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ADJUNCTION FOR VARIETIES WITH A $\mathbb{C}^*$ ACTION

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Abstract. Let X be a complex projective manifold, L an ample line bundle on X, and assume that we have a $\mathbb{C}^*$ action on (X, L). We classify such triples $(X, L, \mathbb{C}^*)$ for which the closure of a general orbit of the $\mathbb{C}^*$ action is of degree ≤ 3 with respect to L and, in addition, the source and the sink of the action are isolated fixed points, and the $\mathbb{C}^*$ action on the normal bundle of every fixed point component has weights ± 1. We treat this situation by relating it to the classical adjunction theory. As an application, we prove that contact Fano manifolds of dimension 11 and 13 are homogeneous if their group of automorphisms is reductive of rank ≥ 2.

Contents

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1. Introduction

1.1. A view on manifolds with a $\mathbb{C}^*$ action

Let us recall that Amplitude Modulation (AM) and Frequency Modulation (FM) are two different technologies of broadcasting radio signals. AM works by modulating the amplitude of the signal with constant frequency. In FM technology the information is encoded by varying the frequency of the wave with amplitude being constant. In the present paper we adopt the idea of passing the information via either AM or FM technology to deal with varieties with a $\mathbb{C}^*$ action.

Given a complex projective variety X with an ample line bundle L and an action of $\mathbb{C}^*$ on (X, L), we can study this setup in two ways: (1) by examining the amplitude of L on curves on X (AM technology) and (2) by understanding the weights of a linearization of the action of $\mathbb{C}^*$ on L over the connected components of the fixed point set of this action (FM technology).

The structure of X with a $\mathbb{C}^*$ action can be encoded in a graph whose vertices are components of the fixed point locus, and the edges are orbits whose closures meet the respective components. Given a linearization μ_L of the line bundle L, to each component of the fixed point locus one can associate the weight in $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*) = \mathbb{Z}$ with which $\mathbb{C}^*$ acts on fibers of L over the component in question. Now, the radio analogy goes as follows: one can relate the values of μ_L (frequencies of L) on the components of the fixed point set of the $\mathbb{C}^*$ action with the degree (the volume) of L on the closures of orbits joining the respective fixed point set components.

Namely, given a $\mathbb{C}^*$ equivariant morphism $f : \mathbb{P}^1 \to X$ we get the following identity (see Lemma 3.1):

$$\delta \cdot \deg f^*L = \mu_L(f(0)) - \mu_L(f(\infty)) \qquad \text{(AM}\leftrightarrow\text{FM)}$$

where $0, \infty$ are the fixed points of the action of $\mathbb{C}^*$ on $\mathbb{P}^1$, and $\delta = \delta(T_0\mathbb{P}^1)$ is the weight of the $\mathbb{C}^*$ action on the tangent of $\mathbb{P}^1$ at 0. Thus, the left-hand side of the above equality measures the amplitude of the line bundle L, while the right-hand side measures the difference of the weights of the $\mathbb{C}^*$ action on the fibers of f^*L

over the fixed points. In view of the (AM↔FM) equality we define the *bandwidth* of a pair (X, L) as the degree of the closure of a general orbit of the $\mathbb{C}^*$ action with respect to L, and we are interested in classifying some pairs (X, L) admitting a $\mathbb{C}^*$ action of small bandwidth.

In [11], [16] Ionescu and Fujita proved classification results for polarized pairs (X, L) by looking at the *nef value* $\tau = \tau(X, L) := \min\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}$ (see Theorem 2.1). In this paper, assuming that we have a nontrivial $\mathbb{C}^*$ action on (X, L), we will make use of the (AM↔FM) equality to study the positivity of the divisor $K_X + tL$, so that we are able to compute the nef value of (X, L) or find an estimate of it. Combining this information with classical results from adjunction theory, we obtain a first classification result for bandwidth one and two varieties (see Theorem 4.1). As a main application of our approach we study pairs (X, L) of bandwidth three which emerged naturally in the context of the LeBrun–Salamon conjecture (see Theorem 4.5). To this end, the technique consists again in relating new methods and properties due to the torus actions arising from Białynicki-Birula decomposition (cf. Theorem 2.3) with the more classical adjunction machinery.

1.2. Motivation and contents of the paper

The celebrated LeBrun–Salamon conjecture in Riemannian geometry asserts that the only positive quaternion-Kähler manifolds are Wolf spaces. Its algebro-geometric counterpart asserts that the closed orbits in projectivizations of adjoint representations of simple algebraic groups are the only Fano contact manifolds. Recently, in [5] the combinatorics of torus action have been used to prove the conjecture in low dimensions. In the present paper we use the techniques of a $\mathbb{C}^*$ action on pairs (X, L) as above, to prove the following extension of previous results; see also Theorem 6.2 for a more detailed formulation.

Theorem. *Let X_σ be a Fano contact manifold of dimension ≤ 13 and $\operatorname{Pic} X_\sigma = \mathbb{Z}L_\sigma$. If the group of contact automorphisms G is reductive of rank ≥ 2 then X_σ is the closed orbit in the projectivization of the adjoint representation of a simple algebraic group.*

It is known that the contact manifold coming from a quaternion-Kähler manifold admits Kähler–Einstein metric, so that when dealing with the LeBrun–Salamon conjecture the varieties in question have the group of the contact automorphisms reductive, hence this assumption on G is not restrictive (see [32]). Moreover, earlier results were for contact Fano manifolds X with $\dim X \leq 9$ and without lower bound on the rank of the group of its automorphisms. Notice that, being the dimension of the Lie algebra of G equal to $h^0(X, L)$ (see for instance [5, Lem. 4.5]), then the assumption on the rank is true if, e.g., $h^0(X, L) > 3$. We refer to Section 6, where after recalling past and recent results in the context of the LeBrun–Salamon conjecture, we apply new methods from adjunction theory for varieties with a $\mathbb{C}^*$ action to solve the conjecture under the assumptions of the above theorem.

Indeed, following the strategy of [5], to deal with the LeBrun–Salamon conjecture we need to classify polarized pairs (X, L) of small bandwidth. As will be explained in Subsection 6.3, such pairs (X, L) appear in our analysis as subvarieties of the initial Fano contact manifold, and we need to study them to collect all the combinatorial data of the action as a crucial step to show the above theorem. To this end, we use tools and new methods developed in the previous sections, concerning adjunction theory for varieties admitting a $\mathbb{C}^*$ action. In this framework, the main technical result of the paper is Theorem 4.5 describing polarized pairs (X, L) with an action of $\mathbb{C}^*$ of bandwidth three which satisfies some technical assumptions that are natural for the application to contact manifolds. Denoting by n the dimension of X with $n \geq 2$, the result is the following list of possibilities:

(1) $(X, L) = (\mathbb{P}(\mathcal{V}), \mathcal{O}(1))$ is a scroll over $\mathbb{P}^1$, where $\mathcal{V}$ is either $\mathcal{O}(1)^{n-1} \oplus \mathcal{O}(3)$ or $\mathcal{O}(1)^{n-2} \oplus \mathcal{O}(2)^2$, or
(2) (X, L) is a quadric bundle $(\mathbb{P}^1 \times \mathcal{Q}^{n-1}, \mathcal{O}(1,1))$, or
(3) $n \geq 6$ is divisible by 3 and X is Fano, $\rho_X = 1$, $-K_X = (2/3)nL$.

In order to obtain the above classification, in Section 3 we relate the classical adjunction theory (see [2], [12], [16]) and Mori theory (see [20], [24]) with a combinatorial description of a manifold with a $\mathbb{C}^*$ action. In fact, types (1) and (2) of pairs (X, L) in the above list are described in terms of their adjunction morphism. Type (2) in the above list leads to contact manifolds which are homogeneous with respect to SO groups, as described in the Appendix of the present paper. Type (3) in the same list has been recently classified in [28] by using different methods from birational and projective geometry. In total, there are four of these varieties, all of them rational homogeneous; we refer to [28, Thm. 6.8] for their complete list. In the recent preprint [29] the varieties of type (3) are related to contact manifolds homogeneous with respect to the four exceptional simple groups of F_4, E_6, E_7 and E_8 type. Dealing also with such cases in which G is of exceptional type, in [28, Thm. 6.1] the LeBrun–Salamon conjecture has been proved in arbitrary dimension, under certain assumptions on the rank of the maximal torus.

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1.3. Notation

The following notation is used throughout the article.

- X is a complex projective normal variety of dimension n. For the largest part of the paper we assume X smooth with an ample line bundle L, so that (X, L) is a *polarized pair*.
- Given a polarized pair (X, L) we denote by $\tau = \tau(X, L)$ the *nef value*, namely $\tau(X, L) := \min\{t \in \mathbb{R} : K_X + tL \text{ is nef}\}$, moreover

$$\phi_\tau := \phi_{K_X + \tau L} \colon X \to X'$$

  is the *adjunction* (or *adjoint*) *morphism.*
- We denote by $H = (\mathbb{C}^*)^r$ an algebraic torus of arbitrary rank r, acting on X. Moreover, we denote by $M = \mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{C}^*) \cong \mathbb{Z}^r$ the set of characters (or weights) of H.

- $X^H = \bigsqcup_{i\in I} Y_i$ is the fixed locus of the H action, where I is a set indexing its connected components; by $\mathcal{Y} = \{Y_i\}$ we denote the set of the irreducible fixed point components of X^H.
- For an arbitrary line bundle $\mathcal{L} \in \operatorname{Pic} X$ we denote by $\mu_{\mathcal{L}} \colon H \times \mathcal{L} \to \mathcal{L}$ (or simply by μ) a linearization of the action of H on $\mathcal{L}$. By abuse, we continue to denote by $\mu_{\mathcal{L}} \colon \mathcal{Y} \to M \cong \mathbb{Z}^r$ the associated map on the set of fixed point components, which we call the *fixed point weight map*, see Definition 3.
- Given a $\mathbb{C}^*$ action on X, and a nef line bundle $\mathcal{L} \in \operatorname{Pic} X$ admitting a linearization $\mu = \mu_{\mathcal{L}}$, the *bandwidth* of the triple $(X, \mathcal{L}, \mathbb{C}^*)$ is defined as $|\mu| = \mu_{\max} - \mu_{\min}$ where $\mu_{\max}$ and $\mu_{\min}$ denote the maximal and minimal value of the function $\mu_{\mathcal{L}}$, see Definition 4.

2. Preliminaries

In the present section we recall basic definitions and properties of adjunction and Mori theory as well as regarding varieties with a $\mathbb{C}^*$ action. We refer the reader to [24] for a detailed exposition on Mori theory, and to [2], [12], [16] for an account on adjunction theory. We work over the field of complex numbers, with projective, irreducible, reduced varieties.

2.1. Adjunction and Mori theory

Let X be a normal projective variety of arbitrary dimension n. Let us denote by $\mathbf{N}^1(X)$ (respectively $\mathbf{N}_1(X)$) the $\mathbb{R}$-spaces of Cartier divisors (respectively, 1-cycles on X), modulo numerical equivalence. We denote by $\rho_X := \dim \mathbf{N}_1(X) = \dim \mathbf{N}^1(X)$ the *Picard number* of X, and by $[\cdot]$ the numerical equivalence class in $\mathbf{N}_1(X)$, and in $\mathbf{N}^1(X)$. The intersection of divisors and curves determines a nondegenerate bilinear pairing of these two $\mathbb{R}$-spaces. We consider cones $\mathcal{C}(X) \subset \mathbf{N}_1(X)$ and $\mathcal{A}(X) \subset \mathbf{N}^1(X)$ spanned by classes of effective curves and classes of ample divisors, respectively. Their closures (in the standard topology on $\mathbb{R}$-spaces) are dual in terms of the intersection product.

A *contraction* of X is a surjective morphism with connected fibers $\phi\colon X \to Y$ onto a normal projective variety. Any contraction yields a surjective linear map $\phi_* \colon \mathbf{N}_1(X) \to \mathbf{N}_1(Y)$ given by the push-forward of 1-cycles, and the pull-back of Cartier divisors $\phi^* \colon \mathbf{N}^1(Y) \to \mathbf{N}^1(X)$ such that $\phi^*([D]) = [\phi^*(D)]$.

The case of our main interest is the following situation.

Assumption 1. Let (X, L) be a polarized manifold, namely X is a smooth projective variety of dimension n and L is an ample line bundle on it. In addition we assume that the variety X admits a nontrivial $\mathbb{C}^*$ action, that is $\mathbb{C}^* \times X \to X$, with a linearization $\mu\colon \mathbb{C}^* \times L \to L$.[1]

For the polarized pair (X, L) we define its *nef value* as follows:

$$\tau = \tau(X, L) := \min\{t \in \mathbb{R} : K_X + tL \ \text{ is nef}\}.$$

We note that if X admits a $\mathbb{C}^*$ action then it is uniruled, hence K_X is not nef, so that $\tau > 0$. Thus, by the Kawamata rationality theorem (see [20, Thm. 4.1.1]) one

[1] Note that in Section 6 we consider the case when the variety is a contact manifold X_σ of dimension $2n+1$ with an action of a torus $\widehat{H}$ of rank ≥ 2.

has $\tau \in \mathbb{Q}$. Moreover, the Kawamata–Shokurov Base Point free Theorem provides the adjunction morphism

$$\phi_\tau := \phi_{K_X+\tau L} : X \to X'$$

such that $K_X + \tau L = \phi_\tau^* L'$ for some $\mathbb{Q}$-Cartier ample divisor L' on X'. The variety X' is normal and ϕ_τ has connected fibers, namely ϕ_τ is a contraction of X. In fact

$$X' = \operatorname{Proj}\left(\bigoplus_{m\geq 0} \mathrm{H}^0(X, m(K_X + \tau L))\right) \tag{1}$$

where m is such that $m(K_X + \tau L)$ is Cartier.

The following result is due to Ionescu and Fujita (see [16] and also [12]), and will be crucial for proving the results in Sections 4 and 5.

Theorem 2.1. *Let (X, L) be a polarized pair. Then $\tau \leq n+1$ with equality only for the projective space, that is if $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$.*

(1) *Suppose that $n \geq 2$ and $\tau < n+1$. Then $\tau \leq n$ with equality only if*
   1. *either X is a smooth quadric, that is $(X, L) = (\mathcal{Q}^n, \mathcal{O}(1))$, or*
   2. *(X, L) is a $\mathbb{P}^{n-1}$-bundle over a smooth curve with L relative $\mathcal{O}(1)$.*

(2) *Suppose that $n \geq 3$ and $\tau < n$. Then $\tau \leq n-1$ with equality only if one of the following holds:*
   1. *(X, L) is a del Pezzo manifold, that is $-K_X = (n-1)L$; see* [11], [17] *for their complete classification.*
   2. *(X, L) is a quadric bundle over a smooth curve with L relative $\mathcal{O}(1)$.*
   3. *(X, L) is a $\mathbb{P}^{n-2}$-bundle over a smooth surface with L relative $\mathcal{O}(1)$.*
   4. *The adjoint morphism $\phi_{n-1}\colon X \to X'$ is a birational morphism contracting a finite number of disjoint divisors $E_i \cong \mathbb{P}^{n-1}$ to smooth points of X' and $L_{|E_i} \cong \mathcal{O}(1)$; there exists an ample line bundle L' over X' such that $\phi_{n-1}^* L' = K_X + (n-1)L$.*

The following observation follows easily by taking a rational curve $C \subset X$ which spans an extremal ray contained in the extremal face contracted by the adjoint morphism $\phi_\tau\colon X \to X'$, and using that $\tau = -(K_X \cdot C)/(L \cdot C) \leq (n+1)/(L \cdot C)$.

Remark 1. Let (X, L) be a polarized pair with $n \geq 3$. Assume that $\tau > n-2$. Then $\tau \geq n-1$, and $\tau \in \mathbb{Z}$ except for $(X, L) = (\mathbb{P}^4, \mathcal{O}(2))$, $(X, L) = (\mathbb{P}^3, \mathcal{O}(3))$, and $(X, L) = (\mathcal{Q}^3, \mathcal{O}(2))$.

2.2. Varieties with a $\mathbb{C}^*$ action

Let us consider an effective (i.e. nontrivial) action of an algebraic torus $H = (\mathbb{C}^*)^r$ on a smooth projective variety X, that is $H \times X \ni (t, x) \to t \cdot x \in X$. Given a subtorus $H' \subseteq H$ we can consider the resulting action $H' \times X \to X$; this operation will be called *downgrading* the action of H to H' (see [5, §2.2] for further details). The action is called *almost faithful* if the resulting homomorphism $H \to \mathrm{Aut}(X)$ has a finite kernel.

Except for Section 6, we will be primarily interested in the case $r = 1$.

We consider the fixed locus of the action X^H and its decomposition into connected components:

$$X^H = \bigsqcup_{i\in I} Y_i$$

where I is a set of indices, and each component Y_i is a smooth subvariety (see, e.g., the main theorem in [18]). By $\mathcal{Y} = \{Y_i : i \in I\}$ we denote the set of the irreducible fixed point components of X^H.

Remark 2. We stress that if X is smooth then the connected components of X^H are smooth, hence irreducible. If X is not smooth then the connected components of X^H may not be irreducible as the following example shows (thanks to Joachim Jelisiejew): consider the quadric cone $\mathcal{Q} = \{z_1z_2 + z_2z_4 = 0\}$ in the projective space with coordinates $[z_0, z_1, \ldots, z_4]$ and a $\mathbb{C}^*$ action with weights $(0, 0, 0, 1, -1)$. Then the fixed point set consists of two isolated points $[0, 0, 0, 1, 0]$, $[0, 0, 0, 0, 1]$, and the reducible conic $\mathcal{Q} \cap \{z_3 = z_4 = 0\}$.

We have the following standard observation.

Lemma 2.2. *Let X be a variety with an effective $\mathbb{C}^*$ action. Then the cone of curves $\mathcal{C}(X)$ is generated by classes of closures of orbits and by classes of curves contained in the fixed locus of the action.*

Proof. The result follows by applying standard Mori breaking technique using the $\mathbb{C}^*$ action, see, e.g., [36, p. 253] for details. Let us take an arbitrary irreducible curve $C \subset X$ with normalization $f : \widehat{C} \to C \subset X$. We consider the morphism $F : \mathbb{C}^* \times \widehat{C} \to X$ defined by setting

$$\mathbb{C}^* \times \widehat{C} \ni (t, p) \mapsto F(t, p) = t \cdot f(p) \in X.$$

We extend the morphism F to a rational map $\mathbb{C} \times \widehat{C} \dashrightarrow X$ which we resolve to a regular morphism $\widehat{F} : \widehat{S} \to X$ blowing up the product over $0 = \mathbb{C} \setminus \mathbb{C}^*$. The image (as a 1-cycle) under $\widehat{F}$ of the fiber of $\widehat{S} \to \mathbb{C} \times \widehat{C}$ over 0 is the sum of curves which are stable under the $\mathbb{C}^*$ action and it is numerically equivalent to C. $\square$

For every $Y \in \mathcal{Y}$ the torus H acts on $TX_{|Y}$ so that we get the decomposition $TX_{|Y} = T^+ \oplus T^0 \oplus T^-$, where T^+, T^0, T^- are respectively the subbundles of $TX_{|Y}$ on which H acts with positive, zero or negative weights. Then, by local linearization, $T^0 = TY$ and

$$T^+ \oplus T^- = \mathcal{N}_{Y/X} = \mathcal{N}^+(Y) \oplus \mathcal{N}^-(Y)$$

is the decomposition of the normal bundle $\mathcal{N}_{Y/X}$ into the part on which H acts with positive, respectively, negative weights.

Definition 1. Setting as above. We say that the $\mathbb{C}^*$ action on X is *equalized* if for every component $Y \in \mathcal{Y}$ the torus acts on $\mathcal{N}^+(Y)$ with all the weights equal to $+1$ and on $\mathcal{N}^-(Y)$ with all the weights equal to -1.

It is a basic fact (see [33]) that for $x \in X$ the action $\mathbb{C}^* \times \{x\} \to X$ extends to a holomorphic map $\mathbb{P}^1 \times \{x\} \to X$, hence there exist $\lim_{t\to 0} t \cdot x$, and $\lim_{t\to\infty} t \cdot x$. Moreover, since the orbits are locally closed, and the closure of an orbit is an invariant subset, then both the limit points of an orbit lie in $\mathcal{Y}$. We will call these limits the *source* and the *sink* of the orbit of x, respectively.

For every $Y \in \mathcal{Y}$ we can define the Białynicki-Birula cells in the following way:

$$X^+(Y) = \{x \in X : \lim_{t\to 0} t \cdot x \in Y\} \text{ and } X^-(Y) = \{x \in X : \lim_{t\to \infty} t \cdot x \in Y\}.$$

The following result is due to Białynicki-Birula and known as BB decomposition. We use this argument as presented in [6]. See [3] for the original exposition. A vast generalization of this result, which is also valid for singular varieties, can be found in a recent paper [19] and references therein.

Theorem 2.3. *In the situation described above the following hold:*

- $X_i^\pm$ *are locally closed subsets and there are two decompositions*

$$X = \bigsqcup_{i\in I} X^+(Y_i) = \bigsqcup_{i\in I} X^-(Y_i)$$

  which we call X^+ *or* X^- *BB decomposition, respectively.*
- *For every* $Y \in \mathcal{Y}$ *there are* $\mathbb{C}^*$*-isomorphisms* $X^+(Y) \cong \mathcal{N}^+(Y)$ *and* $X^-(Y) \cong \mathcal{N}^-(Y)$ *lifting the natural maps* $X^\pm(Y) \to Y$*. Moreover, the map* $X^\pm(Y) \to Y$ *is algebraic and is a* $\mathbb{C}^{\mathrm{rk}^\pm(Y)}$ *fibration, where we set* $\mathrm{rk}^\pm(Y) := \mathrm{rank}\,\mathcal{N}^\pm(Y)$.
- *There is a decomposition in homology*

$$H_m(X, \mathbb{Z}) = \bigoplus_{i\in I} H_{m-2\,\mathrm{rk}^+(Y_i)}(Y_i, \mathbb{Z}) = \bigoplus_{i\in I} H_{m-2\,\mathrm{rk}^-(Y_i)}(Y_i, \mathbb{Z}).$$

The unique Y such that $X^+(Y)$ is dense in X is called the *source* of the action. The unique Y such that $X^-(Y)$ is dense in X is called the *sink*.

We have a partial order on $\mathcal{Y}$ in the following way:

$$Y_i \prec Y_j \Leftrightarrow \exists\, x \in X : \lim_{t\to 0} t \cdot x \in Y_i \text{ and } \lim_{t\to\infty} t \cdot x \in Y_j \tag{2}$$

Definition 2. An effective $\mathbb{C}^*$ action on a smooth variety X is said to have *one pointed end* if its source or sink is a single point. The action is said to have *two pointed ends* if both the source and the sink are isolated points.

We note that replacing t with t^{-1} we change the action to the opposite and X^+ decomposition into X^- decomposition. When we refer to a one pointed end action we are assuming that the source is given by an isolated point.

Using BB decomposition we can describe the Picard group of our varieties in terms of the source of the action.

Proposition 2.4. *Let us keep the same notation of Theorem* 2.3*. Suppose that a* $\mathbb{C}^*$ *action on a smooth variety* X *has one pointed end with source* y_0*. Then* X *is rational, and* $\mathrm{Pic}\,X$ *is finitely generated with no torsion. Moreover, the divisors* $D_i^+ = \overline{X^+(Y_i)}$ *for* Y_i *such that* $\mathrm{rk}^-(Y_i) = 1$*, are irreducible and their classes make the basis of* $\mathrm{Pic}\,X$*.*

Proof. Applying Theorem 2.3, we get $H_2(X, \mathbb{Z}) = \bigoplus_{i\in I} H_{2-2\,\mathrm{rk}^-(Y_i)}(Y_i, \mathbb{Z})$. Being y_0 an isolated point with $\mathrm{rk}^-(y_0) = 0$, then the only fixed components Y_i which contribute to the homology are those having $\mathrm{rk}^-(Y_i) = 1$. Therefore $H_2(X, \mathbb{Z}) \cong \mathbb{Z}^\rho$ with $\rho \in \mathbb{Z}_{\geq 0}$. By Theorem 2.3, one has $X^+(y_0) \cong \mathbb{C}^n$, hence X is rational. In particular, X being simply connected, one has

$$\mathrm{Pic}\,X \cong H^2(X, \mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_2(X, \mathbb{Z}), \mathbb{Z}) \cong \mathbb{Z}^\rho$$

and our claim follows. $\square$

2.3. Linearization

Let $p\colon \mathcal{L} \to X$ be a line bundle over a normal projective variety with an action of an algebraic torus $H = (\mathbb{C}^*)^r$. We recall that a linearization μ of $\mathcal{L}$ is an H equivariant action on $\mathcal{L}$ which is linear on the fibers of p, that is for every $t \in H$ and $x \in X$ the restriction $\mu\colon \mathcal{L}_x \to \mathcal{L}_{t\cdot x}$ is linear. In this case we say that $(\mathcal{L}, \mu)$ is an H linearized line bundle on X. See [27, §1.3], [4, §2.2] or [22, §2] for details on linearizations. From now on, we denote by $\mu_{\mathcal{L}}$ or simply by μ a chosen linearization of the line bundle $\mathcal{L}$.

By [22, Prop. 2.4] and the subsequent Remark in [22], we know that there exists a linearization of the action of an algebraic torus H on $\mathcal{L}$. Using [4, Lem. 3.2.4] we deduce that given two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ with linearizations $\mu_{\mathcal{L}_1}$ and $\mu_{\mathcal{L}_2}$, their product $\mathcal{L}_1 \otimes \mathcal{L}_2$ has a natural linearization $\mu_{\mathcal{L}_1\otimes\mathcal{L}_2} = \mu_{\mathcal{L}_1} + \mu_{\mathcal{L}_2}$, where for H linearized line bundles we will use the additive notation. Also the dual of any H linearized line bundle on X is H-linearized as well. Thus the isomorphism classes of H-linearized line bundles form an abelian group relative to the tensor product, which we denote by $\mathrm{Pic}^H(X)$. We have a short exact sequence:

$$0 \longrightarrow \mathrm{Hom}(H, \mathbb{C}^*) = M \xrightarrow{\gamma} \mathrm{Pic}^H(X) \xrightarrow{\varphi} \mathrm{Pic}(X) \longrightarrow 0 \tag{3}$$

where φ forgets the linearization and γ are linearizations of the trivial bundle. In particular, any two linearizations of a line bundle differ by a character. Moreover, TX and ΩX have natural linearizations, hence K_X too.

Remark 3. Given a line bundle $\mathcal{L} \to X$ with a linearization $\mu : H \times \mathcal{L} \to \mathcal{L}$ we get the action on $\mathrm{H}^0(X, \mathcal{L})$ such that

$$H \times \mathrm{H}^0(X, \mathcal{L}) \ni (t, \sigma) \to \big(x \mapsto (t\cdot\sigma)(x) := \mu(t, \sigma(t^{-1}\cdot x))\big) \in \mathrm{H}^0(X, \mathcal{L})$$

for every $\sigma \in \mathrm{H}^0(X, \mathcal{L})$, $t \in H$, and $x \in X$. When $\mathcal{L}$ is semiample we can consider the graded finitely generated $\mathbb{C}$-algebra $\mathcal{R} = \bigoplus_{m\geq 0} \mathrm{H}^0(X, m\mathcal{L})$. As we have already observed, each line bundle $m\mathcal{L}$ has an induced linearization, so then there is an induced H action on $\mathrm{H}^0(X, m\mathcal{L})$, hence on $\mathcal{R}$.

Proposition 2.5. *Let (X, L) be as in Assumption 1. Then the target of the adjunction morphism $\phi_\tau : X \to X'$ admits an action of $\mathbb{C}^*$ (possibly non effective) such that ϕ_τ is $\mathbb{C}^*$ equivariant.*

Proof. Taking the natural linearization for K_X it follows that $K_X + \tau L$ admits a linearization, and by Remark 3 we deduce that the torus acts on the variety given by (1). In fact, taking a sufficiently large multiple m of the divisor $K_X + \tau L$, we may assume that $m(K_X + \tau L)$ is the pullback of a very ample divisor on X', and the action on X' is induced by equivariant embedding into $\mathbb{P}(\mathrm{H}^0(X, m(K_X + \tau L)))$. □

The following construction was used in [5, §2.1]. Let X be a normal projective variety with an action of an algebraic torus H of rank r whose set of fixed point components is $\mathcal{Y}$. Let us consider a linearization $\mu_{\mathcal{L}}$ of a line bundle $p\colon \mathcal{L} \to X$, and $Y \in \mathcal{Y}$. Given $y \in Y$ we associate $\mu_{\mathcal{L}}(y) \in M = \mathrm{Hom}_{\mathrm{alg}}(H, \mathbb{C}^*) \cong \mathbb{Z}^r$ which is the weight of the action of H on $p^{-1}(y)$. If y_1, y_2 belong to the same connected component Y, then $\mu_{\mathcal{L}}(y_1) = \mu_{\mathcal{L}}(y_2)$, and we will denote this weight by $\mu_{\mathcal{L}}(Y)$. In this way we get a homomorphism of abelian groups $\operatorname{Pic}^H(X) \to M^{\mathcal{Y}}$, with $M^{\mathcal{Y}}$ denoting the additive group of functions $\mathcal{Y} \to M$, which to linearized line bundle $(\mathcal{L}, \mu_{\mathcal{L}})$ associates the function

$$\mathcal{Y} \ni Y \mapsto \mu_{\mathcal{L}}(Y) \in M$$

Definition 3. The above constructed function, which by abuse we continue to denote by $\mu_{\mathcal{L}}$ (or simply by μ), will be called *fixed point weight map*

$$\mu_{\mathcal{L}} : \mathcal{Y} \to M = \operatorname{Hom}_{\mathrm{alg}}(H, \mathbb{C}^*) \cong \mathbb{Z}^r$$

Remark 4. Suppose that an algebraic torus H acts on X and it contains an algebraic subtorus $\iota : H' \to H$. Then the action of H induces via ι the action of H'. Given any line bundle $\mathcal{L}$ over X with H linearization $\mu_{\mathcal{L}}$, we have a unique induced linearization $\mu'_{\mathcal{L}}$ of the action of H'. Moreover, we have the inclusion of the fixed point locus $X^H \subset X^{H'}$, and hence the map of the fixed point components $\iota_\bullet : \mathcal{Y} \to \mathcal{Y}'$. Then for the associated fixed point weight maps one has

$$\mu'_{\mathcal{L}} \circ \iota_\bullet = \iota^* \circ \mu_{\mathcal{L}}$$

where $\iota^* : M \to M'$ is the homomorphism of lattices of characters of the respective tori.

In the case of a $\mathbb{C}^*$ action on X, we distinguish the sink Y_∞ of the action and say that the linearization is *normalized* if $\mu_{\mathcal{L}}(Y_\infty) = 0$. That is, a normalized line bundle $(\mathcal{L}, \mu_{\mathcal{L}})$ is in the kernel of the homomorphism

$$\operatorname{Pic}^{\mathbb{C}^*}(X) \ni (\mathcal{L}, \mu_{\mathcal{L}}) \mapsto \mu_{\mathcal{L}}(Y_\infty) \in \mathbb{Z}.$$

In other words, the choice of a normalized linearization splits the exact sequence (3).

Using the map $\mu_{\mathcal{L}}$ for a $\mathbb{C}^*$ action, we introduce the following new definition.

Definition 4. Let X be a normal projective variety admitting a $\mathbb{C}^*$ action. Suppose that $\mathcal{L}$ is a nef line bundle over X with the fixed point weight map $\mu_{\mathcal{L}} : \mathcal{Y} \to \mathbb{Z}$. We denote by $\mu_{\min}$ and $\mu_{\max}$ the minimal and maximal value of μ. The *bandwidth* $|\mu|$ of the triple $(X, \mathcal{L}, \mathbb{C}^*)$ is $|\mu| := \mu_{\max} - \mu_{\min}$. For short, we also say that X and $\mathcal{L}$ have bandwidth $|\mu|$.

3. Adjunction, Mori theory for varieties with a $\mathbb{C}^*$ action

In this section we describe the main ideas regarding adjunction theory for varieties with a $\mathbb{C}^*$ action.

3.1. AM vs FM

We begin with an easy example which we discuss in detail. Then we will prove AM vs FM equality in Lemma 3.1. We refer to [29, §2.3] for some consequences of this equality, and for its generalization to vector bundles.

Example 1. Let us consider the standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ which in homogeneous coordinates $[x_0, x_1]$ is defined as follows

$$\mathbb{C}^* \times \mathbb{P}^1 \ni (t, [x_0, x_1]) \to t \cdot [x_0, x_1] = [tx_0, x_1] \in \mathbb{P}^1$$

The two fixed points are $y_0 = [0, 1]$ and $y_\infty = [1, 0]$ with local coordinates x_0/x_1 and x_1/x_0 on which $\mathbb{C}^*$ acts with weights $+1$ and -1, respectively. If we write $t = x_0/x_1$ and $t^{-1} = x_1/x_0$, then the action extends the action of $\mathbb{C}^*$ on itself. Moreover, if $y \in \mathbb{P}^1 \setminus \{y_0, y_\infty\}$ then $\lim_{t\to 0} t \cdot y = y_0$ and $\lim_{t\to\infty} t \cdot y = y_\infty$. Thus y_0 is the *source* and y_∞ is the *sink* of the action.

We recall that the universal bundle $\mathcal{L} = \mathcal{O}(-1)$ is embedded into the trivial bundle $V \times \mathbb{P}^1$, where V is the vector space with coordinates (x_0, x_1) and $V \setminus \{0\} \to \mathbb{P}^1$ is the projection. The vector space has the obvious $\mathbb{C}^*$ action $t \cdot (x_0, x_1) = (tx_0, x_1)$, and the composition

$$\mathcal{L} \hookrightarrow V \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$$

is $\mathbb{C}^*$ equivariant. The fiber of $\mathcal{L} \to \mathbb{P}^1$ over $y_\infty = [1, 0]$ is a line with coordinate x_0, and over $y_0 = [0, 1]$ is a line with coordinate x_1. This yields a linearization $\mu_{\mathcal{L}}$ of $\mathcal{L}$ such that $\mu_{\mathcal{L}}(y_\infty) = 1$ and $\mu_{\mathcal{L}}(y_0) = 0$. Then, if we replace $\mathcal{L} = \mathcal{O}(-1)$ with $\mathcal{L}^\vee = \mathcal{O}(1)$ we get $\mu_{\mathcal{L}^\vee}(y_\infty) = -1$ and $\mu_{\mathcal{L}^\vee}(y_0) = 0$.

Lemma 3.1. *Let $\mathbb{C}^* \times \mathbb{P}^1 \to \mathbb{P}^1$ be an effective action with fixed points y_0 and y_∞, which are respectively the source and the sink of the action. Consider a line bundle $\mathcal{L}$ over $\mathbb{P}^1$ with linearization $\mu_{\mathcal{L}}$. Then*

$$\mu_{\mathcal{L}}(y_0) - \mu_{\mathcal{L}}(y_\infty) = \delta(y_0) \cdot \deg \mathcal{L}$$

where $\delta(y_0)$ denotes the weight of the $\mathbb{C}^*$ action on the tangent space $T_{y_0}\mathbb{P}^1$.

Proof. If $\mathcal{L} := \mathcal{O}(-1)$ and the action is standard, then the statement follows by Example 1. As observed in Subsection 2.3, a linearization of a line bundle implies a linearization of its multiples and of its dual. Similarly, a multiple of the standard action multiplies the weights, both δ and μ. Hence the claim follows. $\square$

Now let us apply the above observation to any manifold X with a $\mathbb{C}^*$ action. Given a nontrivial orbit $\mathbb{C}^* \cdot x \hookrightarrow X$ and its closure $C \subset X$ we can take either a normalization $f : \mathbb{P}^1 \to C \subset X$ or a *parametrization* $f_{\mathbb{C}^*} : \mathbb{P}^1 \to C \subset X$. The latter is defined by the formula $f_{\mathbb{C}^*}(t) = t \cdot x$ for $t \in \mathbb{C}^*$, so that the action of $\mathbb{C}^*$ on $\mathbb{P}^1$ is standard.

The morphism $f_{\mathbb{C}^*}$ factors through the normalization f:

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d, "\pi_\delta"'] \arrow[dr, "f_{\mathbb{C}^*}"] & \\ \mathbb{P}^1 \arrow[r, "f"] & C \subset X \end{tikzcd} .$$

That is $f_{\mathbb{C}^*} = \pi_\delta \circ f$ where π_δ is $\mathbb{C}^*$ equivariant cover $\mathbb{P}^1 \to \mathbb{P}^1$ of degree δ associated to the weight of the $\mathbb{C}^*$ action on the tangent space $T_{y_0}\mathbb{P}^1$. Equivalently, δ is the order of stabilizer of x in $\mathbb{C}^*$ acting on X.

Finally, if the action of $\mathbb{C}^*$ on X is equalized then, by the local description of the action around the fixed point components (see Theorem 2.3), we conclude that C is smooth and $f = f_{\mathbb{C}^*}$.

Having the above in mind we obtain the following:

Corollary 3.2. *Let X be a smooth variety with an effective $\mathbb{C}^*$ action, and let $f\colon \mathbb{P}^1 \to X$ be a non-constant $\mathbb{C}^*$ equivariant map. Let y_∞ and y_0 be respectively the sink and source of the action on $\mathbb{P}^1$. Take $\mathcal{L}$ a line bundle on X with linearization $\mu_{\mathcal{L}}$. Then the following hold:*

(a) $\deg f^*\mathcal{L}$ *has the same sign (or it is zero) as the difference*

$$\mu_{\mathcal{L}}(f(y_0)) - \mu_{\mathcal{L}}(f(y_\infty)).$$

(b) *If $\mathcal{L}$ is nef and the action of H is faithful, then the bandwidth of the triple $(X, \mathcal{L}, \mathbb{C}^*)$ is equal to the degree of $\mathcal{L}$ on the closure of a general orbit of $\mathbb{C}^*$.*

(c) *If the action of $\mathbb{C}^*$ is equalized and f is the normalization of the closure of a nontrivial orbit $C \subset X$, then* $\deg f^*\mathcal{L} = \mu_{\mathcal{L}}(f(y_0)) - \mu_{\mathcal{L}}(f(y_\infty))$.

Example 2. In what follows, we discuss an easy example of a non-equalized action which explains the assumptions in the preceding corollary. Let us consider an action of $\mathbb{C}^*$ on $\mathbb{P}^2$ with weights $(0,1,2)$, that is

$$\mathbb{C}^* \times \mathbb{P}^2 \ni (t, [z_0, z_1, z_2]) \to [z_0, tz_1, t^2 z_2] \in \mathbb{P}^2$$

with three fixed points $y_0 = [1,0,0]$, $y_1 = [0,1,0]$, $y_2 = [0,0,1]$, where y_0 is the source of this action, and y_2 is the sink. If $L = \mathcal{O}(1)$ then $\mu_L(y_i) = -i$ for $i = 0,1,2$, so that $(\mathbb{P}^2, L, \mathbb{C}^*)$ has bandwidth two. Lines $z_0 = 0$ through y_1 and y_2, and $z_2 = 0$ through y_0 and y_1 are closures of orbits with the standard action of $\mathbb{C}^*$. Take the line $z_1 = 0$ through y_0 and y_2. By Lemma 3.1, we deduce that this line is the closure of the orbit with the action of $\mathbb{C}^*$ of weight 2 (and thus the isotropy of rank two), so that Corollary 3.2(c) fails. A general orbit is a conic $z_0 z_2 = a \cdot z_1^2$ with $a \neq 0$.

3.2. Graph of the action, cone theorem

Let us start this subsection with a simple version of the localization theorem, see, e.g., [10], [31] for more details. We are interested in the description of $\operatorname{Pic} X$ in terms of normalized linearizations.

Proposition 3.3. *Let X be a projective manifold with an action of the torus $\mathbb{C}^*$. Take the decomposition of the fixed locus into irreducible components $X^{\mathbb{C}^*} = \bigsqcup_{i\in I} Y_i$ with Y_∞ denoting the sink component. Suppose that any effective curve on X is numerically equivalent to a sum of closures of orbits of $\mathbb{C}^*$. Consider a function $\Upsilon : \operatorname{Pic} X \to \bigoplus_{Y_i \neq Y_\infty} \mathbb{Z} \cdot Y_i$ such that*

$$\Upsilon(\mathcal{L}) = \sum_{Y_i \neq Y_\infty} \mu^\infty_{\mathcal{L}}(Y_i) \cdot Y_i$$

where $\mu^\infty_{\mathcal{L}}$ is the normalized linearization of $\mathcal{L}$, i.e., $\mu^\infty_{\mathcal{L}}(Y_\infty) = 0$. Then Υ is a homomorphism of groups with the kernel equal to numerically trivial line bundles.

Proof. In the discussion following Remark 4, we noted that normalized linearization splits the sequence (3). Therefore, Υ is the composition

$$\operatorname{Pic} X \to \operatorname{Pic}^{\mathbb{C}^*} X \to \mathbb{Z}^{\mathcal{Y}} \to \bigoplus_{Y_i \neq Y_\infty} \mathbb{Z} \cdot Y_i$$

where the arrow in the middle is the fixed point weight map, and the right arrow is the projection. Thus Υ is a homomorphism of groups. By Lemma 3.1, if $\mu^0_{\mathcal{L}}$ is zero then the degree of $\mathcal{L}$ on the closure of every orbit is zero, hence the claim. $\square$

Definition 5. Let X be a smooth projective variety with an effective $\mathbb{C}^*$ action. We define a *directed graph* $\mathcal{G} = \mathcal{G}(X, \mathbb{C}^*) := (\mathcal{Y}, \mathcal{E})$ with the set of vertices being the set of the fixed point components $\mathcal{Y} = \{Y_i\}$ of the $\mathbb{C}^*$ action, and the set of directed edges $\mathcal{E}$ defined as follows: $\epsilon(Y_{i_1}, Y_{i_2}) = \overrightarrow{Y_{i_1}Y_{i_2}} \in \mathcal{E}$ is a directed edge joining components $Y_{i_1}, Y_{i_2} \in \mathcal{Y}$ if and only if there exists a nontrivial orbit $\mathbb{C}^* \cdot x$ such that $\lim_{t\to 0} t \cdot x \in Y_{i_1}$ and $\lim_{t\to\infty} t \cdot x \in Y_{i_2}$. Note that this edge is directed from Y_{i_1} to Y_{i_2} and by (2) we have $Y_{i_1} \prec Y_{i_2}$. In this case, we say that the fixed point components Y_{i_1} and Y_{i_2} are joined by an orbit of the $\mathbb{C}^*$ action; Y_{i_1} precedes Y_{i_2}, and Y_{i_2} succeeds Y_{i_1} in the graph $\mathcal{G}$.

Example 3. This is an extension of Example 1. Let us consider an action of $\mathbb{C}^*$ on a vector space W of dimension $d = d_1 + \cdots + d_s$, with $d_j > 0$, given by weights $a_1 > \cdots > a_s$, and eigenspaces of dimensions $d_1, \ldots, d_s$. Namely, if $t \in \mathbb{C}^*$ then in some coordinates on W we have

$$\begin{aligned} t \cdot (z_1, \ldots, z_{d_1}, &z_{d_1+1}, \ldots, z_{d_1+d_2}, \ldots) \\ &= (t^{a_1} z_1, \ldots, t^{a_1} z_{d_1}, t^{a_2} z_{d_1+1}, \ldots, t^{a_2} z_{d_1+d_2}, \ldots). \end{aligned}$$

The $\mathbb{C}^*$ action descends to $\mathbb{P}^{d-1}$, the quotient of W via homotheties. The fixed locus of this action has s components $Y_1 \cong \mathbb{P}^{d_1-1}, \ldots, Y_s \cong \mathbb{P}^{d_s-1}$ associated to eigenspaces of weights $a_1, \ldots, a_s$ respectively. The action of $\mathbb{C}^*$ on the fiber of $W \setminus \{0\} \to \mathbb{P}^{d-1}$ over Y_i is of weight a_i. Thus, the induced linearization μ_L of the ample line bundle $L = \mathcal{O}(1)$ maps Y_i to $-a_i$. The graph $\mathcal{G}$ is a complete graph with vertices in $\mathcal{Y} = \{Y_1, \ldots, Y_s\}$ directed, so that we have

$$\epsilon(Y_{i_1}, Y_{i_2}) = \overrightarrow{Y_{i_1}Y_{i_2}} \in \mathcal{E} \;\Leftrightarrow\; a_{i_1} < a_{i_2} \;\Leftrightarrow\; \mu(Y_{i_1}) > \mu(Y_{i_2}). \tag{4}$$

We note that any polarized pair (X, L) with a $\mathbb{C}^*$ action can be embedded equivariantly into some projective space $\mathbb{P}^N$, so that mL is the restriction of $\mathcal{O}(1)$, for some $m \gg 0$. Accordingly, the graph $\mathcal{G}$ of fixed points and orbits for X is mapped to the graph of $\mathbb{P}^N$. Thus, in particular, the graph $\mathcal{G}$ has no directed cycles nor loops.

An edge $\epsilon(Y_{i_1}, Y_{i_2}) = \overrightarrow{Y_{i_1}Y_{i_2}} \in \mathcal{E}$ is called *minimal* if there is no sequence of length > 1 of directed edges joining Y_{i_1} to Y_{i_2}. The set of minimal edges for the graph $\mathcal{G} = (\mathcal{Y}, \mathcal{E})$ is denoted by $\mathcal{E}^0$.

In the situation of Proposition 3.3, we consider a vector space $\mathbb{R}^{|\mathcal{Y}|-1} = \bigoplus_{i\neq\infty} \mathbb{R}\cdot Y_i$ with the dual basis of functionals Y_i^*. We define functionals $\widehat{\epsilon}(Y_i, Y_\infty) = Y_i^*$ and $\widehat{\epsilon}(Y_{i_1}, Y_{i_2}) = Y_{i_1}^* - Y_{i_2}^*$ for $i_2 \neq \infty$. For a functional $\widehat{\epsilon}$, we denote by $\widehat{\epsilon}_{\geq 0}$ the halfspace on which the functional is non-negative.

The following is an effective version of the nef cone for varieties with a $\mathbb{C}^*$ action.

Theorem 3.4. *In the situation of Proposition* 3.3*, we assume that the cone of 1-cycles $\mathcal{C}(X)$ is generated by classes of closures of orbits of the $\mathbb{C}^*$ action. Let us* *consider the map* $\Upsilon_{\mathbb{R}} : \mathbf{N}^1(X) \to \bigoplus_{Y_i \neq Y_\infty} \mathbb{R} \cdot Y_i$ *which comes from the morphism defined in Proposition* 3.3. *Then*

$$\Upsilon_{\mathbb{R}}(\overline{\mathcal{A}}(X)) = \Upsilon_{\mathbb{R}}(\mathbf{N}^1(X)) \cap \left(\bigcap_{\epsilon(Y_{i_1}, Y_{i_2}) \in \mathcal{E}^0} \widehat{\epsilon}(Y_{i_1}, Y_{i_2})_{\geq 0} \right).$$

Proof. In view of Corollary 3.2, since $\mathcal{C}(X)$ is generated by classes of closures of orbits of $\mathbb{C}^*$, we need to prove that a line bundle $\mathcal{L} \in \operatorname{Pic} X$ is nef if and only if the fixed point weight map $\mu_{\mathcal{L}}^{\infty} : \mathcal{Y} \to \mathbb{Z}$ is non-increasing on the vertices of the directed graph $\mathcal{G}$. That is, the partial linear order given by the function $\mu_{\mathcal{L}}^{\infty}$ is opposite to the order $\prec$ coming from the directed graph $\mathcal{G}$. Given $\epsilon(Y_{i_1}, Y_{i_2}) \in \mathcal{E}$, then $\mu_{\mathcal{L}}^{\infty}(Y_{i_1}) \geq \mu_{\mathcal{L}}^{\infty}(Y_{i_2})$ if and only if $\widehat{\epsilon}(Y_{i_1}, Y_{i_2})(\mu_{\mathcal{L}}^{\infty}) \geq 0$. It is enough to check this inequality for the minimal edges $\epsilon(Y_{i_1}, Y_{i_2}) \in \mathcal{E}^0$ to conclude the proof. □

3.3. When orbits generate the cone of 1-cycles

In Proposition 3.3 and Theorem 3.4 we assume that the classes of closures of orbits generate $\mathbf{N}_1(X)$ and $\mathcal{C}(X)$, respectively. On the other hand, from Lemma 2.2 we know that $\mathcal{C}(X)$ is generated by the classes of closures of orbits and classes of curves contained in the fixed locus of the action. Hence the assumptions of Proposition 3.3 and Theorem 3.4 are satisfied when the fixed locus consists of a finite number of points. In this subsection we extend this observation for a broader class of varieties which turns out to be very useful in our applications.

Lemma 3.5. *Assume that* $\mathbb{C}^*$ *acts effectively on a projective manifold* X. *Suppose that* Y *is a connected component of the fixed locus of the action. Then the following conditions are equivalent:*

(1) *The component* Y *is succeeded in the directed graph* $\mathcal{G}$ *by one component consisting of a single point* y.
(2) *The closure of the Białynicki-Birula cell* $X^+(Y)$ *adds a single point:*
$$\overline{X^+(Y)} \setminus X^+(Y) = \{y\}.$$
(3) *The positive weight subbundle* T^+ *of* $TX_{|Y}$ *is an ample line bundle and there exists an* H *equivariant morphism:*
$$\mathbb{P}_Y(T^+ \oplus \mathcal{O}) \longrightarrow X$$
where the action of H *on the* $\mathbb{P}^1$*-bundle has two fixed point components associated to two sections,* Y^0 *and* Y^∞. *The section* Y^0 *has normal bundle* T^+ *and it is mapped isomorphically to* $Y \subset X$; *the section* Y^∞ *has normal bundle* $(T^+)^\vee$ *and it is mapped to a point* $y \in X$.

Proof. The implication (2) ⇒ (1) is clear, because $X^+(Y)$ contains all orbits whose source is in Y. Also the implication (3) ⇒ (2) is obvious. This let us focus on the implication (1) ⇒ (3).

Take L a very ample line bundle on X and consider a $\mathbb{C}^*$ invariant divisor D in $|L|$ which does not contain y. Every orbit $t \cdot x$ of $\mathbb{C}^*$ such that $\lim_{t \to 0} t \cdot x \in Y$ has $\lim_{t \to \infty} t \cdot x = y$. Since the closure of every such orbit has intersection with D, it follows that $D \cap X^+(Y) = Y$ and, as a divisor on $X^+(Y) \cong T^+$ (cf. Theorem 2.3), the restriction of D is a multiple of the zero section in the bundle T^+, that is $D \cdot X^+(Y) = mY$ for some $m > 0$. Thus T^+ is an ample line bundle over Y on which H acts with a weight $\delta > 0$. Moreover, since the argument does not depend on the choice of a very ample L, the restriction $\operatorname{Pic} X \to \operatorname{Pic} Y$ is contained in $\mathbb{Z} \cdot T^+$. On the other hand, because of Lemma 3.1, the degree of any line bundle L on the closure of every orbit joining Y with y is equal to $(\mu_L(Y) - \mu_L(y))/\delta$.

The projective $\mathbb{P}^1$-bundle $\pi : \mathbb{P}_Y(T^+ \oplus \mathcal{O}) \to Y$ has two sections associated to projections to two factors of the decomposable bundle. We denote the one with normal T^+ by Y^0 and the other one, whose normal is dual to T^+, by Y^∞. Since T^+ is ample, we have a contraction morphism

$$\mathbb{P}_Y(T^+ \oplus \mathcal{O}) \to \operatorname{Proj}(\operatorname{Sym}(T^+ \oplus \mathcal{O})) := \mathcal{S}(Y, T^+)$$

which contracts the section Y^∞ to a point y^∞ which is the vertex of the projective cone $\mathcal{S}(Y, T^+)$. We define the action of $\mathbb{C}^*$ on $\mathbb{P}_Y(T^+ \oplus \mathcal{O})$ so that Y^0 and Y^∞ are the source and sink, respectively, and along the fibers of the $\mathbb{P}^1$-bundle the action has weight δ. Therefore, we have a $\mathbb{C}^*$ equivariant embedding $T^+ \hookrightarrow \mathbb{P}_Y(T^+ \oplus \mathcal{O})$ with image equal to $\mathbb{P}_Y(T^+ \oplus \mathcal{O}) \setminus Y^\infty = \mathcal{S}(Y, T^+) \setminus \{y^\infty\}$.

We claim that the $\mathbb{C}^*$ equivariant isomorphism $T^+ \cong X^+(Y)$ (see Theorem 2.3), extends to a regular $\mathbb{C}^*$ equivariant morphism

$$\mathbb{P}_Y(T^+ \oplus \mathcal{O}) \to \mathcal{S}(Y, T^+) \to X$$

which has the properties as in (3). Indeed, any $\mathbb{C}^*$ invariant divisor in $|mY|$ on $T^+ \cong X^+(Y)$ extends to $\mathbb{P}_Y(T^+ \oplus \mathcal{O})$ as the sum $aY^0 + bY^\infty + \pi^*(M)$, where $a + b = m$ and $M \in |bT^+|$. Thus the desired extension exists and maps Y^∞ to y. We note that $\mathcal{S}(Y, T^+) \to \overline{X^+(Y)} \hookrightarrow X$ is the normalization. $\square$

We note that changing the direction of the action of $\mathbb{C}^*$, and therefore the direction of the graph $\mathcal{G}$, we get a similar statement as in the lemma above, with 0 swapped with ∞, source with the sink, and T^+ with T^-.

Corollary 3.6. *Assume that $\mathbb{C}^*$ acts effectively on a projective manifold X, and let Y be a fixed point component which satisfies one of the equivalent conditions of Lemma* 3.5*. Then the curves contained in Y are numerically proportional to classes of closures of orbits joining Y with y.*

Proof. The corollary is a version of a known observation that curves in the base of a cone are numerically proportional to lines in the ruling of the cone. We use Lemma 3.5(3), and keep the same notation there introduced. For an irreducible curve C contained in $\mathbb{P}_Y(T^+) = Y^\infty$, one has $C \equiv \widetilde{C} + \alpha F$, where $\widetilde{C}$ is an irreducible curve contained in $\mathbb{P}_Y(\mathcal{O}) = Y^0$, F is a fiber of the $\mathbb{P}^1$-bundle $\mathbb{P}_Y(T^+ \oplus \mathcal{O}) \to Y$, and $\alpha \in \mathbb{Q}$. Mapping $\mathbb{P}_Y(T^+ \oplus \mathcal{O})$ to X, since $\widetilde{C}$ is contracted to a point, we get the claim. $\square$

3.4. Technical lemmata

This last part of the present section contains technical lemmata which will be used later in our applications.

Lemma 3.7. *Let $\phi\colon X \to Z$ be a surjective $\mathbb{C}^*$ equivariant morphism of two normal projective varieties with an action of $\mathbb{C}^*$. Suppose that X is smooth and Y_0 (resp. Y_∞) is the source (resp. the sink) of X. Then for a general $z \in Z$ we have*

$$\lim_{t \to 0} t \cdot z \in \phi(Y_0) \quad \text{and} \quad \lim_{t \to \infty} t \cdot z \in \phi(Y_\infty).$$

Proof. Since ϕ is equivariant, its restriction to $X^+(Y_0)$ or, respectively, to $X^-(Y_\infty)$ dominates Z, and this implies the claim. □

Corollary 3.8. *Let $\phi\colon X \to Z$ be a surjective $\mathbb{C}^*$ equivariant morphism of two normal projective varieties with an action of $\mathbb{C}^*$. If X is smooth and the action of $\mathbb{C}^*$ on X has one pointed end or two pointed ends, then the action on Z has at least one pointed end or two pointed ends, respectively.*

Lemma 3.9. *Suppose that $\mathbb{C}^*$ acts effectively on a projective manifold X. Let us consider two different components Y_1 and $Y_2 \in \mathcal{Y}$. Assume that both Y_1 and Y_2 are succeded in $\mathcal{G}$ by a single point component $\{y\} \in \mathcal{Y}$. Then we have*

$$\dim Y_1 + \dim Y_2 \leq n - 2$$

Proof. First we observe that $\dim \overline{X^+(Y_1)} + \dim \overline{X^+(Y_2)} \leq n$, otherwise there would be an orbit passing through y and belonging to $\overline{X^+(Y_1)} \cap \overline{X^+(Y_2)}$, against BB decomposition. On the other hand, because $Y_i \subsetneq \overline{X^+(Y_i)}$ for $i = 1, 2$, we obtain that

$$\dim Y_1 + \dim Y_2 + 2 \leq \dim \overline{X^+(Y_1)} + \dim \overline{X^+(Y_2)} \leq n$$

hence the claim. □

4. Varieties with small bandwidth

In the present section we classify polarized varieties (X, L) with an effective $\mathbb{C}^*$ action such that the bandwidth, or the degree of the closure of a general orbit, is ≤ 3.

4.1. Bandwidth ≤ 2

The following has been proved in [5, Prop. 3.12], we reprove it using the notion of adjunction.

Theorem 4.1. *Let (X, L) be a polarized pair satisfying Assumption 1, and $\dim X = n$. Then*

- *if the sink of the action is an isolated point, and $|\mu| = 1$ then $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$;*
- *if $n \geq 2$ and the $\mathbb{C}^*$ action has two pointed ends with $|\mu| = 2$ then either $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$, or $(X, L) = (\mathcal{Q}^n, \mathcal{O}(1))$. Moreover, the $\mathbb{C}^*$ action is equalized only in the latter case.*

Proof. Assume that $|\mu| = 1$. Let $Y_\infty = \{y_\infty\}$ and Y_0 be respectively the sink and the source of the action. We can take $\mu(Y_\infty) = 0$, so that $\mu(Y_0) = 1$. Applying [5, Lem. 3.11] we deduce that $\mu_{K_X}(Y_\infty) \geq n$ and $\mu_{K_X}(Y_0) < 0$. Therefore

$$\mu_{K_X + nL}(Y_\infty) \geq n > \mu_{K_X + nL}(Y_0)$$

and denoting by C the closure of an orbit joining the source and the sink, by Corollary 3.2 (a) we get the inequality $(K_X + nL)\cdot C < 0$, so that $K_X + nL$ is not nef. Therefore, using Remark 1 one has $\tau = n+1$, and by Theorem 2.1 we obtain that $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$.

Similarly, in case $|\mu| = 2$, $Y_\infty = \{y_\infty\}$, $Y_0 = \{y_0\}$, we get

$$\mu_{K_X+nL}(Y_\infty) \geq n \geq \mu_{K_X+nL}(Y_0).$$

If $\mu_{K_X+nL}(Y_\infty) > \mu_{K_X+nL}(Y_0)$, then as above we deduce that $K_X + nL$ is not nef, $\tau = n+1$ and applying Theorem 2.1 we get $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$. Assume that $\mu_{K_X+nL}(Y_\infty) = n = \mu_{K_X+nL}(Y_0)$, and thus $K_X + nL$ is nef. Then the divisor $K_X + nL$ has intersection zero with a general orbit joining the source and the sink, so that the adjoint morphism ϕ_n contracts X to a point. Applying again Theorem 2.1, which in this case coincides with a classical result by Kobayashi and Ochiai (see [23]), we then deduce that either $(X, L) = (\mathbb{P}^n, \mathcal{O}(1))$, or $(X, L) = (\mathcal{Q}^n, \mathcal{O}(1))$. In the first case, the $\mathbb{C}^*$ action in coordinates will be $(t, [z_0, \ldots, z_n]) \to [z_0, tz_1, \ldots, tz_{n-1}, t^2 z_n]$. This action is not equalized because there exists an invariant line joining the sink and the source, and in view of Lemma 3.1 one has that the weight of the tangent bundle of this line at the sink is 2 (the case $n = 2$ has been discussed in Example 2). Consider $(X, L) = (\mathcal{Q}^n, \mathcal{O}(1))$. The action of a maximal torus $\widehat{H}$ on (X, L) is described in [5, Exmpl. 2.20]. Now, the $\mathbb{C}^*$ action is obtained from a downgrading of the action of $\widehat{H}$, that gives a projection of the corresponding lattice of characters $\pi\colon \widehat{M} \to \mathbb{Z}$. We observe that taking a component $Y \subset X^{\mathbb{C}^*}$ corresponding to a weight $i \in \mathbb{Z}$, the weights of the $\mathbb{C}^*$ action on $\mathcal{N}_{Y/X}$ are obtained as projection of the weights of the $\widehat{H}$ action on the normal bundles at all the fixed components of $X^{\widehat{H}}$ which correspond to the weights that are sent to i through π. Using the computations done in [5, Exmpl. 2.20] regarding the weights of the $\widehat{H}$ action on the cotangent bundles at the fixed components, we obtain the weights on the normal bundles at the same components, and using the previous observation we finally conclude that the $\mathbb{C}^*$ action is equalized. □

The following two results concern the action of a torus on a quadric, a case of small bandwidth. Before this discussion, we recall by [5, §2.1] that for any polarized pair (X, L) with an action of an algebraic torus H, the *polytope of fixed points* $\Delta(X, L, H, \mu_L)$ is the convex hull of the image of the weight point map μ_L (see Definition 3).

Lemma 4.2. *Let X be a smooth quadric of dimension $n \geq 3$. Suppose that $\mathbb{C}^*$ acts effectively on X with fixed locus consisting of two components. Then both fixed point components are isomorphic to $\mathbb{P}^m$, with $m = \lfloor n/2 \rfloor$.*

Proof. X being a smooth quadric, $\mathbb{C}^*$ is contained in some maximal torus $\widehat{H}$ of SO_{n+2} with the lattice of characters $\widehat{M} = \bigoplus_{i=0}^m \mathbb{Z}e_i$. Thus, we are in the situation described in [5, Exmpl. 2.20] and the action of $\mathbb{C}^*$ is obtained from some downgrading $\mathbb{C}^* \to \widehat{H}$ which comes with the homomorphism of lattices of characters $\widehat{M} \to \mathbb{Z}$. We know that the polytope $\Delta = \Delta(\mathcal{Q}^n, \mathcal{O}(1), \widehat{H}) = \mathrm{conv}(\pm e_i, i = 0, \ldots, m)$ has $2(m+1)$ vertices associated to fixed points of the action of $\widehat{H}$. Moreover, the polytope Δ is central symmetric and therefore its projection has the same property. In view of Remark 4 all vertices of Δ are mapped via $\widehat{M} \to \mathbb{Z}$ to the set consisting of two points. Therefore the projection contracts two opposite facets of $\Delta(\mathcal{Q}^n, \mathcal{O}(1), \widehat{H})$, each containing at least $m+1$ vertices, thus both symplices are associated to $\mathbb{P}^m$. The last statement follows by [5, Lem. 2.10]. $\square$

Proposition 4.3. *Let X be a smooth quadric of dimension $n \geq 3$ or a quadric cone, that is a cone over the smooth quadric of dimension $n-1$. By L we denote the line bundle $\mathcal{O}(1)$. Suppose that the torus $\mathbb{C}^*$ acts effectively on X with one pointed end, and the bandwidth of the action is $|\mu| \leq 2$. Then one of the following holds:*

(1) *X is a smooth quadric, $|\mu| = 2$, the $\mathbb{C}^*$ action has two pointed ends y_0 and y_2, and $X^{\mathbb{C}^*} = \{y_0, y_2\} \sqcup \mathcal{Q}^{n-2}$.*
(2) *X is a quadric cone and $X^{\mathbb{C}^*}$ has two components: the vertex and a divisor $\cong \mathcal{Q}^{n-1}$.*
(3) *X is a quadric cone and $X^{\mathbb{C}^*}$ has three components: the vertex and two components $\cong \mathbb{P}^m$, with $m = \lfloor (n-1)/2 \rfloor$.*

Proof. First, suppose that X is the smooth quadric $\mathcal{Q}^n$, so that we are in the situation described in [5, Exmpl. 2.20]. The torus $\mathbb{C}^*$ is contained in some maximal torus $\widehat{H}$ of SO_{n+2} with the lattice of characters $\widehat{M}$. Denote by r the rank of $\widehat{H}$, and take $e_1, \ldots, e_r$ a basis of $\widehat{M}$. By the downgrading, we see that the linearization of the action is associated to a projection $\pi\colon \widehat{M} \to \mathbb{Z}$. From [5, Exmpl. 2.20] we know that $\Delta(\mathcal{Q}^n, \mathcal{O}(1), \widehat{H}) = \mathrm{conv}(\pm e_i, i = 1, \ldots, r)$, and all the fixed points correspond to vertices of this polytope. Therefore $\pi(\Delta(\mathcal{Q}^n, \mathcal{O}(1), \widehat{H}))$ is a central symmetric polytope, and the action of $\mathbb{C}^*$ has two pointed ends corresponding to $\pi(e_i)$, $\pi(-e_i)$ for some $i = 1, \ldots, r$. Moreover, the elements $\pm e_j$ with $j \neq i$ will be projected to the same point in $\mathbb{Z}$, so that $|\mu| = 2$. We then obtain that the fixed point component associated to such a point is isomorphic to $\mathcal{Q}^{n-2}$, and this settles the smooth case.

Now suppose that X is a quadric cone. Let us choose a section of $L = \mathcal{O}(1)$ which is $\mathbb{C}^*$ equivariant and does not vanish at the vertex of the cone. Thus, the zero set $X' \subset X$ is a smooth quadric invariant with respect to the action of $\mathbb{C}^*$. Then either $X' \in X^{\mathbb{C}^*}$ and we get (2), or the restriction of the action of $\mathbb{C}^*$ to X' has bandwidth 1. In this latter case, applying Lemma 4.2 to X' we obtain (3). $\square$

4.2. Bandwidth 3

In this subsection we will study polarized pairs (X, L) under the following assumption.

Assumption 2. Let (X, L) be a polarized pair, where X is a manifold of dimension $n \geq 2$ with a linearized $\mathbb{C}^*$ action, such that it has two pointed ends and the bandwidth of $(X, L, \mathbb{C}^*)$ is three. In addition, assume that the action is equalized.

We start by discussing the easier case of surfaces, that was first studied in [5, Exmpl. 3.16].

Lemma 4.4. *Assume that (X, L) satisfies Assumption 2, with $n = 2$. Then either $(X, L) = (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 2))$, or $(X, L) = (\mathbb{P}_{\mathbb{P}^1}(\mathcal{O}(1) \oplus \mathcal{O}(3)), \mathcal{O}(1))$.*

Proof. First, using either BB decomposition or the localization theorem (see proofs [5, pp. 29–30]) we get the description of the fixed point components, which are four isolated points, one for every weight $0, \dots, 3$. Applying Proposition 2.4 and its proof, we conclude that X is rational, and $\operatorname{Pic} X \cong \mathbb{Z}^2$. Therefore, $X \cong \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(e))$ for some $e \in \mathbb{Z}_{\geq 0}$. We will observe that either $e = 0$ or $e = 2$.

L being ample on a Hirzebruch surface, we may write $L = aC_0 + bF$, where C_0 is the minimal section, F the fiber of the natural projection, $a > 0$, and $b > ae$. The $\mathbb{C}^*$ action on X is obtained by a downgrading of a $(\mathbb{C}^*)^2$ action on X. The weights of this action on L at the fixed points are the following: $(0, 0)$; $(0, a)$; $(b - ae, a)$; $(b, 0)$ (cf. [7, Ex. 6.1.18]). Then, the weights of the $\mathbb{C}^*$ action are obtained by a projection $\pi \colon \mathbb{Z}^2 \to \mathbb{Z}$, such that $\pi(\triangle(X, L, (\mathbb{C}^*)^2)) = \triangle(X, L, \mathbb{C}^*)$. A straightforward computation shows that the only cases in which we get a bandwidth three $\mathbb{C}^*$ action having one fixed point corresponding to each lattice point, are those listed in the statement. $\square$

We now extend the study of bandwidth three varieties to a high dimension. The following result will be the crucial point to prove the results in Subsection 6.3, and will be shown in Section 5, by using adjunction theory when $n \geq 3$. Here, by inner fixed points components we mean the components which are neither the sink nor the source.

Theorem 4.5. *Suppose that (X, L) satisfies Assumption 2. Then one of the following holds:*

(1) $(X, L) = (\mathbb{P}(\mathcal{V}), \mathcal{O}(1))$ *is a scroll over $\mathbb{P}^1$, with L being relative $\mathcal{O}(1)$ on the projectivisation of the vector bundle $\mathcal{V}$ which is either $\mathcal{O}(1)^{n-1} \oplus \mathcal{O}(3)$ or $\mathcal{O}(1)^{n-2} \oplus \mathcal{O}(2)^2$. The inner fixed points components are two copies of $\mathbb{P}^{n-2}$.*

(2) $(X, L) = (\mathbb{P}^1 \times \mathcal{Q}^{n-1}, \mathcal{O}(1, 1))$ *is a product quadric bundle over $\mathbb{P}^1$. The inner fixed points components are two isolated points and two copies of $\mathcal{Q}^{n-3}$.*

(3) *$n \geq 6$ is divisible by 3 and X is Fano, $\rho_X = 1$, $-K_X = (2/3)nL$. The inner fixed points components are two smooth subvarieties of dimension $2n/3 - 2$.*

In the scroll case, we have the standard action of a rank n algebraic torus on X; in the quadric bundle case one has the standard action of $\mathbb{C}^* \times H_r$, with $\mathbb{C}^*$ acting on $\mathbb{P}^1$, and H_r a maximal rank torus acting on $\mathcal{Q}^{n-1}$. In Examples 4, 5 we will see how the $\mathbb{C}^*$ action is obtained from a downgrading of the standard action of the respective torus of bigger rank. In Example 6, we present a variety satisfying Theorem 4.5(3). Notice that the classification of varieties satisfying part (3) of the above theorem has been recently reached in [28], using tools from birational and projective geometry. In total, there are four of these varieties, all of them are rational homogeneous; we refer to [28, Thm. 6.8] for their complete list.

4.3. Examples

Example 4. Let us consider the standard action of $\mathbb{C}^*$ on $\mathbb{P}^1$ with source at y_0 and sink at y_∞. For any line bundle $\mathcal{L}$ over $\mathbb{P}^1$, we can choose its linearization so that $\mu_{\mathcal{L}}(y_0) = a$ and $\mu_{\mathcal{L}}(y_\infty) = a - \deg \mathcal{L}$, where $a \in \mathbb{Z}$ can be chosen arbitrarily, cf. (3) and Lemma 3.1. Given a decomposable bundle $\mathcal{V}$ over $\mathbb{P}^1$, we can define its linearization by linearizing its components. If $\mathcal{V} = \mathcal{O}(1)^{n-1} \oplus \mathcal{O}(3)$ then we linearize

$\mathcal{O}(1)$'s with $\mu(y_\infty) = 1$ and $\mu(y_0) = 2$, while the component $\mathcal{O}(3)$ is linearized so that $\mu(y_\infty) = 0$ and $\mu(y_0) = 3$. This determines the action of $\mathbb{C}^*$ on $X = \mathbb{P}(\mathcal{V})$ with the linearization of the relative $L = \mathcal{O}(1)$.

Alternatively, the pair (X, L) can be described as a toric variety associated to a polytope $\Delta(L)$ in a lattice M with generators e_i, $i = 1, \dots, n$. We take the vertices of Δ as follows: $0, 3e_1$ and $e_1 + e_i, 2e_1 + e_i$ for $i > 1$. The action of $\mathbb{C}^*$ is defined by a downgrading $M \to \mathbb{Z}$ by the projection to the first coordinate.

A similar construction works for $\mathcal{V} = \mathcal{O}(1)^{n-2} \oplus \mathcal{O}(2)^2$. We linearize $\mathcal{O}(1)$'s as before with $\mu(y_\infty) = 1$ and $\mu(y_0) = 2$, one copy of $\mathcal{O}(2)$ with $\mu(y_\infty) = 0$ and $\mu(y_0) = 2$, and the other with $\mu(y_\infty) = 1$ and $\mu(y_0) = 3$. Or, alternatively we take $\Delta(L)$ in $M = \bigoplus_{i=1}^n \mathbb{Z}e_i$ with vertices as follows: $0, 2e_1$ and $e_1 + e_2, 3e_1 + e_2$, and $e_1 + e_i, 2e_1 + e_i$ for $i > 2$. The action of $\mathbb{C}^*$ is defined by downgrading $M \to \mathbb{Z}$ by the projection to the first coordinate.

The fixed locus has four components: two extremal fixed points and two components isomorphic to $\mathbb{P}^{n-2}$. The chosen linearization of the bundle L associates to them the values $0, 1, 2, 3$.

We note that $K_X + nL = \pi^*\mathcal{O}(n)$, with $\pi\colon \mathbb{P}(\mathcal{V}) \to \mathbb{P}^1$ the natural projection, hence the nef value of the polarized variety (X, L) is n and π is the adjunction map for (X, L).

In Figure 1 we present schematically the scroll situation: the thick black points and line segments are fixed point components, the thin, dotted and dashed line segments are orbits, and the shaded regions are fibers of the adjoint morphism over 0 and ∞.

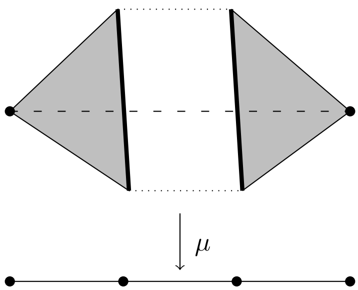


FIGURE 1. Scroll case: fixed points, orbits, linearization

Example 5. For $r \geq 2$, let H_r be a torus $(\mathbb{C}^*)^r$ with lattice of characters $M = \bigoplus_{i=1}^r \mathbb{Z}e_i$. The standard action of $H_r \subset \mathrm{SO}_{2r}, \mathrm{SO}_{2r+1}$ on the quadrics $\mathcal{Q}$ denoting $\mathcal{Q}^{2r-2}$ or $\mathcal{Q}^{2r-1}$ has a natural linearization on $\mathcal{O}(1)$, so that $\Delta(\mathcal{Q}, \mathcal{O}(1), H_r)$ has vertices $\pm e_i$. Take $\mathbb{P}^1$ with the standard action of $\mathbb{C}^*$, and the linearization of $\mathcal{O}(1)$ with weights $(1, 2)$. Consider $X = \mathbb{P}^1 \times \mathcal{Q}$ with the induced action of the product $\mathbb{C}^* \times H_r$, and the lattice of characters $\mathbb{Z}e_0 \oplus M$. If $L = \mathcal{O}(1, 1)$ then the induced linearization yields $\Delta(\mathcal{Q}, L, \mathbb{C}^* \times H_r)$ with vertices $e_0 \pm e_i, 2e_0 \pm e_i$ for $i > 0$.

Now, we take the action of $\mathbb{C}^*$ on $X = \mathbb{P}^1 \times \mathcal{Q}$ which is obtained by the downgrading associated to the projection $\bigoplus_{i=0}^r \mathbb{Z}e_i \to \mathbb{Z}$ such that $e_0, e_1 \mapsto 1$ and $e_i \mapsto 0$ for $i > 1$.

If $\dim X = 3$, then $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}(1,1,1)$, and the downgrading can be described in a symmetric way as a projection of a cube onto one of its diagonals. The action has 8 fixed points. We note that in this case $-K_X = 2L$ and the associated adjoint morphism contracts X to a point. In Figure 2 we present schematically the fixed point set together with the orbits of the action, and the associated value of the linearization μ on the fixed point components.

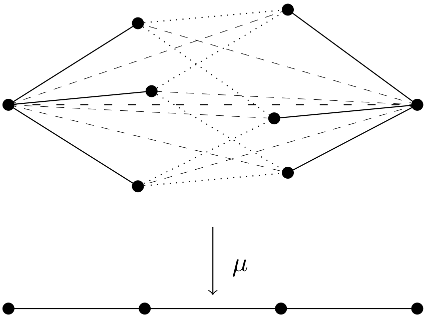


FIGURE 2. $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with diagonal $\mathbb{C}^*$ action

If $\dim X = n > 4$, then the induced action of $\mathbb{C}^*$ has two pointed ends and there are two fixed components associated to each weight 1 and 2: one is given by an isolated point, the other one is isomorphic to $\mathcal{Q}^{n-3}$. In particular, for $n = 4$ the two fixed components associated to the weight 1 are an isolated point, and another one isomorphic to $\mathbb{P}^1$ with the restriction of L being $\mathcal{O}(2)$, and the same holds for the fixed components associated to the weight 2.

The nef value of the pair (X, L) is $n-1$ with the adjoint map being the projection $X \to \mathbb{P}^1$.

In Figure 3 we present schematically the fixed point locus together with the orbits of the action, and the associated value of the linearization μ on the fixed point components. The two shaded regions present the fibers of the adjoint morphism over the fixed points of the $\mathbb{C}^*$ action on $\mathbb{P}^1$, that is 0 and ∞.

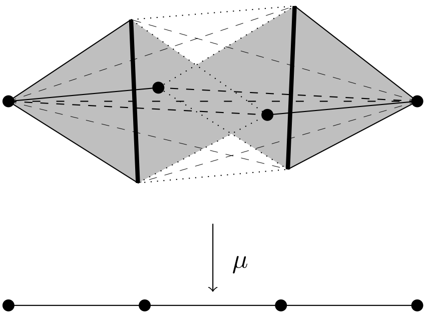


FIGURE 3. $\mathbb{P}^1 \times \mathcal{Q}$ with $\mathbb{C}^*$ action

Example 6. Consider the Sp_6 homogeneous variety, namely the Lagrangian Grassmannian of isotropic planes in $\mathbb{P}^5$. We verify that such a variety satisfies Theorem 4.5(3). It has dimension 6 and lives in $\mathbb{P}^{13}$ (see [13, §17.1] for details about this variety). The representation of dimension 14 is associated to the highest weight $(1,1,1)$. The action of the big torus in Sp_6, which is of rank 3, has 8 fixed points associated to the Weyl group orbit of the dominant weight. The weights associated to fixed points yield a cube in the weight space. We take the downgrading associated to the projection of the cube onto a long diagonal. The resulting $\mathbb{C}^*$ action has fixed point locus which consists of two isolated points (the source and the sink) and two copies of $\mathbb{P}^2$. A schematic picture is presented in Figure 4, with shaded triangles denoting the surface components of the fixed point set. The adjoint morphism contracts the variety to a point. We refer the interested reader to [28, §6] for the complete treatment and classification of all the varieties satisfying Theorem 4.5(3).

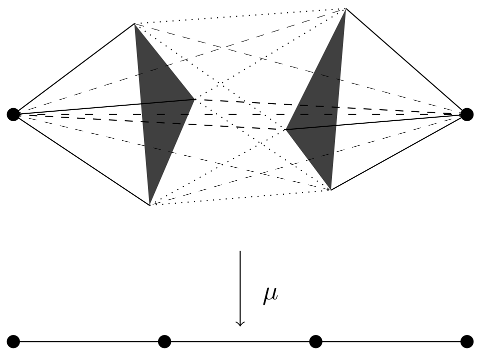


FIGURE 4. The Sp_6 homogeneous variety

Remark 5. In the case in which all fixed points are isolated points, we can apply BB decomposition and equivariant cohomology. Assume that $n \geq 3$. Under the Assumption 2, the equivariant Riemann–Roch gives the formula for $\chi_m(t) = \chi(X, L^m)$ (see [5, Cor. A.3]):

$$\chi_m(t) = \frac{1}{(1-t)^n} + a\frac{t^m}{(1-t^{-1})(1-t)^{n-1}} + a\frac{t^{2m}}{(1-t)(1-t^{-1})^{n-1}} + \frac{t^{3m}}{(1-t^{-1})^n}$$

where $a = \operatorname{rank}\operatorname{Pic} X$ is the number of fixed points associated to the weights 1 and 2. In fact, because of the BB decomposition, X has pure cohomology and $\chi(\mathcal{O}_X) = 1$; therefore

$$\frac{1}{(1-t)^n} + a\frac{1}{(1-t^{-1})(1-t)^{n-1}} + a\frac{1}{(1-t)(1-t^{-1})^{n-1}} + \frac{1}{(1-t^{-1})^n} = 1.$$

From this, multiplying by $(1-t)^n(1-t^{-1})^n$ we get the equality

$$(1-t)^n + (1-t^{-1})^n + a(2-t-t^{-1}) \cdot \left((1-t^{-1})^{n-2} + (1-t)^{n-2}\right) = (2-t-t^{-1})^n.$$

For $n = 2$ we get $a = 1$, while for $n = 3$ we get $a = 3$. Let us assume $n \geq 4$ and write the highest terms of the left-hand side

$$(-t)^n + (n+a) \cdot (-t)^{n-1} + \left(\binom{n}{2} + na\right) \cdot (-t)^{n-2} + \cdots$$

while the highest terms of the right-hand side are

$$(-t)^n + 2n \cdot (-t)^{n-1} + \left(4\binom{n}{2} + n\right) \cdot (-t)^{n-2} + \cdots$$

Comparing the second and the third term we see that for $n \geq 4$ there is no solution. For $n = a = 3$ we have the case of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

5. Classification of bandwidth 3 varieties

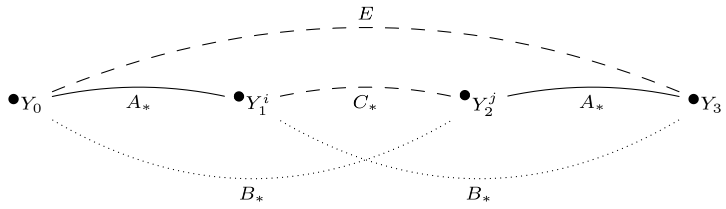
This section is devoted to prove Theorem 4.5. Let us keep Assumption 2, where we consider a normalized linearization μ so that we define $\mathcal{Y}_i := \{Y \in \mathcal{Y} : \mu(Y) = i\}$ for $i = 0, 1, 2, 3$. All the fixed components in $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are called *inner components*. Using this notation, we will denote by $Y_0 = \{y_0\}$ and $Y_3 = \{y_3\}$ respectively the sink and the source of the $\mathbb{C}^*$ action. In view of Lemma 4.4, from now on we suppose that $n \geq 3$.

Lemma 5.1. *In the situation of Assumption* 2 *we have* $\mathcal{Y}_1 \neq \varnothing \neq \mathcal{Y}_2$. *Moreover, in notation of Theorem 2.3, for every* $Y_1 \in \mathcal{Y}_1$, $Y_2 \in \mathcal{Y}_2$ *we have* $\operatorname{rk}^+(Y_1) = 1 = \operatorname{rk}^-(Y_2)$.

Proof. Firstly, arguing as in the proof of Proposition 2.4, we note that for $n > 1$ we have $\mathcal{Y}_1 \cup \mathcal{Y}_2 \neq \varnothing$. So, contrary to what the lemma says, let us assume $\mathcal{Y}_2 = \varnothing$ and take $Y \in \mathcal{Y}_1$. Then, because of Lemma 3.5, one has $\operatorname{rk}^+(Y) = \operatorname{rk}^-(Y) = 1$, hence $\dim Y = n-2$, and both $\overline{X^+(Y)}$ and $\overline{X^-(Y)}$ are divisors. Thus, using Lemma 3.9 we deduce that there are no other fixed point components in $\mathcal{Y}_1$, that is $\mathcal{Y}_1 = \{Y\}$. Again, by Proposition 2.4, divisors $\overline{X^+(Y)}$ and $\overline{X^-(Y)}$ are linearly equivalent and $\operatorname{Pic} X = \mathbb{Z} \cdot D$, where D is their equivalence class. Moreover, if C_1 is the closure of an orbit with source at Y and sink at y_0 and C_2 the closure of an orbit with source at y_3 and sink at Y, then $D \cdot C_1 = D \cdot C_2 = 1$. However, because of Corollary 3.2, $L \cdot C_1 = 1$ while $L \cdot C_2 = 2$, a contradiction. The last statement follows again by Lemma 3.5. $\square$

5.1. Orbits

The following is the graph of closures of possible orbits joining fixed points components with $Y_1^i \in \mathcal{Y}_1$ and $Y_2^j \in \mathcal{Y}_2$. By abuse, the orbits and their closures will be called by the same name. We use the notation A_* to denote that there could be different orbits of type A joining the component Y_0 (or Y_3) with one of the components $Y_1^i \in \mathcal{Y}_1$ (respectively, with one of the components $Y_2^j \in \mathcal{Y}_2$). In the same way, B_* and C_* denote the possible different orbits of type B and C.



Remark 6. For $n \geq 2$ orbits of type A and E always exist. However, not all of the above types of orbits always exist:

(1) if $X = \mathbb{P}^1 \times \mathbb{P}^1$, $L = \mathcal{O}(1,2)$ then there are no orbits of type C,
(2) if $X = \mathbb{P}(\mathcal{O}(1)^{n-1} \oplus \mathcal{O}(3))$ then there are no orbits of type B.

In what follows, if not needed, we will not distinguish curves of different types A or B and, if no confusion is probable, we will write d_* for the respective dimension of a component in $\mathcal{Y}_*$.

Lemma 5.2. *Let (X, L) be as in Assumption 2 and let us keep the above notation for the possible orbits. The first two rows in the following table present the intersection of the closure of the orbit of the respective type (the column) with the divisor L and $-K_X$. The third row presents the resulting estimate on τ.*

	A_*	B_*	C_*	E
L	1	2	1	3
$-K_X$	$d_* + 2$	$2n - d_* - 2$	$2n - 4 - (d_1 + d_2)$	$2n$
$\tau \geq$	$d_* + 2$	$n - 1 - d_*/2$	$2n - 4 - (d_1 + d_2)$	$2n/3$

For curves of type A and B, by d_* we denote the dimension of the corresponding fixed source or sink component in $\mathcal{Y}_1$ or $\mathcal{Y}_2$. For curves of type C, by d_2 and d_1 we denote the dimension of the sink/source component in $\mathcal{Y}_2$, $\mathcal{Y}_1$, respectively.

Proof. In view of Corollary 3.2, the values of the first two rows are obtained by calculating the difference in μ_L and μ_{-K_X} at the source and the sink of each of the one-dimensional orbits of the respective type. The values for μ_L are known. Here, by μ_{-K_X} we denote the natural linearization of $-K_X$. Since the action is equalized, by [5, Lem. 3.11] we have

$$\mu_{-K_X}(Y) = \mathrm{rk}^+(Y) - \mathrm{rk}^-(Y)$$

where we use the notation $\mathrm{rk}^\pm(Y) = \mathcal{N}^\pm(Y)$ as in Theorem 2.3. On the other hand, by Lemma 3.5 one has $\mathrm{rk}^+(Y_1) = \mathrm{rk}^-(Y_2) = 1$ for $Y_1 \in \mathcal{Y}_1, Y_2 \in \mathcal{Y}_2$. This allows us to compute the values of μ_{-K_X}, and thus of the second row in the table. The third row is obtained by calculating the value of $K_X + \tau L$ from the two previous rows. $\square$

Proposition 5.3. *In the situation of Assumption* 2 *we have the following:*

(1) *The space of 1-cycles* $\mathbf{N}_1(X)$ *and the cone of curves* $\mathcal{C}(X)$ *are generated by classes of closures of orbits of the* $\mathbb{C}^*$ *action.*
(2) *If* $-K_X = \tau L$ *then* $\tau = 2n/3$ *and for any* $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ *we have* $\dim Y = 2n/3 - 2$.
(3) X *is a Fano manifold unless there are components* $Y_1 \in \mathcal{Y}_1$, $Y_2 \in \mathcal{Y}_2$, *both of codimension 2, connected by an orbit of type* C.
(4) *If there exists a component of* X^H *of codimension* 2, *then* $\tau \geq n$.
(5) *If a component in* $\mathcal{Y}_1 \cup \mathcal{Y}_2$ *does not meet a curve of type* C, *then it is of codimension* 2.

Proof. Claim (1) follows by Corollary 3.6. To prove (2) we note that the orbits of type E are general, thus they always exist, so, by Lemma 5.2, using the column associated to E, we have

$$-K_X \cdot E = 2n = \tau L \cdot E = 3\tau$$

while using other entries in the table we get $2n - d_* - 2 = \frac{4}{3}n$. In order to show (3) we use Lemma 5.2 again, and look at the intersection of orbits with $-K_X$, where we recall that $d_* \leq n-2$. Part (4) follows by the existence of orbits of type A, and by the respective entries at the last row of the above table. Finally, assume that $Y \in \mathcal{Y}_1 \cup \mathcal{Y}_2$ does not meet a curve of type C. Then, by Lemma 3.5 we get $\operatorname{rank} \mathcal{N}^{\pm}(Y) = 1$, hence $\dim Y = n-2$ and we obtain (5). $\square$

5.2. $\tau \geq n$, scroll over a curve

If $\tau \geq n$, then because of Proposition 5.3(2), we know that (X, L) is neither $(\mathbb{P}^n, \mathcal{O}(1))$ nor $(\mathcal{Q}^n, \mathcal{O}(1))$. Thus by Remark 1 and Theorem 2.1, it follows that (X, L) is a scroll over $\mathbb{P}^1$. This is the first claim in the following.

Lemma 5.4. *If the nef value of the pair* (X, L) *is* $\geq n$ *then this pair is a scroll over* $\mathbb{P}^1$ *as described in case* (1) *of Theorem* 4.5.

Proof. By the above discussion, we deduce that (X, L) is a scroll, so we know that there exist curves contracted by the adjoint morphism, passing through the end points. By looking at the table of Lemma 5.2, we see that the intersection of $K_X + nL$ with curves of type B_* and E is positive, hence curves of type A_* are contracted. Using the same table to compute the intersection number with such curves, we get

$$(K_X + nL) \cdot A_* = -d_* - 2 + nL \cdot A_* = 0.$$

Since by Corollary 3.2(c) we know that $L \cdot A_* = 1$, we deduce that $d_* = n-2$, namely there exist $Y_1 \in \mathcal{Y}_1$ and $Y_2 \in \mathcal{Y}_2$ which have dimension $n-2$. Hence $\operatorname{rank} \mathcal{N}^{\pm}(Y_i) = 1$, and being $\rho_X = 2$, arguing as in the proof of Proposition 2.4 we conclude that $X^{\mathbb{C}^*} = \{y_0\} \sqcup Y_1 \sqcup Y_2 \sqcup \{y_3\}$. Moreover, $\overline{X^+(Y_1)}$ and $\overline{X^-(Y_2)}$ are divisors and fibers of the adjoint morphism $\phi := \phi_n \colon X \to \mathbb{P}^1$, then they are isomorphic to $\mathbb{P}^{n-1}$. Since by Lemma 3.5 these divisors are respectively cones over Y_1 and Y_2, and X is a scroll, it follows that $Y_1 \cong Y_2 \cong \mathbb{P}^{n-2}$.

We observe that there exist orbits of type C joining Y_1 and Y_2, otherwise we reach a contradiction using Lemma 3.9. By Proposition 2.4, we deduce that X is a rational scroll, hence it has another contraction. Therefore, by Theorem 3.4, the curves of type C generate the other ray of the cone $\mathcal{C}(X)$ whose intersection with $-K_X$ is zero, as we see from the table in Lemma 5.2. Let $\mathcal{V} = \phi_* L$, thus $X = \mathbb{P}(\mathcal{V})$ and if we write $\mathcal{V} = \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_n)$ with $0 < a_1 \leq \cdots \leq a_n$, then the other contraction of X contracts sections of ϕ associated to the smallest summand in this decomposition. Hence, $1 = L \cdot C = a_1$ and because $K_X + nL = \phi^* \mathcal{O}(\deg \mathcal{V} - 2)$ one has $0 = K_X \cdot C = \sum a_i - n - 2$ from which we get both possible splitting types of $\mathcal{V}$ as in Theorem 4.5(1). □

Lemma 5.5. *Suppose that $|\mathcal{Y}_i| = 1$ for either $i = 1$ or $i = 2$. Then either $\rho_X = 1$ and X is Fano of index $2n/3$, or (X, L) is a scroll over $\mathbb{P}^1$ as in Lemma 5.4.*

Proof. Suppose that $|\mathcal{Y}_1| = 1$. If $\rho_X > 1$, then by Proposition 2.4 there exists a component $Y_2^j \in \mathcal{Y}_2$ with $\operatorname{rank} \mathcal{N}^+(Y_2^j) = 1$, and since by Assumption 2 one has $\operatorname{rank} \mathcal{N}^-(Y_2^j) = 1$, then this component is of codimension 2. Hence, by Proposition 5.3(4) one has $\tau \geq n$, and by Lemma 5.4 we obtain that (X, L) is a scroll over $\mathbb{P}^1$ as in Lemma 5.4. On the other hand, if (X, L) is not such a scroll, then by what we have already proved it follows that $\tau < n$, and there is no component of $X^{\mathbb{C}^*}$ of codimension 2. Then X is Fano because of Proposition 5.3 (3), and by the claim (2) of the same Proposition its index is $2n/3$. □

5.3. $\tau \leq n - 1$, quadric bundle over a curve

In this subsection we keep Assumption 2 with $\tau \leq n - 1$.

Lemma 5.6. *If $\tau \leq n - 1$ then $|\mathcal{Y}_1| = |\mathcal{Y}_2| = \rho_X$, every inner component is connected to another inner component by a curve of type C, the manifold X is Fano, and the cone $\mathcal{C}(X)$ is generated by classes of curves of type A and C.*

Proof. Firstly, we note that if an inner component Y is not connected to some other inner component by a curve of type C, then we are in the situation of Lemma 3.5 for both y_3 preceding Y and y_0 succeeding Y; therefore Y is of codimension 2 and Proposition 5.3 (4) gives a contradiction. Now we prove that $\rho_X = |\mathcal{Y}_1|$; the equality $\rho_X = |\mathcal{Y}_2|$ follows by the same arguments. By Proposition 2.4, we know that $\rho_X \geq |\mathcal{Y}_1|$; if the inequality is strict then we argue as in the proof of Lemma 5.5 to get a component $Y_2^j \in \mathcal{Y}_2$ with $\operatorname{rank} \mathcal{N}^+(Y_2^j) = 1$, which again leads to $\dim Y_2^j = n - 2$, and by Proposition 5.3(4) we reach a contradiction. The rest of the lemma follows by Proposition 5.3(3) and Theorem 3.4. □

Lemma 5.7. *Suppose $\rho_X > 1$. Then $\tau \geq n - 2$.*

Proof. By Lemma 5.6, we may assume that there are at least two components $Y_i^1 \neq Y_i^2 \in \mathcal{Y}_i$ for each $i = 1, 2$ and, moreover, we can choose these components so that Y_2^i is connected to Y_1^i via a curve of type C, for $i = 1, 2$. The latter follows by a standard argument on finding partial matching in a bipartite graph with vertices $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$. Using Lemma 3.9 we get $d_i^1 + d_i^2 \le n-2$, where $d_i^j = \dim Y_i^j$ for $i,j = 1,2$. We confront this inequality with the estimate on τ for curves of type C from Lemma 5.2 to get

$$2\tau \ge 4n - 8 - (d_1^1 + d_1^2 + d_2^1 + d_2^2) \ge 2n-4$$

hence the claim. $\square$

Remark 7. From the proof of Lemma 5.7 we conclude that in case $\rho_X > 1$ and $\tau = n-2$ all inequalities in the proof become equalities. That is, using the above notation

$$d_1^1 + d_1^2 = d_2^1 + d_2^2 = n-2 = d_1^1 + d_2^1 = d_1^2 + d_2^2.$$

In view of Lemma 5.2 the two right-hand side equalities imply that in this case the curves of type C joining Y_1^i with Y_2^i, for $i = 1,2$, are contracted by ϕ_τ.

Remark 8. Suppose that the adjoint morphism $\phi_\tau \colon X \to X'$ is not the contraction to a point. Using Lemma 5.2 we get $\tau > \frac{2}{3}n$. Assuming $\tau \le n-1$ we obtain $n \ge 4$, and if $\tau = n-2$ then $n \ge 7$.

Lemma 5.8. *Assume that the adjoint morphism $\phi_\tau \colon X \to X'$ is not the contraction to a point. Then $\tau \ge n-1$.*

Proof. By Lemma 5.7 and Remark 1, we need to exclude the case $\tau = n-2$. We argue by contradiction and assume that $\phi_{n-2} \colon X \to X'$ is the adjoint morphism. We use Remarks 7 and 8. By the former, we know that curves of type C which join $Y_1^1 \in \mathcal{Y}_1$ and $Y_2^1 \in \mathcal{Y}_2$ are contracted by ϕ_{n-2}. We may assume $d_1^1 \ge d_1^2$, hence $d_1^1 \ge 3$ and $\dim \overline{X^+(Y_1^1)} \ge 4$. By fiber-locus inequality [35, Thm. 1.1] we deduce that fibers of ϕ_{n-2} have dimension $\ge n-3$, hence a fiber of ϕ_{n-2} has positive dimensional intersection with $\overline{X^+(Y_1^1)}$. Then $\overline{X^+(Y_1^1)}$ is contracted to a point by ϕ_{n-2}, because of Corollary 3.6. Thus ϕ_{n-2} contracts curves of type A joining y_0 and Y_1^1, hence $d_1^1 = n-4$ by Lemma 5.2. Applying Remark 7 we get $d_1^2 = d_2^1 = 2$ and $d_1^1 = d_2^2 = n-4$. Using Lemma 5.2, the curves in $\overline{X^+(Y_1^2)}$ and $\overline{X^-(Y_2^1)}$ are not contracted; therefore no fiber of ϕ_{n-2} of dimension $\ge n-2$ meets these subvarieties. Again, by [35, Thm. 1.1] we conclude that $\phi_{n-2} : X \to X'$ is an equidimensional scroll over a smooth threefold; the smoothness follows from [12, Lem. 2.12]. The morphism $\overline{X^+(Y_1^2)} \to X'$ (and, in fact, $\overline{X^-(Y_2^1)} \to X'$) is finite and $\mathbb{C}^*$ equivariant, from which we infer that the $\mathbb{C}^*$ action on X' has two fixed point components, the image of y_0 and of Y_1^2, which is of dimension 2. However, also $X \to X'$ is $\mathbb{C}^*$ equivariant; thus Corollary 3.8 implies that the action of $\mathbb{C}^*$ on X' has two pointed ends, a contradiction. $\square$

Lemma 5.9. *Suppose that $\tau = n-1$, and the adjoint morphism $\phi_{n-1} \colon X \to X'$ is not the contraction to a point. Then $\dim X' = 1$, and ϕ_{n-1} is a quadric bundle.*

Proof. Applying Theorem 2.1(2), since by our assumption X is not as described in point (a), we are left to eliminate also cases (c) and (d) of that theorem. In the former case, because of Corollary 3.8, there exists a fixed point y' of the $\mathbb{C}^*$ action on X' which is not an end point. Hence, if $F \subset X$ is the fiber of ϕ_τ over y', then F

is $\mathbb{C}^*$ invariant with $\mu_{|\mathcal{Y}\cap F}$ assuming values only 1 or 2. In fact $F \cong \mathbb{P}^{n-2}$, hence one of its fixed point components is of positive dimension and contained in, say, a component $Y_1 \in \mathcal{Y}_1$. We note however that by Corollary 3.6 the curves in Y_1 are numerically proportional to orbits joining the sink of the $\mathbb{C}^*$ action with Y_1. Hence the morphism ϕ_{n-1} contracts Y_1 and also $\overline{X^+(Y_1)}$, which contains the sink. This contradicts the fact that F does not contain any of the end points of the $\mathbb{C}^*$ action on X.

Suppose that $\phi_{n-1} : X \to X'$ is birational; thus, by Theorem 2.1 it contracts at least one $F \cong \mathbb{P}^{n-1}$ to a point. By the same arguments as above, we may assume that F contains one of the end points of the $\mathbb{C}^*$ action on X, say $y_0 \in F$. The action of $\mathbb{C}^*$ on F is of bandwidth ≤ 2 and it is equalized. If the action of $\mathbb{C}^*$ on F is of bandwidth 1, then it has a fixed point component of dimension $n-2$, which by Proposition 5.3 implies $\tau \geq n$, which is not possible. If the $\mathbb{C}^*$ action on F is of bandwidth 2 then, by the same argument which was used in the first part of the proof, the other fixed end point component is an isolated point. Now, by Theorem 4.1, this is not possible if the action is equalized. □

Lemma 5.10. *Suppose that $\tau = n-1$ and the adjoint morphism $\phi_{n-1}\colon X \to \mathbb{P}^1$ is a quadric bundle. Then $X = \mathbb{P}^1 \times \mathcal{Q}^{n-1}$, $L = \mathcal{O}(1,1)$, and ϕ_{n-1} is the projection. Moreover, the sets of inner fixed points components $\mathcal{Y}_1$ and $\mathcal{Y}_2$ consist of an isolated point and a copy of $\mathcal{Q}^{n-3}$.*

Proof. By Remark 8 we know that $n \geq 4$. We describe $X^{\mathbb{C}^*}$, by proving that for each $i = 1, 2$ the set of the fixed point components $\mathcal{Y}_i$ contains an isolated point and $\mathcal{Q}^{n-3}$. Let us take the quadrics corresponding to the fibers of ϕ_{n-1} over the end points of the induced action of $\mathbb{C}^*$ on $\mathbb{P}^1$; these fibers are either smooth quadrics or quadric cones. In view of Corollary 5.6 and Proposition 4.3, we deduce that both $\mathcal{Y}_1$ and $\mathcal{Y}_2$ have two components, say $\mathcal{Y}_i = \{Y_i^1, Y_i^2\}$. Using Lemma 3.9 we get $\dim Y_i^1 + \dim Y_i^2 \leq n-2$, and we may assume that $\dim Y_i^1 \leq \dim Y_i^2$. Notice that the fibers cannot be quadric cones. Suppose by contradiction that there is a fiber which is a quadric cone, then curves of type B must be contracted by ϕ_{n-1}, but by Lemma 5.2 we know that $K_X + (n-1)L$ has intersection positive with these curves. Thus, by Proposition 4.3 we deduce that Y_i^1 is an isolated point and $Y_i^2 \cong \mathcal{Q}^{n-3}$ for each $i = 1, 2$, as claimed.

Now we will prove that X is a product. Using Proposition 5.3 we know that X is Fano, then we can consider the other extremal contraction $\Psi\colon X \to Z$. By Theorem 3.4, we recall that the cone $\mathcal{C}(X)$ is generated by classes of curves of type A and C. More specifically, curves of type A may join an end point, say y_0, with a component Y_1^1 which is a point, or with $Y_1^2 \cong \mathcal{Q}^{n-3}$. In the former case, by Lemma 5.2 the intersection with $-K_X$ is 2, in the latter $n-1$. Similarly, using Lemma 5.2 we verify that curves of type C may have intersection with $-K_X$ equal to $2, n-1$ and $2n-4$; the latter is not possible as $\tau = n-1$ and $n \geq 4$. Fibers of Ψ are of dimension ≤ 1 and have intersection ≥ 2 with $-K_X$, hence applying [35, Cor. 1.3] it follows that Ψ is a $\mathbb{P}^1$-bundle. Moreover Ψ has two disjoint sections which correspond to the smooth quadrics that are fibers of ϕ_{n-1} over the two end points of the action on $\mathbb{P}^1$. Therefore Ψ is a trivial bundle over $Z \cong \mathcal{Q}^{n-1}$, and $X \cong \mathbb{P}^1 \times \mathcal{Q}^{n-1}$. □

5.4. Conclusion of the proof of Theorem 4.5

Now fitting together the results of the above subsections, we are able to prove the classification theorem for bandwidth 3 varieties.

Proof of Theorem 4.5. If $n = 2$ we reach the claim by Lemma 4.4. Hence, from now on we consider the case $n \geq 3$. We first assume that $\rho_X \geq 2$. By Lemma 5.7, we know that $\tau \geq n-2$. Moreover, Remark 1 and Theorem 2.1 imply that $\tau \in \{n-2, n-1, n\}$. If $\tau = n$ then by the discussion at the beginning of Subsection 5.2 and Lemma 5.4 we get (1).

Assume that $\tau < n$. We first show that if $\rho_X \geq 2$ and the adjunction morphism ϕ_τ is the contraction to a point, then $(X, L) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1))$. Indeed, if ϕ_τ is the contraction to a point, applying Lemma 5.2 we deduce that $\tau = \frac{2}{3}n$. We analyze what happens for $\tau = 2n/3 = n-1$, and $\tau = 2n/3 = n-2$. In the first case, applying calculations from Remark 5, we see that $\rho_X = 3$. Moreover, by Proposition 5.3(3),(4) X is a Fano 3-fold and from Theorem 2.1(2) it has index 2, then we get $(X, L) = (\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1,1))$. The fixed point locus of the $\mathbb{C}^*$ action is given by 8 isolated points as described in Example 5.

Now, we prove that the case $\rho_X \geq 2$ and $\tau = \frac{2}{3}n = n-2$ is not possible. If this happens, [34, Thm. B] implies that $(X, L) = (\mathbb{P}^3 \times \mathbb{P}^3, \mathcal{O}(1,1))$. Then $L = L_1 \otimes L_2$ where L_i are the pullback of $\mathcal{O}(1)$ via projections on each of the factors $p_i \colon \mathbb{P}^3 \times \mathbb{P}^3 \to \mathbb{P}^3$. Each L_i is nef and nontrivial, therefore, since by our assumption the bandwidth of L is 3, then one of them, say L_1 has bandwidth 1. The contraction p_1 is equivariant and thus the resulting action of $\mathbb{C}^*$ on $(\mathbb{P}^3, \mathcal{O}(1))$ has bandwidth 1, and by Corollary 3.8 it has two pointed ends, a contradiction.

Hence, for $n \geq 4$ we may assume that either $\rho_X = 1$ or ϕ_τ is not the contraction to a point. In the latter case, applying Lemma 5.8, Lemma 5.9, and Lemma 5.10 we obtain (2). If $\rho_X = 1$, using Proposition 5.3 (2), (3) and Remark 5 we obtain part (3) of the statement, hence the claim. $\square$

Remark 9. Let us focus on Theorem 4.5(1), (2), and keep the notation used in the proof of that theorem. Take the corresponding adjunction morphism $\phi_\tau \colon X \to \mathbb{P}^1$. Since ϕ_τ is $\mathbb{C}^*$ equivariant (see Proposition 2.5), the fixed locus $X^{\mathbb{C}^*}$ will be contained in the inverse image of the fixed locus of the $\mathbb{C}^*$ action on $\mathbb{P}^1$. In the scroll case, in the proof of Lemma 5.4, we have shown that the fixed point components $Y_1 \cong Y_2 \cong \mathbb{P}^{n-2} \subset F \cong \mathbb{P}^{n-2}$, with F being the fiber of ϕ_τ. Therefore, we get $\mathcal{N}^+(Y_1) = \mathcal{O}(1)$, $\mathcal{N}^-(Y_1) = \mathcal{O}$, $\mathcal{N}^+(Y_2) = \mathcal{O}$, and $\mathcal{N}^-(Y_2) = \mathcal{O}(1)$. In the quadric bundle case, in Lemma 5.10, we proved that $\mathcal{Y}_i = \{pt\} \sqcup Y_i^2$ for $i = 1, 2$, where $Y_1^2 \cong Y_2^2 \cong \mathcal{Q}^{n-3} \subset \widetilde{F} \cong \mathcal{Q}^{n-2}$, with $\widetilde{F}$ being the fiber of ϕ_τ. Hence, one has $\mathcal{N}^+(Y_1^2) = \mathcal{O}(1)$, $\mathcal{N}^-(Y_1^2) = \mathcal{O}(1) \oplus \mathcal{O}$, $\mathcal{N}^+(Y_2^2) = \mathcal{O}(1) \oplus \mathcal{O}$, and $\mathcal{N}^-(Y_2^2) = \mathcal{O}(1)$.

6. Contact manifolds

6.1. Contact manifolds of dimension 11 and 13

In this section, X_σ is a contact variety of dimension $2n+1$ with L_σ an ample line bundle on it, and $\operatorname{Pic} X_\sigma \cong \mathbb{Z}L_\sigma$. By definition, L_σ is the cokernel of the contact distribution $F_\sigma \to TX_\sigma$ with a rank $2n$ vector subbundle $F_\sigma \subset TX_\sigma$, and $\sigma \in \mathrm{H}^0(X_\sigma, \Omega X_\sigma \otimes L_\sigma)$ such that $d\sigma$ defines a nowhere degenerate pairing $F_\sigma \times F_\sigma \to L_\sigma$. In particular, $-K_{X_\sigma} = (n+1)L_\sigma$.

Contact manifolds appear in the context of quaternion-Kähler manifolds and LeBrun–Salamon conjecture in differential geometry, which asserts that every positive quaternion-Kähler manifold is a Wolf space. The algebro-geometric version of LeBrun–Salamon conjecture predicts that every Fano contact manifold is rational homogeneous and, in fact, isomorphic to the adjoint variety of a simple group, that is the closed orbit in the projectivisation of the adjoint representation of this simple group. The contact manifold coming from a quaternion-Kähler manifold admits a Kähler–Einstein metric, so that in the differential-geometric context it is not restrictive to assume that the group of its contact automorphisms is reductive.

Let us recall that the case when $\operatorname{Pic} X_\sigma \neq \mathbb{Z}L_\sigma$ is known; in such a case $(X_\sigma, L_\sigma) = (\mathbb{P}(TY), \mathcal{O}_{\mathbb{P}(TY)}(1))$ with Y a projective manifold of dimension $n+1$; see [26, Cor. 4.2], [21, Thm. 1.1], [8, Cor. 4]. Also the case in which L_σ have sufficiently many sections is known, see [1, Thm. 0.1]. For $\dim X_\sigma \leq 9$ we have the following theorem; we refer to [9, Thm. 1], [5, Thm. 1.2].

Theorem 6.1. *Let (X_σ, L_σ) be a contact Fano manifold of dimension ≤ 9 whose group of contact automorphisms G is reductive. Then G is simple and X_σ is the closed orbit in the projectivisation of the adjoint representation of G.*

In [5] the proof of the above theorem for $\dim X_\sigma = 7, 9$ is based on the analysis of the action of the maximal torus $\widehat{H}$ in the group G of contact automorphisms of X_σ. The torus $\widehat{H}$ is of rank ≥ 2 by a result of Salamon (see [32, Thm. 7.5]) reproved in [5, Thm. 6.1] in the contact case.

In Theorem 6.2 we will follow the strategy adopted in [5] to extend the above result. To this end, before recalling the main idea of [5], we briefly recall some preliminaries.

For any manifold X with an ample line bundle L and an almost faithful action of a torus H, one analyses data in the lattice M of characters of H. We recall, see [5, §2.1], that a linearization μ of L defines the polytope of fixed point $\Delta(L) := \Delta(X, L, H, \mu)$ that is the convex hull in $M_\mathbb{R}$ of the weights $\mu(Y_i) \in M$ with which H acts on the fiber of L over each point in a fixed component $Y_i \subset X^H$. Moreover, such a linearization gives also the polytope of sections $\Gamma(L) := \Gamma(X, L, H, \mu)$ which is the convex hull in $M_\mathbb{R}$ of the characters (eigenvalues) of the action of H on $H^0(X, L)$.

Fixed point components in $\mathcal{Y}$ are represented by points in M together with vectors representing the weights of the action of H on their conormal bundle; for each $Y \in \mathcal{Y}$ the set of these (possibly multiple) weights is called the *compass* and denoted by $\mathcal{C}(Y, X, H)$ or simply by $\mathcal{C}(Y)$. We refer to [5, §2.3] for details about the compass. In the contact case, because of the pairing coming from the contact form, the vectors in the compass satisfy the associated symmetry (see [5, Lem. 4.1]).

Definition 6. Given a polarized pair (X, L) with an action of an algebraic torus H and linearization μ, we define the *grid data* of the quadruple (X, L, H, μ) as follows:

(1) the isomorphism classes of connected fixed point components Y_i, for $X^H = \bigsqcup_{i\in I} Y_i$ together with the fixed point weight map

$$\mu : \mathcal{Y} = \{Y_i : i \in I\} \to M = \mathrm{Hom}(H, \mathbb{C}^*);$$

(2) the compasses $\mathcal{C}(Y_i)$ for every $Y_i \subset X^H$; and the isomorphism classes of the splitting of the normal bundle

$$\mathcal{N}_{Y_i/X} = \bigoplus \mathcal{N}^{-\nu(Y_i)}(Y_i)$$

where $\nu(Y_i) \in \mathcal{C}(Y_i)$ and $\mathcal{N}^{-\nu(Y_i)}(Y_i)$ are the eigen-subbundles of the respective weights.

The localized version of the Riemann–Roch theorem asserts that the Euler characteristic of L, $\chi^H(X, L)$ as a function graded in M depends only on the grid data under certain assumptions, see [5, Thm. A.1].

The proof of Theorem 6.1 for $\dim X_\sigma = 7, 9$ goes along the following steps:

(0) Prove that there exists a nontrivial action of a (reductive) group G with a maximal torus $\widehat{H}$ of rank r on X_σ; it is enough to show that $h^0(X_\sigma, L_\sigma) > 0$, see [32] and [5, Thm. 6.1].

(1) Prove that $\Delta(X_\sigma, L_\sigma, \widehat{H}, \mu) = \Gamma(X_\sigma, L_\sigma, \widehat{H}, \mu)$, and the vertices of this polytope are associated to isolated fixed point components [5, Lem. 4.7].

(2) Prove that $\Gamma(X_\sigma, L_\sigma, \widehat{H}, \mu)$ is associated to the adjoint representation of the group G [5, Lem. 4.5] and therefore G is semisimple [5, Lem. 4.6].

(3) Prove that G is simple and therefore $\Delta(L_\sigma) = \Gamma(L_\sigma)$ is the root polytope of G in the lattice of weights of G (see [5, Prop. 4.8]).

(4) Examine, case by case, root polytopes of simple groups and eliminate the ones which are not associated to the action on the adjoint contact variety (see [5, §5]).

(5) Once it is shown that the grid data of the quadruple $(X_\sigma, L_\sigma, \widehat{H}, \mu)$ are the same as in the adjoint contact variety case, one can conclude that X_σ is actually the adjoint variety by [5, Prop. 2.23].

We note that the starting point, that is step (0), is essential to launch the whole argument. On the other hand, steps (2), (3) and (5) in this line of argument are fairly general. Step (1) depends on a general lemma about the existence of sections of an ample line bundle L_Y on a arbitrary Fano manifold Y such that $\mathrm{Pic}\, Y = \mathbb{Z}L_Y$, and $\dim Y = n - r + 1$. In [5], a well-known fact for Fano 3-folds is used. In what follows, we present a generalization of this result for Fano 4-folds and 5-folds, that is Lemma 6.3 (see also [14, Cor. 1.3]).

The results of step (4) are summarized in [5, Thm. 5.3]. If $\dim X_\sigma \leq 13$ and $r \geq 2$ then that theorem can be improved by analysing the case of the action of a simple group of type A_2 or G_2 on X_σ. This is done in Subsection 6.3. The classification of bandwidth 3 manifolds given by Theorem 4.5 is the key ingredient in this argument.

As result we obtain the following:

Theorem 6.2. *Let (X_σ, L_σ) be a polarized pair, with X_σ contact Fano manifold of dimension ≤ 13, and $\operatorname{Pic} X_\sigma = \mathbb{Z}L_\sigma$. Assume that the group of contact automorphisms G is reductive of rank ≥ 2 (the latter is true if, e.g., $h^0(X_\sigma, L_\sigma) > 3$). Then X_σ is a rational homogeneous variety, and in particular:*

(1) *if $\dim X_\sigma = 11$ then X_σ is the closed orbit in the projectivisation of the adjoint representation of SO_9;*
(2) *if $\dim X_\sigma = 13$ then X_σ is the closed orbit in the projectivisation of the adjoint representation of SO_{10}.*

Proof. As noted above, the proof goes along the lines established in [5]. Namely, using Corollary 6.4, and applying [5, Prop. 4.8] and [5, Lem. 4.5] we are in the situation of [5, Assumption 5.2]. In particular, by that assumption we recall that the group G is simple. By contradiction, assume that G is of type A_2 or G_2. In such a case, consider the action of a rank two torus $\widehat{H} \subset G$ on (X_σ, L_σ). Due to Proposition 6.9, we find out that the grid data of $(X_\sigma, L_\sigma, \widehat{H}, \mu)$ coincide with the grid data of $(G(1, \mathcal{Q}^{n+2}), L, H_2, \mu)$ with H_2 a rank two torus contained in the maximal torus acting on $G(1, \mathcal{Q}^{n+2})$. From the proof of Propositions 6.9, and A.1 we observe that $L_{\sigma|Y} \cong L_{|Y}$ for every fixed point component Y. The equality of the grid data, together with the isomorphism $L_{\sigma|Y} \cong L_{|Y}$ are equivalent to require that $L_{\sigma|Y}$ is $\widehat{H}$-equivariantly isomorphic to $L_{|Y}$ and that $\mathcal{N}_{Y/X_\sigma}$ is $\widehat{H}$-equivariantly isomorphic to $\mathcal{N}_{Y/G(1,\mathcal{Q}^{n+2})}$, respectively. This gives an equality of $\widehat{H}$-equivariant Euler characteristics (see [5, Thm. A.1]):

$$\chi^{\widehat{H}}(X_\sigma, L_\sigma) = \chi^{H_2}(G(1, \mathcal{Q}^{n+2}), L).$$

Then, being X_σ and $G(1, \mathcal{Q}^{n+2})$ Fano, one has that $H^0(X_\sigma, L_\sigma)$, $H^0(G(1, \mathcal{Q}^{n+2}), L)$ are equal as elements of the representation ring of $\widehat{H}$. Therefore, using again Proposition 6.9, if $n = 5$ then $h^0(X_\sigma, L_\sigma) = \dim \mathrm{SO}_9 = 36$, and if $n = 6$ then $h^0(X_\sigma, L_\sigma) = \dim \mathrm{SO}_{10} = 45$. In both cases, these dimensions are bigger than the dimensions of G_2 and A_2, against [5, Assumption 5.2] for which $H^0(X_\sigma, L_\sigma)$ can be identified with the Lie algebra of G. Thus G is neither G_2 nor A_2. Now, applying [5, Thm. 5.3] we conclude that when $n = 5$ one has $(X_\sigma, L_\sigma) \cong (G(1, \mathcal{Q}^7), \mathcal{O}(1))$ with $G = \mathsf{B}_4$; while for $n = 6$ we get $(X_\sigma, L_\sigma) \cong (G(1, \mathcal{Q}^8), \mathcal{O}(1))$ with $G = \mathsf{D}_5$; hence the claim. $\square$

Remark 10. Notice that the theorem above improves [5, Thm. 5.3], since when $n = 5, 6$ the group of the contact automorphisms G cannot be of type neither G_2 nor A_2. We refer to the recent preprint [28, Thm. 6.1] where, under certain assumptions on the rank of the maximal torus, the LeBrun–Salamon conjecture has been proved in arbitrary dimension, dealing also with the cases in which G is of exceptional type. We note that with our approach of downgrading torus action to $\mathbb{C}^*$ action of bandwidth 3, the assumption that the rank of the group G is at least two is inevitable. The case of rank one group requires understanding bandwidth 4 action of $\mathbb{C}^*$ on contact manifolds. Finally, as noted above, the fact that $h^0(X_\sigma, L_\sigma) > 0$ implies the action of a reductive group of positive rank. On the other hand, $h^0(X_\sigma, L_\sigma) > 3$ implies the action of a reductive group of rank ≥ 2, because the only rank 1 groups are $\mathbb{C}^*$ and $\mathrm{PSL}(2)$. Although, at present we can not verify either of these inequalities, they seem to be almost equally hard to check, hence the assumption on rank of G being ≥ 2 is rather harmless.

6.2. Dimension of anticanonical systems

For the following result see also the recent paper [14] and references therein.

Lemma 6.3. *Let X be a Fano manifold of positive dimension ≤ 5 with* $\operatorname{Pic} X = \mathbb{Z}L$. *Then $h^0(X, L) > 1$.*

Proof. The claim is known if $\dim X \leq 3$, or if X is a Mukai variety of index $\dim X - 2$. Hence, it is enough to prove the claim for $\dim X = 4$ and $L = -K_X$, and for $\dim X = 5$ and $L = -K_X$, or $L = -K_X/2$. Moreover, because of Kodaira vanishing, we are left to prove that $\chi(X, L) = \sum_i (-1)^i h^i(X, L) > 1$.

First, assume that $\dim X = 4$. We define $\chi(t) = \chi(X, t(-K_X))$, and using the Riemann–Roch for 4-folds we get

$$\chi(t) = \frac{1}{24}c_1^4 \cdot t^4 + \frac{1}{12}c_1^4 \cdot t^3 + \frac{1}{24}(c_1^2c_2 + c_1^4) \cdot t^2 + \frac{1}{24}c_1^2c_2 \cdot t + 1$$

where $c_i = c_i(TX)$ are the Chern classes, and their intersection is evaluated at the fundamental class of X. The last coefficient is 1 because $\chi(0) = 1$. Thus we get

$$\chi(1) = \chi(0) + \frac{1}{6}c_1^4 + \frac{1}{12}c_1^2c_2 > 1.$$

The inequality follows because TX is stable (see [30], [15]) and we can use the Bogomolov inequality [25, Thm. 0.1] to get

$$c_1^2c_2 \geq \frac{\operatorname{rk} TX - 1}{2 \operatorname{rk} TX} \cdot c_1^4 > 0.$$

Now, we consider the case $\dim X = 5$. We use the notation of the previous argument; for simplicity we set $d = c_1^5$. The Hilbert polynomial $\chi(t)$ is invariant with respect to Serre's involution $t \mapsto -t - 1$. Using this involution we get two possible presentations of its decomposition in $\mathbb{R}[t]$:

$$\begin{aligned} \chi(t) &= \frac{d}{120}\left(t + \frac{1}{2}\right)(t^2 + t + a_1)(t^2 + t + a_2) \text{ or} \\ \chi(t) &= \frac{d}{120}\left(t + \frac{1}{2}\right)(t^2 + b_1t + b_2)((t+1)^2 - b_1(t+1) + b_2) \end{aligned}$$

where $da_1a_2 = 240$ and $db_2(b_2 - b_1 + 1) = 240$, respectively, because $\chi(0) = 1$. We can compare it with the Riemann–Roch formula:

$$\begin{aligned} \chi(t) = {} & \frac{1}{120}c_1^5 \cdot t^5 + \frac{1}{48}c_1^5 \cdot t^4 + \frac{1}{72}(c_1^3c_2 + c_1^5) \cdot t^3 + \frac{1}{48}c_1^2c_2 \cdot t^2 \\ & + \frac{1}{720}\left(-c_1^5 + 4c_1^2c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4\right) \cdot t + 1. \end{aligned}$$

So we get respective identities

$$c_1^3c_2 = \frac{d}{5} \cdot (3a_1 + 3a_2 + 1) \quad \text{and} \quad c_1^3c_2 = \frac{d}{5} \cdot (6b_2 - 3b_1^2 + 3b_1 + 1).$$

Using these identities, we verify that

$$\chi(1) = 3\chi(0) + \frac{1}{24} \cdot (c_1^3 c_2 + c_1^5) \quad \text{and} \quad \chi\left(\frac{1}{2}\right) = 2\chi(0) + \frac{1}{96} c_1^3 c_2 + \frac{1}{384} c_1^5$$

which, again, by the stability of TX and the Bogomolov inequality, yields the lemma. □

We now obtain the following result which improves [5, Lem. 4.7].

Corollary 6.4. *Let X_σ be a contact Fano manifold of dimension $2n+1$ with $\operatorname{Pic} X_\sigma = \mathbb{Z}L_\sigma$. Suppose that X_σ admits an almost faithful action of a torus $\widehat{H}$ of rank r. If $r \geq n-4$ then $\Gamma(X_\sigma, L_\sigma, \widehat{H}) = \Delta(X_\sigma, L_\sigma, \widehat{H})$, and every extremal component of $X_\sigma^{\widehat{H}}$ is a point.*

Proof. The proof is the same as the one of [5, Lem. 4.7] but in place of [5, Cor. 3.8] we use the respective version following from Lemma 6.3 of the present paper. □

6.3. SL_3 action on contact manifolds

In this subsection we consider the following situation; compare with [5, Assumption 5.1].

Assumption 3. Let G be a simple group of type A_2 or G_2 with a maximal two dimensional torus $\widehat{H} < G$. Assume that G acts almost faithfully via contactomorphisms on a contact manifold (X_σ, L_σ), with $\dim X_\sigma = 2n+1$ and $\operatorname{Pic} X_\sigma = \mathbb{Z}L_\sigma$. That is, the morphism $G \to \operatorname{Aut}(X_\sigma)$ has finite kernel. The linearization μ comes from the action of G on TX_σ. Assume that all extremal fixed points of the action of $\widehat{H}$ on X_σ are isolated, and the polytope $\Delta(X_\sigma, L_\sigma, \widehat{H}, \mu)$ is the root polytope $\Delta(G)$ in the lattice $\widehat{M}$ of characters of the torus $\widehat{H}$.

The following diagram is copied from [5, §5.5]. We use the notation coming from that paper.

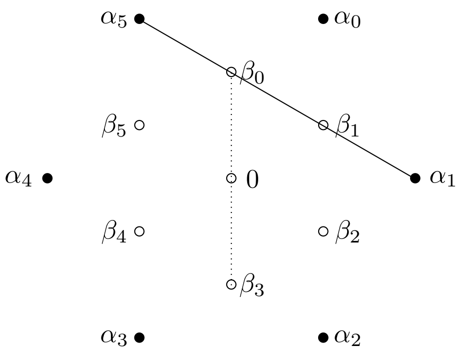


FIGURE 5. Lattice points corresponding to the action of $\widehat{H}$

By y_{α_i} we denote the extremal fixed points of the action of $\widehat{H}$; by $Y^j_{\beta_i}$ we denote the inner fixed point components associated to the weight β_i, while by Y_0 we denote central components associated to the weight 0. The indices of α's and β's are between 0 and 5; by convention they are taken modulo 6.

Fix $i \in \{0, \dots, 5\}$ and let H' be the subtorus corresponding to the projection $\pi_i \colon \widehat{M} \to \mathbb{Z}$ which maps $\alpha_{i-1}, \beta_i, \beta_{i+1}, \alpha_{i+1}$ to $1 \in \mathbb{Z}$. Using that $\rho_{X_\sigma} = 1$, and arguing as in the proof of Proposition 2.4, we deduce that there exists a unique connected component $X_i \subset X_\sigma^{H'}$ which contains the extremal fixed points y_{α_i+1}, y_{α_i-1}, and all inner components of $X_\sigma^{\widehat{H}}$ associated to β_i and β_{i+1}. On the above diagram, we indicate the (solid) line segment associated to X_i. We will use the convention that $y_{\alpha_{i-1}}$ is the sink and $y_{\alpha_{i+1}}$ is the source of the action of $\mathbb{C}^*$ on X_i.

Lemma 6.5. *Let us keep the above notation, and suppose that Assumption 3 is satisfied. Then the following hold:*

1. *X_i is a smooth connected variety of dimension $n-1$ with an ample line bundle $L_i := L_{\sigma|X_i}$,*
2. *X_i admits a natural $\mathbb{C}^*$ action and a natural linearization μ_i of L_i,*
3. *the fixed point components of $X_i^{\mathbb{C}^*}$ are $\{y_{\alpha_{i-1}}\}, Y^j_{\beta_i}, Y^j_{\beta_{i+1}}, \{y_{\alpha_{i+1}}\}$,*
4. *(X_i, L_i) is a bandwidth 3 variety with two end points, and the action of $\mathbb{C}^*$ is equalized.*

Proof. By construction, X_i is smooth and connected. Moreover, X_i admits the restricted action of the 1-dimensional torus $\mathbb{C}^* = \widehat{H}/H'$ as required in (2) (see [5, Lem. 2.10(2)]), and by [5, Lem. 2.10(3)] one has $X_i^{\mathbb{C}^*} = X_\sigma^{\widehat{H}} \cap X_i$; therefore the extremal fixed points of the $\mathbb{C}^*$ action have weights α_{i-1} and α_{i+1}, thus the bandwidth of $(X_i, L_i, \mathbb{C}^*)$ is three. We are left to check that the $\mathbb{C}^*$ action is equalized, and that $\dim X = n-1$. To this end, let us describe the compasses at the fixed point components. Taking a fixed component $Y \subset X_i^{\mathbb{C}^*}$, applying the definition of the compass, and using the projection $\pi_i \colon \widehat{M} \to \mathbb{Z}$, one has

$$\mathcal{C}(Y, X_i, \mathbb{C}^*) = \mathcal{C}(Y, X_\sigma, \widehat{H}) \cap \ker(\pi_i). \tag{5}$$

The description of the compasses for the rank two torus $\widehat{H}$ is obtained following the same proof of [5, Lem. 5.15]. In what follows, we use an exponent to denote the occurrence of the corresponding element in the compass. By [5, Lem. 5.15(1)], we obtain that:

$$\mathcal{C}(y_{\alpha_{i-1}}, X_\sigma, \widehat{H}) = \big(\alpha_i - \alpha_{i-1}, \alpha_{i-2} - \alpha_{i-1}, -\alpha_{i-1}, (\beta_i - \alpha_{i-1})^{n-1}, (\beta_{i-1} - \alpha_{i-1})^{n-1}\big). \tag{6}$$

Using (5) we deduce that $\mathcal{C}(y_{\alpha-1}, X_i, \mathbb{C}^*) = (1^{n-1})$. In a similar way, we obtain that $\mathcal{C}(y_{\alpha+1}, X_i, \mathbb{C}^*) = (-1^{n-1})$. This also implies that $\dim X = n-1$, because by definition of the compass at a fixed component Y, the elements contained in it must be in number equal to $\dim X - \dim Y$; and if we consider $Y = y_{\alpha-1}$, being the sink a point by assumption, we obtain claim (1).

For an irreducible inner fixed point component Y_{β_i} of dimension d, the proof of [5, Lem. 5.15 (2)] allows to compute

$$\begin{aligned}\mathcal{C}(Y_{\beta_i}, X_\sigma, \widehat{H}) = \big(&\alpha_i - \beta_i, \alpha_{i-1} - \beta_i, \beta_{i+2} - \beta_i, \beta_{i-2} - \beta_i, \\ &- \beta_i^{d+1}, (\beta_{i+1} - \beta_i)^{n-d-2}, (\beta_{i-1} - \beta_i)^{n-d-2}\big);\end{aligned} \tag{7}$$

and by (5) we obtain that $\mathcal{C}(Y_{\beta_i}, X_i, \mathbb{C}^*) = (1^{n-d-2}, -1)$. Repeating the same procedure for the other inner fixed point components, we may conclude that the $\mathbb{C}^*$ action on X_i is equalized, and the statement follows. □

Lemma 6.6. *Let us keep the above notation, and suppose that Assumption 3 holds. Then*

$$\mathcal{N}^-_{Y^j_{\beta_i}/X_i} \cong \left(\mathcal{N}^+_{Y^j_{\beta_i}/X_{i-1}}\right)^* \otimes L|_{Y^j_{\beta_i}}$$

Proof. The pairing $d\sigma : F_\sigma \times F_\sigma \to L_\sigma$ is invariant with respect to the action of $\widehat{H}$. Hence it determines the pairing on the normal of the eigencomponents of the normal to any fixed point component. □

Corollary 6.7. *The variety (X_i, L_i) described in Lemma 6.5 is not a scroll over $\mathbb{P}^1$ described in case* (1) *of Theorem* 4.5.

Proof. In the scroll case $\mathcal{N}^-_{Y^j_{\beta_i}/X_i} \cong \mathcal{N}^+_{Y^j_{\beta_i}/X_{i-1}} \cong \mathcal{O}$, see Remark 9, which contradicts Lemma 6.6. □

Lemma 6.8. *Suppose that Assumption* 3 *is satisfied, and keep the same notation there introduced. Then:*

(1) *if $n = 5$ there are no central components;*
(2) *if $n = 6$ one has $Y_0 = \mathbb{P}^1 \sqcup \mathbb{P}^1$, and the compass $\mathcal{C}(\mathbb{P}^1, X_\sigma, \widehat{H})$ is given by all the vectors $\pm\beta_i$ for $i = 0, 1, 2$, where each element occurs with multiplicity two.*

Proof. We first show that if $Y_{0,k}$ is an irreducible central component, then the elements $\alpha_i \notin \mathcal{C}(Y_{0,k}, X_\sigma, \widehat{H})$. Assume by contradiction that one of these elements, say α_0, belongs to the compass. Consider a subtorus $H_1 \subset \widehat{H}$ corresponding to a projection $\pi \colon \widehat{M} \to \mathbb{Z}$ sending α_0 to 0. Take a variety $Z \subset X_\sigma^{H_1}$ which contains $Y_{0,k}$ and the extremal fixed points y_{α_0}, y_{α_3}. Applying [5, Lem. 2.10(2)] such a variety Z admits the action of $\mathbb{C}^* = \widehat{H}/H_1$, with fixed locus $Z^{\mathbb{C}^*} = y_{\alpha_0} \sqcup Y_{0,k} \sqcup y_{\alpha_3}$. Moreover, by Corollary [5, Cor. 4.4] the variety Z is contact. Replacing X_i with Z and π_i with π in the formula (5), and using (6) we get $\mathcal{C}(y_{\alpha_0}, Z, \mathbb{C}^*) = (-\alpha_0)$, therefore $\dim Z = 1$. We then conclude $Z \cong \mathbb{P}^1$, so that $Y_{0,k} = \varnothing$, a contradiction. Hence, if $Y_{0,k} \neq \varnothing$, applying [5, Cor. 2.14], it follows that the only elements which belong to $\mathcal{C}(Y_{0,k}, X_\sigma, \widehat{H})$ are among the β_i's. On the other hand, using Lemma 6.5, Corollary 6.7, and Theorem 4.5 we deduce that if $n = 5$ one has $Y_{\beta_i} = \{pt\} \sqcup \mathbb{P}^1$; if $n = 6$ we have $Y_{\beta_i} = \{pt\} \sqcup \mathbb{P}^1 \times \mathbb{P}^1$.

By the above argument, if $Y_{0,k} \neq \varnothing$, we may assume that the element β_0 belongs to $\mathcal{C}(Y_{0,k}, X_\sigma, \widehat{H})$. Being $Y_{0,k}$ contact (see [5, Cor. 4.4]), its compass is symmetric (see [5, Lem. 4.1]), therefore β_3 also has to belong to the compass with the same multiplicity of β_0.

Now, take a subtorus associated to the projection of the lattice $\widehat{M}$ along the dotted line segment in Figure 5. Applying the reduction procedure explained above, we find a contact variety $Z' \subset X_\sigma^{\widehat{H}}$ which contains an irreducible fixed component of Y_{β_0}, and of Y_{β_3}; these fixed components will be respectively the sink and the source of a bandwidth two $\mathbb{C}^*$ action on Z'. We may exclude the case in which the sink (or the source) of such an action is the isolated fixed point in Y_{β_0} (resp. in Y_{β_3}); indeed in such a case, arguing as above, we would again have $Y_{0,k} = \varnothing$. Hence, the two extremal points of the $\mathbb{C}^*$ action on Z' are both isomorphic to $\mathcal{Q}^{n-4}$.

Using (5) and (7), we get $\mathcal{C}(Y_{\beta_0}, Z', \mathbb{C}^*) = (-\beta_0^{n-3})$; therefore $\dim Z' = 2n-7$; so that if $n = 5$ one has $Z' \cong \mathbb{P}^3$; if $n = 6$ we get $Z' \cong \mathbb{P}(T\mathbb{P}^3)$.

In the first case, since the extremal components are isomorphic to $\mathbb{P}^1$ with the restriction of L_σ being $\mathcal{O}(2)$ (see Theorem 4.5(2)), we are considering a $\mathbb{C}^*$ on $(\mathbb{P}^3, \mathcal{O}(2))$ having weights $(0,0,1,1)$, and such an action does not have central components.

If $n = 6$, we consider the corresponding polytope of fixed points of the big torus in SL_4 acting on Z' (see the picture of [13, Exercise 15.10]), and by a downgrading associated to the projection along one of the faces of the cube in which the polytope is inscribed we get $Y_0 = \mathbb{P}^1 \sqcup \mathbb{P}^1$. We have already observed that the compass at the central components is symmetric. Using this fact, and being $\dim Z' = 5$, one has $\mathcal{C}(\mathbb{P}^1, Z', \mathbb{C}^*) = (\beta_0, \beta_0, \beta_3, \beta_3)$. Since $\dim X_\sigma = 13$, the compass $\mathcal{C}(\mathbb{P}^1, X_\sigma, \widehat{H})$ contains 12 elements counted with their multiplicity. Therefore, repeating the same argument with the weights β_1 and β_2, we obtain the claim. $\square$

Proposition 6.9. *Suppose that Assumption 3 is satisfied. Then for $n = 5, 6$ the grid data of the quadruple $(X_\sigma, L_\sigma, \widehat{H}, \mu)$ is the same as for the quadruple $(G(1, \mathcal{Q}^{n+2}), L, H_2, \mu)$ from Proposition A.1.*

Proof. We need to compare the information about the fixed point components for X_σ with the result obtained in Proposition A.1. The information about inner components Y_{β_i} is given by downgrading to subvarieties X_i. Indeed, because of Lemma 6.5 and Corollary 6.7, we are in the situation of case (2) of Theorem 4.5; see also Remark 9. Therefore, the isomorphism classes of the components and their normal subbundles in X_i are uniquely determined. We now observe that the same holds for the central components. To this end, we apply Lemma 6.8 to (X_σ, L_σ) and to $(G(1, \mathcal{Q}^{n+2}), L)$. In particular, when $n = 6$, by the proof of the same lemma we recall that the central component $Y_0 = \mathbb{P}^1 \sqcup \mathbb{P}^1$ is contained in three distinct varieties Z_j admitting a bandwidth two $\mathbb{C}^*$ action whose sink and source are respectively associated to the weights β_0 and β_3; β_1 and β_4; β_2 and β_5. Since the normal bundle of each copy of $\mathbb{P}^1$ in Z_j is uniquely determined, also its decomposition according to the weights of the $\mathbb{C}^*$ action will be uniquely determined, and the statement follows. Thus, for $n = 5$, since the extremal component Y_{β_i} is $\mathbb{P}^1$ with restriction of L being $\mathcal{O}(2)$ it follows that $Z'_i \cong \mathbb{P}^3$ and $L_{\sigma|Z'_i} \cong \mathcal{O}(2)$, and there are no central components. Moreover the eigenbundle $\mathcal{N}^{\beta_i}(Y_{\beta_i})$ is also uniquely determined and equal to $\mathcal{O}(1)^2$. If $n = 6$, since the component Y_{β_i} is $\mathbb{P}^1 \times \mathbb{P}^1$ we get $b_2(Z'_i) = 2$ and one has $Z'_i \cong \mathbb{P}(T\mathbb{P}^3)$. In this latter case, we consider the corresponding polytope of fixed points of the big torus in SL_4 action on Z (see the picture of [13, Exercise 15.10]), and by downgrading associated to the projection along one of the faces of the cube in which the polytope is inscribed we get $Y_0 = \mathbb{P}^1 \sqcup \mathbb{P}^1$. Finally, we note that the normal bundle of each of the fixed point components in Z'_i is uniquely determined together with its decomposition according to the quotient torus action. $\square$

A. Embedding SL_3 into classical linear groups

In this appendix we summarize information about the structure of the adjoint variety X_{adj} of a classical simple algebraic group G from the viewpoint of a linear embedding of the group SL_3 into the group in question. Let us recall that X_{adj} is the closed orbit in the projectivisation of the adjoint representation of G.

We focus on the case of a linear embedding $\mathrm{SL}_3 \hookrightarrow \mathrm{SO}_m$ which yields the action of SL_3 on the adjoint variety of SO_m. The results of this section are stated in Proposition A.1. The conclusion is that the associated bandwidth 3 variety which we described in Subsection 6.3 (see Lemma 6.5) yields the case (2) in Theorem 4.5.

First, let us recall that the root systems of SL_4 and SO_6 coincide and their adjoint variety is $\mathbb{P}(T\mathbb{P}^3)$. We consider a natural embedding $\mathrm{SL}_3 \hookrightarrow \mathrm{SL}_4$ which comes with the linear embedding of the standard representation W_3 of SL_3 into the standard representation of SL_4, that is $W_4 = W_3 \oplus \mathbb{C}$, as representation of SL_3, where $\mathbb{C}$ denotes the trivial representation of SL_3. The adjoint representation $\mathrm{adj}(\mathrm{SL}_4)$ of SL_4 is an irreducible summand of $W_4 \oplus W_4^* = \mathrm{adj}(\mathrm{SL}_4) \oplus \mathbb{C}$; thus as a representation of SL_3 it decomposes as

$$\mathrm{adj}(\mathrm{SL}_4) = W_3 \oplus \mathrm{adj}(\mathrm{SL}_3) \oplus W_3^* \oplus \mathbb{C}$$

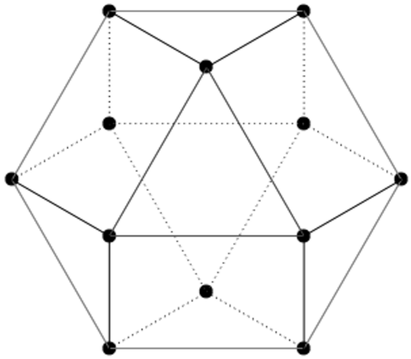
On the other hand, an embedding $\mathrm{SL}_3 \hookrightarrow \mathrm{SO}_6$ is defined as follows

$$\mathrm{SL}_3 \ni A \longrightarrow \widehat{A} = \begin{pmatrix} 0 & (A^{\intercal})^{-1} \\ A & 0 \end{pmatrix} \in \mathrm{SO}_6,$$

where SO_6 is understood as the group of matrices preserving the form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, or a quadric in $\mathbb{P}^5$ given by the equation $x_1y_1 + \cdots + x_3y_3 = 0$. If V_6 is the standard representation of SO_6, then as a representation of $\mathrm{SL}_3 \hookrightarrow \mathrm{SO}_6$ it decomposes as $W_3 \oplus W_3^*$. The second exterior power $\bigwedge^2 V_6$ is the adjoint representation of SO_6 and, as the representation of SL_3, it can be written again as

$$\wedge^2(W_3 \oplus W_3^*) = \wedge^2 W_3 \oplus W_3 \otimes W_3^* \oplus \wedge^2 W_3^* \oplus \mathbb{C} = W_3 \oplus \mathrm{adj}(\mathrm{SL}_3) \oplus W_3^* \oplus \mathbb{C}$$

In terms of the action of the Cartan torus H_3 of both SL_4 and SO_6, the fixed points of the action on the adjoint variety via the standard linearization map are mapped to roots in the associated rank three lattice of weights M_3. As points in the space $M_3 \otimes \mathbb{R}$ they are vertices of a cuboctahedron (rectified cube). The embedding of SL_3 in each of these groups yields an embedding of Cartan tori $H_2 \hookrightarrow H_3$, with H_2 the 2-dimensional torus contained in SL_3. Thus we get the projection of the corresponding lattices of weights $M_3 \to M_2$. The diagram below describes the projection of the roots visualized as a projection of the cuboctahedron. The front and the rear faces are associated to representation W_3 and W_3^*, while the hexagonal cross-section is associated to the representation $\mathrm{adj}(\mathrm{SL}_3)$.



By [5, Lem. 4.5], the vertices of the root polytopes are associated to the fixed points of the action of the Cartan torus on X_{adj} under the fixed point weight map for the standard linearization. Thus, it follows that the dots $\bullet$ are also the images of these fixed points.

We can write the above description directly in terms of the coordinates of the action of the torus $H_2 \subset \mathrm{SL}_3$. Its associated lattice of characters is $M_2 = \bigoplus_{i=1}^3 \mathbb{Z}e_i / \sum e_i$. We choose the coordinates (x_i, y_i) on V_6 so that the weights of the action of H_2 on $V_6 = W_3 \oplus W_3^*$ are e_i on x_i and $-e_i$ on y_i. There are six fixed points of the action of H on the quadric $\mathcal{Q}^4 = \{\sum x_i y_i = 0\} \subset \mathbb{P}(V_6)$, each associated to the weight $\pm e_i$.

Recall that X_{adj} for SO_6 parametrizes the lines contained in the quadric. If $\alpha \wedge \beta$ denotes the line for which only coordinates α and β do not vanish, then the H_2 invariant lines contained in the quadric $\sum x_i y_i = 0$ are either of the two types:

- $x_i \wedge y_j$ for $i \neq j$
- $x_i \wedge x_j$, or $y_i \wedge y_j$ for $i \neq j$.

There are six lines of each type; the weight of the action of H_2 on the line of the first type is $e_i - e_j$, while on the latter type it is $\pm e_i$. These are the fixed points of the action described above as a projection of the vertices of the cuboctahedron.

When $m \geq 7$ we take a natural inclusion

$$\mathrm{SL}_3 \hookrightarrow \mathrm{SO}_6 \times \mathrm{SO}_{m-6} \hookrightarrow \mathrm{SO}_m$$

for a suitable decomposition of the standard SO_m representation $V_m = V_6 \oplus V_{m-6}$. As before, we decompose the resulting SL_3 representation:

$$\wedge^2 V_m = W_3 \oplus \mathrm{adj}(\mathrm{SL}_3) \oplus \mathbb{C} \oplus W_3^* \oplus (W_3 \otimes V_{m-6}) \oplus (W_3^* \otimes V_{m-6}) \oplus \wedge^2 V_{m-6}$$

where the representation V_{m-6} is trivial as the SL_3 representation.

We extend the preceding discussion to the SO_m invariant quadric $\widehat{\mathcal{Q}} \subset \mathbb{P}(V_m)$ such that $\widehat{\mathcal{Q}} \cap \mathbb{P}(V_6) = \{\sum x_i y_i = 0\} = \mathcal{Q}$. That is, for suitably chosen coordinates (z_i) in V_{m-6} we have

$$\widehat{\mathcal{Q}} = \{x_1 y_1 + x_2 y_2 + x_3 y_3 + z_1^2 + \cdots + z_{m-6}^2 = 0\}.$$

By $\mathcal{Q}^\perp$ we denote the intersection $\widehat{\mathcal{Q}}$ with $\mathbb{P}(V_{m-6}) = \{x_i = y_j = 0\}$. That is, $\mathcal{Q}^\perp = \{z_1^2 + \cdots + z_{m-6}^2 = 0\}$ is a quadric of dimension $m-8$. Now, apart from the lines contained in $\mathcal{Q}$, we have extra components in the fixed point locus of the action of $H_2 \subset \mathrm{SL}_3$ on the grassmannian of lines on $\widehat{\mathcal{Q}}$. They are as follows:

- lines joining fixed points of the action of H_2 on $\mathcal{Q}$ with any point in $\mathcal{Q}^\perp$; they are contained in the subspaces $W_3 \otimes V_{m-6}$ and $W_3^* \otimes V_{m-6}$ in the decomposition of $\bigwedge^2 V_{m-6}$ above; therefore there are six of such components associated to the weights $\pm e_i$;
- lines contained in $\mathcal{Q}^\perp$ for $m \geq 9$; they are contained in the subspace $\wedge^2 V_{m-6}$ in the above decomposition; therefore these fixed point component(s) are associated to the weight 0.

We summarize the discussion in the following.

Proposition A.1. *Assume the situation as above. The following is the list of the fixed components of the fixed point locus of the action of the 2-dimensional torus $H_2 \subset \mathrm{SL}_3 \subset \mathrm{SO}_m$ on the Grassmannian of lines in the quadric $\mathcal{Q}^{m-2}$ denoted by $G(1, \mathcal{Q}^{m-2})$, which is a contact variety of dimension $2m-7$. Let us denote by L an ample line bundle generating* $\operatorname{Pic} G(1, \mathcal{Q}^{m-2})$.

(1) *For $m \geq 6$ one point extremal components associated to weights $\pm e_i + \pm e_j$, $i \neq j$;*
(2) *for $m \geq 6$ one point components associated to weights $\pm e_i$;*
(3) *for $m \geq 8$ additional components associated to weights $\pm e_i$ which are quadrics of dimension $m-8$; in particular they are*
   1. *two points for each weight, for $m = 8$;*
   2. *a conic, that is $\mathbb{P}^1$ with $L_{|\mathbb{P}^1} \cong \mathcal{O}(2)$, for $m = 9$;*
   3. *a quadric $\mathbb{P}^1 \times \mathbb{P}^1$ with restriction of L being $\mathcal{O}(1,1)$, for $m = 10$;*
   4. *a quadric $\mathcal{Q}^{m-8}$ with restriction of L being $\mathcal{O}(1)$, for $m \geq 11$;*
(4) *central component(s) for $m \geq 10$ which is the grassmannian of lines in the quadric $\mathcal{Q}^{m-8}$, in particular*
   1. *$\mathbb{P}^1 \sqcup \mathbb{P}^1$ for $m = 10$;*
   2. *an irreducible variety for $m \geq 11$.*

A similar discussion can be made in the case of classical linear groups. In the following table we present adjoint varieties as well as bandwidth 3 varieties and central components associated to the downgrading of the action to a linear embedding of SL_3. Notation: G is the group, X_{adj} is the adjoint variety for the group, $\dim X_{\mathrm{adj}} = 2n+1$. In the spirit of Subsection 6.3, X_i is the bandwidth 3 variety associated to restricting and downgrading of the group action following the embedding $\mathrm{SL}_3 \hookrightarrow G$, $\dim X_i = n-1$; moreover $\mathcal{Y}_*$ is the set of fixed point components in $\mathcal{Y}_1$ or $\mathcal{Y}_2$. Finally, Y_0 is the union of fixed point components associated to the weight 0.

n	G	$\operatorname{rk} G$	X_{adj}	X_i	Y_*	Y_0
3	SO_7	3	$G(1, \mathcal{Q}^5)$	$\mathbb{P}^1 \times \mathbb{P}^1$	$\bullet$	$\varnothing$
4	SO_8	4	$G(1, \mathcal{Q}^6)$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$\bullet \sqcup \bullet \sqcup \bullet$	$\varnothing$
5	SO_9	4	$G(1, \mathcal{Q}^7)$	$\mathbb{P}^1 \times \mathcal{Q}^3$	$\bullet \sqcup \mathbb{P}^1$	$\varnothing$
6	SO_{10}	5	$G(1, \mathcal{Q}^8)$	$\mathbb{P}^1 \times \mathcal{Q}^4$	$\bullet \sqcup \mathbb{P}^1 \times \mathbb{P}^1$	$\mathbb{P}^1 \sqcup \mathbb{P}^1$
7	SO_{11}	5	$G(1, \mathcal{Q}^9)$	$\mathbb{P}^1 \times \mathcal{Q}^5$	$\bullet \sqcup \mathcal{Q}^3$	$\mathbb{P}^3$
≥ 8	SO_{n+4}	$\lfloor n/2 \rfloor$	$G(1, \mathcal{Q}^{n+2})$	$\mathbb{P}^1 \times \mathcal{Q}^{n-2}$	$\bullet \sqcup \mathcal{Q}^{n-4}$	$G(1, \mathcal{Q}^{n-4})$
≥ 3	Sp_{2n+2}	$n+1$	$\mathbb{P}^{2n+1}$	$\mathbb{P}^1$	$\varnothing$	$\mathbb{P}^{2n-5}$
≥ 3	SL_{n+2}	$n+1$	$\mathbb{P}(T\mathbb{P}^{n+1})$	$\mathbb{P}^{n-2} \sqcup \mathbb{P}^{n-2}$	$\mathbb{P}^{n-3}$	$\mathbb{P}(T\mathbb{P}^{n-2})$

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