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Symmetry-breaking bifurcation for the Moore–Nehari differential equation

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Abstract. We study the bifurcation problem of positive solutions for the Moore-Nehari differential equation, $u'' + h(x, \lambda)u^p = 0$, $u > 0$ in $(-1, 1)$ with $u(-1) = u(1) = 0$, where $p > 1$, $h(x, \lambda) = 0$ for $|x| < \lambda$ and $h(x, \lambda) = 1$ for $\lambda \le |x| \le 1$ and $\lambda \in (0, 1)$ is a bifurcation parameter. We shall show that the problem has a unique even positive solution $U(x, \lambda)$ for each $\lambda \in (0, 1)$. We shall prove that there exists a unique $\lambda_* \in (0, 1)$ such that a non-even positive solution bifurcates at λ_* from the curve $(\lambda, U(x, \lambda))$, where λ_* is explicitly represented as a function of p.

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1. Introduction

In this paper, we study the bifurcation problem of positive solutions for the Moore–Nehari differential equation

$$u'' + h(x, \lambda)u^p = 0, \quad u > 0 \quad \text{in } (-1, 1), \quad u(-1) = u(1) = 0, \tag{1.1}$$

where $p > 1$, $h(x, \lambda) = 0$ for $|x| < \lambda$ and $h(x, \lambda) = 1$ for $\lambda \le |x| \le 1$ and $\lambda \in (0, 1)$ is a bifurcation parameter.

We first state the regularity of solutions for (1.1). Since $h(x, \lambda)$ is discontinuous at $x = \pm\lambda$, no solution belongs to $C^2[-1, 1]$. Since $h(x, \lambda)$ is a $L^\infty(-1, 1)$ function of x, any solution belongs to $W^{2,\infty}(-1, 1)$. It is known that $W^{2,\infty}(-1, 1)$ coincides with the set of functions u of class $C^1[-1, 1]$ such that $u'(x)$ is Lipschitz continuous (for example, see [2, Proposition 8.4]). Since

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$h(x,\lambda)$ is smooth except for $x=\pm\lambda$, a positive solution u is a C^∞ function on $(-1,1)$ except for $x=\pm\lambda$.

We introduce a result due to Moore and Nehari [14].

Theorem 1.1. *(Moore and Nehari [14]) For some $\lambda\in(0,1)$, (1.1) has at least three positive solutions: an even solution $u(x)$, a non-even solution $v(x)$ and its reflection $v(-x)$.*

The theorem above is similar to a result by Smets, Willem and Su [18], who studied the Hénon equation

$$-\Delta u=|x|^\lambda u^p,\quad u>0\quad\text{in } B,\qquad u=0,\quad\text{on }\partial B, \tag{1.2}$$

where B is a unit ball in $\mathbb{R}^N$ and $1<p<\infty$ when $N=1,2$ and $1<p<(N+2)/(N-2)$ when $N\geq 3$. They proved that if $\lambda>0$ is large enough, no least energy solution of (1.2) is radial. Therefore (1.2) has both a positive radial solution and a positive non-radial solution. Here, a least energy solution is defined by the minimizer of the Rayleigh quotient $R(u)$ on the Nehari manifold $\mathcal{N}$, which are defined by

$$R(u):=\left(\int_\Omega|\nabla u|^2dx\right)\Big/\left(\int_\Omega|x|^\lambda|u|^{p+1}dx\right)^{2/(p+1)},$$

$$\mathcal{N}:=\{u\in H_0^1(\Omega)\backslash\{0\}:\ \int_\Omega(|\nabla u|^2-|x|^\lambda|u|^{p+1})dx=0\}.$$

Kajikiya [6,7] proved that a non-even solution given in Theorem 1.1 can be obtained as a least energy solution of $R(u)$ in which $|x|^\lambda$ is replaced by $h(x,\lambda)$. Sim and Tanaka [17] studied (1.2) when $N=1$, i.e.,

$$u''+|x|^\lambda u^p=0,\quad u>0\quad\text{in }(-1,1),\qquad u(-1)=u(1)=0. \tag{1.3}$$

They investigated the bifurcation problem, in which they took the exponent λ as a bifurcation parameter. They proved that (1.3) has a unique even positive solution for each $\lambda>0$. Denote this solution by $U(x,\lambda)$. Hence the set of even positive solutions draws a curve $(\lambda,U(x,\lambda))$ in $(0,\infty)\times C^2[-1,1]$. They proved that a non-even positive solution bifurcates from this curve at a certain $\lambda=\lambda_*$.

On the other hand, Amadori and Gladiali [1] studied (1.2) with $N\geq 3$. They fix $\lambda\in(0,1]$ and take p as a bifurcation parameter. They proved that there exists a bifurcation point $\overline{p}\in(1,p_\lambda)$ with $p_\lambda:=(N+2+2\lambda)/(N-2)$ such that a positive non-radial solution bifurcates from a unique positive radial solution and the bifurcation branch is unbounded in the Hölder space $C_0^{1,\gamma}(\overline{B})$.

Gritsans and Sadyrbaev [5] investigated (1.1) when $p=3$ and $h(x,\lambda)=0$ for $|x|<\lambda$ and $h(x,\lambda)=2$ for $\lambda\leq|x|\leq 1$. They proved that for any $\lambda\in(0,1)$, (1.1) has infinitely many sign-changing solutions.

In (1.1), the weight function $h(x,\lambda)$ vanishes when $|x|\leq\lambda$. A similar function is studied by López-Gómez and Rabinowitz [10]. They studied the bifurcation problem

$$du''+\lambda u-a(x)f(u)=0\quad\text{in }(0,L),\qquad u(0)=u(L)=0,$$

where $a(x) \geq 0$, $a \in C[0,L]$ and $a(x) \equiv 0$ on $[\alpha,\beta] \subset [0,L]$. They proved in [10] the existence of a positive solution and nodal solutions when λ is in a certain range. See also [11–13].

The purpose of the present paper is to prove that (1.1) has a unique even positive solution (denoted by $U(x,\lambda)$) and that a non-even positive solution bifurcates from $U(x,\lambda)$.

Proposition 1.2. *For any $\lambda \in (0,1)$, (1.1) has a unique even positive solution $U(x,\lambda)$. Moreover, $U(x,\lambda)$ is strictly increasing with respect to λ and it is continuous in the following sense: for each fixed $\lambda_0 \in (0,1)$, $U(x,\lambda)$ converges to $U(x,\lambda_0)$ in $C^1[-1,1]$ as $\lambda \to \lambda_0$.*

The curve $(\lambda, U(x,\lambda))$ in $(0,1) \times C^1[-1,1]$ represents all even positive solutions. We shall show that a non-even positive solution bifurcates from this curve. To state the main result, we define a constant $\lambda_*(p)$ by

$$\lambda_*(p) := \frac{4}{4+(p^2-1)\tau(p)^2}, \tag{1.4}$$

$$\tau(p) := \int_0^1 (1-t^{p+1})^{-1/2}dt = \frac{1}{p+1}B(2/(p+1),1/2). \tag{1.5}$$

Here $B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}dt$ is the beta function. Our main result is as follows.

Theorem 1.3. *There exists a connected closed unbounded set $\mathcal{C} \subset (0,1) \times C^1[-1,1]$ of positive solutions for (1.1) such that $\mathcal{C}$ emanates from the point $(\lambda_*, U(x,\lambda_*))$ and $(\lambda, U(x,\lambda))$ is not a bifurcation point when $\lambda \neq \lambda_*$, where $\lambda_* = \lambda_*(p)$ is defined by (1.4). The point $(\lambda_*, U(x,\lambda_*))$ is a unique even positive solution in $\mathcal{C}$, and all points in $\mathcal{C}\backslash\{(\lambda_*, U(x,\lambda_*))\}$ are non-even positive solutions. Moreover, for every $\lambda \in (\lambda_*,1)$, there exists a $u(x)$ such that $(\lambda,u) \in \mathcal{C}$. The $C^1[-1,1]$ norm of u diverges to infinity as $\lambda \to 1$ with $(\lambda,u) \in \mathcal{C}$. The bifurcation point $\lambda_*(p)$ is a decreasing function of p, $\lim_{p\to1}\lambda_*(p) = 1$ and $\lim_{p\to\infty}\lambda_*(p) = 0$.*

We sketch our idea of the proof for Theorem 1.3. Let $U(x,\lambda)$ denote the unique even positive solution of (1.1). We define the linearized operator,

$$L(\lambda) := -\frac{d^2}{dx^2} - ph(x,\lambda)U(x,\lambda)^{p-1}.$$

Consider the eigenvalue problem

$$L(\lambda)\phi = -\phi'' - ph(x,\lambda)U(x,\lambda)^{p-1}\phi = \mu\phi, \quad \phi(-1) = \phi(1) = 0.$$

We denote the k-th eigenvalue of the problem above by $\mu_k(\lambda)$. We shall show that

(i) $\mu_1(\lambda) < 0$ for all $\lambda \in (0,1)$,
(ii) for $\lambda_* = \lambda_*(p)$ given by (1.4), $\mu_2(\lambda) > 0$ in $(0,\lambda_*)$, $\mu_2(\lambda_*) = 0$ and $\mu_2(\lambda) < 0$ in $(\lambda_*,1)$,
(iii) $\mu_3(\lambda) > 0$ for all $\lambda \in (0,1)$.

The above assertions ensure that the Morse index of $U(x, \lambda)$ (the number of the negative eigenvalues of the linearized operator $L(\lambda)$) changes at λ_*. Using this result and applying the bifurcation theorem due to Rabinowitz [15] and Schmitt and Thompson [16], we shall prove that a non-even positive solution bifurcates at $\lambda = \lambda_*$ from the curve $(\lambda, U(x, \lambda))$. To prove this assertion, the present paper is organized into six sections. In Sect. 2, we prove the existence and uniqueness of the even positive solution $U(x, \lambda)$ and prove that $U(x, \lambda)$ is continuous with respect to λ. Moreover we show the assertion (i) $\mu_1(\lambda) < 0$ for all $\lambda \in (0, 1)$. Assertion (iii) will be proved in Sect. 3. Assertion (ii) will be shown in Sect. 4. In Sect. 5, we give some a priori estimates of positive solutions. In Sect. 6, we prove Theorem 1.3.

2. First eigenvalue

In this section, we prove the existence and uniqueness of the even positive solution for (1.1) and show that the first eigenvalue $\mu_1(\lambda)$ is positive for all λ. We first note that the graph of a solution on $[-\lambda, \lambda]$ for (1.1) must be a line segment because $u''(x) = 0$ for $|x| < \lambda$ by the first equation of (1.1).

To prove Proposition 1.2, we consider the Emden–Fowler equation

$$u'' + |u|^{p-1}u = 0. \tag{2.1}$$

The next lemma is well known (for example, see [4] or [8, Lemma 3.1]).

Lemma 2.1. *Let u be a nontrivial solution of (2.1). Then $u(x)$ is a periodic solution having zeros. For any $\alpha > 0$, $\alpha^{2/(p-1)}u(\alpha x)$ is also a solution of (2.1).*

For $a > 0$, we consider the problem

$$u'' + |u|^{p-1}u = 0, \quad u > 0 \quad \text{in } (0, a), \quad u(0) = 0, \quad u'(a) = 0. \tag{2.2}$$

To represent the solution of (2.2), we consider the ρ-Laplace Emden–Fowler equation

$$(|u'|^{\rho-2}u')' + |u|^{\sigma-2}u = 0 \quad \text{in } \mathbb{R},$$

with $\rho, \sigma \in (1, \infty)$. According to Drábek and Manásevich [4], Takeuchi [19–21], the solution of the equation above is represented by using the the generalized sine function $\sin_{\rho,\sigma} x$, which is defined below. We put

$$g(x) := \int_0^x (1 - t^\sigma)^{-1/\rho} dt \quad \text{for } 0 \le x \le 1.$$

Then $g(x)$ has an inverse function g^{-1}. We define $\sin_{\rho,\sigma} x := g^{-1}(x)$ and put

$$\pi_{\rho,\sigma} := 2g(1) = 2\int_0^1 (1 - t^\sigma)^{-1/\rho} dt. \tag{2.3}$$

Note that $\sin_{\rho,\sigma}(\pi_{\rho,\sigma}/2) = 1$. Since $\sin_{\rho,\sigma} x$ is increasing in $[0, \pi_{\rho,\sigma}/2]$ onto $[0, 1]$, we extend it by

$$\sin_{\rho,\sigma} x := \sin_{\rho,\sigma}(\pi_{\rho,\sigma} - x) \quad \text{in } (\pi_{\rho,\sigma}/2, \pi_{\rho,\sigma}].$$

Furthermore we extend it to the whole of $\mathbb{R}$ as an odd $2\pi_{\rho,\sigma}$-periodic function. Then $u(x) := \sin_{\rho,\sigma} x$ satisfies

$$(|u'|^{\rho-2}u')' + \frac{\sigma(\rho-1)}{\rho}|u|^{\sigma-2}u = 0.$$

Put $\rho := 2$, $\sigma := p+1$ and define

$$S_p(x) := \sin_{2,p+1} x, \tag{2.4}$$

which satisfies

$$S_p'' + \frac{p+1}{2}|S_p|^{p-1}S_p = 0 \quad \text{in } \mathbb{R}. \tag{2.5}$$

By (1.5) and (2.3), it holds that $\tau(p) = \pi_{2,p+1}/2$. This identity and the definition of $S_p(x)$ imply that

$$S_p(0) = 0, \quad S_p'(0) = 1, \quad S_p(\tau(p)) = 1, \quad S_p'(\tau(p)) = 0. \tag{2.6}$$

Lemma 2.2. *For each $a > 0$, (2.2) has a unique solution $u(x,a)$, which is given by*

$$u(x,a) := a^{-2/(p-1)}M(p)S_p(\tau(p)a^{-1}x), \tag{2.7}$$

where

$$M(p) := \left([(p+1)/2]\tau(p)^2\right)^{1/(p-1)}. \tag{2.8}$$

Moreover, if $0 < a < b$, then $u(x,b) < u(x,a)$ for $0 < x \le a$ and $u(b,b) < u(a,a)$.

Proof. Let $v(x)$ be a unique solution of the initial value problem

$$v'' + |v|^{p-1}v = 0, \quad v(0) = 0, \quad v'(0) = 1.$$

By (2.4)–(2.6), $v(x)$ is represented as

$$v(x) = qS_p(q^{-1}x), \quad q := ((p+1)/2)^{1/(p+1)}.$$

We define $t_0 := q\tau(p)$, which is the first critical point of $v(x)$, i.e., $v'(t_0) = 0$ and $v'(x) > 0$ in $[0, t_0)$. Define $v(x,\alpha) := \alpha^{2/(p-1)}v(\alpha x)$. By Lemma 2.1, we know that functions $v(x,\alpha)$ for all $\alpha > 0$ represent all the solutions satisfying (2.2) except for the condition $u'(a) = 0$. Therefore $u(x) = \alpha^{2/(p-1)}v(\alpha x)$ satisfies (2.2) if and only if $v'(\alpha a) = 0$, i.e., $\alpha = t_0/a$. Hence (2.2) has a unique solution $u(x,a) := t_0^{2/(p-1)}a^{-2/(p-1)}v(t_0a^{-1}x)$, which is rewritten as (2.7). Since $S_p(x)$ is increasing in $[0, \tau(p)]$, it follows from (2.7) that if $0 < a < b$, then $u(x,b) < u(x,a)$ for $0 < x \le a$ and $u(b,b) < u(a,a)$. The proof is complete. □

Using Lemma 2.2, we prove Proposition 1.2.

Proof of Proposition 1.2. Let $0 < \lambda < 1$. Consider the equation

$$u'' + u^p = 0, \quad u > 0 \quad \text{in } (-1,-\lambda), \quad u(-1) = 0, \quad u'(-\lambda) = 0. \tag{2.9}$$

Since (2.2) is autonomous, Lemma 2.2 ensures that (2.9) has a unique solution $U(x)$. Extend it to $[-1,1]$ by putting $U(x) := U(-\lambda)$ for $-\lambda \le x \le 0$ and $U(x) := U(-x)$ for $0 \le x \le 1$. Then $U(x)$ is an even solution of (1.1). Conversely, let $v(x)$ be any even solution of (1.1). Since $v''(x) \equiv 0$ in $(-\lambda,\lambda)$ and $v'(0) = 0$, it holds that $v'(-\lambda) = 0$. Then $v(x)$ satisfies (2.9). The uniqueness of solutions for (2.9) shows that $v(x) = U(x)$ in $[-1,-\lambda]$ and hence these are identically equal on $[-1,1]$. Thus (1.1) has a unique even solution $U(x,\lambda)$. By Lemma 2.2, $U(x,\lambda)$ is strictly increasing with respect to λ.

We shall show the continuity of $U(x,\lambda)$ with respect to λ. Let $0 < L_1 < L_2 < 1$. We have already proved the order relation

$$U(x,L_1) \le U(x,\lambda) \le U(x,L_2) \quad \text{for } \lambda \in [L_1,L_2]. \tag{2.10}$$

Therefore, $h(x,\lambda)U(x,\lambda)^p$ is bounded in $L^\infty(-1,1)$ for $\lambda \in [L_1,L_2]$, and so is $U''(x,\lambda)$ by (1.1). Thus $U(x,\lambda)$ is bounded in $W^{2,\infty}(-1,1)$. Let $\lambda_0 \in (0,1)$ and let λ_n be any sequence converging to λ_0. Put $U_n(x) := U(x,\lambda_n)$ and $h_n(x) := h(x,\lambda_n)$. Then

$$U_n'' + h_n(x)U_n(x)^p = 0, \quad U_n > 0 \quad \text{in } (-1,1), \quad U_n(-1) = U_n(1) = 0. \tag{2.11}$$

Since U_n is bounded in $W^{2,\infty}(-1,1)$, it has a subsequence (again denoted by U_n) converging in $C^1[-1,1]$ by the Sobolev embedding. Denote its limit by $U_0(x)$. Integrating (2.11) over $[0,x]$ and using the evenness of $U_n(x)$, we have

$$U_n'(x) + \int_0^x h_n(t)U_n(t)^p dt = 0.$$

As $n \to \infty$, we obtain

$$U_0'(x) + \int_0^x h(t,\lambda_0)U_0(t)^p dt = 0,$$

which shows that

$$U_0'' + h(x,\lambda_0)U_0(x)^p = 0, \quad \text{in } (-1,1), \quad U_0(-1) = U_0(1) = 0.$$

By (2.10), we have $U_0(x) \ge U(x,L_1) > 0$, where $0 < L_1 < \lambda_0$. Therefore $U_0(x)$ is an even positive solution of (1.1) with $\lambda = \lambda_0$. The uniqueness of such a solution ensures that $U_0(x) = U(x,\lambda_0)$. The uniqueness of the limit implies that $U_n(x)$ itself (without extracting a subsequence) converges to $U(x,\lambda_0)$. The proof is complete. □

Let $y(x)$ be a unique solution of (2.2) with $a = 1$. By (2.7), it is written as

$$y(x) = M(p)S_p(\tau(p)x), \tag{2.12}$$

which satisfies

$$y'' + |y|^{p-1}y = 0, \quad y > 0 \quad \text{in } (0,1), \quad y(0) = 0, \quad y'(1) = 0.$$

We define

$$z(x,\lambda) := (1-\lambda)^{-2/(p-1)} y((1-\lambda)^{-1}x). \tag{2.13}$$

By the proof of Proposition 1.2, the unique even positive solution $U(x,\lambda)$ of (1.1) can be defined by

$$U(x,\lambda) := z(x+1,\lambda) \quad \text{for } -1 \le x \le -\lambda, \tag{2.14}$$

$$U(x,\lambda) := z(1-\lambda,\lambda) \quad \text{for } -\lambda \le x \le 0, \tag{2.15}$$

$$U(x,\lambda) := U(-x,\lambda) \quad \text{for } 0 \le x \le 1. \tag{2.16}$$

Putting $x = 1$ in (2.12), we have

$$y(1) = M(p) = [((p+1)/2)\tau(p)^2]^{1/(p-1)}. \tag{2.17}$$

The expression above will be used later on.

Proposition 1.2 says that an even solution of (1.1) is unique for any $\lambda \in (0, 1)$, that is, a solution of (1.1) is unique in a class of even solutions. However, the next lemma ensures that a solution of (1.1) is unique in the set of all solutions when $\lambda > 0$ is small.

Lemma 2.3. *([8, Theorem 1.2]) For $\lambda > 0$ small enough, (1.1) has a unique positive solution. Moreover, it is even.*

We denote the unique even solution of (1.1) by $U(x, \lambda)$. Since $h(x, \lambda)$ converges to 1 except for $x = 0$ as $\lambda \to +0$, we define $h(x, 0) \equiv 1$ for $x \in [-1, 1]$. Therefore $h(x, \lambda)$ is defined for all $\lambda \in [0, 1)$. Consider the problem

$$U'' + U^p = 0, \quad U > 0 \quad \text{in } (-1, 1), \quad U(-1) = U(1) = 0. \tag{2.18}$$

It is well known that the problem above has a unique solution and it becomes even (for example, see [8] or [17]). Clearly, this solution $U(x)$ is written as $U(x) = y(x+1)$, where $y(x)$ is given by (2.12). Moreover, $U(x)$ is concave and hence $U'(x) > 0$ in $[-1, 0)$, $U'(0) = 0$ and $U'(x) < 0$ in $(0, 1]$. Denote a unique solution of the problem above by $U(x, 0)$. Hence we have

$$U'(x, 0) > 0 \quad \text{in } [-1, 0), \quad U'(0, 0) = 0, \quad U'(x, 0) < 0 \quad \text{in } (0, 1]. \tag{2.19}$$

Therefore $U(x, \lambda)$ is defined for all $\lambda \in [0, 1)$. Let $\|\cdot\|_q$ denote the $L^q(-1, 1)$ norm. Since $\|h(\cdot, \lambda) - h(\cdot, 0)\|_q \to 0$ as $\lambda \to +0$ for any $q \in [1, \infty)$, the same method as in the proof of Proposition 1.2 ensures that $U(x, \lambda)$ converges to $U(x, 0)$ in $C^1[-1, 1]$ as $\lambda \to +0$. Therefore $U(x, \lambda)$ is continuous in $C^1[-1, 1]$ for $\lambda \in [0, 1)$.

We define the linearized operator as

$$L(\lambda) := -\frac{d^2}{dx^2} - ph(x, \lambda)U(x, \lambda)^{p-1}.$$

Consider the eigenvalue problem

$$L(\lambda)\phi = -\phi'' - ph(x, \lambda)U(x, \lambda)^{p-1}\phi = \mu\phi, \quad \phi(-1) = \phi(1) = 0. \tag{2.20}$$

We denote the k-th eigenvalue of (2.20) by $\mu_k(\lambda)$. It is well known that each eigenvalue is simple, i.e., each eigenspace is one dimensional, and each eigenfunction corresponding to $\mu_k(\lambda)$ has exactly $k - 1$ interior zeros in $(-1, 1)$.

Let $\phi_k(x, \lambda)$ be an eigenfunction corresponding to $\mu_k(\lambda)$. We constrain it by the conditions $\|\phi_k\|_\infty = 1$ and $\phi_k'(1) < 0$. Here, $\|\cdot\|_\infty$ denotes the $L^\infty(-1, 1)$ norm. Then $\phi_k(x, \lambda)$ is uniquely determined and satisfies

$$\begin{aligned} &-\phi_k'' - ph(x, \lambda)U(x, \lambda)^{p-1}\phi_k = \mu_k(\lambda)\phi_k \quad \text{in } (-1, 1), \\ &\phi_k(-1) = \phi_k(1) = 0, \quad \|\phi_k\|_\infty = 1, \quad \phi_k'(1) < 0. \end{aligned} \tag{2.21}$$

Lemma 2.4. *For each k, $\mu_k(\lambda)$ and $\phi_k(x, \lambda)$ are continuous for $\lambda \in [0, 1)$ in the spaces $\mathbb{R}$ and $C^1[-1, 1]$, respectively, that is, $\phi_k(x, \lambda)$ converges to $\phi_k(x, \lambda_0)$ in $C^1[-1, 1]$ as $\lambda \to \lambda_0 \in [0, 1)$.*

Proof. Let $0 < \Lambda < 1$. Then the potential $-ph(x, \lambda)U(x, \lambda)^{p-1}$ is uniformly bounded for $\lambda \in [0, \Lambda]$. The boundedness of the potential implies that of the eigenvalue. Indeed, choose a constant $M > 0$ such that

$$-M \le -ph(x, \lambda)U(x, \lambda)^{p-1} \le 0 \quad \text{for } x \in [-1, 1],\ \lambda \in [0, \Lambda].$$

The order relation of potentials also implies that of the eigenvalues (see [3]). Hence the k-th eigenvalue $\mu_k(\lambda)$ is greater than or equal to that of the operator $-d^2/dx^2 - M$ and $\mu_k(\lambda)$ is less than or equal to that of $-d^2/dx^2$. Therefore for each k, $\mu_k(\lambda)$ is bounded for $\lambda \in [0, \Lambda]$. We rewrite (2.21) as

$$-\phi'' = ph(x, \lambda)U(x, \lambda)^{p-1}\phi + \mu_k(\lambda)\phi, \tag{2.22}$$

where we have written ϕ instead of ϕ_k. The right hand side is bounded in $L^\infty(-1, 1)$ and so is $\phi''(x, \lambda)$ for $\lambda \in [0, \Lambda]$. Therefore $\phi(\cdot, \lambda)$ is bounded in $W^{2,\infty}(-1, 1)$. Let $\lambda_0 \in [0, 1)$ and let λ_n be any sequence converging to λ_0. Put $\phi_n(x) := \phi(x, \lambda_n)$, $U_n(x) := U(x, \lambda_n)$, $h_n(x) := h(x, \lambda_n)$ and $\mu_n := \mu_k(\lambda_n)$ and substitute them in (2.22). Integrating (2.22), we have

$$-\phi_n'(x) + \phi_n'(-1) = \int_{-1}^{x} \left(ph_n(t)U_n(t)^{p-1} + \mu_n\right) \phi_n(t)dt.$$

Since ϕ_n is bounded in $W^{2,\infty}(-1, 1)$, it converges to a limit ϕ_0 along a subsequence in $C^1[-1, 1]$. Moreover μ_n also converges to a limit μ_0 along a subsequence. As $n \to \infty$, we have

$$-\phi_0'(x) + \phi_0'(-1) = \int_{-1}^{x} \left(ph(t, \lambda_0)U(t, \lambda_0)^{p-1} + \mu_0\right) \phi_0(t)dt,$$

which is rewritten as

$$-\phi_0'' - ph(x, \lambda_0)U(x, \lambda_0)^{p-1}\phi_0 = \mu_0\phi_0, \quad \phi_0(-1) = \phi_0(1) = 0.$$

By $\|\phi_n\|_\infty = 1$, we have $\|\phi_0\|_\infty = 1$. Therefore $\phi_0(x)$ is an eigenfunction. Since $\phi_n(x)$ is an eigenfunction corresponding to $\mu_k(\lambda_n)$, it has exactly $k-1$ interior zeros in $(-1, 1)$. Denote them by $t_{n,i}$ with $1 \le i \le k-1$ such that

$$-1 < t_{n,1} < t_{n,2} < \cdots < t_{n,k-1} < 1.$$

Put $t_{n,0} := -1$ and $t_{n,k} := 1$. We claim that $t_{n,i} - t_{n,i-1} \ge c$ for $1 \le i \le k$ with some $c > 0$ independent of n. Suppose to the contrary that there exists a sequence $\{n_j\} \subset \mathbb{N}$ such that $t_{n_j,i} - t_{n_j,i-1}$ converges to zero as $j \to \infty$. Along a subsequence, $t_{n_j,i}$ converges to a point t_0. Let r_j be a critical point of $\phi_{n_j}(x)$ in $(t_{n_j,i-1}, t_{n_j,i})$, i.e., $\phi_{n_j}'(r_j) = 0$. Then $r_j \to t_0$ as $j \to \infty$. Therefore $\phi_0(t_0) = \phi_0'(t_0) = 0$ and hence $\phi_0(x) \equiv 0$. This contradicts $\|\phi_0\|_\infty = 1$. Hence $t_{n,i} - t_{n,i-1} \ge c$ with some $c > 0$. Since $\phi_n(x)$ converges to $\phi_0(x)$ in $C^1[-1, 1]$, $\phi_0(x)$ has exactly $k-1$ interior zeros in $(-1, 1)$. Accordingly, it becomes an eigenfunction corresponding to $\mu_k(\lambda_0)$. Hence $\mu_0 = \mu_k(\lambda_0)$. The uniqueness of the limit guarantees that $\mu_k(\lambda) \to \mu_k(\lambda_0)$ and $\phi(x, \lambda) \to \phi(x, \lambda_0)$ as $\lambda \to \lambda_0$. The proof is complete. □

Proposition 2.5. *For all $\lambda \in [0, 1)$, the first eigenvalue $\mu_1(\lambda)$ is negative.*

Proof. We rewrite (1.1) with $u = U(x, \lambda)$ as

$$\left(-\frac{d^2}{dx^2} - h(x, \lambda)U(x, \lambda)^{p-1}\right) U = 0, \quad U(-1) = U(1) = 0.$$

This equation shows that the operator $-d^2/dx^2 - h(x, \lambda)U(x, \lambda)^{p-1}$ has zero as the first eigenvalue because $U > 0$. Compare the equation above with (2.20) and note that $-ph(x, \lambda)U^{p-1} \leq -h(x, \lambda)U^{p-1}$ by $p > 1$ and the strict inequality holds for $\lambda \leq |x| < 1$. From the order relation of potentials, we conclude that the first eigenvalue $\mu_1(\lambda)$ is negative. The proof is complete. □

3. Third eigenvalue

In this section, we investigate the third eigenvalue $\mu_3(\lambda)$.

Proposition 3.1. *The third eigenvalue* $\mu_3(\lambda)$ *is positive for all* $\lambda \in [0, 1)$.

To prove the proposition above, we consider the eigenvalue problem in the interval $(0, 1)$,

$$-\psi'' - ph(x, \lambda)U(x, \lambda)^{p-1}\psi = \nu\psi, \quad \psi(0) = \psi(1) = 0. \tag{3.1}$$

Denote the k-th eigenvalue of the problem above by $\nu_k(\lambda)$. Recall that $\mu_k(\lambda)$ denotes the k-th eigenvalue in the interval $(-1, 1)$. Then we have the next lemma, which is a well-known result. However we give a proof for the sake of completeness.

Lemma 3.2. $\mu_2(\lambda) = \nu_1(\lambda)$.

Proof. Let ψ be the first eigenfunction of (3.1). Put $\psi(x) := -\psi(-x)$ for $x \in [-1, 0]$. Since $h(x, \lambda)$ and $U(x, \lambda)$ are even functions, $\psi(x)$ becomes an eigenfunction of (2.20). Since $\psi(x)$ has exactly one zero in $(-1, 1)$, it must be the eigenfunction corresponding to the second eigenvalue $\mu_2(\lambda)$. Therefore $\mu_2(\lambda) = \nu_1(\lambda)$. □

Recall that $h(x, 0) \equiv 1$, $U(x, 0)$ is a unique solution of (2.18) and $\mu_k(0)$ denotes the k-th eigenvalue of (2.20) with $\lambda = 0$.

Lemma 3.3. $\mu_2(0) > 0$.

Proof. Instead of $\mu_2(0)$, we write μ_2. Let $\phi(x)$ be an eigenfunction corresponding to μ_2. Then it has exactly one interior zero. It must be the origin by Lemma 3.2. Therefore it satisfies

$$\phi'' + (pU(x, 0)^{p-1} + \mu_2)\phi = 0, \quad \phi(-1) = \phi(0) = \phi(1) = 0. \tag{3.2}$$

Put $V(x) := U'(x, 0)$. Then V satisfies

$$V'' + pU(x, 0)^{p-1}V = 0. \tag{3.3}$$

We shall show that $\mu_2 > 0$. Suppose to the contrary that $\mu_2 \leq 0$. Compare Eqs. (3.2) and (3.3). The Sturm comparison theorem (see Lemma 4.2 later) shows that either $V(x)$ has a zero in $(0, 1)$ or $V(x) \equiv C\phi(x)$ on $[0, 1]$ for some constant $C \neq 0$. By (2.19), $V(x) = U'(x, 0) \neq 0$ in $(0, 1)$. Moreover,

$V(1) = U'(1,0) < 0 = \phi(1)$. A contradiction occurs. Therefore $\mu_2 > 0$. The proof is complete. □

Since $\mu_k(\lambda)$ is continuous for $\lambda \in [0,1)$, Lemma 3.3 implies the next one.

Lemma 3.4. *There exists a* $\lambda_0 \in (0,1)$ *such that* $\mu_2(\lambda) > 0$ *for* $\lambda \in [0,\lambda_0)$.

We consider the eigenvalue problem in the interval $(0,1)$,

$$-\psi'' - ph(x,\lambda)U(x,\lambda)^{p-1}\psi = \rho\psi \quad \text{in } (0,1), \quad \psi'(0) = \psi(1) = 0.$$

Note that $\psi'(0) = 0$. Denote the k-th eigenvalue of the problem above by $\rho_k(\lambda)$. Recall that $\mu_k(\lambda)$ denotes the k-th eigenvalue in the interval $(-1,1)$ under the Dirichlet boundary condition. Then we have the next lemma.

Lemma 3.5. $\mu_3(\lambda) = \rho_2(\lambda)$.

Proof. Let ψ be an eigenfunction corresponding to $\rho_2(\lambda)$. Then it has a unique interior zero $x_0 \in (0,1)$. Since $\psi'(0) = 0$ and $h(x,\lambda)$ and $U(x,\lambda)$ are even, we can define $\psi(x) := \psi(-x)$ for $x \in [-1,0]$. Then $\psi(x)$ is defined on $[-1,1]$ and satisfies

$$-\psi'' - ph(x,\lambda)U(x,\lambda)^{p-1}\psi = \rho_2(\lambda)\psi, \quad \psi(-1) = \psi(1) = 0.$$

Therefore it becomes an eigenfunction having exactly two interior zeros, $\pm x_0$, in $(-1,1)$. Hence it is an eigenfunction corresponding to $\mu_3(\lambda)$. Consequently, $\mu_3(\lambda) = \rho_2(\lambda)$. □

Proof of Proposition 3.1. By Lemma 3.4, $\mu_3(\lambda) > \mu_2(\lambda) > 0$ in $[0,\lambda_0)$. We claim that $\mu_3(\lambda) \neq 0$ for all λ. If this claim would be proved, then the proposition follows. Suppose to the contrary that $\mu_3(\lambda) = 0$ at some λ. By Lemma 3.5, $\rho_2(\lambda) = 0$. Let $\psi(x)$ be an eigenfunction corresponding to $\rho_2(\lambda) = 0$, i.e.,

$$\psi'' + ph(x,\lambda)U(x)^{p-1}\psi = 0 \quad \text{in } (0,1), \quad \psi'(0) = \psi(1) = 0,$$

where $U(x) := U(x,\lambda)$. Denote the unique interior zero of $\psi(x)$ by $x_0 \in (0,1)$. We can assume that $\psi(0) > 0$ after replacing ψ by $-\psi$ if necessary. Then $\psi(x) > 0$ in $(0,x_0)$ and $\psi(x) < 0$ in $(x_0,1)$ and hence $\psi'(1) > 0$. Since $h(x,\lambda) = 0$ in $(0,\lambda)$, $\psi'(x) = 0$ in this interval. Accordingly, we have

$$\psi'(x) = 0 \quad \text{on } [0,\lambda], \quad \psi'(1) > 0. \tag{3.4}$$

We employ the comparison function $v(x)$, which was developed in [17],

$$v(x) := xU'(x) + \frac{2}{p-1}U(x) \quad \text{for } x \in [\lambda,1],$$

where $U(x) := U(x,\lambda)$. Note that $v(x)$ is Lipschitz continuous on $[\lambda,1]$ and is a C^∞ function in $(\lambda,1)$. Since $U(x)$ belongs to $C^3(\lambda,1]$ by $p > 1$, $v(x)$ belongs to $C^2(\lambda,1]$. Moreover, $v(x)$ satisfies

$$v'' + pU(x)^{p-1}v = 0 \quad \text{in } (\lambda,1).$$

We define the Wronskian $w(x)$ by

$$w(x) := v'(x)\psi(x) - v(x)\psi'(x).$$

Since $v(x)$ and $\psi(x)$ satisfy the same linear differential equation in $(\lambda, 1)$, $w(x)$ is constant. Indeed, we have

$$w'(x) = v''(x)\psi(x) - v(x)\psi''(x) = -pU^{p-1}v\psi + pU^{p-1}v\psi = 0.$$

Thus $w(x)$ is constant in $(\lambda, 1]$. Recall that $U'(x) = 0$ on $[0, \lambda]$ and $U'(x) < 0$ on $(\lambda, 1]$. Since $\psi(1) = 0$, we use (3.4) to find

$$w(1) = -v(1)\psi'(1) = -U'(1)\psi'(1) > 0. \tag{3.5}$$

We compute $v'(x)$ as

$$v'(x) = xU''(x) + \frac{p+1}{p-1}U'(x) = -xU(x)^p + \frac{p+1}{p-1}U'(x) \quad \text{in } (\lambda, 1].$$

Since $U'(\lambda) = 0$, it holds that $\lim_{x\to\lambda+0} v'(x) = -\lambda U(\lambda)^p$. Since $\psi(\lambda) > 0$ and $\psi'(\lambda) = 0$ by (3.4), we have

$$\lim_{x\to\lambda+0} w(x) = -\lambda U(\lambda)^p \psi(\lambda) < 0. \tag{3.6}$$

Inequalities (3.5) and (3.6) contradict the fact that $w(x)$ is constant. Therefore $\rho_2(\lambda) = \mu_3(\lambda)$ must not be zero. The proof is complete. □

4. Second eigenvalue

We shall show that the second eigenvalue $\mu_2(\lambda)$ changes its sign exactly once as λ varies in $[0, 1)$.

Proposition 4.1. *Let $\lambda_* = \lambda_*(p)$ be given by (1.4). Then $\mu_2(\lambda) > 0$ in $[0, \lambda_*)$, $\mu_2(\lambda_*) = 0$ and $\mu_2(\lambda) < 0$ in $(\lambda_*, 1)$.*

To prove Proposition 4.1, we need the Sturm comparison theorem in the space $W^{2,1}(a, b)$. Let us consider

$$u'' + q(x)u = 0, \quad v'' + Q(x)v = 0 \quad \text{in } (a, b), \tag{4.1}$$

with a finite interval (a, b). The Sturm comparison theorem usually requires the assumption that $u, v \in C^2(a, b)$. However it is enough to assume that they belong to $W^{2,1}(a, b)$. Indeed, the standard proof of the theorem is still valid even if $q, Q \in L^1(a, b)$ and $u, v \in W^{2,1}(a, b)$. A function $u(x)$ belongs to $W^{2,1}(a, b)$ if and only if $u \in C^1[a, b]$ and $u'(x)$ is absolutely continuous on $[a, b]$.

Lemma 4.2. *Let $q, Q \in L^1(a, b)$ and $q(x) \leq Q(x)$ a.e. in (a, b). Let $u, v \in W^{2,1}(a, b)$, $u, v \not\equiv 0$ in (a, b) and assume that they satisfy (4.1). If $u(a) = u(b) = 0$, then either* (i) *or* (ii) *below holds:*

(i) *$v(x)$ has a zero in (a, b),*
(ii) *$u(x) \equiv cv(x)$ with some $c \neq 0$.*

If the second alternative holds, then $q = Q$ a.e. in (a, b). Therefore, if $Q(x) - q(x) \geq 0$ a.e. in (a, b) and $Q(x) - q(x) > 0$ in a subset with positive measure in (a, b) and if $u(a) = u(b) = 0$, then only assertion (i) *holds.*

To study the second eigenvalue $\mu_2(\lambda)$, we investigate the equation

$$\phi'' + ph(x,\lambda)U(x,\lambda)^{p-1}\phi = 0 \quad \text{in } (-1,1), \quad \phi(-1) = 0,\ \phi'(-1) \neq 0. \tag{4.2}$$

We shall construct a solution of the equation above. To this end, for the even positive solution $U(x) = U(x,\lambda)$ of (1.1), we define

$$v(x) := xU'(x) + \frac{2}{p-1}U(x), \qquad w(x) := U'(x).$$

Note that

$$v, w \in W^{1,\infty}(-1,1) \cap C^2([-1,1]\backslash\{\pm\lambda\}),$$

which satisfy

$$v'' + phU^{p-1}v = 0, \quad w'' + phU^{p-1}w = 0 \quad \text{in } (-1,1)\backslash\{\pm\lambda\}. \tag{4.3}$$

We put

$$\phi_1(x) := v(x) + w(x) \quad \text{on } [-1,-\lambda], \tag{4.4}$$

$$\phi_2(x) := -(1-\lambda)U(-\lambda)^p(x+\lambda) + \frac{2}{p-1}U(-\lambda) \quad \text{on } [-\lambda,\lambda], \tag{4.5}$$

$$\phi_3(x) := \alpha v(x) + \beta w(x) \quad \text{on } [\lambda, 1], \tag{4.6}$$

where α and β are constants to be determined later. We here note that the graph of $\phi_2(x)$ is a line segment. Define

$$\phi(x) := \begin{cases} \phi_1(x) & \text{on } [-1,-\lambda], \\ \phi_2(x) & \text{on } [-\lambda,\lambda], \\ \phi_3(x) & \text{on } [\lambda,1]. \end{cases} \tag{4.7}$$

It follows from an easy computation that

$$\phi_1'(x) = -(1+x)U^p + \frac{p+1}{p-1}U'(x) \quad \text{on } [-1,-\lambda], \tag{4.8}$$

$$\phi_2'(x) = -(1-\lambda)U(-\lambda)^p \quad \text{on } [-\lambda,\lambda], \tag{4.9}$$

$$\phi_3'(x) = \alpha\left(-xU^p + \frac{p+1}{p-1}U'(x)\right) - \beta U^p \quad \text{on } [\lambda,1]. \tag{4.10}$$

Then it is easy to verify that $\phi(x)$ is a C^1 function at $x = -\lambda$. Indeed, by (4.4), (4.5), (4.8), (4.9) and by using $U'(-\lambda) = 0$, we have

$$\phi_1(-\lambda) = \phi_2(-\lambda), \quad \phi_1'(-\lambda) = \phi_2'(-\lambda).$$

We here determine α and β such that $\phi(x)$ is a C^1 function at $x = \lambda$. To this end, we impose the conditions

$$\phi_2(\lambda) = \phi_3(\lambda), \quad \phi_2'(\lambda) = \phi_3'(\lambda).$$

Recall that $U'(\lambda) = U'(-\lambda) = 0$ and $U(\lambda) = U(-\lambda)$. We put $\eta := U(\lambda) = U(-\lambda)$. Using (4.5), (4.6), (4.9) and (4.10), we rewrite the equations above as

$$-2\lambda(1-\lambda)\eta^p + \frac{2}{p-1}\eta = \frac{2}{p-1}\eta\alpha,$$
$$-(1-\lambda)\eta^p = -\lambda\eta^p\alpha - \beta\eta^p.$$

Solving the equations, we have

$$\alpha = 1 - \lambda(1-\lambda)(p-1)\eta^{p-1}, \tag{4.11}$$
$$\beta = 1 - 2\lambda + \lambda^2(1-\lambda)(p-1)\eta^{p-1}. \tag{4.12}$$

After defining α and β as above, $\phi(x)$ belongs to $C^1[-1,1]$ and moreover $\phi'(x)$ is Lipschitz continuous. Therefore $\phi(x)$ belongs to $W^{2,\infty}(-1,1)$. Since $v(x)$ and $w(x)$ satisfy (4.3), $\phi(x)$ fulfills (4.2). Then we obtain the lemma below.

Lemma 4.3. *Define α and β by (4.11) and (4.12), respectively, and $\phi(x)$ by (4.7). Then ϕ belongs to $W^{2,\infty}(-1,1)$ and satisfies (4.2).*

The value $\alpha+\beta$ will play an important role to determine the sign of the second eigenvalue $\mu_2(\lambda)$. By (4.11) and (4.12), it is computed as

$$\alpha + \beta = 2(1-\lambda) - \lambda(1-\lambda)^2(p-1)\eta^{p-1}. \tag{4.13}$$

We use (2.13) and (2.14) to obtain

$$\eta = U(-\lambda,\lambda) = z(1-\lambda,\lambda) = (1-\lambda)^{-2/(p-1)}y(1).$$

Substituting this relation in (4.13) and using (2.17), we obtain

$$\alpha + \beta = 2(1-\lambda) - \lambda(p-1)y(1)^{p-1} = 2 - 2^{-1}[4 + (p^2-1)\tau(p)^2]\lambda.$$

By (1.4), we have

$$\alpha + \beta = 2(\lambda_*(p) - \lambda)/\lambda_*(p). \tag{4.14}$$

We investigate the number of zeros of $\phi(x)$ given by (4.7).

Lemma 4.4. *Let $\phi(x)$ be given by (4.7). Then $\phi(x) > 0$ in $(-1,-\lambda]$ and $\phi(x)$ has either one zero or two zeros in $(-\lambda,1]$.*

Proof. It is clear that for $x \in (-1,-\lambda]$,

$$\phi(x) = \phi_1(x) = v(x) + w(x) = (1+x)U'(x) + \frac{2}{p-1}U(x) > 0.$$

We choose an eigenfunction $\psi(x)$ corresponding to $\mu_1(\lambda)$, which satisfies

$$\psi'' + (phU^{p-1} + \mu_1(\lambda))\psi = 0, \quad \psi(-1) = \psi(1) = 0.$$

Recall that $\mu_1(\lambda) < 0$ by Proposition 2.5. Compare the equation above with (4.2) and use Lemma 4.2. Then ϕ has a zero in $(-1,1)$.

We shall show that ϕ has at most two zeros in $(-1,1]$. Let $\psi(x)$ be an eigenfunction corresponding to $\mu_3(\lambda)$, which has exactly two interior zeros in $(-1,1)$ and satisfies

$$\psi'' + (phU^{p-1} + \mu_3(\lambda))\psi = 0, \quad \psi(-1) = \psi(1) = 0.$$

Recall that $\mu_3(\lambda) > 0$ for all $\lambda \in (0,1)$ by Proposition 3.1. Suppose to the contrary that ϕ has at least three zeros in $(-1,1]$, say $-1 < x_1 < x_2 < x_3 \le 1$. Then

$$\phi(-1) = \phi(x_1) = \phi(x_2) = \phi(x_3) = 0.$$

By Lemma 4.2 with $\mu_3(\lambda) > 0$, $\psi(x)$ has at least three zeros in $(-1,x_3)$. A contradiction occurs. Therefore $\phi(x)$ has at most two zeros. □

By the definition of ϕ and (4.14), we have

$$\phi(1) = \phi_3(1) = (\alpha + \beta)U'(1) = 2U'(1)(\lambda_*(p) - \lambda)/\lambda_*(p).$$

Since $U'(1) < 0$, $\phi(1)$ and $\lambda_*(p) - \lambda$ have the opposite signs. By Lemma 4.4, $\phi(x) > 0$ in $(-1, -\lambda]$ and the number of zeros of $\phi(x)$ in $(-1, 1]$ is either one or two. Hence $\phi(1) > 0$ (equivalently, $\lambda > \lambda_*(p)$) holds if and only if $\phi(x)$ has exactly two zeros in $(-1, 1)$. The condition $\phi(1) < 0$ (i.e., $\lambda < \lambda_*(p)$) holds if and only if $\phi(x)$ has exactly one zero in $(-1, 1)$. Thus we have the next lemma.

Lemma 4.5. *Let $\phi(x)$ be given by (4.7). If $\lambda < \lambda_*(p)$, then $\phi(x)$ has exactly one zero in $(-1, 1)$ and $\phi(1) \neq 0$. If $\lambda = \lambda_*(p)$, then $\phi(x)$ has exactly one zero in $(-1, 1)$ and $\phi(1) = 0$. If $\lambda > \lambda_*(p)$, then $\phi(x)$ has exactly two zeros in $(-1, 1)$ and $\phi(1) \neq 0$.*

Using the lemma above with Lemma 4.2, we show Proposition 4.1.

Proof of Proposition 4.1. Let $\psi(x)$ be an eigenfunction corresponding to $\mu_2(\lambda)$, i.e.,

$$\psi'' + (ph(x, \lambda)U(x, \lambda)^{p-1} + \mu_2(\lambda))\psi = 0, \quad \psi(-1) = \psi(1) = 0. \tag{4.15}$$

By the proof of Lemma 3.2, $\psi(x)$ has a unique interior zero in $(-1, 1)$ and it is the origin. Hence

$$\psi(-1) = \psi(0) = \psi(1) = 0.$$

We now show that if $\lambda < \lambda_*(p)$, then $\mu_2(\lambda) > 0$. Suppose to the contrary that $\mu_2(\lambda) \leq 0$ at some $\lambda \in [0, \lambda_*(p))$. Compare (4.15) with (4.2). By Lemma 4.2, either ϕ has at least two zeros in $(-1, 1)$ or $\phi(x) \equiv c\psi(x)$ with some $c \neq 0$. The former assertion contradicts Lemma 4.5 because $\lambda < \lambda_*(p)$. The latter means that $\phi(1) = 0$, which contradicts Lemma 4.5. Therefore $\mu_2(\lambda) > 0$ when $\lambda < \lambda_*(p)$. In the same discussion, we can prove that if $\lambda > \lambda_*(p)$, then $\mu_2(\lambda) < 0$. By the continuity of $\mu_2(\lambda)$, $\mu_2(\lambda_*)$ must be zero. The proof is complete. □

5. Estimates of positive solutions

In this section, we give some a priori estimates for positive solutions of (1.1). When $h(x, \lambda)$ is a general weight function, an a priori estimate for the L^∞ norm was obtained in [22, Theorem 4.1] by using the integral of $h(x, \lambda)$. In the present paper, since $h(x, \lambda) = 1$ for $\lambda \leq |x| \leq 1$, we use this definition to get an optimal estimate as below.

Theorem 5.1. *There exist constants $c, C > 0$ independent of λ such that any positive solution $u(x)$ of (1.1) satisfies*

$$\|u'\|_\infty = (2/(p+1))^{1/2}\|u\|_\infty^{(p+1)/2}, \tag{5.1}$$

$$c(1-\lambda)^{-2/(p-1)} \leq \|u\|_\infty \leq C(1-\lambda)^{-2/(p-1)}, \tag{5.2}$$

$$c(1-\lambda)^{-(p+1)/(p-1)} \leq \|u'\|_\infty \leq C(1-\lambda)^{-(p+1)/(p-1)}, \tag{5.3}$$

$$c \leq \|u\|_\infty \leq \|u\|_{C^1}, \tag{5.4}$$

where $\|u\|_{C^1}$ *denotes the* $C^1[-1,1]$ *norm of* u*.*

Let u be any positive solution of (1.1) and let x_0 be a maximum point of $u(x)$. If u is even, then it attains its maximum at all points on $[-\lambda,\lambda]$. In this case, we choose $x_0=\lambda$. If u is not even, then it has a unique maximum point x_0, which lies in $(-1,-\lambda)$ or in $(\lambda,1)$. We assume that $x_0\in(\lambda,1)$ after replacing $u(x)$ by $u(-x)$ if necessary. Note that this replacement leaves the norms $\|u\|_\infty$ and $\|u'\|_\infty$ invariant. Therefore we assume that $x_0\in[\lambda,1)$ even if u is even or not.

Lemma 5.2. *Let* $u(x)$ *be any positive solution of* (1.1) *and let* $x_0\in[\lambda,1)$ *be a maximum point of* $u(x)$*. Define* $\tau(p)$ *by* (1.5)*. Then*

$$u(x_0)=[((p+1)/2)\tau(p)^2]^{1/(p-1)}(1-x_0)^{-2/(p-1)}.$$

Proof. Since $h(x,\lambda)=1$ in $[\lambda,1]$, we have

$$u''+u^p=0 \quad \text{in } [\lambda,1], \quad u'(x_0)=0,\ u(1)=0. \tag{5.5}$$

By (2.7), the solution $u(x)$ of the equation above is written as

$$u(x)=(1-x_0)^{-2/(p-1)}M(p)S_p\left(\tau(p)(1-x_0)^{-1}(1-x)\right) \quad \text{in } [x_0,1]. \tag{5.6}$$

Therefore, $u(x_0)=(1-x_0)^{-2/(p-1)}M(p)$. This identity with (2.8) proves the lemma. □

Proof of Theorem 5.1. Combining (5.1) with (5.2), we have (5.3). Inequality (5.4) follows readily from (5.2). Therefore it is enough to show (5.1) and (5.2). Let $u(x)$ be any positive solution of (1.1) with a maximum point $x_0\in[\lambda,1)$. Since u is concave, the maximum of $|u'(x)|$ is achieved at $x=1$ or $x=-1$. We shall show that it is attained at $x=1$ when $x_0\in[\lambda,1)$. If u is even, then $|u'(1)|=|u'(-1)|$. Hence our claim is valid. Let u be non-even. Then $x_0\in(\lambda,1)$. Since $x_0>\lambda$, it holds that $u'(x)=u'(\lambda)=u'(-\lambda)>0$ in $[-\lambda,\lambda]$ and $u(-\lambda)<u(\lambda)$. We here define the energy $E(x)$ by

$$E(x):=\frac{1}{2}u'(x)^2+\frac{1}{p+1}u(x)^{p+1}.$$

Multiplying both sides of (1.1) by $u'(x)$, we find that $E(x)$ is constant in $[-1,-\lambda]$ and in $[\lambda,1]$. Since $u(-\lambda)<u(\lambda)$ and $u'(-\lambda)=u'(\lambda)$, we have $E(-\lambda)<E(\lambda)$. Therefore $E(-1)=E(-\lambda)<E(\lambda)=E(1)$. This shows that $|u'(-1)|<|u'(1)|$ and hence $\|u'\|_\infty=|u'(1)|$. Differentiating (5.6) at $x=1$ and using (2.8) and Lemma 5.2, we have

$$u'(1)=-(1-x_0)^{-(p+1)/(p-1)}M(p)\tau(p)=-\left(\frac{2}{p+1}\right)^{1/2}u(x_0)^{(p+1)/2}.$$

This proves (5.1).

We shall show (5.2). If u is even, we take $x_0=\lambda$ in Lemma 5.2. Then

$$\|u\|_\infty=[((p+1)/2)\tau(p)]^{2/(p-1)}(1-\lambda)^{-2/(p-1)}.$$

Thus (5.2) holds. Let u be non-even. Then $x_0\in(\lambda,1)$. We claim that $x_0<(1+\lambda)/2$. Suppose to the contrary that $x_0\geq(1+\lambda)/2$. Since $u(x)$ satisfies (5.5), $u(x)$ is symmetric with respect to the line $x=x_0$, i.e., $u(x_0-x)=$

$u(x_0+x)$ when $\lambda \le x_0 - x < x_0 + x \le 1$. Substituting $x = 1 - x_0$, we have $u(2x_0-1) = u(1) = 0$. This is impossible. Accordingly, $x_0 < (1+\lambda)/2$. Hence $(1-\lambda)/2 < 1 - x_0 < 1 - \lambda$. Combining this inequality with Lemma 5.2, we have (5.2). The proof is complete. □

From Theorem 5.1, we derive the next result.

Lemma 5.3. *Let u be a positive solution of* (1.1). *If v is a nonnegative solution of* (1.1) *satisfying $\|u-v\|_{C^1} < c$, then v is a positive solution. Here, $c > 0$ is given by* (5.4).

Proof. Let u be a positive solution of (1.1). Let v be a nonnegative solution of (1.1) satisfying $\|u-v\|_{C^1} < c$. Then $\|v\|_{C^1} > 0$ by (5.4). Hence $v \not\equiv 0$ in $(-1,1)$. By the strong maximum principle, $v(x) > 0$ in $(-1,1)$. □

6. Proof of the main result

Since $h(x,\lambda)$ is not differentiable with respect to λ, the standard bifurcation method based on the Lyapunov–Schmidt reduction does not seem to work well. Instead of such a method, we will make use of the following result to prove Theorem 1.3. See [16, p.58, Theorem 12], [15] or [9].

Proposition 6.1. *Let E be a real Banach space and $T\colon \mathbb{R}\times E \to E$ completely continuous such that $T(l,0) = 0$ for all $l \in \mathbb{R}$. Suppose that there exist constants a, $b \in \mathbb{R}$ with $a < b$ such that $(a,0)$ and $(b,0)$ are not bifurcation points for the equation*

$$v - T(l,v) = 0. \tag{6.1}$$

Furthermore, assume that

$$\deg(I - T(a,\cdot), B_r(0), 0) \neq \deg(I - T(b,\cdot), B_r(0), 0),$$

where I is the identity operator, $B_r(0) = \{v \in E : \|v\|_E < r\}$ is an isolating neighborhood of the trivial solution for both constants a and b and $\deg(\cdot)$ denotes the Leray–Schauder degree. Define

$$\mathcal{S} := \overline{\{(l,v) : (l,v) \text{ is a solution of (6.1) with } v \neq 0\}} \cup ([a,b]\times\{0\})$$

and let $\mathcal{C}$ be the maximal connected subset of $\mathcal{S}$ containing $[a,b]\times\{0\}$. Then either

(i) *$\mathcal{C}$ is unbounded in $\mathbb{R}\times E$, or*
(ii) *$\mathcal{C} \cap [(\mathbb{R}\backslash[a,b])\times\{0\}] \neq \emptyset$.*

Let $\Lambda \in C(\mathbb{R})$ be a strictly increasing function such that

$$\lim_{l\to-\infty}\Lambda(l) = 0, \quad \lim_{l\to\infty}\Lambda(l) = 1.$$

(For example, $\Lambda(l) = 1/(1+e^{-l})$.) Then $0 < \Lambda(l) < 1$ for $l \in \mathbb{R}$.

We define T: $\mathbb{R}\times C^1[-1,1] \to C^1[-1,1]$ by

$$T(l,v) = \int_{-1}^{1} G(x,y)h(y,\Lambda(l))|U(y,\Lambda(l)) + v(y)|^p dy - U(x,\Lambda(l)), \tag{6.2}$$

where $G(x, y)$ is a Green's function of the operator $F[v] = -v''$ with $v(-1) = v(1) = 0$:

$$G(x, y) = \frac{1}{2} \begin{cases} (1 + x)(1 - y), \; -1 \le x \le y \le 1, \\ (1 - x)(1 + y), \; -1 \le y \le x \le 1. \end{cases}$$

By the standard argument, we can prove that T is completely continuous. We note that $T(l, 0) = 0$ for $l \in \mathbb{R}$ and hence (6.1) has a solution $v = 0$. If v is a solution of (6.1), then $u(x) := U(x, \Lambda(l)) + v(x)$ is a nonnegative solution of (1.1) with $\lambda = \Lambda(l)$. Indeed,

$$\begin{aligned} u(x) &= U(x, \Lambda(l)) + v(x) \\ &= \int_{-1}^{1} G(x, y) h(y, \Lambda(l)) |U(y, \Lambda(l)) + v(y)|^p dy \\ &= \int_{-1}^{1} G(x, y) h(y, \Lambda(l)) |u(y)|^p dy \ge 0, \end{aligned}$$

or equivalently,

$$u(x) = \int_{-1}^{1} G(x, y) h(y, \Lambda(l)) |u(y)|^p dy \ge 0.$$

Hence $u(x)$ is nonnegative and satisfies

$$-u''(x) = h(x, \Lambda(l))|u(x)|^p = h(x, \Lambda(l)) u(x)^p, \quad u(-1) = u(1) = 0.$$

Thus it is a nonnegative solution of (1.1) with $\lambda = \Lambda(l)$. Moreover, if $u(x) \not\equiv 0$, then the strong maximum principle (or the uniqueness of solutions for the initial value problem $(u(x_0), u'(x_0)) = (0, 0)$) ensures that $u(x)$ is a positive solution. Hence the next lemma follows.

Lemma 6.2. *Define $T(l, v)$ by (6.2). If $v(x)$ satisfies (6.1), then $u(x) := U(x, \Lambda(l)) + v(x)$ is a nonnegative solution of (1.1) with $\lambda = \Lambda(l)$. Moreover, if $u(x) \not\equiv 0$, then it becomes a positive solution of (1.1).*

Let $m(\lambda)$ be the Morse index of $U(x, \lambda)$, that is, the number of negative eigenvalues of (2.20). Recall that λ_* defined by (1.4) is a unique zero of $\mu_2(\lambda)$. By Propositions 2.5, 3.1 and 4.1, if $0 < \lambda < \lambda_*$, then $m(\lambda) = 1$ and $U(x, \lambda)$ is nondegenerate, and if $\lambda_* < \lambda < 1$, then $m(\lambda) = 2$ and $U(x, \lambda)$ is nondegenerate. Here, $U(x, \lambda)$ is said to be nondegenerate if $\mu = 0$ is not an eigenvalue of (2.20).

In the same way as in [17, Lemma 7.2], we have the following result.

Lemma 6.3. *If $U(x, \Lambda(l))$ is nondegenerate, then $(l, 0)$ is not a bifurcation point for (6.1).*

To get the bifurcation branch, we use Proposition 6.1 with the Whyburn lemma (see [23, p.12, (9.4)]).

Lemma 6.4. (Whyburn [23]) *Let (X, d) be a metric space and M_n be a sequence of subsets of X. Suppose that each M_n is nonempty, compact, connected, and satisfies*

$$M_1 \supset M_2 \supset M_3 \supset \cdots.$$

Then $M := \cap_{n=1}^{\infty} M_n$ is nonempty, compact, and connected.

We take $l_* \in \mathbb{R}$ such that $\Lambda(l_*) = \lambda_*$. Lemma 6.3 implies that if $l \neq l_*$, then $(l, 0)$ is not a bifurcation point for (6.1).

Let $a,\ b \in \mathbb{R}$ satisfy $a < l_* < b$. Then $0 < \Lambda(a) < \lambda_* < \Lambda(b) < 1$, $m(\Lambda(a)) = 1$, $m(\Lambda(b)) = 2$ and $\deg(I - T(c, \cdot), B_r(0), 0)$ with $c = a, b$ is well-defined for some sufficiently small $r > 0$. By the same argument as in [17, Section 7], we conclude that

$$\deg(I - T(a, \cdot), B_r(0), 0) = (-1)^{m(\Lambda(a))} = (-1)^1 = -1,$$

and

$$\deg(I - T(b, \cdot), B_r(0), 0) = (-1)^{m(\Lambda(b))} = (-1)^2 = 1.$$

We use Proposition 6.1 with $E = C^1[-1, 1]$. For each $n \in \mathbb{N}$, there exists a maximal connected subset $\mathcal{C}_n \subset \mathbb{R} \times C^1[-1, 1]$ of $\mathcal{S}$ containing $[a, b] \times \{0\}$ with $a := l_* - 1/n$ and $b := l_* + 1/n$. Here, we recall that $(l, 0)$ is not a bifurcation point if $l \neq l_*$. Therefore, the second of the alternatives in Proposition 6.1 can be eliminated, that is, $\mathcal{C}_n$ is unbounded. We define

$$\overline{B}(R) := \{u \in C^1[-1, 1] : \|u\|_{C^1} \leq R\}.$$

For each $R > 0$ and $n \in \mathbb{N}$, we denote by $\mathcal{C}_n(R)$ the maximal connected subset of $\mathcal{C}_n \cap ([l_* - R, l_* + R] \times \overline{B}(R))$ containing the point $(l_*, 0) \in \mathcal{C}_n$. This set is compact because each point in $\mathcal{C}_n(R)$ is a nonnegative solution satisfying $\|u\|_{C^1} \leq R$. This estimate implies an a priori estimate in $W^{2.\infty}(-1, 1)$, i.e., for each $(l, u) \in \mathcal{C}_n(R)$, it holds that $\|u\|_{W^{2,\infty}} \leq C_R$ with some $C_R > 0$ (see the proof of Lemma 6.5 later). Here C_R is a constant independent of n. This estimate with the Sobolev embedding shows the compactness of $\mathcal{C}_n(R)$. Then $\mathcal{C}_n(R)$ is nonempty compact connected and satisfies $\mathcal{C}_n(R) \supset \mathcal{C}_{n+1}(R)$ for $n \geq 1$. We define $\mathcal{C}(R) := \cap_{n=1}^{\infty} \mathcal{C}_n(R)$. By Lemma 6.4, this is nonempty compact and connected. We define

$$\mathcal{C}_* := \bigcup_{R>0} \mathcal{C}(R). \tag{6.3}$$

Since $(l_*, 0) \in \mathcal{C}(R) \cap \mathcal{C}(R')$ for any $R, R' > 0$, $\mathcal{C}_*$ is connected.

Lemma 6.5. *$\mathcal{C}_*$ is unbounded.*

Proof. Let $R > 0$ be any number. Since $\mathcal{C}_n$ is unbounded, $\mathcal{C}_n(R)$ intersects the boundary of $[l_* - R, l_* + R] \times \overline{B}(R)$. Choose an intersection point (l_n, v_n). Then either

$$|l_n - l_*| = R \text{ and } \|v_n\|_{C^1} \leq R, \tag{6.4}$$

or

$$|l_n - l_*| \leq R \text{ and } \|v_n\|_{C^1} = R. \tag{6.5}$$

Put $u_n(x) := U(x, \Lambda(l_n)) + v_n(x)$. This is a nonnegative solution of (1.1) by Lemma 6.2. By (6.4) or (6.5) and $\Lambda(l_* - R) \leq \Lambda(l_n) \leq \Lambda(l_* + R)$, $u_n(x)$ is bounded in $C^1[-1, 1]$, i.e., $\|u_n\|_{C^1} \leq C_R$ with a certain constant $C_R > 0$ independent of n. By (1.1), we have

$$\|u_n''\|_\infty = \|h(x, \lambda) u_n^p\|_\infty \leq C_R^p.$$

Thus u_n is bounded in $W^{2,\infty}(-1,1)$. By the Sobolev embedding, we can extract a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ which converges in $C^1[-1,1]$. Then v_{n_j} converges to a limit v_∞ in $C^1[-1,1]$ and l_{n_j} also converges to a limit l_∞ along a subsequence. Since $(l_{n_j}, v_{n_j}) \in \mathcal{C}_{n_j}(R)$, $\mathcal{C}_{n_j}(R) \supset \mathcal{C}_{n_{j+1}}(R)$ and each $\mathcal{C}_{n_j}(R)$ is compact, the definition of $\mathcal{C}(R)$ shows that $(l_\infty, v_\infty) \in \mathcal{C}(R)$. Hence $(l_\infty, v_\infty) \in \mathcal{C}(R) \subset \mathcal{C}_*$. By (6.4) or (6.5), we have $|l_\infty - l_*| = R$ or $\|v_\infty\|_{C^1} = R$. Since $R > 0$ is arbitrary, $\mathcal{C}_*$ is unbounded. □

Remark 6.6. *We can define $\mathcal{C}_*$ another way. We fix $a, b \in \mathbb{R}$ such that $a < l_* < b$. Let $\mathcal{C}_0$ be the maximal connected subset of $\mathcal{S}$ containing $[a,b] \times \{0\}$. We remove the set $([a,b]\backslash\{l_*\}) \times \{0\}$ from $\mathcal{C}_0$ and define $\mathcal{C}_*$ by*

$$\mathcal{C}_* := \mathcal{C}_0 \backslash \mathcal{L}, \quad \mathcal{L} := ([a,b]\backslash\{l_*\}) \times \{0\}.$$

The definition above is the same as in (6.3). We can prove directly that $\mathcal{C}_*$ defined above is connected. However this proof is longer than that for (6.3).

We conclude this paper by proving Theorem 1.3.

Proof of Theorem 1.3. It follows from the definition of $\mathcal{C}_*$ that

$$\mathcal{C}_* \cap (\mathbb{R} \times \{0\}) = \{(l_*, 0)\}. \tag{6.6}$$

Define

$$\mathcal{C} := \{(\lambda, U(\,\cdot\,, \lambda) + v) : (\Lambda^{-1}(\lambda), v) \in \mathcal{C}_*\},$$

where Λ^{-1} is the inverse function of Λ. Then $\mathcal{C}$ is a closed connected subset of $(0,1) \times E$. Observe that $(l_*, 0) \in \mathcal{C}_*$ by (6.6). This point corresponds to $(\lambda_*, U(x, \lambda_*))$, and hence $(\lambda_*, U(x, \lambda_*)) \in \mathcal{C}$, which is an even positive solution. Therefore $\mathcal{C}$ contains a positive solution. Recall that each element of $\mathcal{C}$ is a nonnegative solution by Lemma 6.2. Let $\mathcal{D}$ be a set of points $(\lambda, u) \in \mathcal{C}$ which are positive solutions of (1.1). Then it is nonempty and relatively open in $\mathcal{C}$ by Lemma 5.3. It is also relatively closed in $\mathcal{C}$ by (5.4) with the strong maximum principle. Since $\mathcal{C}$ is connected, $\mathcal{D}$ coincides with $\mathcal{C}$. Thus all elements of $\mathcal{C}$ are positive solutions of (1.1). By Proposition 1.2 with (6.6), any point $(\lambda, u) \in \mathcal{C}$ except for $(\lambda_*, U(\cdot, \lambda_*))$ must be non-even.

Next we will prove that, for $l > l_*$, there exists a v such that $(l, v) \in \mathcal{C}_*$. Assume to the contrary that there exists an $L_1 > l_*$ such that any $(l, v) \in \mathcal{C}_*$ satisfies $l \le L_1$. On the other hand, by Lemma 2.3, there exists an $L_0 \in \mathbb{R}$ such that (1.1) has no positive non-even solution with $\lambda = \Lambda(l)$ if $l < L_0$. Therefore, $\mathcal{C}_* \subset [L_0, L_1] \times C^1[-1,1]$. By Theorem 5.1, there exists a constant $M > 0$ such that if u is a positive solution of (1.1) with $\lambda = \Lambda(l)$ and $l \in [L_1, L_2]$, then $\|u\|_{C^1} \le M$. We observe that if $(l, v) \in \mathcal{C}_*$, then $U(x, \Lambda(l)) + v(x)$ and $U(x, \Lambda(l))$ are positive solutions of (1.1), and therefore

$$\begin{aligned} \|v\|_{C^1} &= \|U(\,\cdot\,, \Lambda(l)) + v - U(\,\cdot\,, \Lambda(l))\|_{C^1} \\ &\le \|U(\,\cdot\,, \Lambda(l)) + v\|_{C^1} + \|U(\,\cdot\,, \Lambda(l))\|_{C^1} \\ &\le 2M, \end{aligned}$$

which means that $\mathcal{C}_*$ is bounded. This contradicts Lemma 6.5. Hence, for every $l > l_*$, there exists a v such that $(l, v) \in \mathcal{C}_*$. This result shows that for every

$\lambda \in (\lambda_*, 1)$, there exists a u such that $(\lambda, u) \in \mathcal{C}$. By Theorem 5.1, $\|u\|_{C^1}$ diverges to infinity as $\lambda \to 1$ with $(\lambda, u) \in \mathcal{C}$. Therefore $\mathcal{C}$ is unbounded.

We shall show that $\lambda_*'(p) < 0$. We put $f(p) := (p^2-1)\tau(p)^2$. It is enough to show that $f'(p) > 0$. We compute

$$f'(p) = 2p\tau(p)^2 + 2(p^2-1)\tau(p)\tau'(p) = 2\tau(p)[p\tau(p) + (p^2-1)\tau'(p)].$$

Differentiating (1.5), we have

$$\tau'(p) = \frac{1}{2}\int_0^1 (1-t^{p+1})^{-3/2} t^{p+1} \log t\, dt.$$

This integral is finite because $\log t \fallingdotseq t-1$ near $t=1$. Using the identity above, we have

$$\begin{aligned} &p\tau(p) + (p^2-1)\tau'(p) \\ &\quad = p\int_0^1 (1-t^{p+1})^{-1/2} dt + \frac{p^2-1}{2}\int_0^1 (1-t^{p+1})^{-3/2} t^{p+1}\log t\, dt \\ &\quad = \frac{1}{2}\int_0^1 (1-t^{p+1})^{-3/2}\left\{2p(1-t^{p+1}) + (p^2-1)t^{p+1}\log t\right\} dt. \end{aligned}$$

For fixed $p \in (1, \infty)$, we define

$$g(t) := 2p(1-t^{p+1}) + (p^2-1)t^{p+1}\log t.$$

If we would prove that $g(t) > 0$ for $t \in (0,1)$, then it follows that $f'(p) > 0$. Differentiating $g(t)$, we have

$$g'(t) = -(p+1)^2 t^p + (p^2-1)(p+1)t^p \log t < 0 \quad \text{for } t \in (0,1).$$

Hence $g(t)$ is decreasing. Since $g(1) = 0$, $g(t)$ is positive. Consequently, $f'(p) > 0$ and $\lambda_*'(p) < 0$. Observe that

$$\lim_{p\to 1} \tau(p) = \frac{\pi}{2}, \quad \lim_{p\to\infty} \tau(p) = 1.$$

This proves that $\lim_{p\to 1} \lambda_*(p) = 1$ and $\lim_{p\to\infty} \lambda_*(p) = 0$. The proof is complete. □

References

[1] Amadori, A.L., Gladiali, F.: Bifurcation and symmetry breaking for the Henon equation. Adv. Differ. Equ. **19**, 755–782 (2014)

[2] Brezis, H.: Functional Analysis, Sobolev Spaces and Partial Differential Equations (Universitext). Springer, New York (2011)

[3] Courant, R., Hilbert, D.: Methods of Mathematical Physics, vol. 1. Interscience, New York (1989)

[4] Drábek, P., Manásevich, R.: On the closed solution to some nonhomogeneous eigenvalue problems with p-Laplacian. Differ. Integral Equ. **12**, 773–788 (1999)

[5] Gritsans, A., Sadyrbaev, F.: Extension of the example by Moore–Nehari. Tatra Mt. Math. Publ. **63**, 115–127 (2015)

[6] Kajikiya, R.: Non-even least energy solutions of the Emden–Fowler equation. Proc. Am. Math. Soc. **140**, 1353–1362 (2012)

[7] Kajikiya, R.: Non-radial least energy solutions of the generalized Hénon equation. J. Differ. Equ. **252**, 1987–2003 (2012)

[8] Kajikiya, R.: Nonradial positive solutions of the p-Laplace Emden–Fowler equation with sign-changing weight. Math. Nachr. **289**, 290–299 (2016)

[9] Le, V.K., Schmitt, K.: Global Bifurcation in Variational Inequalities: Applications to obstacle and unilateral problems (Applied Mathematical Sciences). Springer, Berlin (1997)

[10] López-Gómez, J., Rabinowitz, P.H.: Nodal solutions for a class of degenerate boundary value problems. Adv. Nonlinear Stud. **15**, 253–288 (2015)

[11] López-Gómez, J., Rabinowitz, P.H.: The effects of spatial heterogeneities on some multiplicity results. Discrete Contin. Dyn. Syst. **36**, 941–952 (2016)

[12] López-Gómez, J., Rabinowitz, P.H.: Nodal solutions for a class of degenerate one dimensional BVP's. Topol. Methods Nonlinear Anal. **49**, 359–376 (2017)

[13] López-Gómez, J., Molina-Meyer, M., Rabinowitz, P.H.: Global bifurcation diagrams of one node solutions in a class of degenerate boundary value problems. Discrete Contin. Dyn. Syst. Ser. B **22**, 923–946 (2017)

[14] Moore, R.A., Nehari, Z.: Nonoscillation theorems for a class of nonlinear differential equations. Trans. Am. Math. Soc. **93**, 30–52 (1959)

[15] Rabinowitz, P.: Some aspects of nonlinear eigenvalue problems. Rocky Mt. J. Math. **3**, 161–202 (1973)

[16] Schmitt, K., Thompson, R.: Nonlinear Analysis and Differential Equations: An introduction. University of Utah Lecture Note, Salt Lake City (2004)

[17] Sim, I., Tanaka, S.: Symmetry-breaking bifurcation for the one-dimensional Hénon equation. Commun. Contemp. Math. (to appear)

[18] Smets, D., Willem, M., Su, J.: Non-radial ground states for the Hénon equation. Commun. Contemp. Math. **4**, 467–480 (2002)

[19] Takeuchi, S.: Generalized Jacobian elliptic functions and their application to bifurcation problems associated with p-Laplacian. J. Math. Anal. Appl. **385**, 24–35 (2012)

[20] Takeuchi, S.: The basis property of generalized Jacobian elliptic functions. Commun. Pure Appl. Anal. **13**, 2675–2692 (2014)

[21] Takeuchi, S.: Multiple-angle formulas of generalized trigonometric functions with two parameters. J. Math. Anal. Appl. **444**, 1000–1014 (2016)

[22] Tanaka, S.: Morse index and symmetry-breaking for positive solutions of one-dimensional Hénon type equations. J. Differ. Equ. **255**, 1709–1733 (2013)

[23] Whyburn, G.T.: Topological Analysis, revised edn. Princeton University Press, Princeton (1964)

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