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TOPOLOGICAL LOOPS HAVING SOLVABLE INDECOMPOSABLE LIE GROUPS AS THEIR MULTIPLICATION GROUPS

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Abstract. We prove that the solvability of the multiplication group $\mathrm{Mult}(L)$ of a connected simply connected topological loop L of dimension three forces that L is classically solvable. Moreover, L is congruence solvable if and only if either L has a non-discrete centre or L is an abelian extension of a normal subgroup $\mathbb{R}$ by the 2-dimensional non-abelian Lie group or by an elementary filiform loop. We determine the structure of indecomposable solvable Lie groups which are multiplication groups of three-dimensional topological loops. We find that among the six-dimensional indecomposable solvable Lie groups having a four-dimensional nilradical there are two one-parameter families and a single Lie group which consist of the multiplication groups of the loops L. We prove that the corresponding loops are centrally nilpotent of class 2.

1. Introduction

The multiplication group $\mathrm{Mult}(L)$ and the inner mapping group $\mathrm{Inn}(L)$ of a loop L are important tools for the investigation of the structure of L since there are strong connections between the structure of the groups $\mathrm{Mult}(L)$ and $\mathrm{Inn}(L)$ and that of L. In [B1] R. H. Bruck proved that if the group $\mathrm{Mult}(L)$ is nilpotent, then the loop L is centrally nilpotent and the group $\mathrm{Inn}(L)$ is abelian. In [V] A. Vesanen showed that if the loop L is finite and the group $\mathrm{Mult}(L)$ is solvable, then L is classically solvable; this means there exists a series of subloops of L of the form $\{e\} = L_0 \le L_1 \le \cdots \le L_n = L$ such that L_{i-1} is a normal subloop in L_i and L_i/L_{i-1} is an abelian group for all $i = 1, \dots, n$. Since the variety of loops is congruence modular in [SV1], D. Stanovský and P. Vojtěchovský

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developed commutator theory for loops following universal algebra. A loop L is called congruence solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of normal subloops of L such that every factor L_i/L_{i-1} is abelian in L/L_{i-1}. For loops, congruence solvability is strictly stronger than classical solvability (cf. [SV1, Construction 9.1 and Example 9.3, pp. 27, 29] and [F, Exercise 10, pp. 44–45]). In [SV2] D. Stanovský and P. Vojtěchovský proved that a loop L is congruence solvable if and only if it is obtained by iterated abelian extensions.

In [NS1] P. T. Nagy and K. Strambach investigated consistently topological and differentiable loops as topological and differentiable sections in Lie groups. In this paper we follow their treatment and study topological loops L having a solvable Lie group K as their multiplication group. In this case K is a Lie transformation group acting transitively and effectively on the topological space L. Using the transitive actions of Lie groups on the space $\mathbb{R}^n$, $n \leq 3$, we show that topological loops L of dimension 3 with solvable multiplication group are classically solvable. In connection to abelian extensions we find, necessary and sufficient condition for L to be congruence solvable (see Theorem 8).

The question, which groups can be realized as the multiplication groups and the inner mapping groups of loops motivates a lot of research on loops and their relation to groups (cf. [C], [D], [Ma], [NV], [NK2]). The key concept for answering this question is the connected transversals which were introduced by T. Kepka and M. Niemenmaa (cf. [NK1]).

The criteria in [NK1] are applied successfully for Lie groups to be the multiplication groups of topological loops (cf. [F1]–[F5]). In [F1] we proved that only special nilpotent Lie groups, the elementary filiform Lie groups of dimension ≥ 4, are the multiplication groups of 2-dimensional connected topological proper loops. In [F3] we determined the solvable non-nilpotent connected simply connected Lie groups of dimension ≤ 5 which are the multiplication groups for 3-dimensional topological loops. Since this classification did not give any example of a topological loop L having an indecomposable solvable Lie group (i.e., a Lie group which is not the direct product of proper connected Lie groups) as the group $\mathrm{Mult}(L)$ of L in the present paper, we turn our attention to this type of group.

In Theorems 9, 11, 12 we give the precise structure of the 3-dimensional connected simply connected loops L and their multiplication groups if $\mathrm{Mult}(L)$ are solvable indecomposable Lie groups. Since the isomorphism classes of the 6-dimensional solvable Lie algebras are fully known (cf. [Mu], [ST], [T]) we applied our results on the one hand for Lie algebras having 2-dimensional centre and on the other hand for those which have 4-dimensional nilradical. We show that the 6-dimensional solvable indecomposable Lie groups with one of the following properties:

- they have discrete centre and correspond to 4-dimensional nilradicals,
- they have 1-dimensional centre and belong to 4-dimensional non-abelian nilradicals,
- they have 2-dimensional centre

are not the multiplication groups of 3-dimensional topological loops (cf. Propositions 10 and 14).

In Section 5 we find that among the 6-dimensional solvable indecomposable Lie algebras with 1-dimensional centre and 4-dimensional abelian nilradicals there are two classes of Lie algebras depending on a real parameter and a single Lie algebra which consist of the Lie algebras of the multiplication groups of 3-dimensional simply connected topological loops. All these Lie algebras have 3-dimensional abelian commutator subalgebras and their nilradical has an abelian complement in the Lie algebra. We prove that the corresponding loops L are centrally nilpotent of class 2 and determine their inner mapping groups.

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2. Preliminaries

A set L with a binary operation $(x, y) \mapsto x \cdot y$ is called a loop if there exists an element $e \in L$ such that $x = e \cdot x = x \cdot e$ holds for all $x \in L$ and for each $x \in L$ the left translations $\lambda_x : L \to L, \lambda_x(y) = x \cdot y$ and the right translations $\rho_x : L \to L$, $\rho_x(y) = y \cdot x$ are bijections of L. A loop L is proper if it is not a group. The left and right division operations on L are defined by the maps $(x, y) \mapsto x\backslash y = \lambda_x^{-1}(y)$, respectively $(x, y) \mapsto y/x = \rho_x^{-1}(y)$, $x, y \in L$. Let $\mu_x : L \to L$ be the map $\mu_x(y) = y\backslash x$ and hence $\mu_x^{-1}(y) = x/y$. The permutation groups $\mathrm{Mult}(L) = \langle \lambda_x, \rho_x;\ x \in L\rangle$, $T\mathrm{Mult}(L) = \langle \lambda_x, \rho_x, \mu_x;\ x \in L\rangle$ are called the multiplication group and the total multiplication group of L. Let $\mathrm{Inn}(L)$, respectively $T\mathrm{Inn}(L)$ be the stabilizer of the identity element $e \in L$ in $\mathrm{Mult}(L)$, respectively in $T\mathrm{Mult}(L)$. They form a subgroup of $\mathrm{Mult}(L)$, respectively in $T\mathrm{Mult}(L)$ and call the inner mapping group, respectively the total inner mapping group of L.

The kernel of a homomorphism $\alpha : (L, \cdot) \to (L', *)$ of a loop L into a loop L' is a normal subloop N of L. A word W is a formal product of letters $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ and their inverses, where $t(\bar{x}) = t(x_1, \dots, x_n)$ is a loop term. Upon substituting elements u_i of a particular loop L for the variables x_i in a word W and upon interpreting $\lambda_{t(\bar{x})}$, $\rho_{t(\bar{x})}$ as translations of L, we obtain $W_{\bar{u}}$, an element of $\mathrm{Mult}(L)$. If $W_{\bar{u}}(e) = e$ for every loop L with identity element e and every assignment of elements $u_i \in L$ we say that W is an inner word. The concept of tot-inner word is defined similarly allowing $\mu_{t(\bar{x})}$ as generating letters.

The following result describes the commutator of two normal subloops in purely loop theoretical fashion (cf. [SV1]).

Theorem 1. *Let $\mathcal{W}$ be a set of tot-inner words such that for every loop L one has $T\mathrm{Inn}(L) = \langle W_{\bar{u}} : W \in \mathcal{W}, u_i \in L\rangle$. Let L be a loop and N_1, N_2 be two normal subloops of L. The commutator $[N_1, N_2]_L$ is the smallest normal subloop of L containing the set $\{W_{\bar{u}}(a)/W_{\bar{v}}(a) : W \in \mathcal{W}, a \in N_1, u_i, v_i \in L, u_i/v_i \in N_2\}$.*

Let $T_x = \rho_x^{-1}\lambda_x$, $U_x = \rho_x^{-1}\mu_x$, $L_{x,y} = \lambda_{xy}^{-1}\lambda_x\lambda_y$, $R_{x,y} = \rho_{yx}^{-1}\rho_x\rho_y$, $M_{x,y} = \mu_{y\backslash x}^{-1}\mu_x\mu_y$. A suitable set of tot-inner words in Theorem 1 is for instance $\mathcal{W} = \{T_x, U_x, L_{x,y}, R_{x,y}, M_{x,y}\}$.

A normal subloop N of L is called central, respectively abelian in L if $[N, L]_L = \{e\}$, respectively $[N, N]_L = \{e\}$. The centre $Z(L)$ of a loop L consists of all elements z which satisfy the equations $zx \cdot y = z \cdot xy,\ x \cdot yz = xy \cdot z,\ xz \cdot y = x \cdot zy,\ zx = xz$

for all $x, y \in L$. A normal subloop is central in L if and only if it is a subloop of $Z(L)$. If we put $Z_0 = e$, $Z_1 = Z(L)$ and $Z_i/Z_{i-1} = Z(L/Z_{i-1})$, then we obtain a series of normal subloops of L. If Z_{n-1} is a proper subloop of L but $Z_n = L$, then L is centrally nilpotent of class n. A loop L is called classically solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of subloops of L such that, for every $i = 1, 2, \ldots, n$, L_{i-1} is normal in L_i and the factor L_i/L_{i-1} is a commutative group. A loop L is said to be congruence solvable if there is a series $\{e\} = L_0 \leq L_1 \leq \cdots \leq L_n = L$ of normal subloops of L such that every factor loop L_i/L_{i-1} is abelian in L/L_{i-1}. Every centrally nilpotent loop is congruence solvable. Let $(A, +, 0)$ be a commutative group, let $(F, \cdot, e)$ be a loop, let $\varphi, \phi : F \times F \to \mathrm{Aut}(A)$ be functions with $\varphi(y, e) = \mathrm{Id} = \phi(e, y)$ and $\theta : F \times F \to A$ be a function with $\theta(e, y) = 0 = \theta(y, e)$ for every $y \in F$. The multiplication

$$(x, a) \oplus (y, b) = (x \cdot y, \varphi(x, y)(a) + \phi(x, y)(b) + \theta(x, y))$$

defines a loop on $F \times A$, denoted by $L = F \oplus_\Gamma A$, which is called the abelian extension of the normal subgroup A by F over the cocycle $\Gamma = (\varphi, \phi, \theta)$. A loop L is called an iterated abelian extension if it has the form

$$((((A_0 \oplus_{\Gamma_1} A_1) \oplus_{\Gamma_2} A_2) \oplus_{\Gamma_3} \cdots \oplus_{\Gamma_{k-2}} A_{k-2}) \oplus_{\Gamma_{k-1}} A_{k-1}) \oplus_{\Gamma_k} A_k,$$

where A_i, $i = 0, \ldots, k$, are abelian groups and all extensions are abelian (cf. [SV2], Section 5 and [Mo, Def., p. 380]).

The following assertion is proved in [SV2, Cor. 5.1, p. 380].

Lemma 2. *A loop L is congruence solvable if and only if it is an iterated abelian extension.*

The next assertion was proved in [A, Thms. 3, 4, and 5], in [B2, IV.1, Lem. 1.3], and in [F5, Lem. 2.3].

Lemma 3. *Let L be a loop with multiplication group $\mathrm{Mult}(L)$ and identity element e.*

(i) *Let α be a homomorphism of the loop L onto the loop $\alpha(L)$ with kernel N. Then α induces a homomorphism of the group $\mathrm{Mult}(L)$ onto the group $\mathrm{Mult}(\alpha(L))$. Let $M(N)$ be the set $\{m \in \mathrm{Mult}(L);\ xN = m(x)N \text{ for all } x \in L\}$. Then $M(N)$ is a normal subgroup of $\mathrm{Mult}(L)$ containing the multiplication group $\mathrm{Mult}(N)$ of the loop N and the multiplication group of the factor loop L/N is isomorphic to $\mathrm{Mult}(L)/M(N)$.*

(ii) *For every normal subgroup $\mathcal{N}$ of $\mathrm{Mult}(L)$ the orbit $\mathcal{N}(e)$ is a normal subloop of L and $\mathcal{N} \leq M(\mathcal{N}(e))$.*

Let K be a group, let $S \leq K$, and let A and B be two left transversals to S in K. We say that A and B are S-connected if $a^{-1}b^{-1}ab \in S$ for every $a \in A$ and $b \in B$. The core $\mathrm{Co}_K(S)$ of S in K is the largest normal subgroup of K contained in S. If L is a loop, then $\Lambda(L) = \{\lambda_a;\ a \in L\}$ and $R(L) = \{\rho_a;\ a \in L\}$ are $\mathrm{Inn}(L)$-connected transversals in the group $\mathrm{Mult}(L)$. In [NK1] the authors established a purely group theoretical characterization for a group K to be the multiplication group of L.

Lemma 4. *A group K is isomorphic to the multiplication group of a loop if and only if there exists a subgroup S with $\mathrm{Co}_K(S) = 1$ and S-connected transversals A and B satisfying $K = \langle A, B\rangle$.*

Lemma 5. *Let L be a loop with multiplication group $\mathrm{Mult}(L)$ and inner mapping group $\mathrm{Inn}(L)$. Then the normalizer $N_{\mathrm{Mult}(L)}(\mathrm{Inn}(L))$ is the direct product $\mathrm{Inn}(L) \times Z$, where Z is the centre of $\mathrm{Mult}(L)$, and $\mathrm{Co}_{\mathrm{Mult}(L)}(\mathrm{Inn}(L)) = \{1\}$.*

A loop L is called topological if L is a topological space and the binary operations $(x, y) \mapsto x \cdot y$, $(x, y) \mapsto x\backslash y$, $(x, y) \mapsto y/x : L \times L \to L$ are continuous. In general the multiplication group of L is a topological transformation group that does not have a natural (finite dimensional) differentiable structure. If the multiplication group of L is a Lie group K, then K is a Lie transformation group acting transitively and effectively on L. Moreover, there is a Lie subgroup S of K with $\mathrm{Co}_K(S) = 1$ and S-connected continuous transversals A and B with $K = \langle A, B\rangle$.

We often use the following lemma. Its first assertion is proved in [H, IX.1], the second assertion is showed in [F2, Lem. 3.3, p. 390].

Lemma 6. *For every connected topological loop there exists the universal covering loop L. If L is a 3-dimensional connected simply connected topological loop having a solvable Lie group as its multiplication group, then it is homeomorphic to $\mathbb{R}^3$.*

The elementary filiform Lie group $\mathcal{F}_n$ is the simply connected Lie group of dimension $n \geq 3$ such that its Lie algebra has a basis $\{e_1, \dots, e_n\}$ with $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n - 1$. A 2-dimensional simply connected loop $L_{\mathcal{F}}$ is called an elementary filiform loop if its multiplication group is an elementary filiform group $\mathcal{F}_n$, $n \geq 4$. Every elementary filiform loop is centrally nilpotent of class 2 ([F1, p. 420]). A transitive action of a Lie group G on a manifold M is called primitive, if on M there is no G-invariant foliation with connected fibres of positive dimension smaller than dim M. A Lie algebra is called indecomposable, if it is not the direct sum of two proper ideals.

Now we collect the known results about the 3-dimensional topological loops having solvable Lie groups as their multiplication groups (cf. [F2, Lems. 3.4, 3.5, 3.6 and Props. 3.7, 3.8], [A, pp. 392–393, Thm. 11], [F3, Thm. 6, Sects. 4 and 5], [F5, Props. 2.6, 2.7]).

Lemma 7. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $\mathrm{Mult}(L)$ is a solvable Lie group.*

a) *Then the centre Z of the group $\mathrm{Mult}(L)$ and the centre $Z(L) = Z(e)$ of the loop L, where e is the identity of L, are isomorphic. The centre Z is either discrete or it has dimension 1 or 2.*
b) *If dim$(Z(L)) = 1$ and the factor loop $L/Z(L)$ is isomorphic to the group $\mathbb{R}^2$ or if dim$(Z(L)) = 2$, then L is centrally nilpotent of class 2 and the inner mapping group $\mathrm{Inn}(L)$ of L is abelian.*
c) *If dim$(Z(L)) = 2$, then $\mathrm{Mult}(L)$ is a semidirect product of the group $V \cong \mathbb{R}^m$, $m \geq 3$, by a group $Q \cong \mathbb{R}$ such that $V = Z \times \mathrm{Inn}(L)$, where $\mathbb{R}^2 = Z \cong Z(L)$ is the centre of $\mathrm{Mult}(L)$. If $\mathrm{Mult}(L)$ is indecomposable, then for every 1-dimensional connected subgroup N of Z the orbit $N(e)$ is a connected central subgroup of L such that the factor loop $L/N(e)$ is not isomorphic to $\mathbb{R}^2$.*

d) *If L has a* 1*-dimensional connected normal subloop N, then N is isomorphic to the group $\mathbb{R}$ and we have the following possibilities:*
   (i) *The factor loop L/N is isomorphic to $\mathbb{R}^2$. Then N is contained in the centre of L and the group* $\mathrm{Mult}(L)$ *is a semidirect product of the group $P \cong \mathbb{R}^m$, $m \geq 2$ by a group $Q \cong \mathbb{R}^2$ such that $P = C \times \mathrm{Inn}(L)$, where $\mathbb{R} = C \cong N$ is a central subgroup of* $\mathrm{Mult}(L)$.
   (ii) *The loop L/N is isomorphic either to the non-abelian* 2*-dimensional Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_{\mathcal{F}}$. Then the group* $\mathrm{Mult}(L)$ *has a normal subgroup S containing* $\mathrm{Mult}(N) \cong \mathbb{R}$ *such that the factor group* $\mathrm{Mult}(L)/S$ *is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ if $L/N \cong \mathcal{L}_2$ or to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_{\mathcal{F}}$.*

e) *The indecomposable solvable non-nilpotent Lie groups of dimension ≤ 5 are not the multiplication groups of* 3*-dimensional topological loops. The centre of every* 3*-dimensional connected topological proper loop having an at most* 6*-dimensional indecomposable nilpotent Lie group as its multiplication group has dimension* 1.

3. The structure of indecomposable solvable multiplication groups of 3-dimensional topological loops

Let L be a 3-dimensional connected simply connected topological proper loop such that its group $\mathrm{Mult}(L)$ is a solvable Lie group. By Lemma 6 the loop L is homeomorphic to $\mathbb{R}^3$. The solvable Lie group $\mathrm{Mult}(L)$ has a minimal non-trivial connected normal subgroup K of dimension 1 or 2. By Lemma 3 the orbit $K(e)$ is a connected normal subloop of L. Since the core $\mathrm{Co}_{\mathrm{Mult}(L)}(\mathrm{Inn}(L))$ is trivial one has $K(e) \neq \{e\}$. Hence the dimension of $K(e)$ is 1 or 2. Therefore the group $\mathrm{Mult}(L)$ acts transitively, effectively and imprimitively on the topological space L homeomorphic to $\mathbb{R}^3$. According to [L], p. 141, there are three classes of Lie groups G acting imprimitively on $\mathbb{R}^3$:

I. In $\mathbb{R}^3$ there is a G-invariant foliation $\mathcal{F}$ with 2-dimensional connected fibres D, but there is no G-invariant foliation of D with 1-dimensional connected fibres.

II. In $\mathbb{R}^3$ there is a G-invariant foliation $\mathcal{F}$ with 1-dimensional connected fibres C, but there is no G-invariant foliation with 2-dimensional fibres D which are unions of fibres C.

III. In $\mathbb{R}^3$ there is a G-invariant foliation $\mathcal{F}$ with 1-dimensional connected fibres C and there is a G-invariant foliation with 2-dimensional fibres D which are unions of fibres C.

If the group $\mathrm{Mult}(L)$ belongs to the I. class, then the loop L has a 2-dimensional connected normal subloop M such that M has no 1-dimensional connected normal subloop. Since M has a Lie group as its multiplication group, M is either a 2-dimensional Lie group or an elementary filiform loop. All these loops have a 1-dimensional normal subgroup (cf. [F1, p. 420]). This contradiction yields that $\mathrm{Mult}(L)$ is not in the I. class.

If the group $\mathrm{Mult}(L)$ belongs to the II. class, then the loop L has a 1-dimensional connected normal subloop N but there does not exist any 2-dimensional connected normal subloop M of L which contains N. Among the Lie algebras acting locally primitively on $\mathbb{R}^2$ only

$$\mathbf{g}_1 = \langle \partial/\partial x, \partial/\partial y, \alpha(x\partial/\partial x + y\partial/\partial y) + y\partial/\partial x - x\partial/\partial y\rangle, \alpha \geq 0, \text{ and}$$
$$\mathbf{g}_2 = \langle \partial/\partial x, \partial/\partial y, x\partial/\partial x + y\partial/\partial y, y\partial/\partial x - x\partial/\partial y\rangle$$

are solvable (cf. [G, p. 341], also [L, Thm. 34, p. 378]). Hence the Lie algebra $\mathbf{mult(L)}$ of $\mathrm{Mult}(L)$ is either isomorphic to one of the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, or it has a proper subalgebra isomorphic to $\mathbf{g}_i$, $i = 1, 2$. The first case is impossible since none of the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, is the Lie algebra of the multiplication group of a 3-dimensional topological loop (cf. [F3, Sect. 4]). In the second case one has

$$\mathbf{mult(L)} = \langle X_1 + \phi_1(x, y, z)\partial/\partial z, \ldots, X_k + \phi_k(x, y, z)\partial/\partial z, \\ F_1(x, y, z)\partial/\partial z, \ldots, F_{n-k}(x, y, z)\partial/\partial z\rangle.$$

where $X_1, \ldots, X_k$ are the basis elements of $\mathbf{g}_i$, $i = 1, 2$, according to whether $\mathbf{mult(L)}$ contains the subalgebra isomorphic to $\mathbf{g}_i$. Moreover, the Lie subgroup A of $\mathrm{Mult}(L)$ corresponding to the $(n-k)$-dimensional subalgebra $\mathbf{a} = \langle F_i(x, y, z)\partial/\partial z\rangle$, $i = 1, \ldots, n-k$, leaves every 1-dimensional connected left coset xN, $x \in L$, invariant (cf. [L, p. 155]). By Lemma 3 the subgroup A is the normal subgroup $M(N)$ of $\mathrm{Mult}(L)$ and the multiplication group $\mathrm{Mult}(L/N)$ of the 2-dimensional connected factor loop L/N is isomorphic to $\mathrm{Mult}(L)/A$. The factor loop L/N is isomorphic either to a 2-dimensional Lie group or to an elementary filiform loop (cf. Lemma 7d)). The factor Lie algebra $\mathbf{mult(L)}/\mathbf{a}$ is isomorphic to $\mathbf{g}_i$, $i = 1$ or 2. But the Lie algebras $\mathbf{g}_i$, $i = 1, 2$, are not the Lie algebra of the multiplication group of a 2-dimensional topological loop (cf. [F1, Thm. 1, p. 420]).

Hence the group $\mathrm{Mult}(L)$ belongs to the III. class and the loop L has a 2-dimensional connected normal subloop M containing a 1-dimensional connected normal subloop N of L. Since $\mathrm{Mult}(L)$ does not belong to the II. class every 1-dimensional normal subloop of L lies in a 2-dimensional normal subloop of L. By Lemma 7d) the loop N is isomorphic to $\mathbb{R}$ and every orbit of N is homeomorphic to $\mathbb{R}$. By [NS1, Thm. 18.18], the 1-dimensional connected factor loop L/M is isomorphic either to the Lie group $\mathbb{R}$ or to $\mathrm{SO}_2(\mathbb{R})$. The normal subloop M and the factor loop L/N are 2-dimensional connected loops having a Lie group as their multiplication groups (cf. Lemma 3). Hence M and L/N are homeomorphic either to $\mathbb{R}^2$ or to $S^1 \times \mathbb{R}$ or to $S^1 \times S^1$ (cf. [NS1, Thm. 19.1, p. 249]). The manifold L is a fibering of $\mathbb{R}^3$ over L/N with fibers homeomorphic to N and it is also a fibering of $\mathbb{R}^3$ over L/M with fibers homeomorphic to M. Hence the first fundamental group $\pi_1(\mathbb{R}^3)$ of $\mathbb{R}^3$ is isomorphic to the sum $\pi_1(L/N) + \pi_1(N)$ and also to the sum $\pi_1(L/M) + \pi_1(M)$. Since $\pi_1(\mathbb{R}^n) = 0$, $\pi_1(S^1) = \mathbb{Z}$ and N is homeomorphic to $\mathbb{R}$ we obtain that the loops L/N and M are homeomorphic to $\mathbb{R}^2$ and L/M is homeomorphic to $\mathbb{R}$. Every 2-dimensional topological loop which is homeomorphic to $\mathbb{R}^2$ and having a Lie group as its multiplication group is isomorphic either to an elementary filiform loop or to one of the Lie groups $\{\mathbb{R}^2, \mathcal{L}_2\}$ (cf. [F1, Thm. 1]). The series $\{e\} = L_0 \leq N = L_1 \leq M = L_2 \leq L = L_3$ of normal subloops of L has the properties that every factor loop L_i/L_{i-1}, $i \in \{1, 2, 3\}$, is isomorphic to $\mathbb{R}$. The above discussion yields case (a) of the following theorem

Theorem 8. *Let L be a proper connected simply connected topological loop of dimension* 3 *having a solvable Lie group as its multiplication group* $\mathrm{Mult}(L)$*.*

(a) *Then L is classically solvable. There is a normal subgroup $N \cong \mathbb{R}$ of L. Every normal subgroup $N \cong \mathbb{R}$ of L lies in a 2-dimensional normal subloop M of L. The factor loop L/M is isomorphic to $\mathbb{R}$, whereas M and L/N are isomorphic either to a 2-dimensional simply connected Lie group or to an elementary filiform loop.*

(b) *The loop L is congruence solvable if and only if either L has a non-discrete centre or L is an abelian extension of a 1-dimensional normal subgroup $N \cong \mathbb{R}$ by the factor loop L/N isomorphic either to the Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_{\mathcal{F}}$.*

Proof. It remains to prove case (b). By Lemma 2 the loop L is congruence solvable if and only if it is an iterated abelian extension. Among the loops L with solvable multiplication group the following are iterated abelian extensions: If the centre $Z(L)$ of L is non-discrete, then it has dimension 1 or 2 (cf. Lemma 7a)). If $\dim(Z(L)) = 2$, then L is centrally nilpotent of class 2 (see Lemma 7b)) and hence it is congruence solvable. If $Z(L)$ has dimension 1, then L is an extension of the centre $Z(L) \cong \mathbb{R}$ by a loop F isomorphic to the factor loop $L/Z(L)$. The centre $Z(L)$ is central in L (cf. [SV2, p. 370]) hence it is abelian in L. By [SV2, Thm. 4.1, p. 375], the loop L is an abelian extension of $Z(L)$ by $L/Z(L)$. The factor loop $L/Z(L)$ is isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to an elementary filiform loop $L_{\mathcal{F}}$ (cf. case (a)). Since $\mathcal{L}_2$ is a solvable Lie group and every elementary filiform loop is centrally nilpotent of class 2 the factor loop $L/Z(L)$ is an abelian extension of the group $\mathbb{R}$ by $\mathbb{R}$ (cf. [Mo, Lems. 10, 11, pp. 380–381]). Therefore L is an iterated abelian extension. If the loop L has a discrete centre, then by case (a) L has a normal subgroup $N \cong \mathbb{R}$ such that the factor loop L/N is isomorphic either to the Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_{\mathcal{F}}$ (Lemma 7d)(ii)). Since the factor loop L/N is an abelian extension L is an iterated abelian extension precisely if L is an abelian extension of N by L/N. □

Schreier's extensions of the normal subgroup $N \cong \mathbb{R}$ by the Lie group $F = \mathcal{L}_2$ or by an elementary filiform loop $F = L_{\mathcal{F}}$ are special examples of abelian extensions of N by F (cf. [NS2, p. 761]). Hence they are congruence solvable. Now we give a construction for topological loops which yields non-abelian extensions.

Example 1. Let $(S, \cdot)$ be a topological loop of dimension n having a normal subloop S_1 such that the factor loop S/S_1 is isomorphic to the group $\mathbb{R}$ and let $\phi : (S, \cdot) \to (\mathbb{R}, +)$ be a homomorphism. Consider a one-parameter family of loops $\Gamma_t : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(a, b) \mapsto \Gamma_t(a, b) = a *_t b$, $t \in \mathbb{R}$, such that $\Gamma_0(a, b) = a + b$ and Γ_t is not commutative for some $t \in \mathbb{R}$. Assume that all loops Γ_t on the line $\mathbb{R}$ have the same identity element 0 and denote by $\Delta_t(a, b) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ the right division map $(a, b) \mapsto \Delta_t(a, b) = a/_t b$, $t \in \mathbb{R}$, in the loop Γ_t. For the loops Γ_t, $t \neq 0$, we can take loops defined by the sharply transitive continuous section $\sigma_t : \widetilde{PSL_2}(\mathbb{R})/\mathcal{L}_2 \to \widetilde{PSL_2}(\mathbb{R})$ given by the continuous functions $f(u) = \exp[\frac{1}{6} \sin^2 t \cos u(\cos u - 1)]$, $g(u) = (f(u)^{-1} - f(u)) \cot u$ (cf. [NS1, Prop. 18.15 and its proof, pp. 244–245]). All these loops Γ_t, $t \neq 0$, are proper and hence they are not commutative (cf. Cor.

18.19., p. 248). The multiplication

$$(x,a)\circ(y,b)=(x\cdot y,\Gamma_{\phi(x\cdot y)}(a,b))$$

on $S\times\mathbb{R}$ defines a loop L_ϕ which is an extension of the group $\mathbb{R}$ by the loop S. The loop L_ϕ has the identity element $(1,0)$, where 1 is the identity element of the loop $(S,\cdot)$, because of $(1,0)\circ(y,b)=(y,\Gamma_{\phi(y)}(0,b))=(y,b)=(y,b)\circ(1,0)$. Hence the loop L_ϕ is an Albert extension of the group $\mathbb{R}$ by the loop $(S,\cdot)$ given by the one-parameter family Γ_t of the loop multiplications on $\mathbb{R}$ (cf. [N, p. 4]). Let $x\in S$ such that $\phi(x)\neq 0$. For the tot-inner word $T(x,a)=\rho^{-1}_{(x,a)}\lambda_{(x,a)}$ one has $T(x,a)(1,c)=((x,a)\circ(1,c))/(x,a)=(x,\Gamma_{\phi(x)}(a,c))/(x,a)=(1,\Delta_{\phi(x)}(\Gamma_{\phi(x)}(a,c),a))$, which is not independent of $a\in\mathbb{R}$ since the loop $\Gamma_{\phi(x)}$ is not commutative. Hence the normal subgroup $\mathbb{R}$ is not abelian in the loop L_ϕ (cf. [SV2, Proof of Thm. 4.1, p. 377]). Taking for the loop $(S,\cdot)$ the Lie group $\mathcal{L}_2$ or an elementary filiform loop $L_{\mathcal{F}}$ this construction gives a non-abelian extension of the group $\mathbb{R}$ by the loop $(S,\cdot)$.

Theorem 9. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group* $\mathrm{Mult}(L)$ *is an indecomposable solvable Lie group with 2-dimensional centre Z. Then L is centrally nilpotent of class 2 and* $\mathrm{Mult}(L)$ *has dimension ≥ 6. The group* $\mathrm{Mult}(L)$ *is a semidirect product of the subgroup $V=Z\times\mathrm{Inn}(L)\cong\mathbb{R}^m$, $m\geq 5$, by a group $Q\cong\mathbb{R}$, where $\mathbb{R}^2=Z\cong Z(L)$. For every 1-dimensional connected subgroup N of Z the orbit $N(e)$ is a connected central subgroup of L and the factor loop $L/N(e)$ is isomorphic to an elementary filiform loop $L_{\mathcal{F}}$. The group* $\mathrm{Mult}(L)$ *has a normal subgroup S containing $N\cong\mathbb{R}$ such that the factor group* $\mathrm{Mult}(L)/S$ *is isomorphic to an elementary filiform Lie group $\mathcal{F}_n$ with $n\geq 4$.*

Proof. According to Lemma 7a),b),c),e), the loop L is centrally nilpotent of class 2, $\dim(\mathrm{Mult}(L))\geq 6$ and $\mathrm{Mult}(L)$ is a semidirect product as in the assertion. Since N is a subgroup of Z the orbit $N(e)$ lies in the centre $Z(L)$ of L and hence $N(e)$ is a 1-dimensional central subgroup of L. The multiplication group of the 2-dimensional connected simply connected factor loop $L/N(e)$ is a factor group of $\mathrm{Mult}(L)$. According to Lemma 7c) the loop $L/N(e)$ is not isomorphic to $\mathbb{R}^2$. If $L/N(e)$ would be isomorphic to $\mathcal{L}_2$, then by Lemma 7d)(ii) the group $\mathrm{Mult}(L)$ would have a proper factor group isomorphic to $\mathcal{L}_2\times\mathcal{L}_2$. A semidirect product $V\rtimes Q$, where V is an abelian normal subgroup of codimension 1 does not have such a factor group. This contradiction yields that $L/N(e)$ is isomorphic to a loop $L_{\mathcal{F}}$ and the remaining part of the assertion follows from Lemma 7d)(ii). □

Proposition 10. *There does not exist any 3-dimensional proper connected topological loop L having a 6-dimensional indecomposable solvable Lie group with 2-dimensional centre as the group* $\mathrm{Mult}(L)$ *of L.*

Proof. We may assume that L is simply connected and hence homeomorphic to $\mathbb{R}^3$ (cf. Lemma 6). If $\mathrm{Mult}(L)$ is nilpotent, then the assertion follows from Lemma 7 e). According to Theorem 9 the group $\mathrm{Mult}(L)$ has the form $Q\ltimes V$ with the 5-dimensional abelian normal subgroup V. Hence the Lie algebra $\mathbf{mult}(\mathbf{L})$ of $\mathrm{Mult}(L)$ has a 5-dimensional abelian nilradical. The unique Lie algebra with 2-dimensional centre in the list given in [ST, p. 37], is the Lie algebra $\mathbf{g}_{6,6}$ with

$a = 0 = b$ defined by the Lie brackets: $[e_1, e_6] = e_1$, $[e_3, e_6] = e_2$, $[e_5, e_6] = e_4$. The Lie algebra $\mathbf{n}$ of a 1-dimensional central subgroup $N < Z$ has either the form $\mathbf{n}_\alpha = \langle e_2 + \alpha e_4 \rangle$, $\alpha \in \mathbb{R}$, or $\mathbf{n} = \langle e_4 \rangle$. There does not exist any ideal $\mathbf{s}$ of $\mathbf{g}_{6,6}$ containing $\mathbf{n}_\alpha$ or $\mathbf{n}$ such that the factor algebra $\mathbf{g}_{6,6}/\mathbf{s}$ is isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n = \{4, 5\}$. This contradiction to Theorem 9 proves the assertion. $\square$

Theorem 11. *Let L be a 3-dimensional proper connected simply connected topological loop having a solvable indecomposable Lie group with a discrete centre as its multiplication group $\mathrm{Mult}(L)$. The loop L has a connected normal subgroup N isomorphic to $\mathbb{R}$ and the factor loop L/N is isomorphic either to the Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_\mathcal{F}$. The group $\mathrm{Mult}(L)$ has dimension ≥ 6 and it has a normal subgroup S containing $\mathrm{Mult}(N) \cong \mathbb{R}$ such that the factor group $\mathrm{Mult}(L)/S$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ if $L/N \cong \mathcal{L}_2$ or to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/N \cong L_\mathcal{F}$. For every 1-dimensional connected normal subgroup N of L the loop L has a normal subloop M isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_\mathcal{F}$ such that N lies in M and L/M is isomorphic to $\mathbb{R}$. The group $\mathrm{Mult}(L)$ has a normal subgroup V such that the orbit $V(e)$ is the loop M, $\mathrm{Mult}(L)/V \cong \mathbb{R}$, V contains the inner mapping group $\mathrm{Inn}(L)$ of L and the group $\mathrm{Mult}(M)$ of M.*

Proof. By Theorem 8(a) there exists a normal subgroup N of L isomorphic to $\mathbb{R}$ and there is a 2-dimensional normal subloop M of L containing N. As the group $\mathrm{Mult}(L)$ has a discrete centre the factor loop L/N is not isomorphic to $\mathbb{R}^2$ (cf. Lemma 7d)(i)). Hence it is isomorphic either to the Lie group $\mathcal{L}_2$ or to a loop $L_\mathcal{F}$ and the group $\mathrm{Mult}(L)$ has a normal subgroup S as in the assertion (cf. Lemma 7 d (ii)). By Lemma 7 e) we have $\dim(\mathrm{Mult}(L)) \geq 6$. Since L/M is isomorphic to $\mathbb{R}$ there is a normal subgroup $V = \{v \in \mathrm{Mult}(L); xM = v(x)M \text{ for all } x \in L\} < \mathrm{Mult}(L)$ such that $V(e) = M$, $\mathrm{Mult}(L)/V \cong \mathbb{R}$ and V contains the multiplication group $\mathrm{Mult}(M)$ of M (cf. Lemma 3). As the group $\mathrm{Mult}(L)/V$ operates sharply transitively on the orbits of M in L the inner mapping group $\mathrm{Inn}(L)$ is a subgroup of V. $\square$

Theorem 12. *Let L be a 3-dimensional proper connected simply connected topological loop such that its multiplication group $\mathrm{Mult}(L)$ is an indecomposable solvable Lie group with 1-dimensional centre Z. For every 1-dimensional connected normal subgroup K of $\mathrm{Mult}(L)$ the orbit $K(e)$ is a normal subgroup of L isomorphic to $\mathbb{R}$. We have one of the following possibilities:*

(a) *The factor loop $L/K(e)$ is isomorphic to $\mathbb{R}^2$. Then L is centrally nilpotent of class 2, $K(e)$ coincides with the centre $Z(L)$ of L and the group $\mathrm{Mult}(L)$ is a semidirect product of the normal subgroup $P = Z \times \mathrm{Inn}(L) \cong \mathbb{R}^m$, $m \geq 4$, by a group $Q \cong \mathbb{R}^2$ and the orbit $P(e)$ is $Z(L)$.*
(b) *The loop $L/K(e)$ is isomorphic either to the Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_\mathcal{F}$. The group $\mathrm{Mult}(L)$ has a normal subgroup S containing K such that the orbit $S(e)$ coincides with $K(e)$, the factor group $\mathrm{Mult}(L)/S$ is isomorphic to the direct product $\mathcal{L}_2 \times \mathcal{L}_2$ if $L/K(e) \cong \mathcal{L}_2$ or to an elementary filiform Lie group $\mathcal{F}_n$, $n \geq 4$, if $L/K(e) \cong L_\mathcal{F}$.*

The loop L has a 2-dimensional normal subloop M containing $K(e)$ and the group $\mathrm{Mult}(L)$ *has a normal subgroup V as in Theorem* 11*. In particular, if $K(e) = Z(L)$ and $L/Z(L)$ is an elementary filiform loop $L_{\mathcal{F}}$, then L is centrally nilpotent of class* 3 *and M is not isomorphic to the group $\mathcal{L}_2$.*

Proof. By Lemmata 3, 5 the normal subloop $K(e)$ is different from $\{e\}$. Applying Lemma 7d) for the case $N = K(e)$ assertion (a) and (b) is proved. By Theorem 8(a) there is a normal subloop M of L containing $K(e)$ and the remaining part of the assertion follows from the proof of Theorem 11. If $K(e) = Z(L)$ and $L/Z(L)$ is an elementary filiform loop $L_{\mathcal{F}}$, then L is an iterated central extension, since every elementary filiform loop is centrally nilpotent of class 2. By [SV2, Cor. 5.2, p. 380], the loop L is centrally nilpotent of class 3. Moreover, M is not isomorphic to $\mathcal{L}_2$ since $\mathcal{L}_2$ has a trivial centre. □

4. Six-dimensional indecomposable solvable Lie algebras with four-dimensional nilradical having trivial centre or non-abelian nilradical

From now on we deal with 6-dimensional indecomposable solvable Lie algebras having 4-dimensional nilradical. Firstly we formulate the main technical tool which we systematically use to exclude those Lie algebras which are not the Lie algebra of the multiplication group of a 3-dimensional topological loop.

Proposition 13. *Let L be a* 3*-dimensional connected simply connected topological loop having a* 6*-dimensional solvable indecomposable Lie algebra $\mathbf{g}$ with* 4*-dimensional nilradical $\mathbf{n}_{\mathrm{rad}}$ as the Lie algebra of its multiplication group.*

a) *For each* 1*-dimensional ideal $\mathbf{i}$ of $\mathbf{g}$ the orbit $I(e)$, where I is the simply connected Lie group of $\mathbf{i}$ and e is the identity element of L, is a normal subgroup of L isomorphic to $\mathbb{R}$. We have one of the following possibilities:*
   (i) *The factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ and for the nilradical one has $\mathbf{n}_{\mathrm{rad}} = \mathbf{z} \oplus \mathbf{inn(L)} \cong \mathbb{R}^4$, where $\mathbf{z}$ is the* 1*-dimensional centre of $\mathbf{g}$ and $\mathbf{inn(L)}$ is the Lie algebra of the group* $\mathrm{Inn}(L)$*.*
   (ii) *The factor loop $L/I(e)$ is isomorphic either to the Lie group $\mathcal{L}_2$ or to an elementary filiform loop $L_{\mathcal{F}}$. Then there exists a* 2*-dimensional ideal $\mathbf{s}$ of $\mathbf{g}$ such that $\mathbf{i} < \mathbf{s}$ and the factor Lie algebra $\mathbf{g}/\mathbf{s}$ is isomorphic either to the direct sum $\mathbf{l}_2 \oplus \mathbf{l}_2$, where $\mathbf{l}_2$ is the* 2*-dimensional non-abelian Lie algebra or to the elementary filiform Lie algebra $\mathbf{f}_4$.*

   Assume that the centre of $\mathbf{g}$ is trivial or the nilradical of $\mathbf{g}$ is not abelian.

b) *For every* 2*-dimensional abelian ideal $\mathbf{a}$ of $\mathbf{g}$ such that the factor Lie algebra $\mathbf{g}/\mathbf{a}$ is isomorphic neither to $\mathbf{l}_2 \oplus \mathbf{l}_2$ nor to $\mathbf{f}_4$ and for each nilpotent ideal $\mathbf{s}$ of $\mathbf{g}$ having dimension > 2 the orbits $A(e)$, $S(e)$, where A, respectively S is the simply connected Lie group of $\mathbf{a}$, respectively $\mathbf{s}$ and e is the identity element of L, are the same* 2*-dimensional normal subloop M of L. There exists a* 5*-dimensional ideal $\mathbf{v}$ of $\mathbf{g}$ containing the Lie algebra $\mathbf{inn(L)}$, the Lie algebra $\mathbf{mult}(M)$ of the multiplication group of M and the nilradical $\mathbf{n}_{\mathrm{rad}}$. For every ideal $\mathbf{a}$, respectively $\mathbf{s}$ one has $\mathbf{a} \cap \mathbf{inn(L)} = \{0\}$ and the intersection $\mathbf{s} \cap \mathbf{inn(L)}$ has dimension $dim(\mathbf{s}) - 2$. In particular, if $\mathbf{g}$ is not the Lie algebra $N_{6,28}$ in* [T, Table III, p. 1349]*, then the loop M is isomorphic to $\mathbb{R}^2$.*

Proof. Each 1-dimensional ideal $\mathbf{i}$ of $\mathbf{g}$ lies in $\mathbf{n}_{\mathrm{rad}}$, which is isomorphic either to $\mathbb{R}^4$ or to $\mathbf{f}_3 \oplus \mathbb{R}$ or to $\mathbf{f}_4$ (cf. [T]). If the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$, then the orbit $I(e)$ coincides with the 1-dimensional centre $Z(L)$ of L (cf. Proposition 10 and Theorem 12). The Lie algebra $\mathbf{p}$ of the normal subgroup P in Theorem 12(a) is a 4-dimensional abelian ideal $\mathbf{p} = \mathbf{z} \oplus \mathbf{inn(L)}$ of $\mathbf{g}$ such that the commutator ideal $\mathbf{g}'$ is contained in $\mathbf{p}$. The ideal $\mathbf{p}$ is nilpotent hence it coincides with the nilradical $\mathbf{n}_{\mathrm{rad}}$ of $\mathbf{g}$. This proves assertion (i). Since $\mathbf{g}$ has no factor Lie algebra isomorphic to the 5-dimensional elementary filiform Lie algebra $\mathbf{f}_5$ assertion (ii) follows from Theorems 11, 12(b).

Assume that the centre of $\mathbf{g}$ is trivial or the nilradical of $\mathbf{g}$ is not abelian. Taking into account [T, Tables I, III, IV, V, pp. 1347–1350], the commutator Lie algebra $\mathbf{g}'$ of $\mathbf{g}$ coincides with the nilradical $\mathbf{n}_{\mathrm{rad}}$ of $\mathbf{g}$. Let $\mathbf{a}$ be a 2-dimensional abelian ideal of $\mathbf{g}$. The orbit $A(e)$ is a normal subloop of L (cf. Lemma 3) such that $A(e) \neq \{e\}$ and $\dim(A(e)) \neq 1$ (cf. Lemmata 5, 7 d) and Proposition 13 a). Hence $A(e)$ is a 2-dimensional normal subloop M of L. The 5-dimensional ideal $\mathbf{v}$ is the Lie algebra of the normal subgroup V in Theorem 11 and hence one has $V(e) = M$. The ideal $\mathbf{a}$ is contained in $\mathbf{n}_{\mathrm{rad}}$. Let N be the simply connected Lie group of $\mathbf{n}_{\mathrm{rad}}$. The orbit $N(e)$ is a normal subloop of L having dimension 2 or 3 since $A(e) \subseteq N(e)$. Therefore $N(e)$ is either the subloop M or the loop L. As $\mathbf{g}' = \mathbf{n}_{\mathrm{rad}}$ one has $\mathbf{n}_{\mathrm{rad}} \subset \mathbf{v}$. Hence we obtain that $V(e) = N(e) = A(e) := M$. Since each nilpotent ideal $\mathbf{a}$ and $\mathbf{s}$ in assertion b) is contained in $\mathbf{n}_{\mathrm{rad}}$ one has $A(e) = S(e) = N(e) = M$. Since $\dim(A(e)) = 2$ the 2-dimensional abelian Lie group A acts sharply transitively on $A(e)$. Hence one has $A \cap \mathrm{Inn}(L) = \{1\}$. As $\dim(\mathbf{s}) > 2$ and $\dim(S(e)) = 2$ there is a subgroup of S of dimension $\dim(\mathbf{s}) - 2$, which fixes the identity element e of L. The ideal $\mathbf{v}$ contains the Lie algebra $\mathbf{mult}(M)$ of the group $\mathrm{Mult}(M)$ of the 2-dimensional normal subloop M of L. The loop M is isomorphic either to $\mathbb{R}^2$ or to $\mathcal{L}_2$ or to a loop $L_{\mathcal{F}}$ (cf. Theorem 11). Since $\mathbf{n}_{\mathrm{rad}} \subset \mathbf{v}$ and $\dim(\mathbf{v}) = 5$ the intersection of $\mathbf{v}$ with the complement of $\mathbf{n}_{\mathrm{rad}}$ in $\mathbf{g}$ has dimension 1. Therefore $\mathbf{v}$ does not contain a subalgebra isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. The radical of the Lie algebras $\mathbf{g}$ which are different from $N_{6,28}$ does not contain an elementary filiform Lie algebra of dimension ≥ 4. Hence one has $M = V(e) = \mathbb{R}^2$. This proves assertion b). □

Now we prove that the 6-dimensional solvable indecomposable Lie algebras with 4-dimensional nilradical having trivial centre or non-abelian nilradical are not the Lie algebras of the multiplication groups of 3-dimensional topological loops.

Proposition 14. *Let $\mathbf{g}$ be a 6-dimensional solvable indecomposable Lie algebra with 4-dimensional nilradical $\mathbf{n}_{\mathrm{rad}}$ such that $\mathbf{g}$ has trivial centre or $\mathbf{n}_{\mathrm{rad}}$ is not abelian. There does not exist a 3-dimensional connected topological loop L having $\mathbf{g}$ as the Lie algebra of the multiplication group* $\mathrm{Mult}(L)$ *of L.*

Proof. We may assume that L is simply connected and hence it is homeomorphic to $\mathbb{R}^3$ (cf. Lemma 6). The 6-dimensional solvable indecomposable Lie algebras with 4-dimensional nilradical having trivial centre or non-abelian nilradical are listed [T, Tables I, III, IV, V, pp. 1347–1350]. The Lie algebras $N_{6,i}$, $i = 4, 7, 30, 39, 40$, have the ideal $\mathbf{i} = \langle n_4 \rangle$. The Lie algebras $N_{6,i}$, $i = 5, 16, 17$, have the ideal $\mathbf{i} = \langle n_2 \rangle$. The Lie algebras $N_{6,i}$, $i = 8, 9, 10, 13, 14, 28, 35, 36, 37$, have the ideal $\mathbf{i} = \langle n_1 \rangle$.

There does not exist any ideal $\mathbf{s}$ of the above Lie algebras $N_{6,i}$ which contains $\mathbf{i}$ and the factor Lie algebras $N_{6,i}/\mathbf{s}$ are isomorphic either to $\mathbf{f}_4$ or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For $i = 39, 40$, the nilradical of $N_{6,i}$ is not abelian. Hence the factor loop $L/I(e)$ is not isomorphic to $\mathbb{R}^2$. By Proposition 13a)(i),(ii) these Lie algebras are not the Lie algebras of the multiplication groups of 3-dimensional topological loops. The Lie algebras $N_{6,j}$, $j = 12, 15, 18, 19$, have no 1-dimensional ideal. The unique 2-dimensional abelian ideal of $N_{6,12}$, respectively $N_{6,19}$ is $\mathbf{s}_1 = \langle n_2, n_4\rangle$, respectively $\mathbf{s}_2 = \langle n_3, n_4\rangle$. The Lie algebras $N_{6,15}$, $N_{6,18}$ have two 2-dimensional abelian ideals $\mathbf{s}_2$ and $\mathbf{s}_3 = \langle n_1, n_2\rangle$. None of the factor algebras $N_{6,12}/\mathbf{s}_1$, $N_{6,19}/\mathbf{s}_2$, $N_{6,j}/\mathbf{s}_k$, $j = 15, 18$, $k = 2, 3$, are isomorphic to $\mathbf{f}_4$ or to $\mathbf{l}_2 \oplus \mathbf{l}_2$. Hence the orbits $S_i(e)$, where $S_i = \exp(\mathbf{s_i})$, $i = 1, 2, 3$, are 2-dimensional normal subloops of L (cf. Proposition 13b)). If $N_{6,j}$, $j = 12, 15, 18, 19$, would be the Lie algebra of the multiplication group of a 3-dimensional topological loop, then L would have no 1-dimensional normal subgroup. This contradiction to Theorem 8(a) excludes these Lie algebras.

The Lie algebras $N_{6,i}$, $i \in \{1, 2, 3, 6, 11\}$, have trivial centre and neither a subalgebra nor a factor Lie algebra are isomorphic to an elementary filiform Lie algebra. The Lie algebra $N_{6,1}$ depends on four real parameters α, β, γ, δ with $\alpha\beta \neq 0$, $\gamma^2 + \delta^2 \neq 0$. It has the ideals $\mathbf{i}_1 = \langle n_3\rangle$, $\mathbf{i}_2 = \langle n_4\rangle$. If $N_{6,1}$ is the Lie algebra of the multiplication group of a 3-dimensional topological loop, then there are 2-dimensional ideals $\mathbf{s}_j$ of $N_{6,1}$ containing $\mathbf{i}_j$, $j = 1, 2$, such that the factor Lie algebras $N_{6,1}/\mathbf{s}_j$, $j = 1, 2$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ (cf. Theorem 11 and Proposition 13a)(ii). This is the case if and only if $\gamma = \delta = 0$. This contradiction excludes the Lie algebra $N_{6,1}$.

The Lie algebra $N_{6,2}$ depends on three real parameters α, β, γ and the Lie algebra $N_{6,6}$ depends on α, β such that in both cases one has $\alpha^2 + \beta^2 \neq 0$. The Lie algebras $N_{6,3}$, $N_{6,11}$ depend on the real parameter α. The Lie algebra $N_{6,2}$ has the ideals $\mathbf{i}_1 = \langle n_1\rangle$, $\mathbf{i}_2 = \langle n_2\rangle$, $\mathbf{i}_3 = \langle n_4\rangle$ and the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, have the ideals $\mathbf{i}_k$, $k = 2, 3$. If $N_{6,j}$, $j = 2, 3, 6, 11$, would be the Lie algebra of the multiplication group of a 3-dimensional topological loop, then applying Theorem 11 and Proposition 13a)(ii) there are 2-dimensional ideals $\mathbf{s}$ of $N_{6,j}$, $j = 2, 3, 6, 11$, containing $\mathbf{i}_k$, $k = 1, 2, 3$, such that the factor Lie algebras $N_{6,j}/\mathbf{s}$, $j = 2, 3, 6, 11$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For the ideals $\mathbf{s}_1 = \langle n_1, n_4\rangle$, $\mathbf{s}_2 = \langle n_2, n_4\rangle$ of $N_{6,2}$, respectively for the ideal $\mathbf{s}_2$ of the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, the factor Lie algebras $N_{6,2}/\mathbf{s}_i$, $i = 1, 2$, respectively $N_{6,j}/\mathbf{s}_2$, $j = 3, 6, 11$, are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ precisely if $\beta = \gamma = 0$, respectively $\alpha = 0$. Hence we have to consider the Lie algebras $N_{6,2}$ with $\beta = \gamma = 0$, $\alpha \neq 0$, $N_{6,j}$, $j = 3, 11$, with $\alpha = 0$ and $N_{6,6}$ with $\alpha = 0$, $\beta \neq 0$. These Lie algebras have the abelian ideals $\mathbf{s}_3 = \langle n_1, n_2\rangle$, $\mathbf{s}_4 = \langle n_3, n_4\rangle$ such that the factor Lie algebras $N_{6,j}/\mathbf{s}_l$, $j = 2, 3, 6, 11$, $l = 3, 4$, are not isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. The 3-dimensional abelian ideals $\mathbf{s}_5 = \langle n_1, n_2, n_4\rangle$, $\mathbf{s}_6 = \langle n_2, n_3, n_4\rangle$, $\mathbf{s}_7 = \langle n_1, n_3, n_4\rangle$ of $N_{6,2}$ and the ideals $\mathbf{s}_m$, $m = 5, 6$, of $N_{6,j}$, $j = 3, 6, 11$, are in $\mathbf{n}_{\mathrm{rad}}$. According to Proposition 13 b) the orbits $S_l(e)$, where $S_l = \exp(\mathbf{s_l})$, $l \in \{3, 4, 5, 6, 7\}$, and the orbit $N(e)$, where N is the simply connected Lie group of $\mathbf{n}_{\mathrm{rad}}$, are the same normal subgroup $M \cong \mathbb{R}^2$ of L. Since $\mathbf{i}_k \subset \mathbf{n}_{\mathrm{rad}}$, $k = 1, 2, 3$, the group M contains the 1-dimensional normal subgroups $I_k(e)$ of L, where I_k are the simply connected Lie groups of $\mathbf{i}_k$, $k \in \{1, 2, 3\}$. The ideal $\mathbf{v}$ in Proposition 13 b) has one of the following forms: $\mathbf{v}_{1,k} = \langle n_1, n_2, n_3, n_4, x_1 + kx_2\rangle$, $k \in \mathbb{R}$,

$\mathbf{v}_2 = \langle n_1, n_2, n_3, n_4, x_2 \rangle$. For $l = 3, 4$ one has $\mathbf{s}_l \cap \mathbf{inn}(\mathbf{L}) = \{0\}$, for $m = 5, 6, 7$ the intersections $\mathbf{s}_m \cap \mathbf{inn}(\mathbf{L})$ have dimension 1 and $\dim(\mathbf{n}_{\mathrm{rad}} \cap \mathbf{inn}(\mathbf{L})) = 2$. Hence the Lie subalgebra $\mathbf{inn}(\mathbf{L})$ of $N_{6,j}$, $j = 2, 3, 6, 11$, has either the basis elements $b_1 = n_2 + a_1 n_4$, $b_2 = n_1 + a_2 n_3 + a_4 n_4$, where $a_1 a_2 \neq 0$ or the basis elements $b_1' = n_1 + a_1 n_2 + a_2 n_4$, $b_2' = n_2 + a_3 n_3 + a_4 n_4$, where $a_2 a_3 \neq 0$. In the second case for the Lie algebra $N_{6,2}$ we have $a_1 = 0$. The third basis element of $\mathbf{inn}(\mathbf{L})$ is either $b_3 = x_2 + c_1 n_3 + c_2 n_4$ or $b_{3,k} = x_1 + k x_2 + c_1 n_3 + c_2 n_4$, $k, c_1, c_2 \in \mathbb{R}$. Only the subspace $\langle b_1, b_2, b_{3,k} \rangle$ is a 3-dimensional Lie algebra only in the Lie algebras $N_{6,j}$, $j = 3, 6, 11$. Then the Lie subalgebra $\mathbf{inn}(\mathbf{L})$ has the form: $\mathbf{inn}(\mathbf{L})_{a,a_4} = \langle n_2 + a(1+\beta) n_4, n_1 + a n_3 + a_4 n_4, x_1 + x_2 \rangle$, where $a \neq 0$, $a_4 \in \mathbb{R}$, $\beta \neq -1$ for $N_{6,6}$ and $\beta = 0$ for $N_{6,j}$, $j = 3, 11$. Using the automorphism $\alpha(n_i) = a n_i$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_4) = n_4$, $\alpha(n_3) = n_3 - \frac{a_4}{a} n_4$ of the Lie algebras $N_{6,j}$, $j = 3, 6, 11$, we can change the Lie algebra $\mathbf{inn}(\mathbf{L})_{a,a_4}$ onto $\mathbf{inn}(\mathbf{L})_\beta = \langle n_2 + (1+\beta) n_4, n_1 + n_3, x_1 + x_2 \rangle$ in the case $N_{6,6}^{\beta \neq -1}$ and $\beta = 0$ for the Lie algebras $N_{6,j}$, $j = 3, 11$. Linear representations of the simply connected Lie groups G_j of $N_{6,j}$, $j = 3, 6, 11$, are given by:

for $N_{6,3}^{\alpha=0}$:

$$\begin{aligned} g(x_1, x_2, x_3, x_4, x_5, x_6)g(y_1, y_2, y_3, y_4, y_5, y_6) \\ = g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_5 y_1 e^{x_6}, x_3 + y_3 e^{x_5}, \\ x_4 + y_4 e^{x_5} + y_3 x_6 e^{x_5}, x_5 + y_5, x_6 + y_6), \end{aligned}$$

for $N_{6,11}^{\alpha=0}$:

$$g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_5 y_1 e^{x_6}, x_3 + y_3 e^{x_5}, x_4 + y_4 e^{x_5} + y_3 x_5 e^{x_5}, x_5 + y_5, x_6 + y_6),$$

for $N_{6,6}^{\alpha=0,\beta\neq-1}$:

$$\begin{aligned} g(x_1 + y_1 e^{x_6}, x_2 + y_2 e^{x_6} + x_6 y_1 e^{x_6}, \\ x_3 + y_3 e^{x_5}, x_4 + y_4 e^{x_5} + y_3 (x_5 + \beta x_6) e^{x_5}, x_5 + y_5, x_6 + y_6). \end{aligned}$$

One has $\mathrm{Inn}(L) = \{g(u_1, u_2, u_1, (1+\beta) u_2, s, s); u_i, s \in \mathbb{R}\}$, $i = 1, 2$, for $N_{6,6}^{\beta \neq -1}$ and $\beta = 0$ in the cases $N_{6,j}$, $j = 3, 11$. Two arbitrary left transversals to the group $\mathrm{Inn}(L)$ in G_j, $j = 3, 6, 11$, are

$$A = \{g(f_1(k,l,m), f_2(k,l,m), k, l, m, f_3(k,l,m)), k, l, m \in \mathbb{R}\},$$
$$B = \{g(h_1(u,v,w), h_2(u,v,w), u, v, w, h_3(u,v,w)), u, v, w \in \mathbb{R}\},$$

where $f_i(k,l,m) : \mathbb{R}^3 \to \mathbb{R}$ and $h_i(u,v,w) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0,0,0) = h_i(0,0,0) = 0$. For all $a \in A$, $b \in B$ the condition $a^{-1} b^{-1} a b \in \mathrm{Inn}(L)$ holds if and only if in all three cases the equation

$$\begin{aligned} e^{-h_3(u,v,w)} h_1(u,v,w)(1 - e^{-f_3(k,l,m)}) - e^{-f_3(k,l,m)} f_1(k,l,m)(1 - e^{-h_3(u,v,w)}) \\ = u e^{-w}(1 - e^{-m}) - k e^{-m}(1 - e^{-w}), \end{aligned} \tag{1}$$

and for $N_{6,3}^{\alpha=0}$,

$$\begin{aligned} & e^{-h_3(u,v,w)}(1 - e^{-f_3(k,l,m)})(h_2(u,v,w) - w h_1(u,v,w)) \\ & \quad - e^{-f_3(k,l,m)}(1 - e^{-h_3(u,v,w)})(f_2(k,l,m) - m f_1(k,l,m)) \\ & \quad + e^{-f_3(k,l,m) - h_3(u,v,w)}(m h_1(u,v,w) - w f_1(k,l,m)) \\ & = e^{-w}(1 - e^{-m})(v - h_3(u,v,w) u) - e^{-m}(1 - e^{-w})(l - f_3(k,l,m) k) \\ & \quad + e^{-m-w}(f_3(k,l,m) u - h_3(u,v,w) k), \end{aligned} \tag{2}$$

for $N_{6,11}^{\alpha=0}$,

$$\begin{aligned}
&e^{-h_3(u,v,w)}(1-e^{-f_3(k,l,m)})(h_2(u,v,w)-wh_1(u,v,w))\\
&\quad -e^{-f_3(k,l,m)}(1-e^{-h_3(u,v,w)})(f_2(k,l,m)-mf_1(k,l,m))\\
&\quad +e^{-f_3(k,l,m)-h_3(u,v,w)}(mh_1(u,v,w)-wf_1(k,l,m))\\
&=e^{-w}(1-e^{-m})(v-wu)-e^{-m}(1-e^{-w})(l-mk)+e^{-m-w}(mu-wk),
\end{aligned} \tag{3}$$

for $N_{6,6}^{\alpha=0,\beta\neq -1}$,

$$\begin{aligned}
&(1+\beta)[e^{-h_3(u,v,w)}(1-e^{-f_3(k,l,m)})(h_2(u,v,w)-h_3(u,v,w)h_1(u,v,w))\\
&\quad -e^{-f_3(k,l,m)}(1-e^{-h_3(u,v,w)})(f_2(k,l,m)-f_1(k,l,m)f_3(k,l,m))\\
&\quad +e^{-f_3(k,l,m)-h_3(u,v,w)}(f_3(k,l,m)h_1(u,v,w)-h_3(u,v,w)f_1(k,l,m))]\\
&=e^{-w}(1-e^{-m})[v-u(w+\beta h_3(u,v,w))]\\
&\quad -e^{-m}(1-e^{-w})[l-k(m+\beta f_3(k,l,m))]\\
&\quad +e^{-m-w}[mu-wk+\beta(f_3(k,l,m)u-h_3(u,v,w)k)]
\end{aligned} \tag{4}$$

are satisfied for all $u,v,w,k,l,m\in\mathbb{R}$. Equation (1) is satisfied precisely if one has $h_3(u,v,w)=w$, $h_1(u,v,w)=u$, $f_1(k,l,m)=k$, $f_3(k,l,m)=m$. Putting this into equations (2), (3), (4) we obtain in case (4)

$$e^{-w}(1-e^{-m})(v-(1+\beta)h_2(u,v,w))=e^{-m}(1-e^{-w})(l-(1+\beta)f_2(k,l,m)) \tag{5}$$

and in cases (2), (3) equation (5) with $\beta=0$. Equation (5) holds if and only if one has $h_2(u,v,w)=v/(1+\beta)$, $f_2(k,l,m)=l/(1+\beta)$, where $\beta=0$ in the cases $N_{6,j}^{\alpha=0}$, $j=3,11$, and $\beta\in\mathbb{R}\setminus\{-1\}$ in the case $N_{6,6}^{\alpha=0,\beta\neq-1}$. In all these cases $A\cup B$ does not generate the group G_j, $j=3,6,11$. By Lemma 4 the Lie algebras $N_{6,j}$, $j=3,6,11$, are not the Lie algebras of groups $\mathrm{Mult}(L)$ of 3-dimensional topological loops L.

The Lie algebras $N_{6,j}$, $j\in\{29,31,32,33,34,38\}$, have non-abelian nilradical, and neither a subalgebra nor a factor Lie algebra of $N_{6,j}$ are isomorphic to an elementary filiform Lie algebra $\mathbf{f}_n$, $n\geq 4$. The Lie algebras $N_{6,31}$ and $N_{6,32}^{\alpha}$ have the ideal $\mathbf{i}=\langle n_1\rangle$. Both Lie algebras contain the nilpotent ideals: $\mathbf{s}_1=\langle n_1,n_3\rangle$, $\mathbf{s}_2=\langle n_1,n_4\rangle$, $\mathbf{s}_3=\langle n_1,n_2\rangle$, $\mathbf{s}_4=\langle n_1,n_2,n_3\rangle$, $\mathbf{s}_5=\langle n_1,n_2,n_4\rangle$, $\mathbf{s}_6=\langle n_1,n_3,n_4\rangle$, $\mathbf{n}_{\mathrm{rad}}$. If $N_{6,j}$, $j=31,32$, would be the Lie algebra of the multiplication group of a 3-dimensional topological loop, then by Theorem 11 and Proposition 13a)(ii) there exist 2-dimensional ideals $\mathbf{s}$ of $N_{6,j}$, $j=31,32$, containing $\mathbf{i}$ such that the factor Lie algebras $N_{6,j}/\mathbf{s}$, $j=31,32$, are isomorphic to $\mathbf{l}_2\oplus\mathbf{l}_2$. The factor Lie algebra $N_{6,31}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2}\oplus\mathbf{l_2}$, whereas the factor Lie algebras $N_{6,31}/\mathbf{s_i}$, $i=2,3$, are not so. The factor Lie algebra $N_{6,32}^{\alpha}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2}\oplus\mathbf{l_2}$ if and only if $\alpha=0$, but the factor Lie algebras $N_{6,32}^{\alpha=0}/\mathbf{s_i}$, $i=2,3$, are not so. The factor Lie algebra $N_{6,32}^{\alpha}/\mathbf{s}_3$ is isomorphic to $\mathbf{l_2}\oplus\mathbf{l_2}$ precisely if $\alpha=1$, whereas the factor Lie algebras $N_{6,32}^{\alpha=1}/\mathbf{s_i}$, $i=1,2$, are not so. Let S_k, respectively N be the simply connected Lie groups of $\mathbf{s_k}$, $k=1,2,\ldots,6$, respectively $\mathbf{n}_{\mathrm{rad}}$. For $N_{6,31}$,

$N_{6,32}^{\alpha=0}$ the orbits $S_i(e)$, $i=2,3,\dots,6$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L and for $N_{6,32}^{\alpha=1}$ we have $S_j(e) = N(e) := M$, $j=1,2,4,5,6$ (cf. Proposition 13 b). The subgroup M contains the normal subgroup $I(e) \cong \mathbb{R}$, where I is the simply connected Lie group of $\mathbf{i}$, of L. For $m=4,5,6$ the intersections $\mathbf{s}_m \cap \mathbf{inn(L)}$ have dimension 1 and $\mathbf{n}_{\mathrm{rad}} \cap \mathbf{inn(L)}$ has dimension 2 (see Proposition 13 b). Since for $N_{6,31}$ and $N_{6,32}^{\alpha=0}$ one has $\mathbf{s}_i \cap \mathbf{inn(L)} = \{0\}$, $i=2,3$ and for $N_{6,32}^{\alpha=1}$ we have $\mathbf{s}_j \cap \mathbf{inn(L)} = \{0\}$, $j=1,2$, the Lie algebra $\mathbf{inn(L)}$ contains the elements $b_1 = n_1 + a_1 n_3$, $b_2 = n_2 + a_2 n_1 + a_3 n_4$, $a_1 a_3 \neq 0$, in the cases $N_{6,31}$ and $N_{6,32}^{\alpha=0}$ and the elements $b_1 = n_1 + a_1 n_2$, $b_2 = n_3 + a_2 n_1 + a_3 n_4$, $a_1 a_3 \neq 0$ in the case $N_{6,32}^{\alpha=1}$. As $[b_1, b_2] = a_1 n_1$, $a_1 \neq 0$ in both cases $\mathbf{inn(L)}$ would contain the ideal $\langle n_1 \rangle$ of $N_{6,j}$, $j = 31, 32$. This contradicts Lemma 5.

The Lie algebras $N_{6,33}$, $N_{6,38}$, $N_{6,34}^{\alpha}$ and $N_{6,29}^{\alpha,\beta}$ have the ideals $\mathbf{i}_1 = \langle n_1 \rangle$, $\mathbf{i}_2 = \langle n_4 \rangle$. The Lie algebras $N_{6,i}$, $i = 29, 38$, have the nilpotent ideals $\mathbf{s}_1 = \langle n_1, n_2 \rangle$, $\mathbf{s}_2 = \langle n_1, n_4 \rangle$, $\mathbf{s}_3 = \langle n_1, n_3 \rangle$, $\mathbf{s}_4 = \langle n_1, n_2, n_4 \rangle$, $\mathbf{s}_5 = \langle n_1, n_3, n_4 \rangle$, $\mathbf{s}_6 = \langle n_1, n_2, n_3 \rangle$, $\mathbf{n}_{\mathrm{rad}}$ and the nilpotent ideals of $N_{6,j}$, $j = 33, 34$, are $\mathbf{s}_1$, $\mathbf{s}_2$, $\mathbf{s}_4$, $\mathbf{s}_5$, $\mathbf{n}_{\mathrm{rad}}$. Denote by I_k, S_i and N the simply connected Lie groups of the ideals $\mathbf{i}_k$, $k = 1, 2$, $\mathbf{s}_i$, $i = 1, 2, \dots, 6$ and $\mathbf{n}_{\mathrm{rad}}$. The factor Lie algebras $N_{6,k}/\mathbf{s}_2$, $k \in \{29, 33, 38\}$, are isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ and $N_{6,34}/\mathbf{s}_2$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ precisely if $\alpha = 0$. If $N_{6,j}$, $j = 29, 33, 34, 38$, would be the Lie algebra of the multiplication group of a 3-dimensional topological loop, then the orbits $I_k(e)$, $k = 1, 2$, are normal subgroups of L isomorphic to $\mathbb{R}$ and the factor loops $L/I_k(e)$, $k = 1, 2$, are isomorphic to $\mathcal{L}_2$ since the nilradical of $N_{6,j}$ are not abelian (cf. Proposition 13a)(i),(ii).

For $j = 33, 34$, the factor Lie algebras $N_{6,j}/\mathbf{s}_1$ are not isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$. By Proposition 13 b) the orbits $S_l(e)$, $l = 1, 4, 5$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L such that $S_1 \cap \mathrm{Inn}(L) = \{1\}$, the intersections $S_l \cap \mathrm{Inn}(L)$ have dimension 1, $l = 4, 5$, and $\dim(N \cap \mathrm{Inn}(L)) = 2$. For $N_{6,29}$ the factor Lie algebra $N_{29}/\mathbf{s}_1$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ precisely if $\beta = 0$ and $N_{29}/\mathbf{s}_3$ is isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ if and only if $\alpha = 0$. If $\alpha \neq 0$, respectively $\beta \neq 0$ the orbits $S_l(e)$, $l = 3, 4, 5, 6$, and $N(e)$, respectively the orbits $S_k(e)$, $k = 1, 4, 5, 6$, and $N(e)$ are the normal subgroup $M \cong \mathbb{R}^2$ of L. For $\alpha \neq 0$ one has $S_3 \cap \mathrm{Inn}(L) = \{1\}$, whereas for $\beta \neq 0$ we have $S_1 \cap \mathrm{Inn}(L) = \{1\}$, for $l = 4, 5, 6$ the intersections $S_l \cap \mathrm{Inn}(L)$ have dimension 1 and $N \cap \mathrm{Inn}(L)$ has dimension 2 (cf. Proposition 13b). Since the factor Lie algebras $N_{6,38}/\mathbf{s}_k$, $k = 1, 3$, are not isomorphic to $\mathbf{l_2} \oplus \mathbf{l_2}$ the orbits $S_l(e)$, $l = 1, 3, 4, 5, 6$, and $N(e)$ are the same normal subgroup $M \cong \mathbb{R}^2$ of L and for $l = 1, 3$, one has $S_l \cap \mathrm{Inn}(L) = \{1\}$, for $l = 4, 5, 6$, the intersections $S_l \cap \mathrm{Inn}(L)$ have dimension 1, and $\dim(N \cap \mathrm{Inn}(L)) = 2$ (cf. Proposition 13b). In all cases the normal subgroup $I_k(e)$, $k = 1, 2$, are in M. For $j = 29, 33, 34, 38$, the Lie algebra $\mathbf{inn(L)}$ lies in one of the following ideals: $\mathbf{v}_1 = \langle n_1, n_2, n_3, n_4, x_1 \rangle$, $\mathbf{v}_{2,k} = \langle n_1, n_2, n_3, n_4, x_2 + k x_1 \rangle$, $k \in \mathbb{R}$. If for $N_{6,j}$, $j = 33, 34$, the Lie algebra $\mathbf{inn(L)}$ would contain the basis elements $b_1 = n_2 + a_1 n_4 + a_2 n_1$, $b_2 = n_3 + a_3 n_4 + a_4 n_1$, and for $N_{6,29}$ either the basis elements $b_1 = n_1 + a_1 n_2$, $b_2 = n_3 + a_2 n_4 + a_3 n_1$, or the basis elements $b_1 = n_1 + a_1 n_3$, $b_2 = n_2 + a_2 n_4 + a_3 n_1$ with $a_1 \neq 0$ would be in $\mathbf{inn(L)}$, then $\mathbf{inn(L)}$ would contain the ideal $\langle n_1 \rangle$ of $N_{6,j}$, $j = 29, 33, 34$, since one has $[b_1, b_2] = c n_1$, $c \neq 0$. This is a contradiction to Lemma 5. Otherwise for $N_{6,j}$, $j = 29, 33, 34, 38$, the Lie algebra $\mathbf{inn(L)}$ would contain the basis elements either $b_1' = n_1 + a_1 n_4$, $b_2' = n_2 + a_2 n_3 + a_3 n_4$, $b_3' = x_1 + c_1 n_3 + c_2 n_4$ or b_1', b_2',

$b'_{3,k} = x_2 + kx_1 + c_1n_3 + c_2n_4$, where $a_1a_2 \neq 0$, $k, c_1, c_2, a_3 \in \mathbb{R}$. The subspaces $\langle b'_1, b'_2, b'_3 \rangle$, $\langle b'_1, b'_2, b'_{3,k} \rangle$ are not 3-dimensional Lie algebras. This proves that none of the Lie algebras $N_{6,j}$, $j = 29, 31, 32, 33, 34, 38$, are the Lie algebras of the group $\mathrm{Mult}(L)$ of a 3-dimensional topological loop L. □

5. Three-dimensional topological loops corresponding to six-dimensional solvable Lie algebras with four-dimensional abelian nilradical and one-dimensional centre

Theorem 15. *If L is a connected topological proper loop of dimension* 3 *such that the Lie algebra of its multiplication group* $\mathrm{Mult}(L)$ *is a* 6*-dimensional solvable indecomposable Lie algebra having* 4*-dimensional nilradical, then L is centrally nilpotent of class* 2.

Proof. By Lemma 6 we may assume that L is homoemorphic to $\mathbb{R}^3$. By Proposition 14 it remains to deal with the 6-dimensional solvable indecomposable Lie algebras $N_{6,i}$, $i = 20, \dots, 27$, with abelian nilradical and 1-dimensional centre (cf. [T, Table II, p. 1348]). By Theorem 12(a) we have to prove that there is a normal subgroup $N \cong \mathbb{R}$ of L such that the factor loop L/N is isomorphic to $\mathbb{R}^2$. The Lie algebra $N_{6,20}^{a,b}$, $a^2 + b^2 \neq 0$, has the ideals $\mathbf{i}_1 = \langle n_1 \rangle$, $\mathbf{i}_2 = \langle n_2 \rangle$, $\mathbf{i}_3 = \langle n_3 \rangle$, $\mathbf{i}_4 = \langle n_4 \rangle$. If $N_{6,20}^{a,b}$ is the Lie algebra of the multiplication group of a 3-dimensional connected topological loop L, then the orbits $I_k(e)$, $k \in \{1, 2, 3, 4\}$, are normal subgroups of L isomorphic to $\mathbb{R}$ (cf. Lemma 3). The Lie algebra $N_{6,20}^{a,b}$ has no factor Lie algebra isomorphic to an elementary filiform Lie algebra. Hence the factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, are isomorphic either to $\mathcal{L}_2$ or to $\mathbb{R}^2$ (cf. Proposition 13(i),(ii)). If all factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, are isomorphic to $\mathcal{L}_2$, then by Proposition 13(ii) there are 2-dimensional ideals $\mathbf{s}_k$, $k \in \{1, 2, 3, 4\}$ such that $\mathbf{i}_k \subset \mathbf{s}_k$ and the factor Lie algebras $N_{6,20}^{a,b}/\mathbf{s}_k$ are isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$. For the ideal $\mathbf{s}_1 = \mathbf{s}_2 = \langle n_1, n_2 \rangle$ one has $N_{6,20}^{a,b}/\mathbf{s}_k \cong \mathbf{l}_2 \oplus \mathbf{l}_2$, $k = 1, 2$. The factor Lie algebra $N_{6,20}^{a,b}/\langle n_1, n_3 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ if and only if $a = 0$ and $N_{6,20}^{a,b}/\langle n_1, n_4 \rangle$ is isomorphic to $\mathbf{l}_2 \oplus \mathbf{l}_2$ precisely if $b = 0$. This contradiction to $a^2 + b^2 \neq 0$ yields that at least one of the factor loops $L/I_k(e)$, $k \in \{1, 2, 3, 4\}$, is isomorphic to $\mathbb{R}^2$. For such $k \in \{1, 2, 3, 4\}$ the orbit $I_k(e)$ is the requested normal subgroup of L in Theorem 12(a).

The Lie algebras $N_{6,21}^{a}$, $N_{6,22}^{\varepsilon,a}$, $N_{6,24}$, $N_{6,25}^{a,b}$, $N_{6,26}^{a}$, $N_{6,27}^{\varepsilon}$ have the ideal $\mathbf{i} = \langle n_2 \rangle$ and the unique 1-dimensional ideal of the Lie algebra $N_{6,23}^{a,\varepsilon}$ is its centre $\mathbf{i} = \langle n_4 \rangle$. There does not exist any ideal $\mathbf{s}$ of these Lie algebras $N_{6,i}$ containing $\mathbf{i}$ such that the factor Lie algebras $N_{6,i}/\mathbf{s}$ are isomorphic either to $\mathbf{l}_2 \oplus \mathbf{l}_2$ or to $\mathbf{f}_4$. If $N_{6,i}$, $i = 21, \dots, 27$, would be the Lie algebra of the multiplication group of a 3-dimensional connected topological loop L, then the factor loop $L/I(e)$ is isomorphic to $\mathbb{R}^2$ (cf. Proposition 13(i)). Hence the orbit $I(e)$ satisfies the assertion of Theorem 12(a). □

Proposition 16. *Let $\mathbf{g}$ be a* 6*-dimensional solvable indecomposable Lie algebra having* 4*-dimensional abelian nilradical $\mathbf{n}_{\mathrm{rad}}$ and* 1*-dimensional centre. Let $\mathbf{k}$ be a* 3*-dimensional abelian subalgebra of $\mathbf{g}$ which does not contain any non-zero ideal* *of* $\mathbf{g}$ *and the normalizer* $N_{\mathbf{g}}(\mathbf{k})$ *of* $\mathbf{k}$ *in* $\mathbf{g}$ *is* $\mathbf{n}_{\mathrm{rad}}$. *Then for the Lie algebra* $\mathbf{g}$ *and up to automorphisms of* $\mathbf{g}$ *for the subalgebra* $\mathbf{k}$ *we have one of the following cases:*

(a) $\mathbf{g}_1 := N^{a,b}_{6,20}$, $\mathbf{k}_1 = \langle n_2 + n_1, n_3 + n_1, n_4 + n_1 \rangle$.

(b) $\mathbf{g}_2 := N^{a}_{6,21}$, $\mathbf{g}_3 := N_{6,24}$, $\mathbf{k}_2 = \mathbf{k}_3 = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + n_1 \rangle$, $\varepsilon_1 = 0, 1$.

(c) $\mathbf{g}_4 := N^{a,b}_{6,25}$, $\mathbf{g}_5 := N^{a}_{6,26}$, $\mathbf{k}_4 = \mathbf{k}_5 = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + \varepsilon_2 n_1 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2$, *at least one of* $\{\varepsilon_1, \varepsilon_2\}$ *is different from* 0.

(d) $\mathbf{g}_6 := N^{\varepsilon}_{6,27}$, $\mathbf{k}_6 = \langle n_1 + \varepsilon_1 n_2, n_3 + \varepsilon_2 n_2, n_4 + \varepsilon_3 n_2 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, *such that at least one of* $\{\varepsilon_2, \varepsilon_3\}$ *is different from* 0.

(e) $\mathbf{g}_7 := N^{a,\varepsilon}_{6,23}$, $\mathbf{k}_7 = \langle n_1 + \varepsilon_1 n_4, n_2 + \varepsilon_2 n_4, n_3 + \varepsilon_3 n_4 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, *such that at least one of* $\{\varepsilon_1, \varepsilon_2\}$ *is different from* 0.

(f) $\mathbf{g}_8 := N^{a,\varepsilon}_{6,22}$, $\mathbf{k}_3 = \langle n_1 + n_4, n_2 + n_4, n_3 + \varepsilon_1 n_4 \rangle$, $\varepsilon_1 = 0, 1$.

Proof. The 6-dimensional indecomposable solvable Lie algebras with abelian nilradical and 1-dimensional centre in [T, Table II, p. 1348] are the Lie algebras $\mathbf{g}_i$, $i = 1, \dots, 8$. The Lie algebras $\mathbf{g}_i$, $i = 1, \dots, 5$, have the centre $\mathbf{z} = \langle n_1 \rangle$. For these Lie algebras the subalgebra $\mathbf{k}$ has the form

$$\mathbf{k}_{a_2,a_3,a_4} = \langle n_2 + a_2 n_1, n_3 + a_3 n_1, n_4 + a_4 n_1 \rangle,$$

such that

in the case $\mathbf{g}_1$: $a_2 a_3 a_4 \neq 0$, since $\langle n_2 \rangle$, $\langle n_3 \rangle$, $\langle n_4 \rangle$ are ideals of $\mathbf{g}_1$,

in the cases $\mathbf{g}_2$, $\mathbf{g}_3$: $a_2 a_4 \neq 0$ because $\langle n_4 \rangle$ and $\langle n_2 \rangle$ are ideals of $\mathbf{g}_i$, $i = 2, 3$,

in the cases $\mathbf{g}_4$, $\mathbf{g}_5$: $a_2 \neq 0$ and at least one of the constants $\{a_3, a_4\}$ is different from 0 since $\langle n_2 \rangle$ and $\langle n_3, n_4 \rangle$ are ideals of $\mathbf{g}_i$, $i = 4, 5$.

For the Lie algebras $\mathbf{g}_i$, $i = 1, \dots, 5$, using the automorphism $\alpha(n_1) = n_1$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_2) = a_2 n_2$, $\alpha(n_i) = a_i n_i$, $i = 3, 4$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, we can change $\mathbf{k}_{a_2,a_3,a_4}$ onto $\mathbf{k} = \langle n_2 + n_1, n_3 + \varepsilon_1 n_1, n_4 + \varepsilon_2 n_1 \rangle$, such that ε_1, respectively ε_2 is equal to 0 or 1, according whether a_3, respectively a_4 is 0 or $\neq 0$. The Lie algebra $\mathbf{g}_6$ has the centre $\mathbf{z} = \langle n_2 \rangle$ and hence for the subalgebra $\mathbf{k}$ one has $\mathbf{k}_{a_1,a_3,a_4} = \langle n_1 + a_1 n_2, n_3 + a_3 n_2, n_4 + a_4 n_2 \rangle$, such that $a_3 \neq 0$ or $a_4 \neq 0$ because $\langle n_3, n_4 \rangle$ is an ideal of $\mathbf{g}_6$. Using the automorphism $\alpha(n_2) = n_2$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_i) = a_i n_i$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, $i = 1, 3, 4$, we can reduce the Lie algebra $\mathbf{k}_{a_1,a_3,a_4}$ to $\mathbf{k} = \langle n_1 + \varepsilon_1 n_2, n_3 + \varepsilon_2 n_2, n_4 + \varepsilon_3 n_2 \rangle$, $\varepsilon_i = 0, 1$, $i = 1, 2, 3$, such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is different from 0. The centre of the Lie algebras $\mathbf{g}_i$, $i = 7, 8$, is $\langle n_4 \rangle$. For the subalgebra $\mathbf{k}$ of $\mathbf{g}_i$, $i = 7, 8$, we obtain

$$\mathbf{k}_{a_1,a_2,a_3} = \langle n_1 + a_1 n_4, n_2 + a_2 n_4, n_3 + a_3 n_4 \rangle,$$

such that

in the case $\mathbf{g}_7$: $a_1 \neq 0$ or $a_2 \neq 0$, since $\langle n_1, n_2 \rangle$ is an ideal of $\mathbf{g}_7$,

in the case $\mathbf{g}_8$: $a_1 a_2 \neq 0$ because $\langle n_1 \rangle$ and $\langle n_2 \rangle$ are ideals of $\mathbf{g}_8$.

For $\mathbf{g}_i$, $i = 7, 8$, using the automorphism $\alpha(n_4) = n_4$, $\alpha(x_i) = x_i$, $i = 1, 2$, $\alpha(n_i) = a_i n_i$, if $a_i \neq 0$, otherwise $\alpha(n_i) = n_i$, $i = 1, 2, 3$, we can change $\mathbf{k}_{a_1,a_2,a_3}$ onto $\mathbf{k} = \langle n_1 + \varepsilon_1 n_4, n_2 + \varepsilon_2 n_4, n_3 + \varepsilon_3 n_4 \rangle$, such that ε_i is equal to 0 or 1, according whether $a_i = 0$ or $a_i \neq 0$, $i = 1, 2, 3$. $\square$

Theorem 17. *Let L be a connected simply connected topological proper loop of dimension 3 such that the Lie algebra of its multiplication group* $\mathrm{Mult}(L)$ *is a 6-dimensional solvable indecomposable Lie algebra having 4-dimensional nilradical. Then the following Lie groups are the multiplication groups* $\mathrm{Mult}(L)$ *and the following subgroups are the inner mapping groups* $\mathrm{Inn}(L)$ *of L:*

1) $\mathrm{Mult}(L)_1$ *is given by the multiplication*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&\quad= g(x_1+y_1, x_2+y_2+x_5y_1, x_3+y_3e^{x_6}\cos(x_5)-y_4e^{x_6}\sin(x_5),\\&\qquad\qquad x_4+y_4e^{x_6}\cos(x_5)+y_3e^{x_6}\sin(x_5), x_5+y_5, x_6+y_6),\end{aligned}$$

$\mathrm{Inn}(L)_1$ *is the subgroup* $\{g(u_1,\varepsilon_1u_1+\varepsilon_2u_2+\varepsilon_3u_3,u_2,u_3,0,0); u_1,u_2,u_3\in\mathbb{R}\}$, $\varepsilon_k\in\{0,1\}$, $k=1,2,3$, *such that* $\varepsilon_2^2+\varepsilon_3^2\neq 0$.

2) *The multiplication of the group* $\mathrm{Mult}(L)_2$ *is defined by*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&\quad= g(x_1+y_1e^{x_5}\cos(x_6)-y_2e^{x_5}\sin(x_6), x_2+y_2e^{x_5}\cos(x_6)+y_1e^{x_5}\sin(x_6),\\&\qquad\qquad x_3+y_3, x_4+y_4+(ax_6+x_5)y_3, x_5+y_5, x_6+y_6), a\in\mathbb{R}\end{aligned}$$

$\mathrm{Inn}(L)_2$ *is the subgroup* $\{g(u_1,u_2,u_3,\varepsilon_1u_1+\varepsilon_2u_2+\varepsilon_3u_3,0,0); u_1,u_2,u_3\in\mathbb{R}\}$, $\varepsilon_k\in\{0,1\}$, $k=1,2,3$, *such that* $\varepsilon_1^2+\varepsilon_2^2\neq 0$.

3) *The multiplication of the group* $\mathrm{Mult}(L)_3$ *is given by*

$$\begin{aligned}&g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&\quad= g(x_1+y_1e^{x_5+ax_6}, x_2+y_2e^{x_6}, x_3+y_3,\\&\qquad\qquad x_4+y_4+x_5y_3, x_5+y_5, x_6+y_6), a\in\mathbb{R}\setminus\{0\}\end{aligned}$$

and $\mathrm{Inn}(L)_3$ *is* $\{g(u_1,u_2,u_3,u_1+u_2+\varepsilon u_3,0,0);\ u_1,u_2,u_3\in\mathbb{R}\}$, $\varepsilon=0,1$.

Proof. According to Theorem 15 the centre $Z(L)$ of the simply connected loop L is isomorphic to $\mathbb{R}$ and the factor loop $L/Z(L)$ is isomorphic to $\mathbb{R}^2$. By Proposition 13(i) the Lie algebra $\mathbf{g}$ of the group $\mathrm{Mult}(L)$ of L has abelian nilradical and $\mathbf{n}_{\mathrm{rad}} = \mathbf{z}\oplus\mathbf{inn(L)}$, where $\mathbf{z}$ is the centre of $\mathbf{g}$ and $\mathbf{inn(L)}$ is the Lie algebra of the group $\mathrm{Inn}(L)$. Hence $\mathbf{inn(L)}$ is a 3-dimensional abelian subalgebra of $\mathbf{g}$ which does not contain any non-zero ideal of $\mathbf{g}$ and the normalizer $N_{\mathbf{g}}(\mathbf{inn(L)})$ coincides with $\mathbf{n}_{\mathrm{rad}}$ (cf. Lemma 5). It follows from Proposition 16 that the pairs $(\mathbf{g}_i,\mathbf{k}_i)$, $i=1,\dots,8$, can occur as the Lie algebras of the group $\mathrm{Mult}(L)$ and the subgroup $\mathrm{Inn}(L)$. Linear representations of the simply connected Lie groups G_i of $\mathbf{g}_i$ are given in this order by

$$\begin{aligned}i=1: &\ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+y_1+x_6y_5, x_2+y_2e^{ax_5+bx_6}, x_3+y_3e^{x_6},\\&\quad x_4+y_4e^{x_5}, x_5+y_5, x_6+y_6),\\i=2: &\ g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\&= g(x_1+y_1+x_6y_5, x_2+y_2e^{x_5+ax_6}, x_3+y_3e^{x_6},\\&\quad x_4+y_4e^{x_6}+x_3y_5, x_5+y_5, x_6+y_6),\end{aligned}$$

$$\begin{aligned}
i=3: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1+x_5y_6, x_2+y_2e^{x_6}, x_3+y_3e^{x_5},\\
&\quad x_4+y_4e^{x_5}+x_5e^{x_5}y_3, x_5+y_5, x_6+y_6),\\
i=4: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1-x_6y_5, x_2+y_2e^{ax_5+bx_6},\\
&\quad y_3+x_3\cos(y_5)e^{y_6}-x_4\sin(y_5)e^{y_6},\\
&\quad y_4+x_4\cos(y_5)e^{y_6}+x_3\sin(y_5)e^{y_6}, x_5+y_5, x_6+y_6),\\
i=5: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1+x_6y_5, x_2+y_2e^{x_6},\\
&\quad x_3+y_3\cos(x_5)e^{ax_5}+y_4\sin(x_5)e^{ax_5},\\
&\quad x_4+y_4\cos(x_5)e^{ax_5}-y_3\sin(x_5)e^{ax_5}, x_5+y_5, x_6+y_6),\\
i=6:\varepsilon=0: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1, x_2+y_2+x_5y_1,\\
&\quad x_3+y_3e^{x_6}\cos(x_5)-y_4e^{x_6}\sin(x_5),\\
&\quad x_4+y_4e^{x_6}\cos(x_5)+y_3e^{x_6}\sin(x_5), x_5+y_5, x_6+y_6),\\
i=6:\varepsilon=1: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1+x_5y_6, x_2+y_2+x_5y_1\\
&\quad +\tfrac{1}{2}x_5^2y_6, y_3+x_3e^{y_6}\cos(y_5)-x_4e^{y_6}\sin(y_5),\\
&\quad y_4+x_4e^{y_6}\cos(y_5)+x_3e^{y_6}\sin(y_5), x_5+y_5, x_6+y_6),\\
i=7:\varepsilon=0: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1e^{x_5}\cos(x_6)-y_2e^{x_5}\sin(x_6),\\
&\quad x_2+y_2e^{x_5}\cos(x_6)+y_1e^{x_5}\sin(x_6),\\
&\quad x_3+y_3, x_4+y_4+(ax_6+x_5)y_3, x_5+y_5, x_6+y_6),\\
i=7:\varepsilon=1, a=0: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1e^{x_5}\cos(x_6)+y_2e^{x_5}\sin(x_6),\\
&\quad x_2+y_2e^{x_5}\cos(x_6)-y_1e^{x_5}\sin(x_6), x_3+y_3+x_5y_6,\\
&\quad x_4+y_4+x_5y_3+\tfrac{1}{2}x_5^2y_6, x_5+y_5, x_6+y_6),\\
i=7:\varepsilon=1, a\neq 0: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1e^{x_5}\cos(x_6)+y_2e^{x_5}\sin(x_6),\\
&\quad x_2+y_2e^{x_5}\cos(x_6)-y_1e^{x_5}\sin(x_6),\\
&\quad x_3+y_3+(ax_6+x_5)y_5,\\
&\quad x_4+y_4+(ax_6+x_5)y_3+\tfrac{1}{2}(ax_6+x_5)^2y_5,\\
&\quad x_5+y_5, x_6+y_6),\\
i=8:\varepsilon=0: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1e^{x_5+ax_6}, x_2+y_2e^{x_6}, x_3+y_3,\\
&\quad x_4+y_4+x_5y_3, x_5+y_5, x_6+y_6),\\
i=8:\varepsilon=1: {}& g(x_1,x_2,x_3,x_4,x_5,x_6)g(y_1,y_2,y_3,y_4,y_5,y_6)\\
&= g(x_1+y_1e^{x_5+ax_6}, x_2+y_2e^{x_6}, x_3+y_3+x_5y_6,\\
&\quad x_4+y_4+x_5y_3+\tfrac{1}{2}x_5^2y_6, x_5+y_5, x_6+y_6),
\end{aligned}$$

where $a, b \in \mathbb{R}$ such that for $i = 1, 4$ one has $a^2 + b^2 \neq 0$ and for $i = 8$, $\varepsilon = 0$ we have $a \neq 0$ (cf. [RT, pp. 16–21]). Using these linear representations the Lie groups of the Lie algebras $\mathbf{k}_i$ are:

for $i = 1$: $\mathrm{Inn}(L) = \{g(u_1 + u_2 + u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$;

for $i = 2, 3$: $\mathrm{Inn}(L) = \{g(u_1 + \varepsilon u_2 + u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon = 0, 1$;

for $i = 4, 5$: $\mathrm{Inn}(L) = \{g(u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, u_1, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon_k = 0, 1$, $k = 2, 3$ such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is different from 0;

for $i = 6$: $\mathrm{Inn}(L) = \{g(u_1, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, u_2, u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon_k = 0, 1$, $k = 1, 2, 3$, such that at least one of $\{\varepsilon_2, \varepsilon_3\}$ is different from 0;

for $i = 7$: $\mathrm{Inn}(L) = \{g(u_1, u_2, u_3, \varepsilon_1 u_1 + \varepsilon_2 u_2 + \varepsilon_3 u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon_k = 0, 1$, $k = 1, 2, 3$, such that at least one of $\{\varepsilon_1, \varepsilon_2\}$ is different from 0;

for $i = 8$: $\mathrm{Inn}(L) = \{g(u_1, u_2, u_3, u_1 + u_2 + \varepsilon u_3, 0, 0); u_j \in \mathbb{R}\}$, $j = 1, 2, 3$, $\varepsilon = 0, 1$.

Two arbitrary left transversals to the group $\mathrm{Inn}(L)$ in G_i, $i = 1, \ldots, 5$, are

$$\begin{aligned} A &= \{g(k, f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), l, m), k, l, m \in \mathbb{R}\}, \\ B &= \{g(u, h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), v, w), u, v, w \in \mathbb{R}\}, \end{aligned}$$

those to the group $\mathrm{Inn}(L)$ in G_6 are

$$\begin{aligned} A &= \{g(f_1(k, l, m), k, f_2(k, l, m), f_3(k, l, m), l, m), k, l, m \in \mathbb{R}\}, \\ B &= \{g(h_1(u, v, w), u, h_2(u, v, w), h_3(u, v, w), v, w), u, v, w \in \mathbb{R}\}, \end{aligned}$$

those to the group $\mathrm{Inn}(L)$ in G_i, $i = 7, 8$, are

$$\begin{aligned} A &= \{g(f_1(k, l, m), f_2(k, l, m), f_3(k, l, m), k, l, m), k, l, m \in \mathbb{R}\}, \\ B &= \{g(h_1(u, v, w), h_2(u, v, w), h_3(u, v, w), u, v, w), u, v, w \in \mathbb{R}\}, \end{aligned}$$

where $f_i(k, l, m) : \mathbb{R}^3 \to \mathbb{R}$ and $h_i(u, v, w) : \mathbb{R}^3 \to \mathbb{R}$, $i = 1, 2, 3$, are continuous functions with $f_i(0, 0, 0) = h_i(0, 0, 0) = 0$. We prove that none of the groups G_i, $i = 1, \ldots, 5$, and $G_j^{\varepsilon=1}$, $j = 6, 7, 8$, satisfies the condition that for all $a \in A$ and $b \in B$ one has $a^{-1}b^{-1}ab \in \mathrm{Inn}(L)$. By Lemma 4 these groups are not multiplication groups of L. Taking the elements

$$\begin{aligned} a &= g(0, f_1(0, 0, m), f_2(0, 0, m), f_3(0, 0, m), 0, m) \in A, \\ b &= g(0, h_1(0, v, 0), h_2(0, v, 0), h_3(0, v, 0), v, 0) \in B \end{aligned}$$

in G_i, $i = 1, 3, 4, 5$, the products $a^{-1}b^{-1}ab$ are contained in $\mathrm{Inn}(L)$ if and only if the equation

$$\begin{aligned} i = 1: \ vm &= (1 - e^{-m})h_2(0, v, 0) + (e^{-v} - 1)f_3(0, 0, m) \\ &\quad + h_1(0, v, 0)e^{-av}(1 - e^{-bm}) + f_1(0, 0, m)e^{-bm}(e^{-av} - 1), \end{aligned} \tag{6}$$

$$\begin{aligned} i = 3: \ -vm &= (1 - e^{-m})h_1(0, v, 0) - ve^{-v}f_2(0, 0, m) \\ &\quad + (f_3(0, 0, m) + \varepsilon f_2(0, 0, m))(e^{-v} - 1), \end{aligned} \tag{7}$$

$$\begin{aligned} i=4:\ \ -vm &= h_1(0,v,0)e^{-av}(1-e^{-bm}) + f_1(0,0,m)e^{-bm}(e^{-av}-1)\\ &\quad + (1-e^m)(\varepsilon_1 h_2(0,v,0)+\varepsilon_2 h_3(0,v,0))\\ &\quad + (\cos v-1)(\varepsilon_1 f_2(0,0,m)+\varepsilon_2 f_3(0,0,m))\\ &\quad + \sin v(\varepsilon_2 f_2(0,0,m)-\varepsilon_1 f_3(0,0,m)) \end{aligned} \tag{8}$$

$$\begin{aligned} i=5:\ \ vm &= \varepsilon_1[e^{-av}(\cos(v)f_2(0,0,m)-\sin(v)f_3(0,0,m))-f_2(0,0,m)]\\ &\quad + \varepsilon_2[e^{-av}(\sin(v)f_2(0,0,m)+\cos(v)f_3(0,0,m))-f_3(0,0,m)]\\ &\quad + h_1(0,v,0)(1-e^{-m}) \end{aligned} \tag{9}$$

holds for all $m, v \in \mathbb{R}$. On the left hand side of (6), (7), (8), (9) there is the term vm hence there does not exist any function $h_i(0,v,0)$, $f_i(0,0,m)$, $i=1,2,3$, satisfying equations (6), (7), (8), (9).
Taking the elements $a = g(0, f_1(0,0,m), f_2(0,0,m), f_3(0,0,m), 0, m) \in A$, $b = g(0, h_1(0,v,w), h_2(0,v,w), h_3(0,v,w), v, w) \in B$ of G_2 the products $a^{-1}b^{-1}ab$ are contained in $\mathrm{Inn}(L)$ if and only if the equation

$$\begin{aligned} mv &= e^{-m-w}f_2(0,0,m)v + h_1(0,v,w)e^{-aw-v}(1-e^{-am})\\ &\quad + f_1(0,0,m)e^{-am}(e^{-aw-v}-1)\\ &\quad + (h_3(0,v,w)+\varepsilon h_2(0,v,w))e^{-w}(1-e^{-m})\\ &\quad + (f_3(0,0,m)+\varepsilon f_2(0,0,m))e^{-m}(e^{-w}-1) \end{aligned} \tag{10}$$

holds for all $m, v, w \in \mathbb{R}$. The left hand side is mv. But there does not exist any function $h_i(0,v,w)$, $f_i(0,0,m)$, $i=1,2,3$, satisfying equation (10).
Taking the elements $a = g(f_1(0,0,m), 0, f_2(0,0,m), f_3(0,0,m), 0, m) \in A$, $b = g(h_1(0,v,0), 0, h_2(0,v,0), h_3(0,v,0), v, 0) \in B$ of $G_6^{\varepsilon=1}$, respectively the elements

$$\begin{aligned} a &= g(f_1(0,0,m), f_2(0,0,m), f_3(0,0,m), 0, 0, m) \in A,\\ b &= g(h_1(0,v,0), h_2(0,v,0), h_3(0,v,0), 0, v, 0) \in B \end{aligned}$$

of $G_7^{\varepsilon=1,a=0}$ and of $G_8^{\varepsilon=1}$ the subgroup $\mathrm{Inn}(L)$ contains the products $a^{-1}b^{-1}ab$ if and only if in $G_6^{\varepsilon=1}$ the equation

$$\begin{aligned} \tfrac{1}{2}v^2m - vf_1(0,0,m) &= (1-e^m)(\varepsilon_2 h_2(0,v,0)+\varepsilon_3 h_3(0,v,0)) - \varepsilon_1 vm\\ &\quad + (\cos(v)-1)(\varepsilon_2 f_2(0,0,m)+\varepsilon_3 f_3(0,0,m))\\ &\quad + \sin(v)(\varepsilon_3 f_2(0,0,m)-\varepsilon_2 f_3(0,0,m)), \end{aligned} \tag{11}$$

respectively in $G_7^{\varepsilon=1,a=0}$ the equation

$$\begin{aligned} \tfrac{1}{2}v^2m - vf_3(0,0,m) &= (f_1(0,0,m)-h_1(0,v,0))e^{-v}(\varepsilon_1\cos(m)+\varepsilon_2\sin(m))\\ &\quad + (f_2(0,0,m)-h_2(0,v,0))e^{-v}(\varepsilon_2\cos(m)-\varepsilon_1\sin(m))\\ &\quad + \sin(m)(\varepsilon_1 f_2(0,0,m)-\varepsilon_2 f_1(0,0,m))\\ &\quad - \cos(m)(\varepsilon_1 f_1(0,0,m)+\varepsilon_2 f_2(0,0,m))\\ &\quad + e^{-v}(\varepsilon_1 h_1(0,v,0)+\varepsilon_2 h_2(0,v,0)) - \varepsilon_3 vm, \end{aligned} \tag{12}$$

respectively in $G_8^{\varepsilon=1}$ the equation

$$\begin{aligned}\tfrac{1}{2}v^2m - vf_3(0,0,m) = h_1(0,v,0)e^{-v}(1-e^{-am}) + h_2(0,v,0)(1-e^{-m})\\ + f_1(0,0,m)e^{-am}(e^{-v}-1) - \varepsilon_1 vm\end{aligned} \tag{13}$$

holds for all $m, v \in \mathbb{R}$. On the left hand side of equations (11), (12), (13) there is the term $\frac{1}{2}v^2m$. Hence there does not exist any function $h_i(0,v,0)$, $f_i(0,0,m)$, $i = 1,2,3$, satisfying equations (11), (12), (13).
The products $a^{-1}b^{-1}ab$ with $a = g(f_1(0,l,m), f_2(0,l,m), f_3(0,l,m), 0, l, m)$, $b = g(h_1(0,v,0), h_2(0,v,0), h_3(0,v,0), 0, v, 0)$ in $G_7^{a\neq 0,\varepsilon=1}$ are contained in $\mathrm{Inn}(L)$ if and only if the equation

$$\begin{aligned}&\tfrac{1}{2}(vl^2 - v^2l - a^2vm^2) + (am+l)h_3(0,v,0) - vf_3(0,l,m) - amv^2\\ &\quad= \varepsilon_3 vam + (f_1(0,l,m) - h_1(0,v,0))e^{-v-l}(\varepsilon_1\cos m + \varepsilon_2 \sin m)\\ &\qquad + e^{-v}\varepsilon_1 h_1(0,v,0) + e^{-v}\varepsilon_2 h_2(0,v,0)\\ &\qquad + (f_2(0,l,m) - h_2(0,v,0))e^{-v-l}(\varepsilon_2\cos m - \varepsilon_1\sin m)\\ &\qquad + f_2(0,l,m)e^{-l}(\varepsilon_1\sin m + \varepsilon_2\cos m)\\ &\qquad - f_1(0,l,m)e^{-l}(\varepsilon_2\sin m + \varepsilon_1\cos m)\end{aligned} \tag{14}$$

holds for all $l, m, v \in \mathbb{R}$, where $\varepsilon_i \in \{0,1\}$, $i = 1,2,3$, such that $\varepsilon_1 \neq 0$ or $\varepsilon_2 \neq 0$. Since on the left hand side of (14) there is the term $-\frac{1}{2}v^2l$ and $a \neq 0$ there does not exist any function $f_i(0,l,m)$, $h_i(0,v,0)$, $i = 1,2,3$, such that equation (14) holds.

The set

$$\begin{aligned}A_1 = B_1 = \Big\{g\Big(&e^{-m}\cos(l) - 1, k, \frac{1}{\varepsilon_2^2+\varepsilon_3^2}\big(le^m(\varepsilon_2\cos(l) - \varepsilon_3\sin(l)) + \sin(l)\big)\\ &\cdot\big(\varepsilon_3\cos(l) + \varepsilon_2\sin(l)\big), \frac{1}{\varepsilon_2^2+\varepsilon_3^2}\big(le^m(\varepsilon_2\sin(l) + \varepsilon_3\cos(l))\big)\\ &+ \sin(l)\big(\varepsilon_3\sin(l) - \varepsilon_2\cos(l)\big), l, m\Big); k, l, m \in \mathbb{R}\Big\},\end{aligned}$$

with $\varepsilon_2, \varepsilon_3 \in \{0,1\}$ and $\varepsilon_2^2 + \varepsilon_3^2 \neq 0$ is an $\mathrm{Inn}(L)_6$-connected left transversal in $G_6^{\varepsilon=0}$ which generates the group $G_6^{\varepsilon=0}$.

The set

$$\begin{aligned}A_2 = B_2 = \Big\{g\Big(&\frac{1}{\varepsilon_1^2+\varepsilon_2^2}\big(e^l(l+am)(\varepsilon_1\cos(m) - \varepsilon_2\sin(m)) + \sin(m)\big)\\ &\cdot\big(\varepsilon_1\sin(m) + \varepsilon_2\cos(m)\big), \frac{1}{\varepsilon_1^2+\varepsilon_2^2}\big(e^l(l+am)(\varepsilon_1\sin(m) + \varepsilon_2\cos(m))\\ &- \sin(m)(\varepsilon_1\cos(m) - \varepsilon_2\cos(m)\big), e^{-l}\cos(m) - 1, k, l, m\Big); k, l, m \in \mathbb{R}\Big\},\end{aligned}$$

where $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and $\varepsilon_1^2 + \varepsilon_2^2 \neq 0$ is an $\mathrm{Inn}(L)_7$-connected left transversal in $G_7^{\varepsilon=0}$ which generates the group $G_7^{\varepsilon=0}$.
The sets

$$\begin{aligned}A &= \{g(0, le^m, 1 - e^{-l-am}, k, l, m); k, l, m \in \mathbb{R}\},\\ B &= \{g(-ve^{v+aw}, 0, e^{-w} - 1, u, v, w); u, v, w \in \mathbb{R}\}\end{aligned}$$

are $\mathrm{Inn}(L)_8$-connected left transversals in the group $G_8^{\varepsilon=0}$ such that $A \cup B$ generates $G_8^{\varepsilon=0}$. This proves the assertion. $\square$

References

[A] A. A. Albert, *Quasigroups* I, Trans. Amer. Math. Soc. **54** (1943), 507–519.

[B1] R. H. Bruck, *Contributions to the theory of loops*, Trans. Amer. Math. Soc. **60** (1946), 245–354.

[B2] R. H. Bruck, *A Survey of Binary Systems*, Springer-Verlag, Berlin, 1971.

[C] P. Csörgő, M. Niemenmaa, *On connected transversals to nonabelian subgroups*, Europ. J. Comb. **23** (2002), 179–185.

[D] A. Drapal, *Orbits of inner mapping groups*, Monatsh. Math. **134** (2002), 191–206.

[F1] Á. Figula, *The multiplication groups of* 2*-dimensional topological loops*, J. Group Theory **12** (2009), 419–429.

[F2] Á. Figula, *On the multiplication groups of three-dimensional topological loops*, J. Lie Theory **21** (2011), 385–415.

[F3] Á. Figula, *Three-dimensional topological loops with solvable multiplication groups*, Comm. in Algebra **42** (2014), 444–468.

[F4] Á. Figula, *Quasi-simple Lie groups as multiplication groups of topological loops*, Adv. in Geometry **15** (2015), 315–331.

[F5] Á. Figula, M. Lattuca, *Three-dimensional topological loops with nilpotent multiplication groups*, J. Lie Theory **25** (2015), 787–805.

[F] R. Freese, R. McKenzie, *Commutator Theory for Congruence Modular Varieties*, London Mathematical Society Lecture Notes Series, Vol. 125, Cambridge University Press, Cambridge 1987.

[G] A. González-López, N. Kamran, P. J. Olver, *Lie algebras of vector fields in the real plane*, Proc. London Math. Soc. **64** (1992), no. 2, 339–368.

[H] K. H. Hofmann, K. Strambach, *Topological and analytical loops*, in: *Quasigroups and Loops: Theory and Applications*, O. Chein, H. O. Pflugfelder, J. D. H. Smith eds., Heldermann-Verlag, Berlin, 1990, pp. 205–262.

[L] S. Lie, F. Engels, *Theorie der Transformationsgruppen*, Bd. 3. Verlag von B.G. Teubner, Leipzig, 1893.

[Ma] M. Mazur, *Connected transversals to nilpotent groups*, J. Group Theory **10** (2007), 195–203.

[Mo] C. Moore, D. Thérien, F. Lemieux, J. Berman, A. Drisko, *Circuits and expressions with nonassociative gates*, J. of Computer and System Sciences **60** (2000), 368–394.

[Mu] Г. М. Мубаракзянов, *Классификация разрешимых алгебр Ли шестого порядка с одним ненильпотентным базисным элементом* , Изв. вузов, Матем. **4** (1963), 104–116. [G. M. Mubarakzyanov, *Classification of solvable Lie algebras in dimension six with one non-nilpotent basis element*, Izv. Vyssh. Uchebn. Zaved. Mat. **4** (1963), 104–116 (in Russian)].

[NV] G. P. Nagy, P. Vojtěchovský, *Moufang loops with commuting inner mappings*, J. Pure Appl. Algebra **213** (2009), 2177–2186.

[N] P. T. Nagy, *Nuclear properties of loop extensions*, Results in Math. **74** (2019), Art. number: 100 (2019).

[NS1] P. T. Nagy, K. Strambach, *Loops in Group Theory and Lie Theory*, Walter de Gruyter, Vol. 35, Berlin, 2002.

[NS2] P. T. Nagy, K. Strambach, *Schreier loops*, Czech. Math. J. **58** (2008), 759–786.

[NK1] M. Niemenmaa, T. Kepka, *On multiplication groups of loops*, J. Algebra **135** (1990), 112–122.

[NK2] M. Niemenmaa, T. Kepka, *On connected transversals to abelian subgroups*, Bull. Austral. Math. Soc. **49** (1994), 121–128.

[RT] M. Rawashdeh, G. Thompson, *The inverse problem for six-dimensional codimension two nilradical Lie algebras*, J. Math. Phys. **47** (2006), 112901–112929.

[ST] A. Shabanskaya, G. Thompson, *Six-dimensional Lie algebras with a five-dimensional nilradical*, J. of Lie Theory **23** (2013), 313–355.

[SV1] D. Stanovský, P. Vojtěchovský, *Commutator theory for loops*, J. Algebra **399** (2014), 290–322.

[SV2] D. Stanovský, P. Vojtěchovský, *Abelian Extensions and Solvable Loops*, Results. Math. **66** (2014), 367–384.

[T] P. Turkowski, *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (1990), 1344–1350.

[V] A. Vesanen, *Solvable loops and groups*, J. Algebra **180** (1996), 862–876.

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