We have $L(\lambda)^{[1]} = L(p\lambda)$ for all $\lambda \in X$.*

Hence $L(p\lambda)_\mu = 0$ unless $\mu \in pX$.

8. Direct decompositions

The goal of this section is to show that we have a Krull–Remak–Schmidt decomposition for all finitely generated objects in \mathcal{O} . Once the analogue of the Fitting lemma is proven, the arguments for this result are standard.

Lemma 8. *Suppose that M is an object in \mathcal{O} that is finitely generated as a U -module. Then any endomorphism f of M induces a Fitting-decomposition, i.e., there exists some $n > 0$ such that $M = \text{im } f^n \oplus \text{ker } f^n$.*

Proof. If M is finitely generated, then every weight space of M is finite dimensional and there exists a finite subset B of X such that $M_B := \bigoplus_{\lambda \in B} M_\lambda$ is finite dimensional and generates M . We denote by f_B the vector space endomorphism on M_B that is induced by f . Then there is some $n \gg 0$ such that $\text{im } f_B^n = \text{im } f_B^{n+1} = \text{im } f_B^{n+2} = \dots$. As M_B generates M , $\text{im } f_B^l$ generates $\text{im } f^l \subset M$ for all $l > 0$ and we deduce that the image of f stabilizes: $\text{im } f^n = \text{im } f^{n+1} = \text{im } f^{n+2} = \dots$. Now let $\mu \in X$ and denote by f_μ the endomorphism on M_μ induced by f . Then $\text{im } f_\mu^n = \text{im } f_\mu^{n+1} = \dots$. As M_μ is finite dimensional we also deduce $\text{ker } f_\mu^n = \text{ker } f_\mu^{n+1} = \dots$. We also obtain the Fitting-decomposition $M_\mu = \text{ker } f_\mu^n \oplus \text{im } f_\mu^n$ on each weight space. Hence $M = \text{im } f^n \oplus \text{ker } f^n$. \square

Lemma 9. *Suppose that M is a finitely generated, indecomposable object in \mathcal{O} . Then every endomorphism of M is either nilpotent or an automorphism. In particular, $\text{End}_{\mathcal{O}}(M)$ is a local ring.*

Proof. Let f be an endomorphism of M that is not an automorphism. The finite dimensionality of the weight spaces allows us to deduce that f is not injective. Hence $\text{ker } f \neq \{0\}$ and we deduce $M = \text{ker } f^n$ for some $n \gg 0$ from the indecomposability of M and Lemma 8. Hence f is nilpotent. So every endomorphism is either nilpotent or an automorphism.

If f is nilpotent and g is an automorphism, then $g - f$ is invertible, hence an automorphism. It follows that the sum of two nilpotent endomorphisms of M is again nilpotent. Moreover, $g \circ f$ and $f \circ g$ are not injective, hence nilpotent. So the nilpotent endomorphisms form an ideal in $\text{End}_{\mathcal{O}}(M)$ and each element in its complement is invertible. Hence $\text{End}_{\mathcal{O}}(M)$ is a local ring. \square

Proposition 10. *Let M be a finitely generated object in \mathcal{O} . Then M can be written as a finite direct sum of indecomposable objects in \mathcal{O} . Moreover, such a decomposition is unique up to reordering and isomorphisms.*

Proof. As in the proof of Lemma 8, we denote by B a subset of X such that $M_B = \bigoplus_{\lambda \in B} M_\lambda \subset M$ is finite dimensional and generates M . Then every direct summand of M must intersect M_B nontrivially. In particular, the number of direct summands in a decomposition of M is bounded by the dimension of M_B . So a decomposition of M into indecomposable objects exists. The uniqueness of this decomposition can be proven (with Lemma 9) using standard arguments. \square

Verma flags and projective objects in \mathcal{O}

In this section, we will approximate the category \mathcal{O} via *truncated subcategories* \mathcal{O}^J and show that in these subcategories, every simple object admits a projective cover. This treatment is analogous to the situation of complex Kac–Moody algebra (cf. [F, Sect. 2.2]) We will also prove an appropriate version of the BGGH-reciprocity theorem that was first proven in the context of modular Lie algebras by Humphreys in [H1] (Theorems 4.4 and 4.5). Later, the characteristic 0 case was treated in [BGG, Prop. 2]. Many of the ideas used in the following originate in the article [RCW].

9. Verma flags

Let M be an object in \mathcal{O} .

Definition 2. M is said to *admit a Verma flag* if there is a (finite) filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

and $\lambda_1, \dots, \lambda_n \in X$ such that $M_i/M_{i-1} \cong \Delta(\lambda_i)$ for all $i = 1, \dots, n$.

A filtration like the one in the definition above is sometimes, for example in [H2], called a *standard filtration*.

In the situation of the definition above we set

$$(M : \Delta(\mu)) = \#\{i \in \{1, \dots, n\} \mid \lambda_i = \mu\}.$$

This number is independent of the chosen filtration and is called the *multiplicity of $\Delta(\mu)$ in M* .

Lemma 11. *Suppose that M admits a Verma flag. Then there exists a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$$

and $\lambda_1, \dots, \lambda_n \in X$ such that $M_i/M_{i-1} \cong \Delta(\lambda_i)$ for $i = 1, \dots, n$, and such that $\lambda_i > \lambda_j$ implies $i < j$.

Proof. This follows, by induction, from the fact that a surjective homomorphism $N \rightarrow \Delta(\lambda)$ splits if λ is maximal among the weights of N (by Proposition 2). \square

Lemma 12. *Let $M = A \oplus B$ be a decomposition in \mathcal{O} . Then M admits a Verma flag if and only if A and B admit Verma flags. In this case,*

$$(M : \Delta(\mu)) = (A : \Delta(\mu)) + (B : \Delta(\mu)).$$

Proof. If A and B admit Verma flags, then it is easily shown that M admits a Verma flag and that the claim about the multiplicities holds. It is hence enough to show that if M admits a Verma flag, then A and B do as well. This is proven using the same arguments as in the characteristic 0 case that can be found, for example, in [H2, Sect. 3.7]. \square

10. Some functors on \mathcal{O}

It is now useful to endow the set X with a topology.

Definition 3.

- (1) We say that a subset J of X is *open* if $\lambda \in J$ and $\mu \leq \lambda$ imply $\mu \in J$.
- (2) We say that a subset I of X is *closed* if $\lambda \in I$ and $\lambda \leq \mu$ imply $\mu \in I$.
- (3) We say that a subset K of X is *locally closed* if $\lambda, \nu \in K$ and $\lambda \leq \mu \leq \nu$ imply $\mu \in K$.

This indeed defines a topology. Note that arbitrary unions of closed subsets are closed again.

Let M be an object in \mathcal{O} , and let $I \subset X$ be a closed subset with open complement J . We set

$$M_I := \left\langle \bigoplus_{\lambda \in I} M_\lambda \right\rangle_U \subset M,$$

$$M^J := M/M_I,$$

i.e., $M_I \subset M$ is the minimal submodule of M that contains the weight spaces M_λ with $\lambda \in I$, and M^J the largest quotient of M with all weights contained in J . Clearly, these definitions yield functors $(\cdot)_I, (\cdot)^J : \mathcal{O} \rightarrow \mathcal{O}$.

Proposition 13. *Let M be an object in \mathcal{O} . Then the following are equivalent.*

- (1) M admits a Verma flag.
- (2) M^J admits a Verma flag for any open set J .
- (3) M_I admits a Verma flag for any closed set I .

If either statement above is true, then we have

$$(M^J : \Delta(\nu)) = \begin{cases} (M : \Delta(\nu)), & \text{if } \nu \in J, \\ 0, & \text{otherwise,} \end{cases}$$

$$(M_I : \Delta(\nu)) = \begin{cases} (M : \Delta(\nu)), & \text{if } \nu \in I, \\ 0, & \text{otherwise} \end{cases}$$

for all open subset J and all closed subsets I of X .

Proof. As $M = M_X = M^X$, either (2) or (3) imply (1). Suppose M admits a Verma flag. Let J be an open subset of X with closed complement I . Using Lemma 11 we deduce that there is a submodule M' of M , appearing in a Verma flag of M , such that M' and M/M' admit Verma flags and such that

$$(M' : \Delta(\lambda)) = \begin{cases} (M : \Delta(\lambda)), & \text{if } \lambda \in I, \\ 0, & \text{if } \lambda \in J, \end{cases}$$

and

$$(M/M' : \Delta(\lambda)) = \begin{cases} (M : \Delta(\lambda)), & \text{if } \lambda \in J, \\ 0, & \text{if } \lambda \in I. \end{cases}$$

In particular, it follows that $M' = M_I$ and $M/M' = M^J$. This shows that (1) implies (2) and (3), and that the statement about the multiplicities is true if (1) holds. \square

11. Objects admitting a Verma flag as $U^{\leq 0}$ -modules

Set $U^{\leq 0} := U^-U^0$. Again this is a subalgebra of U . We denote by $U^{\leq 0}\text{-mod}^{wt}$ the full subcategory of the category of $U^{\leq 0}$ -modules that contains all objects that are weight modules, and we denote by Res the restriction functor from the category \mathcal{O} to $U^{\leq 0}\text{-mod}^{wt}$.

Lemma 14. *Let $\lambda \in X$. Then $\text{Res } \Delta(\lambda)$ is a projective object in $U^{\leq 0}\text{-mod}^{wt}$.*

Proof. As $\Delta(\lambda)$ is free as a U^- -module of rank 1 (by Kostant’s version of the PBW-theorem) and as it is generated by any nonzero element in the one-dimensional space $\Delta(\lambda)_\lambda$, we have

$$\text{Hom}_{U^{\leq 0}}(\text{Res } \Delta(\lambda), M) = \text{Hom}_{U^0}(\text{Res } \Delta(\lambda)_\lambda, M_\lambda) = \text{Hom}_{U^0}(k_\lambda, M_\lambda).$$

The functor $M \mapsto \text{Hom}_{U^0}(k_\lambda, M_\lambda)$ is an exact functor even from the category of weight modules to vector spaces. Hence $\text{Res } \Delta(\lambda)$ is a projective object in $U^{\leq 0}\text{-mod}^{wt}$. \square

Lemma 15.

- (1) *Let N be an object in \mathcal{O} that admits a Verma flag. Then $\text{Res } N$ is isomorphic to a direct sum of objects of the form $\text{Res } \Delta(\lambda)$ for various λ .*
- (2) *Let M and N be objects in \mathcal{O} and assume that N admits a Verma flag and let $f : M \rightarrow N$ be a surjective homomorphism. Then $\text{Res } f : \text{Res } M \rightarrow \text{Res } N$ splits.*

Proof. Statement (1) follows from the fact that each $\text{Res } \Delta(\lambda)$ is projective in $U^{\leq 0}\text{-mod}^{wt}$. From (1) and Lemma 14 it then also follows that $\text{Res } N$ is projective in $U^{\leq 0}\text{-mod}^{wt}$, which implies statement (2). \square

12. Tensor products

Let L be an object in \mathcal{O} . Then the functor $M \mapsto M \otimes L$ preserves the category of weight modules and the category of U^+ -locally finite modules, hence it induces a functor from \mathcal{O} to \mathcal{O} .

Lemma 16. *Suppose that L is finite dimensional. If M admits a Verma flag, then so does $M \otimes L$ and we have*

$$(M \otimes L : \Delta(\mu)) = \sum_{\lambda \in X} (M : \Delta(\lambda)) \dim_k L_{\mu-\lambda}.$$

Proof. Note that the functor $(\cdot) \otimes L$ is exact. It is hence enough to prove the statement for $M = \Delta(\lambda)$ for some $\lambda \in X$. In this case, the same arguments can be used as in the characteristic 0 case, which can be found, for example, in [H2, Sect. 3.6]. \square

13. Projective objects in truncated categories

Let J be an open subset of X .

Definition 4. We denote by \mathcal{O}^J the full subcategory of \mathcal{O} that contains all objects M that have the property that $M_\lambda \neq 0$ implies $\lambda \in J$.

Clearly, the functor $M \mapsto M^J$ is actually a functor from \mathcal{O} to \mathcal{O}^J , and we have $M \in \mathcal{O}^J$ if and only if $M = M^J$. Note that $L(\lambda)$ is contained in \mathcal{O}^J if and only if $\Delta(\lambda)$ is contained in \mathcal{O}^J , which is the case if and only if $\lambda \in J$.

Definition 5. We say that an open subset J of X is *quasi-bounded* if for any $\lambda \in X$ the set $\{\mu \in J \mid \lambda \leq \mu\}$ is finite.

Recall that in an abelian category \mathcal{A} an epimorphism $f: A \rightarrow B$ is called *essential* if the following holds: if $g: C \rightarrow A$ is a morphism in \mathcal{A} such that $f \circ g: C \rightarrow B$ is an epimorphism, then g is an epimorphism. A morphism $f: A \rightarrow B$ is called a *projective cover* of B if A is projective and f is an essential epimorphism.

Theorem 17. *Suppose that $J \subset X$ is open and quasi-bounded. For any $\lambda \in J$ there exists a projective cover $p_\lambda: P^J(\lambda) \rightarrow L(\lambda)$ of $L(\lambda)$ in \mathcal{O}^J , and the object $P^J(\lambda)$ admits a Verma flag.*

Proof. Consider the $U^{\geq 0}$ -module $C(\lambda) := U^{\geq 0} \otimes_{U^0} k_\lambda$. Theorem 1 implies that $C(\lambda)$ is free of rank 1 as a U^+ -module. As an U^0 -module it is a weight module and its weights are contained in $\{\mu \mid \mu \geq \lambda\}$. As before, define $C(\lambda)^J$ as the largest quotient $U^{\geq 0}$ -module with all weights contained in J , i.e.,

$$C(\lambda)^J = C(\lambda) / \bigoplus_{\mu \notin J} C(\lambda)_\mu.$$

As J is supposed to be quasi-bounded, $C(\lambda)^J$ is a finite dimensional $U^{\geq 0}$ -module and a weight module supported in $\{\mu \mid \mu \in J, \mu \geq \lambda\}$. It hence has a $U^{\geq 0}$ -module filtration with one-dimensional subquotients isomorphic to k_μ for various $\mu \in J$ (with $\mu \geq \lambda$). Moreover, k_μ occurs with multiplicity $\dim_k C(\lambda)_\mu^J$.

Now consider $Q^J(\lambda) := U \otimes_{U^{\geq 0}} C(\lambda)^J$. This is a weight module, and, as the functor $U \otimes_{U^{\geq 0}} (\cdot)$ is exact (by Theorem 1), it has a finite filtration with subquotients isomorphic to $\Delta(\mu)$ for various $\mu \in J$ (with $\mu \geq \lambda$). Hence it is an object in \mathcal{O}^J that admits a Verma flag and the multiplicities are given by

$$(Q^J(\lambda) : \Delta(\mu)) = \dim_k C(\lambda)_\mu^J.$$

Now let N be an object in \mathcal{O}^J . We then have a natural isomorphisms (that are functorial in N):

$$\begin{aligned} \text{Hom}_{\mathcal{O}^J}(Q^J(\lambda), N) &= \text{Hom}_U(U \otimes_{U^{\geq 0}} C(\lambda)^J, N) \\ &= \text{Hom}_{U^{\geq 0}}(C(\lambda)^J, N) \\ &= \text{Hom}_{U^{\geq 0}}(C(\lambda), N) \quad (\text{as } N = N^J) \\ &= \text{Hom}_{U^{\geq 0}}(U^{\geq 0} \otimes_{U^0} k_\lambda, N) \\ &= \text{Hom}_{U^0}(k_\lambda, N) \\ &\cong N_\lambda. \end{aligned}$$

The functor $N \mapsto N_\lambda$ is exact on \mathcal{O}^J , so the object $Q^J(\lambda)$ is projective in \mathcal{O}^J .

By the weight considerations above, $\Delta(\lambda)$ appears in a Verma flag of $Q^J(\lambda)$ with multiplicity $1 = \dim_k C(\lambda)_\lambda$ and it is minimal among the highest weights of Verma subquotients. So there exists a surjection $Q^J(\lambda) \rightarrow \Delta(\lambda)$ in \mathcal{O}^J . As $Q^J(\lambda)$ is finitely generated (even cyclic) as a U -module we can apply Proposition 10, so $Q^J(\lambda)$ splits into a finite direct sum of indecomposables. As a homomorphism $M \rightarrow \Delta(\lambda)$ is surjective if and only if it is surjective on the λ -weight space, and as $\Delta(\lambda)_\lambda$ is of dimension 1, there must be an indecomposable direct summand $P^J(\lambda)$ of $Q^J(\lambda)$ that admits a surjection onto $\Delta(\lambda)$. Then $P^J(\lambda)$ is projective and admits a Verma flag by Lemma 12.

There exists a surjective homomorphism $f: P^J(\lambda) \rightarrow L(\lambda)$ as $L(\lambda)$ is a quotient of $\Delta(\lambda)$. It remains to show that this is an essential homomorphism. So let $g: M \rightarrow P^J(\lambda)$ be a morphism such that $f \circ g: M \rightarrow L(\lambda)$ is an epimorphism. The projectivity of $P^J(\lambda)$ allows us to find a morphism $h: P^J(\lambda) \rightarrow M$ such that the diagram

$$\begin{array}{ccccc} P^J(\lambda) & \xrightarrow{h} & M & \xrightarrow{g} & P^J(\lambda) \\ & \searrow f & \downarrow f \circ g & \swarrow f & \\ & & L(\lambda) & & \end{array}$$

commutes. As f is an epimorphism, the composition $g \circ h$ cannot be nilpotent, hence must be an automorphism of $P^J(\lambda)$ by Lemma 9. In particular, g is an epimorphism. So f is essential and indeed a projective cover. \square

14. Verma multiplicities and the BGGH-reciprocity

We are now going to prove an analogue of the reciprocity result that appeared in [H1] in the context of representations of modular Lie algebras (see Theorems 4.4 and 4.5 and the sentence before the Remark at the end of [H1, Sect. 4]).

Proposition 18. *Let M be an object in \mathcal{O} that admits a Verma flag. Then*

$$(M : \Delta(\mu)) = \dim \text{Hom}_{\mathcal{O}}(M, \nabla(\mu))$$

for all $\mu \in X$.

Proof. Suppose that $M' \subset M$ is a submodule such that M' admits a Verma flag and $M/M' \cong \Delta(\nu)$ for some $\nu \in X$. Hence

$$(M : \Delta(\mu)) = (M' : \Delta(\mu)) + \delta_{\mu,\nu}$$

for all $\mu \in X$. Since $\text{Ext}_{\mathcal{O}}^1(\Delta(\nu), \nabla(\mu)) = 0$ by Lemma 6, we have an exact sequence

$$0 \rightarrow \text{Hom}(\Delta(\nu), \nabla(\mu)) \rightarrow \text{Hom}(M, \nabla(\mu)) \rightarrow \text{Hom}(M', \nabla(\mu)) \rightarrow 0,$$

and, again using Lemma 6, we deduce

$$\dim_k \text{Hom}(M, \nabla(\mu)) = \dim_k \text{Hom}(M', \nabla(\mu)) + \delta_{\nu, \mu}.$$

The statement follows by induction on the length of a Verma flag. \square

Now we are ready to state and prove the mentioned reciprocity result. It appears as statement (3) in the following.

Proposition 19. *Fix an open subset J of X and $\lambda \in J$.*

- (1) *The only irreducible quotient of $P^J(\lambda)$ is $L(\lambda)$.*
- (2) *Suppose that M is an object in \mathcal{O}^J that is also contained in $\widehat{\mathcal{O}}$ (cf. Section 5). Then*

$$\dim_k \text{Hom}_{\mathcal{O}^J}(P^J(\lambda), M) = [M : L(\lambda)].$$

- (3) *Let $\mu \in X$. Then*

$$(P^J(\lambda) : \Delta(\mu)) = \begin{cases} [\nabla(\mu) : L(\lambda)], & \text{if } \mu \in J, \\ 0, & \text{if } \mu \notin J. \end{cases}$$

Proof. Let $M \subset P^J(\lambda)$ be a submodule with $M \neq P^J(\lambda)$. Let $f: P^J(\lambda) \rightarrow L(\lambda)$ be a projective cover. Then $f(M) = 0$. In particular, if $P^J(\lambda)/M$ is irreducible, then f must induce an isomorphism $P^J(\lambda)/M \cong L(\lambda)$, hence (1). The projectivity of $P^J(\lambda)$ and part (1) imply part (2). Using Proposition 18 we calculate

$$\begin{aligned} (P^J(\lambda) : \Delta(\mu)) &= \dim_k \text{Hom}_{\mathcal{O}^J}(P^J(\lambda), \nabla(\mu)) \\ &= [\nabla(\mu) : L(\lambda)] \quad (\text{by (2)}). \end{aligned}$$

Hence (3). \square

Note that $[\nabla(\mu) : L(\lambda)] = [\Delta(\mu) : L(\lambda)]$ as the characters of the Verma and the dual Verma modules agree.

15. The exactness of $(\cdot)_I$ and $(\cdot)^J$

Let us denote by \mathcal{V} the full subcategory of \mathcal{O} that contains all objects that admit a Verma flag. In general, it does not contain the subobjects and quotients of its objects, so it is not an abelian subcategory. However, it inherits an exact structure from the surrounding category \mathcal{O} . We call a short sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{V} exact if it is exact when considered as a sequence in \mathcal{O} .

Let J be an open subset of X with closed complement I . Proposition 13 implies that $(\cdot)_I$ and $(\cdot)^J$ induce endofunctors on \mathcal{V} that we denote by the same symbol. Then we have the following.

Proposition 20. *Let $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ be a sequence in \mathcal{V} . Then the following are equivalent.*

- (1) *The sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is exact.*
- (2) *For any open subset J of X , the sequence $0 \rightarrow A^J \rightarrow B^J \rightarrow C^J \rightarrow 0$ is exact.*
- (3) *For any closed subset I of X , the sequence $0 \rightarrow A_I \rightarrow B_I \rightarrow C_I \rightarrow 0$ is exact.*

Proof. As X is open in X , property (1) is a special case of either one of the properties (2) or (3). It is hence enough to show that (1) implies (2) and (3). Let M be an object in \mathcal{O} . Then $\bigoplus_{\lambda \in I} M_\lambda$ is stable under the action of the subalgebra $U^{\geq 0}$ (as I is closed). Hence M_I is generated by $\bigoplus_{\lambda \in I} M_\lambda$ not only as a U -submodule, but already as a U^- -submodule, and, in particular, as a $U^{\leq 0}$ -submodule. As the algebra $U^{\leq 0}$ detects the weight decomposition, we deduce that the functor $(\cdot)_I$ factors through the restriction functor $\text{Res} : \mathcal{O} \rightarrow U^{\leq 0}\text{-mod}^{wt}$. But in the category $U^{\leq 0}\text{-mod}^{wt}$ the short exact sequence appearing in (1) splits (cf. Lemma 15). As $(\cdot)_I$ is additive, it produces an exact sequence when applied to a split sequence. Hence (1) implies (3). By the snake lemma, (1) and (3) imply (2). \square

16. Subcategories associated to locally closed subsets

Let K be a locally closed subset of X .

Definition 6. We denote by $\mathcal{V}(K)$ the full subcategory of category \mathcal{V} that contains all objects M that have the property that $(M : \Delta(\lambda)) \neq 0$ implies $\lambda \in K$.

Now let $\gamma \in X$ be a *dominant* weight. Then $L = L(\gamma)$ is finite dimensional. Denote by $\Gamma = \text{Hom}_k(L, k)$ the dual U -module (without the σ -twist!). It is a finite dimensional module contained in \mathcal{O} and the irreducible U -module of lowest weight $-\gamma$. Let $J' \subset J$ be open subsets of X . Then $\tilde{J}' = J' + \gamma \subset \tilde{J} = J + \gamma$ are open in X as well. We define the *shift functors*

$$\mathbf{U} := ((\cdot) \otimes L)_{X \setminus \tilde{J}'} : \mathcal{O} \rightarrow \mathcal{O},$$

$$\mathbf{D} := ((\cdot) \otimes \Gamma)^J : \mathcal{O} \rightarrow \mathcal{O}.$$

Set $K := J \setminus J'$ and $\tilde{K} := \tilde{J} \setminus \tilde{J}'$. Then K and \tilde{K} are locally closed, and the map $(\cdot) + \gamma$ yields a bijection $K \xrightarrow{\sim} \tilde{K}$.

Lemma 21. *Suppose that there exists some $l > 0$ such that the above triple (J, J', γ) satisfies the following assumptions:*

- $\gamma \in p^l X$,
- for all $\lambda \in J$ and $\nu > 0$ we have $\lambda - p^l \nu \in J'$, i.e., $K \cap (K - p^l \nu) = \emptyset$.

Then the following holds.

- (1) *Let $\lambda \in K$. Then $\mathbf{U}(\Delta(\lambda)) \cong \Delta(\lambda + \gamma)$.*
- (2) *Let $\tilde{\lambda} \in \tilde{K}$. Then $\mathbf{D}(\Delta(\tilde{\lambda})) \cong \Delta(\tilde{\lambda} - \gamma)$.*
- (3) *After restriction, \mathbf{U} and \mathbf{D} induce exact functors $\mathbf{U} : \mathcal{V}(K) \rightarrow \mathcal{V}(\tilde{K})$ and $\mathbf{D} : \mathcal{V}(\tilde{K}) \rightarrow \mathcal{V}(K)$.*

Proof. We prove statement (1). Let $\lambda \in K$ and consider the module $\Delta(\lambda) \otimes L$. By Lemma 16 it admits a Verma flag with multiplicities

$$(\Delta(\lambda) \otimes L : \Delta(\mu)) = \dim_k L_{\mu-\lambda}.$$

So Proposition 13 implies that $\mathbf{U}(\Delta(\lambda)) = (\Delta(\lambda) \otimes L)_{X \setminus \tilde{J}'}$ admits a Verma flag with

$$(\mathbf{U}(\Delta(\lambda)) : \Delta(\mu)) = \begin{cases} \dim_k L_{\mu-\lambda}, & \text{if } \mu \in X \setminus \tilde{J}' \\ 0, & \text{else.} \end{cases}$$

Now suppose μ is such that $(\mathbf{U}(\Delta(\lambda)) : \Delta(\mu)) \neq 0$. Then $\mu - \lambda$ is a weight of L and $\mu \notin \tilde{J}'$. As $\gamma = p^l \gamma'$ for some $\gamma' \in X$ we have $L = L(\gamma')^{[l]}$ by Lemma 7 and hence there exists a weight η' of $L(\gamma')$ with $\mu - \lambda = p^l \eta'$. In particular, $\lambda + p^l \eta' \notin \tilde{J}'$. As $\tilde{J}' = J' + p^l \gamma'$, this implies that $\lambda - p^l(\gamma' - \eta') \notin J'$. As $\gamma' - \eta' \geq 0$, our assumption implies that $\eta' = \gamma'$. Hence $\mu = \lambda + p^l \gamma' = \lambda + \gamma$, and, as $\dim_k L_\gamma = 1$ we obtain statement (1). Statement (2) is proven with similar arguments.

Now (3) follows from (1) and (2) and the exactness of the functors $(\cdot) \otimes L$ and $(\cdot) \otimes \Gamma$ together with Proposition 20. \square

Theorem 22. *Suppose that there exists some $l > 0$ such that the triple (J, J', γ) satisfies the following:*

- $\gamma \in p^l X$,
- for all $\lambda \in J$ and $\nu > 0$ we have $\lambda - p^l \nu \in J'$, i.e., $K \cap (K - p^l \nu) = \emptyset$.

Then the functors \mathbf{U} and \mathbf{D} induce mutually inverse equivalences

$$\mathcal{V}(K) \cong \mathcal{V}(\tilde{K})$$

of exact categories.

Proof. We first construct a transformation $\tau : \mathbf{D} \circ \mathbf{U} \rightarrow \text{id}_{\mathcal{V}(K)}$ and show that this is an isomorphism, and then we construct a transformation $\sigma : \text{id}_{\mathcal{V}(\tilde{K})} \rightarrow \mathbf{U} \circ \mathbf{D}$ and show that it is an isomorphism. So first let M be an object in $\mathcal{V}(K)$. We claim that there is a commutative diagram of the following form:

$$\begin{CD} (M \otimes L)_{X \setminus \tilde{J}'} \otimes \Gamma @>b>> ((M \otimes L)_{X \setminus \tilde{J}'} \otimes L)^J \\ @V i \otimes \text{id}_\Gamma VV @VV \tau^M V \\ M \otimes L \otimes \Gamma @>\text{id}_M \otimes \text{ev}>> M \end{CD} .$$

Here, $\text{ev} : L \otimes \Gamma \rightarrow k_{\text{triv}}$ is the evaluation homomorphism, and i is the inclusion $(M \otimes L)_{X \setminus \tilde{J}'} \subset M \otimes L$. The composition of these two homomorphisms yields a homomorphism $(M \otimes L)_{X \setminus \tilde{J}'} \otimes \Gamma \rightarrow M$, and this must factor over the canonical quotient $b : (M \otimes L)_{X \setminus \tilde{J}'} \otimes \Gamma \rightarrow ((M \otimes L)_{X \setminus \tilde{J}'} \otimes L)^J$, as the weights of M are contained in J . The resulting homomorphism τ^M is functorial in M , and hence we obtain a natural transformation $\tau : \mathbf{D} \circ \mathbf{U} \rightarrow \text{id}_{\mathcal{V}(K)}$ of endofunctors on $\mathcal{V}(K)$.

We now show that this is an isomorphism of functors. As both functors are exact it is sufficient to show that τ is an isomorphism when evaluated at a Verma module $\Delta(\mu)$ with $\mu \in K$. By Lemma 21 we have $(\mathbf{D} \circ \mathbf{U})\Delta(\mu) \cong \Delta(\mu)$, and hence it suffices to show that the homomorphism $\tau^{\Delta(\mu)}$ is nonzero. For this, we consider again the diagram above, with $M = \Delta(\mu)$. It suffices to show that the composition $(\text{id}_M \otimes \text{ev}) \circ (i \otimes \text{id}_\Gamma)$ is nonzero. If we denote by $v \in L$ a nonzero vector of (maximal) weight γ and $m \in \Delta(\mu)$ a nonzero vector of weight μ , then $m \otimes v$ is contained in $(\Delta(\mu) \otimes L)_{X \setminus \tilde{J}'}$, as it is of weight $\mu + \gamma$ which is not contained in \tilde{J}' . For an element $\phi \in \Gamma$ with $\phi(v) \neq 0$ we then have $(\text{id}_{\Delta(\mu)} \otimes \text{ev}) \circ (i \otimes \text{id}_\Gamma)(m \otimes v \otimes \phi) = \phi(v)m$, which is nonzero. So $\tau^{\Delta(\mu)}$ is indeed nonzero.

Now let N be an object in $\mathcal{V}(\tilde{K})$. We claim that there is a commutative diagram of the following form:

$$\begin{array}{ccc}
 N & \xrightarrow{\text{id}_N \otimes \text{coev}} & N \otimes \Gamma \otimes L \\
 \sigma^N \downarrow & & \downarrow b \otimes \text{id}_L \\
 ((N \otimes \Gamma)^J \otimes L)_{X \setminus \tilde{J}'} & \xrightarrow{i} & (N \otimes \Gamma)^J \otimes L.
 \end{array}$$

Here, $\text{coev}: k_{\text{triv}} \rightarrow \Gamma \otimes L$ is the coevaluation homomorphism, and b is the canonical quotient $N \otimes \Gamma \rightarrow (N \otimes \Gamma)^J$. The resulting homomorphism $N \rightarrow (N \otimes \Gamma)^J \otimes L$ factors over the inclusion $i: ((N \otimes \Gamma)^J \otimes L)_{X \setminus \tilde{J}'} \rightarrow (N \otimes \Gamma)^J \otimes L$, as the highest weights of the Verma subquotients of N (and hence the weights of some generators of N) are supposed to be contained in \tilde{K} , so are not contained in \tilde{J}' . The resulting homomorphism $\sigma^N: N \rightarrow ((N \otimes \Gamma)^J \otimes L)_{X \setminus \tilde{J}'} = \mathbf{U} \circ \mathbf{D}(N)$ is functorial in N and yields a transformation $\sigma: \text{id}_{\mathcal{V}(\tilde{K})} \rightarrow \mathbf{U} \circ \mathbf{D}$. As both functors are exact, and as $\mathbf{U} \circ \mathbf{D}(\Delta(\tilde{\lambda})) \cong \Delta(\tilde{\lambda})$, it suffices, as before, to show that $\sigma^{\Delta(\tilde{\lambda})}$ is nonzero for all $\tilde{\lambda} \in \tilde{K}$.

Consider again the diagram above. It suffices to show that the composition $\Delta(\tilde{\lambda}) \xrightarrow{\text{id}_{\Delta(\tilde{\lambda})} \otimes \text{coev}} \Delta(\tilde{\lambda}) \otimes \Gamma \otimes L \xrightarrow{b \otimes \text{id}_L} (\Delta(\tilde{\lambda}) \otimes \Gamma)^J \otimes L$ is nonzero. Fix a basis of weight vectors $\{v_0, \dots, v_d\}$ of L and denote by ν_i the weight of v_i . Suppose that $\nu_0 = \gamma$, so $\nu_i \neq \gamma$ for all $i \neq 0$. Let $\{v_0^*, \dots, v_d^*\}$ be the dual basis of Γ . Then $(b \otimes \text{id}_L) \circ (\text{id}_{\Delta(\tilde{\lambda})} \otimes \text{coev})(m)$ is the image of $\sum m \otimes v_i^* \otimes v_i$ in $(\Delta(\tilde{\lambda}) \otimes \Gamma)^J \otimes L$. Suppose that $m \in \Delta(\tilde{\lambda})_{\tilde{\lambda}}$ is nonzero. Then $m \otimes v_i^*$ is of weight $\tilde{\lambda} - \nu_i$. Suppose that the weight $\tilde{\lambda} - \nu_i$ is contained in J . Then $(\tilde{\lambda} - \gamma) + \gamma - \nu_i$ is contained in J . As $\tilde{\lambda} - \gamma \in J$ and as $\gamma = p^l \gamma'$ and $\nu_i = p^l \nu_i'$ our assumption implies that $\nu_i = \gamma$, i.e., $i = 0$. Hence $(b \otimes \text{id}_L) \circ (\text{id}_{\Delta(\tilde{\lambda})} \otimes \text{coev})(m)$ is the image of $m \otimes v_0^* \otimes v_0$ in $(\Delta(\tilde{\lambda}) \otimes \Gamma)^J \otimes L$. As in the proof of Lemma 21 we see that $(m \otimes v_0^*)^J$ is a generator of $(\Delta(\tilde{\lambda}) \otimes \Gamma)^J \cong \Delta(\tilde{\lambda} - \gamma)$, hence nonzero. So $(b \otimes \text{id}_L) \circ (\text{id}_{\Delta(\tilde{\lambda})} \otimes \text{coev})(m)$ is nonzero. This finishes the proof. \square

Subquotient categories

Let us recall the notion of subquotient categories, as developed in [G, Chapitre III.1]. Suppose \mathcal{A} is an abelian category. A full subcategory \mathcal{N} that has the property

that for any short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} we have $B \in \mathcal{N}$ if and only if $A, C \in \mathcal{N}$, is called a *Serre subcategory*. If \mathcal{N} is a Serre subcategory of \mathcal{A} , then we construct the Serre quotient \mathcal{A}/\mathcal{N} as follows. The class of objects of \mathcal{A}/\mathcal{N} is the class of objects in \mathcal{A} , and the set of homomorphisms is constructed as follows. Suppose that $M_2 \subset M_1 \subset M$ are two subobjects of M such that M/M_1 and M/M_2 are contained in \mathcal{N} , and that $N_1 \subset N_2 \subset N$ are two subobjects of N that are both contained in \mathcal{N} . We then have canonical homomorphisms

$$\text{Hom}_{\mathcal{A}}(M, N) \rightarrow \text{Hom}_{\mathcal{A}}(M_1, N/N_1) \rightarrow \text{Hom}_{\mathcal{A}}(M_2, N/N_2).$$

So we can define

$$\text{Hom}_{\mathcal{A}/\mathcal{N}}(M, N) = \varinjlim \text{Hom}_{\mathcal{A}}(M', N/N'),$$

where M' ranges over all subobjects of M such that M/M' is contained in \mathcal{N} , and N' ranges over all subobjects on N that are contained in \mathcal{N} . This construction comes with a canonical map $\text{Hom}_{\mathcal{A}}(M, N) \rightarrow \text{Hom}_{\mathcal{A}/\mathcal{N}}(M, N)$. There is a composition law for morphisms in \mathcal{A}/\mathcal{N} such that we obtain a *quotient functor* $\mathbf{T}: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$. The following statement summarises some of the main results in [G] (see Proposition III.1.1, Lemme III.1.2, Corollaire III.1.3).

Theorem 23.

- (1) *The category \mathcal{A}/\mathcal{N} is abelian and the functor $\mathbf{T}: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{N}$ is exact.*
- (2) *We have $\mathbf{T}(N) = 0$ in \mathcal{A}/\mathcal{N} if and only if $N \in \mathcal{N}$.*
- (3) *For any short exact sequence S in \mathcal{A}/\mathcal{N} there is a short exact sequence S' in \mathcal{A} such that $\mathbf{T}(S')$ is isomorphic to S .*

17. Subquotients for the category \mathcal{O}

Let $J' \subset J \subset X$ be open subsets. Then $\mathcal{O}^{J'}$ is a Serre subcategory of \mathcal{O}^J , and we are interested in the quotient $\mathcal{O}^J/\mathcal{O}^{J'}$. Here we are in a particularly nice situation: let M and N be objects of \mathcal{O}^J . Then there is a *minimal* submodule M_- of M with the property that M/M_- is contained in $\mathcal{O}^{J'}$. It is the submodule $M_{X \setminus J'}$ defined earlier, i.e., the submodule of M generated by all weight spaces M_λ with $\lambda \notin J'$. There is also a *maximal* subobject N_+ of N that is contained in $\mathcal{O}^{J'}$, it is simply the union of all submodules of N contained in $\mathcal{O}^{J'}$. So the limit in the definition of the Hom-spaces in $\mathcal{O}^J/\mathcal{O}^{J'}$ stabilizes and we obtain

$$(*) \quad \text{Hom}_{\mathcal{O}^J/\mathcal{O}^{J'}}(\mathbf{T}M, \mathbf{T}N) = \text{Hom}_{\mathcal{O}^J}(M_-, N/N_+).$$

Proposition 24. *Suppose that J is quasi-bounded. Let λ be an element in $J \setminus J'$.*

- (1) *The image of the projective cover $P^J(\lambda) \rightarrow L(\lambda)$ of $L(\lambda)$ under the functor \mathbf{T} is a projective cover of $\mathbf{T}L(\lambda)$ in $\mathcal{O}^J/\mathcal{O}^{J'}$.*
- (2) *Let N be an object in \mathcal{O}^J . Then the induced homomorphism*

$$\text{Hom}_{\mathcal{O}^J}(P^J(\lambda), N) \rightarrow \text{Hom}_{\mathcal{O}^J/\mathcal{O}^{J'}}(\mathbf{T}P^J(\lambda), \mathbf{T}N)$$

is an isomorphism.

- (3) *The category $\mathcal{O}^J/\mathcal{O}^{J'}$ has enough projectives.*

Proof. First we prove statement (2). We adopt the notation from the paragraph before the statement of this proposition. As λ is not contained in J' and N_+ is an object in $\mathcal{O}^{J'}$, we have $(N_+)_{\lambda} = 0$. Let $f: P^J(\lambda) \rightarrow N_+$ be a homomorphism. Then the weight space $P^J(\lambda)_{\lambda}$ must be contained in the kernel of f . Let $P^J(\lambda) \rightarrow L(\lambda)$ be a projective cover. Then the composition $\ker f \subset P^J(\lambda) \rightarrow L(\lambda)$ is surjective on the λ -weight space, hence is surjective. As $P^J(\lambda) \rightarrow L(\lambda)$ is essential, we deduce $\ker f = P^J(\lambda)$, so $f = 0$, i.e., there is no nontrivial homomorphism from $P^J(\lambda)$ to N_+ . From this and the projectivity of $P^J(\lambda)$ in \mathcal{O}^J we deduce that the canonical homomorphism $\text{Hom}_{\mathcal{O}^J}(P^J(\lambda), N) \rightarrow \text{Hom}_{\mathcal{O}^J}(P^J(\lambda), N/N_+)$ is an isomorphism. Now any submodule A of $P^J(\lambda)$ with the property that $P^J(\lambda)/A$ is contained in $\mathcal{O}^{J'}$ must contain the λ -weight space of $P^J(\lambda)$. As $P^J(\lambda)$ is generated by this weight space, we have $P^J(\lambda)_{-} = P^J(\lambda)$. Equation (*) now yields (2).

As all short exact sequences in $\mathcal{O}^J/\mathcal{O}^{J'}$ can be obtained from short exact sequences in \mathcal{O}^J by applying the quotient functor \mathbf{T} , we deduce that $\mathbf{TP}^J(\lambda)$ is projective in $\mathcal{O}^J/\mathcal{O}^{J'}$ from (2). As \mathbf{T} is exact, the morphism $\mathbf{TP}^J(\lambda) \rightarrow \mathbf{TL}(\lambda)$ is an epimorphism. Now $\text{End}_{\mathcal{O}^J/\mathcal{O}^{J'}}(\mathbf{TP}^J(\lambda)) \cong \text{End}_{\mathcal{O}^J}(P^J(\lambda))$ by (2), so this is a local ring, so $\mathbf{TP}^J(\lambda)$ is indecomposable. The fact that $\mathbf{TP}^J(\lambda) \rightarrow \mathbf{TL}(\lambda)$ is a projective cover now can be proven using the arguments in the last paragraph of the proof of Theorem 17.

Statement (3) follows from (1), since the objects $\{\mathbf{TL}(\lambda)\}_{\lambda \in J \setminus J'}$ represent the simple isomorphism classes in $\mathcal{O}^J/\mathcal{O}^{J'}$. \square

18. Subquotients and locally closed subsets

Again we suppose that J is quasi-bounded. Set $K := J \setminus J'$ and consider the composition of functors

$$\mathbf{V}: \mathcal{V}(K) \subset \mathcal{O}^J \xrightarrow{\mathbf{T}} \mathcal{O}^J/\mathcal{O}^{J'}$$

Note that the projectives $P^J(\nu)$ with $\nu \in K$ are objects in $\mathcal{V}(K)$, and that

$$\text{Hom}_{\mathcal{V}(K)}(P^J(\lambda), P^J(\mu)) = \text{Hom}_{\mathcal{O}^J/\mathcal{O}^{J'}}(\mathbf{TP}^J(\lambda), \mathbf{TP}^J(\mu))$$

for all $\lambda, \mu \in K$ by Proposition 24. As the objects $\{\mathbf{TL}(\lambda)\}_{\lambda \in K}$ represent the simple isomorphism classes in $\mathcal{O}^J/\mathcal{O}^{J'}$, and as the morphisms $\mathbf{T}(P^J(\lambda) \rightarrow L(\lambda))$ are projective covers, we deduce that $\mathcal{O}^J/\mathcal{O}^{J'}$ is the abelianization of the exact category $\mathcal{V}(K)$. In particular, this quotient category does not depend on the choice of J and J' , but only on the locally closed subset K . So henceforth we write $\mathcal{O}_{[K]}$ for this category.

19. The shift functor on subquotients

We return to the setup in Section 16: we fix open subsets $J' \subset J$ of X and suppose that J is quasi-bounded, we choose a dominant weight $\gamma \in X^+$ and set $\tilde{J} := J + \gamma$ and $\tilde{J}' := J' + \gamma$, $K := J \setminus J'$ and $\tilde{K} = \tilde{J} \setminus \tilde{J}' = K + \gamma$. We again consider the functor

$$(\cdot) \otimes L(\gamma): \mathcal{O} \rightarrow \mathcal{O}.$$

It maps the category \mathcal{O}^J , resp. $\mathcal{O}^{J'}$, into the category $\mathcal{O}^{\tilde{J}}$, resp. $\mathcal{O}^{\tilde{J}'}$ and hence induces a functor

$$\mathbf{Z}: \mathcal{O}_{[K]} \rightarrow \mathcal{O}_{[\tilde{K}]}.$$

Theorem 25. *Suppose that there exists some $l > 0$ such that the triple (J, J', γ) satisfies the following:*

- $\gamma \in p^l X$,
- for all $\nu > 0$ we have $K \cap (K - p^l \nu) = \emptyset$.

Then the functor \mathbf{Z} is an equivalence of categories.

Proof. Consider the following composition of functors:

$$\begin{aligned} \mathbf{V}: \mathcal{V}(K) &\rightarrow \mathcal{O}^J \rightarrow \mathcal{O}^J/\mathcal{O}^{J'} = \mathcal{O}_{[K]}, \\ \tilde{\mathbf{V}}: \mathcal{V}(\tilde{K}) &\rightarrow \mathcal{O}^{\tilde{J}} \rightarrow \mathcal{O}^{\tilde{J}}/\mathcal{O}^{\tilde{J}'} = \mathcal{O}_{[\tilde{K}]}. \end{aligned}$$

Let \tilde{I}' be the complement of \tilde{J}' in X . For all objects M in $\mathcal{O}^{\tilde{J}}$ the inclusion $M_{\tilde{I}'} \subset M$ has a quotient in $\mathcal{O}^{\tilde{J}'}$, so this inclusion is an isomorphism in $\mathcal{O}^{\tilde{J}}/\mathcal{O}^{\tilde{J}'}$. Hence the square on the right of the diagram

$$\begin{array}{ccccc} \mathcal{V}(\tilde{K}) & \xrightarrow{\tilde{\mathbf{V}}} & \mathcal{O}^{\tilde{J}}/\mathcal{O}^{\tilde{J}'} & \xrightarrow{\text{id}} & \mathcal{O}^{\tilde{J}}/\mathcal{O}^{\tilde{J}'} \\ \uparrow \mathbf{U} = ((\cdot) \otimes L(\gamma))_{\tilde{I}'} & & \uparrow ((\cdot) \otimes L(\gamma))_{\tilde{I}'} & & \uparrow \mathbf{Z} = (\cdot) \otimes L(\gamma) \\ \mathcal{V}(K) & \xrightarrow{\mathbf{V}} & \mathcal{O}^J/\mathcal{O}^{J'} & \xrightarrow{\text{id}} & \mathcal{O}^J/\mathcal{O}^{J'} \end{array}$$

commutes, making the whole diagram commutative. Now both categories $\mathcal{O}^{\tilde{J}}/\mathcal{O}^{\tilde{J}'}$ and $\mathcal{O}^J/\mathcal{O}^{J'}$ are abelian and have enough projectives. In order to show that \mathbf{Z} is an equivalence, it suffices to show that is essentially surjective on projectives and fully faithful on projectives. Using Proposition 24 and the above commutative diagram, it suffices to show this statement for the functor $\mathbf{U}: \mathcal{V}(K) \rightarrow \mathcal{V}(\tilde{K})$. But \mathbf{U} is an equivalence by Theorem 22. \square

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