Transformation Groups ©Springer Science+Business Media New York (2017)
Vol. 22, No. 1, 2017, pp. 291–293

# ERRATUM TO "MODULI SPACES OF $(G, h)$-CONSTELLATIONS", TRANSFORMATION GROUPS 20 (2015), 2, 335–366

T. BECKER*

Fachbereich Physik
Mathematik und Informatik
Johannes Gutenberg—
Universität Mainz
D–55099 Mainz, Germany
tanja@mathematik.uni-mainz.de

R. TERPEREAU**

Fachbereich Physik
Mathematik und Informatik
Johannes Gutenberg—
Universität Mainz
D–55099 Mainz, Germany
rterpere@uni-mainz.de

Several numerical values given in [BT15, §3.3] are incorrect and should be replaced by the numerical values given in this erratum. More precisely, in [BT15, §3.3] we need to assume that $D_+$ is infinite and to change the definition of the character $\chi$ and the weights $\kappa_\rho$ (as well the $\rho \in \operatorname{Irr} G$ for which they are defined) for the proof of Proposition 3.7 to be correct.

Let us note that, with these corrections, Remark 3.6 of the original paper becomes irrelevant and should be removed. For the sake of simplicity, we rewrote completely the part of the original text where changes have to be made and we added some extra explanations. The changes we make in this erratum have no impact outside [BT15, §3.3].

To compare $\theta$ and $\widetilde{\theta}$ we make the following approach for choosing the character $\chi$ and the weights $\kappa_\rho$ in the definition of our ample line bundle $\mathscr{L}$:

$$\left.\begin{aligned} \kappa_\rho &\in \mathbb{Q}_{>0} \quad \text{arbitrary for } \rho \in D \cap D_-, \\ \kappa_\rho &= \theta_\rho + S_D/(d \cdot h(\rho)) \quad \text{for} \quad \rho \in D \setminus D_-, \\ \chi_\rho &= \theta_\rho - \kappa_\rho + \kappa(\mathcal{F})/\dim A \quad \text{for} \quad \rho \in D_-, \end{aligned}\right\} \tag{17}$$

where $d := \#(D \setminus D_-)$ is the number of non-zero summands in the second sum in the definition of $\widetilde{\theta}$. Moreover if $h(\rho) = 0$, then $\rho$ plays no role in the embedding of the Quot scheme given by (5) and thus we can assume that $h(\rho) \neq 0$ for every $\rho \in D \setminus D_-$. Also, we should from now on assume that $D_+$ is infinite, therefore we always have $S_D > 0$. It is a natural assumption, because if we were working with $D_- \cup D_+$ finite, then $\theta(\mathcal{F}')$ would be a finite sum and thus we would have considered another definition for the $\theta$-(semi)stability; namely the one considered by Craw and Ishii where we ask for every subsheaf $\mathcal{F}'$ (and not only those generated in $D_-$) that $\theta(\mathcal{F}') \underset{(-)}{>} \theta(\mathcal{F}) = 0$. The condition $\theta(\mathcal{F}) = 0$ implies $S_D = -\sum_{\sigma \in D} \theta_\sigma h(\sigma) \in \mathbb{Q}$, so

DOI: 10.1007/s00031-017-9417-x
*Supported by DAAD and SFB/TR 45.
**Supported by SFB/TR 45.
Received December 9, 2016.
Published online February 15, 2017.
Corresponding Author: R. Terpereau, e-mail: rterpere@uni-mainz.de.

$\kappa_\rho \in \mathbb{Q}$. Since $D \supset D_-$ and $D_+$ is infinite, the inequality $S_D > 0$ holds, and thus $\kappa_\rho > 0$ for all $\rho \in D$.

Let us also note the following facts substantiating why the choice (17) for $\chi$ and $\kappa$ is natural:

- Since $\theta_\rho < 0$ and $\kappa_\rho > 0$ for every $\rho \in D_-$, we automatically have

$$\chi_\rho = \theta_\rho - \kappa_\rho + \frac{\kappa(\mathcal{F})}{\dim A} < \frac{\kappa(\mathcal{F})}{\dim A},$$

  so the prerequisites of Lemma 3.1 and Theorem 3.5 are always satisfied.
- One easily calculates $\sum_{\rho\in D_-} \chi_\rho h(\rho) = \theta(\mathcal{F})$, and thus if $\mathcal{F}$ is $\theta$–semistable we obtain $\sum_{\rho\in D_-} \chi_\rho h(\rho) = 0$.
- Let $\mathcal{F}$ be a $(G,h)$–constellation. For any $G$–equivariant coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$, plugging in (17) in Definition 3.3 gives

$$\widetilde{\theta}(\mathcal{F}') \;=\; \sum_{\rho\in D} \theta_\rho h'(\rho) \;+\; \frac{S_D}{d} \sum_{\sigma\in D\setminus D_-} \frac{h'(\sigma)}{h(\sigma)}, \tag{18}$$

  in particular, $\widetilde{\theta}(\mathcal{F}) = \theta(\mathcal{F})$.

For comparing $\theta$ to $\widetilde{\theta}$, we consider $\widetilde{\theta} = \widetilde{\theta}_D$ when the finite subset $D \subset \operatorname{Irr} G$ varies. We obtain the following error terms:

**Proposition 3.7.** *If $\widetilde{D} \supset D$, then for any $G$–equivariant coherent subsheaf $\mathcal{F}'$ of a $(G,h)$–constellation $\mathcal{F}$ we have*

$$|\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}')| \leq \sum_{\tau\in\widetilde{D}\setminus D} \left(\theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}}\right),$$

*where $S_{\widetilde{D}}$ is defined by (16) and $\widetilde{d} := \#(\widetilde{D}\cap D_+)$. Further, we have*

$$|\theta(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}')| \leq \sum_{\tau\in\operatorname{Irr} G\setminus D} \theta_\tau h(\tau).$$

*Proof.* Using (18), we write

$$\widetilde{\theta}_D(\mathcal{F}') = \sum_{\rho\in D_-} \theta_\rho h'(\rho) \;+ \sum_{\sigma\in D\setminus D_-} \theta_\sigma h'(\sigma) \;+\; \frac{S_D}{d} \sum_{\sigma\in D\cap D_+} \frac{h'(\sigma)}{h(\sigma)} \qquad \text{and}$$

$$\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') = \sum_{\rho\in D_-} \theta_\rho h'(\rho) \;+ \sum_{\sigma\in \widetilde{D}\setminus D_-} \theta_\sigma h'(\sigma) \;+\; \frac{S_{\widetilde{D}}}{\widetilde{d}} \sum_{\sigma\in \widetilde{D}\cap D_+} \frac{h'(\sigma)}{h(\sigma)}.$$

In [Bec11b, Prop. 4.3.3], the determination of their difference is carried out:

$$\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}') = \sum_{\tau\in\widetilde{D}\setminus D} \left(\theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}}\right)\left(\frac{h'(\tau)}{h(\tau)} - \frac{1}{d}\sum_{\sigma\in D\setminus D_-} \frac{h'(\sigma)}{h(\sigma)}\right)$$

Further, for every $\tau \in \operatorname{Irr} G$ such that $h(\tau) \neq 0$ (and we assumed it was always the case when $\tau \in \widetilde{D} \setminus D$) we have $0 \leq h'(\tau)/h(\tau) \leq 1$, so it follows $\big|h'(\tau)/h(\tau) - (1/d)\sum_{\sigma \in D \setminus D_-} h'(\sigma)/h(\sigma)\big| \leq 1$. We deduce

$$|\widetilde{\theta}_{\widetilde{D}}(\mathcal{F}') - \widetilde{\theta}_D(\mathcal{F}')| \leq \sum_{\tau \in \widetilde{D} \setminus D} \left| \theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}} \right| = \sum_{\tau \in (\widetilde{D} \setminus D)} \left( \theta_\tau h(\tau) + \frac{S_{\widetilde{D}}}{\widetilde{d}} \right).$$

A similar computation gives the second upper bound. □

The set $\mathcal{D} = \{D \subset \operatorname{Irr} G \mid D \supset D_-\}$ is directed with respect to inclusion. In this sense, we can take the limit over these sets. This allows us to reveal the relation between $\theta$ and $\widetilde{\theta}$:

**Corollary 3.8.** *The function $\theta$ is the pointwise limit of the functions $\widetilde{\theta}_D$ as $D$ converges to* $\operatorname{Irr} G$ (*here it would be more correct to say: as $D$ converges to* $\operatorname{supp} h \cup D_-$ *since we said earlier that we were assuming $h(\rho) \neq 0$ when $\rho \in D \setminus D_-$, but this abuse is without any consequences in the following*):

$$\theta(\mathcal{F}') = \lim_{D \in \mathcal{D}} \widetilde{\theta}_D(\mathcal{F}'), \text{ for all } \mathcal{F}' \subset \mathcal{F}.$$

*Proof.* Since $\theta(\mathcal{F}) = \sum_{\tau \in \operatorname{Irr} G} \theta_\tau h(\tau)$ is convergent, the sum $\sum_{\tau \in \operatorname{Irr} G \setminus D} \theta_\tau h(\tau)$ converges to 0 when $D$ becomes larger. Then the result follows from the second inequality of Proposition 3.7. □

## References

[Bec11b] T. Becker, *Moduli spaces of $(G, h)$-constellations*, Dissertation Johannes Gutenberg–Universität Mainz, `http://ubm.opus.hbz-nrw.de/volltexte/2011/2919/pdf/doc.pdf`, 2011.

[BT15] T. Becker, R. Terpereau, *Moduli spaces of $(G, h)$-constellations*, Transform. Groups **20** (2015), no. 2, 335–366.