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Large-time behaviour of solutions to a class of non-Newtonian compressible fluids

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Abstract. In this paper, we consider the large-time dynamics of weak solutions to a class of compressible fluids with nonlinear constitutive equations in a bounded domain $\Omega \subseteq \mathbb{R}^3$, the global existence of such solutions has been showed by Feireisl et al. (Math Methods Appl Sci 38:3482–3494, 2015). We study the large time behavior of such solutions after discussing the uniqueness of solutions to the stationary problem.

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1. Introduction

Recently, for $T > 0$, a bounded domain $\Omega \subseteq \mathbb{R}^3$, Feireisl et al. [1] showed the large-data existence result of weak solutions to a class of non-Newtonian compressible fluids:

$$\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\frac{\partial (\varrho \mathbf{u})}{\partial t} + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) = \operatorname{div} \mathbb{T} + \varrho \mathbf{b}, \tag{1.2}$$

$$\text{with} \quad \mathbb{T} = -p(\varrho)\mathbb{I} + 2\mu(|\mathbb{D}^d|^2)\mathbb{D}^d + \frac{b \operatorname{div} \mathbf{u}}{(1 - b^a |\operatorname{div} \mathbf{u}|^a)^{1/a}} \mathbb{I}, \tag{1.3}$$

where $t \in (0, T)$ is time, $x \in \Omega$ is the spatial coordinate, $\varrho(t,x), \mathbf{u}(t,x) = (u_1, u_2, u_3)$ represent the density, velocity of the fluid, respectively; $\mu : \mathbb{R} \to (0, \infty)$ is the viscosity function, and a, b are positive model parameters. $\mathbf{b}(t,x) = (b_1, b_2, b_3)$ stands for the density of external body forces. In (1.3),

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symbol $\mathbb{T}$ stands for the cauchy stress tensor, and $\mathbb{D}$ denotes the symmetric part of the velocity gradient $\nabla \mathbf{u}$, we also use $\mathbb{D}^d$ for the deviatoric (traceless) part of $\mathbb{D}$, that is $\mathbb{D}^d = \mathbb{D} - \frac{1}{3}(\operatorname{div}\mathbf{u})\,\mathbb{I}$. In general case, for any tensor quantity $\mathbb{Q}$, we set $\mathbb{Q}^d = \mathbb{Q} - \frac{1}{3}(\operatorname{tr}\mathbb{Q})\,\mathbb{I}$.

In what follows, we shall assume the external force

$$\mathbf{b} = \mathbf{b}(x) = \nabla F$$

where F is a potential and is assumed to be locally Lipschitz continuous on Ω; and the scalar pressure $p(\varrho)$ depends on the density ϱ, satisfying

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \quad p'(\varrho) > 0 \text{ for } \varrho > 0. \tag{1.4}$$

We impose the homogeneous Dirichlet boundary condition on the velocity

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.5}$$

and the initial conditions

$$\varrho(0,\cdot) = \varrho_0 \quad \text{and} \quad (\varrho\mathbf{u})(0,\cdot) = \mathbf{m}_0 \quad \text{in } \Omega, \tag{1.6}$$

where ϱ_0 is the initial density, positive, and $\mathbf{m}_0$ is the initial momentum.

Note that, in the constitutive Eq. (1.3), the relation between $\mathbb{T}$ and $\mathbb{D}$ is nonlinear even if μ does not depend on $\mathbb{D}^d$. More precisely, we assume that

$$\mu(|\mathbb{D}^d|^2) = \mu_0(1 + |\mathbb{D}^d|^2)^{(r-2)/2} \quad \text{with } \mu_0 > 0 \quad \text{and} \quad r \in [11/5, \infty), \tag{1.7}$$

according to [2,3], the monotonicity method applies if

$$r \geq \frac{3n+2}{n+2},$$

specifically, when $n = 3$, we have the lower bound $\frac{11}{5}$ for r.

The system we consider in this paper has both nonlinear constitutive equations and a nonlinear pressure law. As we know, there are few studies concerning compressible fluids with nonlinear relation between the cauchy stress and the velocity gradient, for example [1,4–6], from which, Zhikov and Pastukhova [6] considered the solvability of the Navier–Stokes equations for a compressible Non-Newtonian fluid with general nonlinear constitutive equations and a state equation $p(\varrho) = \varrho^\gamma$, $(\gamma > 1)$. Feireisl et al. [1] showed that for any data fulfilling certain natural conditions concerning their integrability, there exists a weak solution to the problem (1.1)–(1.7) that admits the strictly positive density in $(0,T) \times \Omega$ whenever $\varrho_0 > 0$ in Ω.

In the work of Feireisl and Ptzeltová [7], it is showed that, with some basic hypotheses, any weak solution converged to a fixed stationary state as time goes to infinity. Following the argument of Feireisl and Ptzeltová [7], with the global existence showed by Feireisl et al. [1], we consider the large time behavior of the non-Newtonian fluid. For more results on the problem of large time behavior, see for instance [8–13] and the reference cited there in.

The stationary problem for $\mathbf{u}= 0$

$$\begin{cases} \nabla p(\varrho) = \varrho \nabla F(\varrho), \\ \varrho > 0, \quad \int_\Omega \varrho(x) = m \end{cases} \tag{1.8}$$

where the parameter $m > 0$ represents the total mass conserved by the flow, plays an important role in solving problems of Navier–Stokes equations. From the results such as [14–16], we get that, the uniqueness for (1.8) is of particular interesting, as in that case, the global trajectory of the Navier–Stokes equations for compressible isentropic flow, will converge to the single stationary state. Besides, there are many other contributions about the problem of existence and the uniqueness of a solution of steady compressible flow, for example [14,17–20], and reference therein. In fact, da Veiga [17] obtained a necessary and sufficient conditions for the existence of the rest state, with a positive density. Feireisl and Peteltová [14] showed the optimal condition for the uniqueness of the nonnegative stationary solution.

Hypotheses By $L^p(\Omega)$ and $W^{k,p}(\Omega)$, $1 \leq p, k \leq \infty$, we denote the Lebesgue and Sobolev spaces, respectively, equipped with the standard norm.

We consider the stress tensor $\mathbb{T}$ as

$$\mathbb{T} = -p(\varrho)\mathbb{I} + \mathbb{S}(\mathbf{u}) + \eta(\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u}\,\mathbb{I}, \tag{1.9}$$

where

1. the deviatoric part of the Cauchy stress tensor $\mathbb{S}$ is specified through

$$\begin{aligned}\mathbb{S}(\mathbf{u}) :&= 2\mu(|\mathbb{D}^d|^2)\mathbb{D}^d \\ &= 2\mu_0(1 + |\mathbb{D}^d(\mathbf{u})|^2)^{(r-2)/2}\mathbb{D}^d(\mathbf{u}),\ \mu_0 > 0 \text{ constant},\ r \in [11/5, \infty);\end{aligned} \tag{1.10}$$

2. the bulk viscosity coefficient η is a continuous function of $\operatorname{div}\mathbf{u}$, $\eta(\operatorname{div}\mathbf{u}) : (-\frac{1}{b}, \frac{1}{b}) \to [0, \infty)$, such that there is a convex potential $\Lambda : \mathbb{R} \to [0, \infty]$,

$$\begin{cases} \Lambda(0) = 0, \\ \Lambda'(z) = z\eta(z), \\ \Lambda(z) \to \infty \quad \text{if } z \to \pm\frac{1}{b}, \\ \Lambda(z) = \infty \quad \text{if } |z| \geq \frac{1}{b}; \end{cases} \tag{1.11}$$

3. the pressure $p = p(\varrho)$ and the Helmholtz free energy $\psi = \psi(\varrho)$ satisfy

$$p = \varrho^2\psi'(\varrho), \quad p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \quad p'(\varrho) > 0 \text{ for } \varrho > 0. \tag{1.12}$$

Remark. In the following part, we need to introduce a function P by $P(\varrho) := \varrho\psi(\varrho)$. Using (1.12), it is easy to get that $P''(\varrho) = \frac{p'(\varrho)}{\varrho}$, when $\varrho > 0$, $P''(\varrho) > 0$, which means P is strictly convex on $(0, \infty)$.

Let $\mathbf{b} = \nabla F$, where we shall always assume the potential F satisfies

$$F \in L^\infty(\Omega) \quad \text{and} \quad \text{Lipschitz continuous on } \overline{\Omega}. \tag{1.13}$$

our main result is stated in the following theorem:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with a compact and Lipschitz boundary. Let the potential F satisfy (1.13). Assume, the pressure $p = p(\varrho)$*

satisfy (1.12). *Then for any weak solution* $\varrho, \mathbf{u}$ *of the problem* (1.1)–(1.7), *there exists a stationary state* ϱ_s *such that*

$$\varrho(t) \to \varrho_s \text{ in } L^q(\Omega),\ q \in [1, \infty), \quad ess\sup_{\tau > t} \int_\Omega \varrho(\tau)|\mathbf{u}(\tau)|^2 dx \to 0, \text{ as } t \to \infty. \tag{1.14}$$

The paper is organized as follows. In Sect. 2, we state the global existence results of the weak solutions. Section 3 includes all estimate needed for the convergence of the weak solutions. In Sect. 4, we state some known results about the stationary problem. Section 5 contains the proof of the main result.

2. Global-in-time weak solutions

In this part, we will state the definition of weak solutions of the problem (1.1)–(1.7), which has been established in the work of [1].

Definition 2.1. Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain with a Lipschitz boundary. Suppose that the pressure $p = p(\varrho)$ and the Helmholtz free energy $\psi = \psi(\varrho)$ satisfy (1.12) and that the hypotheses (1.9)–(1.13) hold, let the initial data $(\varrho_0, \mathbf{m}_0)$ satisfy

$$0 < \underline{\varrho} \le \varrho_0(x) \le \overline{\varrho}, \quad \text{for a.e} \quad x \in \Omega, \quad \mathbf{m}_0 \in (L^2(\Omega))^3.$$

Then, for any $T > 0$, a pair of functions $(\varrho, \mathbf{u})$ is called a weak solution of the problem (1.1)–(1.7) if:

(1)

$$\begin{aligned} &\varrho \in C([0,T]; L^1(\Omega)) \cap L^\infty((0,T) \times \Omega), \quad \varrho(0) = \varrho_0, \\ &\sqrt{\varrho}\mathbf{u} \in L^\infty([0,T]; (L^2(\Omega))^3), \quad \mathbf{u} \in L^r([0,T]; (W_0^{1,r}(\Omega))^3), \\ &\eta(\mathrm{div}_x \mathbf{u})|\mathrm{div}_x \mathbf{u}|^2 \in L^1((0,T) \times \Omega); \end{aligned}$$

(2) the equation of continuity (1.1) holds in the following sense:

$$\int_0^\tau \int_{\mathbb{R}^3} (\varrho\varphi_t + \varrho\mathbf{u} \cdot \nabla\varphi)\, dxdt = \left[\int_{\mathbb{R}^3} \varrho\varphi dx\right]\Big|_0^\tau \tag{2.1}$$

for any $\tau \in [0, T]$, for all $\varphi \in \mathscr{D}((0,T) \times \mathbb{R}^3)$ provided $\mathbf{u}$ is prolonged by zero outside Ω. And the renormalized equation

$$\partial_t[b(\varrho)] + \mathrm{div}_x[b(\varrho)\mathbf{u}] + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}_x \mathbf{u} = 0 \tag{2.2}$$

holds in the sense of distributions, for any

$$b \in C^1(\mathbb{R}), \quad |b'(z)z| \le c|z|^{1/2};$$

(3) the following weak formulation of the momentum equation holds

$$\int_0^\tau \int_\Omega \Lambda(\mathrm{div}_x \varphi) - \Lambda(\mathrm{div}_x \mathbf{u})\, dxdt \geq \left[\int_\Omega \varrho|\mathbf{u}|^2 dx\right]\Big|_0^\tau - \left[\int_\Omega \varrho\mathbf{u}\cdot\varphi dx\right]\Big|_0^\tau$$
$$+ \int_0^\tau \int_\Omega \big(\varrho\mathbf{u}\cdot\partial_t\varphi + (\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla\varphi + \mathbb{S}(\mathbf{u}) : \mathbb{D}^d(\mathbf{u}-\varphi)\big)\, dxdt$$
$$+ \int_0^\tau \int_\Omega p(\varrho)\mathrm{div}\,(\varphi - \mathbf{u})\, dxdt + \int_0^\tau \int_\Omega \varrho\mathbf{u}\cdot(\varphi-\mathbf{u})\, dxdt,$$

for a.e $\tau \in [0,T]$, for all $\varphi \in \mathscr{D}((0,T)\times\Omega)$.

3. Energy estimates and local convergence

Firstly, under the assumption that both $\mathbf{u}$ and ϱ are smooth, we will derive the energy estimates. Taking the scalar product of (1.2) with $\mathbf{u}$, we have

$$\varrho_t|\mathbf{u}|^2 + \varrho\left(\frac{1}{2}|\mathbf{u}|^2\right)_t + |\mathbf{u}|^2\mathrm{div}\,(\varrho\mathbf{u}) + \varrho\mathbf{u}\cdot\nabla\left(\frac{|\mathbf{u}|^2}{2}\right) = \mathbf{u}\,\mathrm{div}\,\mathbb{T} + \varrho\nabla F\cdot\mathbf{u},$$

with the help of (1.1), it is easy to get

$$\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right)_t + \frac{|\mathbf{u}|^2}{2}\mathrm{div}\,(\varrho\mathbf{u}) + \varrho\mathbf{u}\cdot\nabla\left(\frac{|\mathbf{u}|^2}{2}\right) = \mathbf{u}\,\mathrm{div}\,\mathbb{T} + \varrho\nabla F\cdot\mathbf{u},$$

it follows that

$$\frac{1}{2}\left[(\varrho|\mathbf{u}|^2)_t + \mathrm{div}\,(\varrho|\mathbf{u}|^2\mathbf{u})\right] + \mathbb{T}\cdot\mathbb{D} = \mathrm{div}\,(\mathbb{T}\mathbf{u}) + \varrho\nabla F\cdot\mathbf{u},$$

integrating over Ω, using boundary condition (1.5) and the Transport Theorem "$\frac{d}{dt}\int_\Omega F\,dx = \int_\Omega F_t + \mathrm{div}\,(F\mathbf{u})\,dx$", we have

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 - \varrho F\right) dx + \int_\Omega \mathbb{T}\cdot\mathbb{D}\,dx = 0. \tag{3.1}$$

On the other hand, without the thermal effects, the dynamics of the process are often carried on by the so-called thermodynamic identity

$$\mathbb{T}\cdot\mathbb{D} - \varrho\dot{\psi} = \xi \tag{3.2}$$

where $\psi = \psi(\varrho)$ is the Helmholtz potential, ξ denotes the rate of dissipation, and $\dot{\psi}$ means the material derivative of ψ,

$$\dot{\psi} = \frac{\partial\psi}{\partial t} + \sum_{k=1}^3 \frac{\partial\psi}{\partial x_k}u_k.$$

Combining (1.1) and the material derivative, we get

$$\varrho\dot{\psi} = \varrho\psi'(\varrho)\dot{\varrho} = -\varrho^2\psi'(\varrho)\mathrm{div}\,\mathbf{u} = -p\,\mathrm{div}\,\mathbf{u} \tag{3.3}$$

where $p =: \varrho^2\psi'(\varrho) := \varrho^2\frac{d\psi(\varrho)}{d\varrho}$. With easy computation, we observe

$$\mathbb{T}\cdot\mathbb{D} = \mathbb{T}^d\cdot\mathbb{D}^d + \frac{1}{3}(\mathrm{tr}\,\mathbb{T})\,\mathrm{div}\,\mathbf{u}. \tag{3.4}$$

Plugging (3.3) (3.4) into (3.2), one gets

$$\xi = \mathbb{T}^d \cdot \mathbb{D}^d + \left(\frac{1}{3}\operatorname{tr}\mathbb{T} + p(\varrho)\right) \operatorname{div}\mathbf{u}.$$

Besides, from (1.9), we have the following relation,

$$\frac{1}{3}\operatorname{tr}\mathbb{T} = -p(\varrho) + \eta(\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u},$$
$$\mathbb{T}^d = 2\mu(|\mathbb{D}^d|^2)\mathbb{D}^d,$$

then,

$$\xi = 2\mu_0(1+|\mathbb{D}^d(\mathbf{u})|^2)^{(r-2)/2}|\mathbb{D}^d(\mathbf{u})|^2 + \eta(\operatorname{div}\mathbf{u})|\operatorname{div}\mathbf{u}|^2. \tag{3.5}$$

Putting (3.2) and (3.5) into (3.1), we have the energy equation,

$$\frac{d}{dt}E(t)+\int_\Omega \left(2\mu_0(1+|\mathbb{D}^d(\mathbf{u})|^2)^{(r-2)/2}|\mathbb{D}^d(\mathbf{u})|^2+\eta(\operatorname{div}\mathbf{u})|\operatorname{div}\mathbf{u}|^2\right)dx = 0, \tag{3.6}$$

where

$$E(t) = \int_\Omega \left(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \varrho(t)\psi - \varrho(t)F\right) dx.$$

Remark. If ϱ, $\mathbf{u}$ are weak solutions, the " $=$" in (3.6) turns to " $\leq$".

Proposition 3.1. *Under the hypotheses of Theorem 1.1, let $\varrho, \mathbf{u}$ be a weak solution of the problem (1.1)–(1.7) on the time interval $(0,\infty)$, and satisfying Definition 2.1.*

Then the mass $m[\varrho(t)]$ is time invariant, i.e

$$m_0 \overset{\text{def}}{=\!=} \int_\Omega \varrho(t)\,dx = \int_\Omega \varrho(s)\,dx \quad \textit{for a.e.} \quad 0 < s \leq t. \tag{3.7}$$

Further more, there exist a constant E_0 such that

$$\begin{aligned} &\mathop{ess\,sup}_{t>1}\left(\|\varrho\psi\|_{L^1(\Omega)} + \|\varrho\|_{L^1(\Omega)} + \|\sqrt{\varrho}\mathbf{u}\|_{L^2(\Omega)}\right)\\ &\quad + \int_1^\infty \int_\Omega \left(|\mathbb{D}^d(\mathbf{u})|^r + |\operatorname{div}\mathbf{u}|^2\right) dxdt \leq E_0. \end{aligned} \tag{3.8}$$

Proof. From the energy inequality (3.6), we know, $\varrho|\mathbf{u}|^2 \in L^\infty_{loc}((0,\infty);L^1(\Omega))$, using Hölder inequality,

$$\int_\Omega \varrho|\mathbf{u}|\,dx = \int_\Omega \varrho^{\frac{1}{2}}\varrho^{\frac{1}{2}}|\mathbf{u}|\,dx \leq \left(\int_\Omega \varrho\,dx\right)^{\frac{1}{2}} \left(\int_\Omega \varrho|\mathbf{u}|^2\,dx\right)^{\frac{1}{2}},$$

we have $\varrho|\mathbf{u}| \in L^\infty_{loc}((0,\infty);L^1(\Omega))$. Choose function $\psi \in \mathscr{D}(0,\infty)$ and a sequence $\phi_n \in \mathscr{D}(\mathbb{R}^3)$ satisfying

$$\begin{cases} 0 \leq \phi_n(x) \leq 1, & |\nabla\phi_n(x)| \leq 1, \\ \phi_n(x) \to 1, & |\nabla\phi_n(x)| \to 0, \text{ as } n \to \infty, \end{cases} \text{for all } x \in \mathbb{R}^3.$$

Taking $\varphi_n = \psi(t)\phi_n(x)$ as the test function in

$$\int_0^\infty \int_\Omega (\varrho\varphi_t + \varrho\mathbf{u}\cdot\nabla\varphi)\,dxdt = 0,$$

we have

$$\int_0^\infty \int_\Omega \left(\varrho\psi'(t)\phi_n + \psi(t)\varrho\mathbf{u}\cdot\nabla\varphi_n(x)\right)\,dxdt = 0.$$

Let $n \to \infty$, with the Lebesgue dominance convergence theorem, we observe

$$\int_0^\infty \int_\Omega \varrho\psi'(t)\,dxdt = 0$$

for any $\psi \in \mathscr{D}(0,\infty)$, yielding (3.7). Combining (3.7), (1.13) and the energy inequality (3.6), we get (3.8). □

Further, we have the following Lemma.

Lemma 3.1. *Under the hypotheses of Theorem 1.1, we have*

$$\lim_{\tau\to\infty}\int_{\tau-1}^{\tau+2} \|\nabla\mathbf{u}\|_{L^r(\Omega)}^r + ||\varrho|\mathbf{u}|^2\|_{L^{\frac{r}{2}}\cap L^1(\Omega)} + \|\varrho|\mathbf{u}|\|_{L^r(\Omega)}^2\,dt = 0. \tag{3.9}$$

Proof. The desired conclusion follows from Definition 2.1 and the results obtained in Proposition 3.1. □

Analogy of Lemma 4.1 in [7], we have

Lemma 3.2. *Let $\phi \in C^\infty(\Omega)$ such that*

$$\overline{supp\,\phi} \subset \Omega,\ 0 \le \phi \le 1, \quad |\nabla\phi| \le M\ \text{ in } \Omega.$$

Let $b \in C^1(\mathbb{R})$ satisfy

$$b,\, b' \ge 0,\, b(z) = 0 \quad \text{for } z \le 0, \quad zb'(z) \le cz^\theta \quad \text{for } z \ge 0,$$

where

$$0 < \theta < \min\left\{\frac{1}{4}, \frac{1}{3} - \frac{1}{r}\right\}. \tag{3.10}$$

Then, under the hypotheses of Theorem 1.1, there exists a constant $Y(b)$ such that

$$\begin{aligned}&\limsup_{\tau\to\infty}\int_\tau^{\tau+1}\int_\Omega p(\varrho)b(\varrho)\phi^2\,dxdt\\ &\quad\le Y(b)\left(\limsup_{\tau\to\infty}\int_{\tau-1}^{\tau+2}\int_\Omega p(\varrho)|\nabla\phi|\,dxdt + ess\sup_{x\in supp\,\phi}|\nabla F(x)|\right).\end{aligned} \tag{3.11}$$

Proof. At first, let us consider the operators

$$\mathcal{A}_i[v] = \Delta^{-1}[\partial_{x_i} v], \quad i = 1,2,3$$

where Δ^{-1} stands for the inverse of the Laplace operator on $\mathbb{R}^3$. To be more specific, the Fourier symbol of $\mathcal{A}_i$ is

$$\hat{\mathcal{A}}_j[\xi] = \frac{-i\xi_j}{|\xi|^2}, \quad j = 1,2,3.$$

Notice that $\mathrm{div}\mathcal{A}[v] = v$ and $\Delta\mathcal{A}_i = \partial_i$, recall the Riesz operators $\mathcal{R}_{ij} = \partial_{x_i}\mathcal{A}_j$, and the Fourier symbols $\mathcal{R}_{ij}(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}$. The classical Mikhlin multiplier theorem yields (see [21])

$$\begin{cases} \|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)} \le C(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & 1 < s < \infty, \\ \|\mathcal{A}_i[v]\|_{L^q(\Omega)} \le C\|\mathcal{A}_i[v]\|_{W^{1,s}(\Omega)} \le C(s,q,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & \frac{1}{q} \ge \frac{1}{s} - \frac{1}{3}, \\ \|\mathcal{A}_i[v]\|_{L^q(\Omega)} \le C(s,\Omega)\|v\|_{L^s(\mathbb{R}^3)}, & s > 3. \end{cases} \tag{3.12}$$

Taking the test functions of the form

$$\varphi^i(t,x) = \psi(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)], \quad i = 1,2,3$$

where

$$\psi \in \mathscr{D}(-1,2), \quad 0 \le \psi \le 1, \quad \psi|_{(0,1)} = 1, \quad |\psi'| \le 2.$$

Since $b(\varrho)$ satisfies the renormalized equation,

$$b(\varrho)_t + \mathrm{div}\,(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\mathrm{div}\,\mathbf{u} = 0,$$

we observe,

$$\phi b(\varrho)_t = \phi(b'(\varrho)\varrho - b(\varrho))\mathrm{div}\,\mathbf{u} - \mathrm{div}\,(\phi b(\varrho)\mathbf{u}) + b(\varrho)\mathbf{u}\cdot\nabla\phi,$$

it follows that

$$\begin{aligned} \varphi^i_t &= \psi'(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)] + \psi(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)_t] \\ &= \psi'(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)] \\ &\quad + \psi(t-\tau)\phi(x)\mathcal{A}_i[\phi(b'(\varrho)\varrho - b(\varrho))\mathrm{div}\,\mathbf{u} \\ &\quad - \mathrm{div}\,(\phi b(\varrho)\mathbf{u}) + b(\varrho)\mathbf{u}\cdot\nabla\phi], \quad i = 1,2,3. \end{aligned}$$

Also, we can get,

$$\partial_j\varphi_i = \psi(t-\tau)\left(\phi\partial_j\mathcal{A}_i[\phi b(\varrho)] + (\partial_j\phi)\mathcal{A}_i[\phi b(\varrho)]\right), \quad i,j = 1,2,3.$$

Especially, since

$$\sum_{i=1}^{3}\phi\partial_i\mathcal{A}_i[\phi b(\varrho)] = \sum_{i=1}^{3}\phi^2 b(\varrho),$$

we have

$$\sum_{i=1}^{3}\partial_i\varphi_i = \psi(t-\tau)\left(\phi^2 b(\varrho) + \sum_{i=1}^{3}(\partial_j\phi)\mathcal{A}_i[\phi b(\varrho)]\right). \tag{3.13}$$

Next, taking φ^i as test function for (1.2), integrating by parts, we get

$$\int_\Omega p(\varrho)\partial_i\varphi^i\,dx = \int_\Omega \Big(- \varrho u_i\varphi^i_t - \varrho u_i\mathbf{u}\cdot\nabla\varphi^i + (\mathbb{S}(\mathbf{u}) : \mathbb{D}^d\varphi)_i \\ + \eta(\mathrm{div}\,\mathbf{u})\mathrm{div}\,\mathbf{u}\,\partial_i\varphi^i - \varrho\partial_i F\varphi^i\Big)\,dx. \tag{3.14}$$

From (3.13), we know

$$\sum_{i=1}^{3}p(\varrho)\partial_i\varphi^i = p(\varrho)\psi(t-\tau)b(\varrho)\phi^2 + \sum_{i=1}^{3}p(\varrho)\psi(t-\tau)(\partial_i\phi)\mathcal{A}_i[\phi b(\varrho)]. \tag{3.15}$$

We observe that (3.14) and (3.15) lead to

$$\begin{aligned}
&\int_\tau^{\tau+1}\int_\Omega p(\varrho)b(\varrho)\phi^2\,dxdt\\
&\quad\le \int_{\tau-1}^{\tau+2}\int_\Omega p(\varrho)b(\varrho)\phi^2\,dxdt\\
&\quad\le \sum_{i=1}^3\int_{\tau-1}^{\tau+2}\int_\Omega\Big(-\varrho u_i\varphi_t^i-\varrho u_i\mathbf{u}\cdot\nabla\varphi^i+\mathbb{S}_i(\mathbf{u})\cdot\nabla\varphi^i\\
&\qquad -p(\varrho)\psi(t-\tau)(\partial_i\phi)\mathcal{A}_i[\phi b(\varrho)]+\eta(\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u}\,\partial_i\varphi^i-\varrho\partial_i F\varphi^i\Big)\,dxdt.
\end{aligned} \tag{3.16}$$

To estimate the first term on the right-hand side of (3.16), we compute

$$\begin{aligned}
&\int_{\tau-1}^{\tau+2}\int_\Omega \varrho u_i\varphi_t^i\,dxdt\\
&\quad=\int_{\tau-1}^{\tau+2}\int_\Omega\Big(\varrho u_i\psi'(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)]\\
&\qquad+\varrho u_i\psi(t-\tau)\phi(x)\mathcal{A}_i[(\phi b(\varrho)-b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\\
&\qquad+b(\varrho)\mathbf{u}\cdot\nabla\phi-\operatorname{div}(\phi b(\varrho)\mathbf{u})]\Big)\,dxdt.
\end{aligned} \tag{3.17}$$

In the next, we will consider some estimates. From the conditions, we find that there is a constant c satisfying

$$b(z)\le cz^\theta,\quad \text{for } z\ge 0,$$

naturally,

$$b(\varrho)^{\frac{1}{\theta}}\le c\varrho,\quad \text{for } \varrho>0.$$

By virtue of Proposition 3.1,

$$\|\phi b(\varrho)\|_{L^{\frac{1}{\theta}}(\mathbb{R}^3)}\le C\int_\Omega\varrho\,dx=Cm_0.$$

In particular, as b' is bounded, we have

$$\int_\Omega b(\varrho)\,dx\le\sup_{z\in\mathbb{R}}b'(z)\int_\Omega\varrho\,dx=m_0\sup_{z\in\mathbb{R}}b'(z),$$

hence,

$$\operatorname*{ess\,sup}_{t>1}\|\phi b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}\le Y_1(b), \tag{3.18}$$

with the help of the classical Miklin multiplier theorem (3.12), we get

$$\begin{aligned}
\operatorname*{ess\,sup}_{t>1}\|\partial_j\mathcal{A}_i\phi b(\varrho)\|_{L^{r_1}(\mathbb{R}^3)}&\le C(r_1)\|\phi b(\varrho)\|_{L^{r_1}(\mathbb{R}^3)}\\
&\le\operatorname*{ess\,sup}_{t>1}\|\phi b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}\\
&\le Y_2(r_1,b),\quad r_1\in\left(1,\frac{1}{\theta}\right),
\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}\int_1^\infty \|\mathcal{A}_i[\operatorname{div}(\phi b(\varrho)\mathbf{u})]\|_{L^{r_2}(\mathbb{R}^3)}^2\,dt &= \int_1^\infty \|\phi b(\varrho)\mathbf{u}\|_{L^{r_2}(\mathbb{R}^3)}^2\,dt \\ &\le \int_1^\infty \|\phi b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}^2 \|\mathbf{u}\|_{L^r(\Omega)}^2\,dt \\ &\le \operatorname*{ess\,sup}_{t>1} \|\phi b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}^2 \int_1^\infty \|\nabla\mathbf{u}\|_{L^r(\Omega)}^2\,dt,\end{aligned} \tag{3.20}$$

where

$$\frac{1}{r_2} = \frac{1}{r} + \theta.$$

As $0 < \theta \le \frac{1}{4}$, the relation $(3.12)_2$ and (3.18) yield:

$$\begin{aligned}\operatorname*{ess\,sup}_{t>1} \|\mathcal{A}_i[\phi b(\varrho)(t)]\|_{L^{r_3}(\mathbb{R}^3)} &\le C(r_3)\operatorname*{ess\,sup}_{t>1} \|\phi b(\varrho)(t)\|_{L^{p_1}(\mathbb{R}^3)} \\ &\le \operatorname*{ess\,sup}_{t>1} \|\phi b(\varrho)(t)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)} \\ &\le Y_4(r_3,b), \quad \text{for any } r_3 > \frac{3}{2}.\end{aligned} \tag{3.21}$$

Further, using the classical Sobolev embedding theorem, from (3.19) and (3.21), we get

$$\operatorname*{ess\,sup}_{t>1} \|\mathcal{A}_i[\phi b(\varrho)(t)]\|_{L^\infty(\mathbb{R}^3)} \le Y_5(b), \quad i = 1,2,3. \tag{3.22}$$

In the same way,

$$\begin{aligned}\int_1^\infty \|\mathcal{A}_i[b(\varrho)\mathbf{u}\cdot\nabla\phi]\|_{L^{r_4}(\mathbb{R}^3)}^2\,dt &\le C(r_4)\int_1^\infty \|b(\varrho)\mathbf{u}\cdot\nabla\phi\|_{L^{p_2}(\mathbb{R}^3)}^2\,dt \\ &\le C(r_4)M^2\int_1^\infty \|b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}^2 \|\mathbf{u}\|_{L^r(\Omega)}^2\,dt \\ &\le C(r_4)M^2\operatorname*{ess\,sup}_{t>1} \|b(\varrho)\|_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}^2 \\ &\quad \int_1^\infty \|\nabla\mathbf{u}\|_{L^r(\Omega)}^2\,dt \\ &\le Y_6(r_4,b),\end{aligned}$$

with

$$\frac{3}{2} < r_4 < \infty \quad \text{as} \quad 0 < \theta \le \frac{1}{3} - \frac{1}{r}.$$

Similarly as before, by virtue of

$$b'(\varrho)\varrho \le C\varrho^\theta,$$

using Proposition 3.1, we observe

$$\|b'(\varrho)\varrho\|_{L^{\frac{1}{\theta}}(\Omega)} \le Cm_0,$$

and

$$\int_\Omega b'(\varrho)\varrho\,dx \le \sup_{z\in\mathbb{R}} b'(z)\int_\Omega \varrho\,dx = m_0\sup_{z\in\mathbb{R}} b'(z),$$

analogously, we get

$$\operatorname*{ess\,sup}_{t>1} \|b'(\varrho)\varrho\|_{L^1\cap L^{\frac{1}{\theta}}(\Omega)} \le Y_7(b).$$

Observe from (3.8), with easy computation,

$$\begin{aligned}
&\int_1^\infty \|\mathcal{A}_i[\operatorname{div}\mathbf{u}\,\phi(b(\varrho)-b'(\varrho)\varrho)]\|^2_{L^{r_5}(\mathbb{R}^3)}\,dt\\
&\quad\le \int_1^\infty \|\operatorname{div}\mathbf{u}\,\phi(b(\varrho)-b'(\varrho)\varrho)\|^2_{L^{p_2}(\mathbb{R}^3)}\,dt\\
&\quad\le \operatorname*{ess\,sup}_{t>1}\left(\|b(\varrho)\phi\|^2_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}+\|b'(\varrho)\varrho\phi\|^2_{L^1\cap L^{\frac{1}{\theta}}(\mathbb{R}^3)}\right)\int_1^\infty \|\operatorname{div}\mathbf{u}\|^2_{L^2(\Omega)}\,dt\\
&\quad\le Y_8(r_5,b),
\end{aligned}$$

where

$$\frac{3}{2} < r_5 \le \frac{6}{6\theta+1}.$$

Combining (3.9) and (3.22) with Hölder inequality, we deduce

$$\begin{aligned}
&\int_{\tau-1}^{\tau+2}\int_\Omega |\varrho u_i\psi'(t-\tau)\phi\mathcal{A}_i[\phi b(\varrho)]|\ dxdt\\
&\quad\le M\operatorname*{ess\,sup}_{t>1}\|\mathcal{A}_i[\phi b(\varrho)]\|_{L^\infty(\mathbb{R}^3)}\int_{\tau-1}^{\tau+2}|\varrho u_i|\,dxdt\\
&\quad\le MY_5(b)\int_{\tau-1}^{\tau+2}\left(\int_\Omega \varrho\,dx\right)^{\frac12}\left(\int_\Omega \varrho|u_i|^2\,dx\right)^{\frac12}\,dt\\
&\quad\le MY_5(b)\int_{\tau-1}^{\tau+2}\|\sqrt{\varrho}u_i\|^{\frac12}_{L^2(\Omega)}\,dt\to 0,\quad \text{as}\quad \tau\to\infty.
\end{aligned}$$

Also, using Hölder inequality, we get

$$\begin{aligned}
&\int_{\tau-1}^{\tau+2}\int_\Omega |\varrho u_i\psi(t-\tau)\phi\mathcal{A}_i[(\phi b(\varrho)-b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\\
&\qquad + b(\varrho)\mathbf{u}\cdot\nabla\phi-\operatorname{div}(\phi b(\varrho)\mathbf{u})]|\,dxdt\\
&\quad\le \|\varrho\|_{L^\infty((0,\infty)\times\Omega)}\int_{\tau-1}^{\tau+2}\|\mathbf{u}\|_{L^r(\Omega)}\|\mathcal{A}_i[(\phi b(\varrho)-b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\\
&\qquad + b(\varrho)\mathbf{u}\cdot\nabla\phi-\operatorname{div}(\phi b(\varrho)\mathbf{u})]\|_{L^{1-\frac{1}{r}}(\mathbb{R}^3)}\,dt\\
&\quad\le \|\varrho\|_{L^\infty(Q)}\big(\int_{\tau-1}^{\tau+2}\|\nabla\mathbf{u}\|^2_{L^r(\Omega)}\,dt\big)^{\frac12}\\
&\qquad\times\big(\int_{\tau-1}^{\tau+2}\|\mathcal{A}_i[\cdots]\|\,dt\big)^{\frac12}_{L^{r_2}\cap L^{r_4}\cap L^{r_5}(\mathbb{R}^3)}\to 0,\quad\text{as}\quad \tau\to\infty.
\end{aligned}$$

Similarly, we have that

$$\int_{\tau-1}^{\tau+2}\int_\Omega\Big(-\varrho u_i\mathbf{u}\cdot\nabla\varphi^i-(\mathbb{S}(\mathbf{u}):\mathbb{D}^d\varphi)_i+\eta(\operatorname{div}\mathbf{u})\operatorname{div}\mathbf{u}\,\partial_i\varphi^i\Big)\,dxdt\to 0,\quad\text{as}\quad \tau\to\infty.$$

In the end, combining (3.7) and (3.22), we obtain

$$\begin{aligned}\int_{\tau-1}^{\tau+2}\int_\Omega |\partial_i F\varrho\varphi^i|\,dxdt &= \int_{\tau-1}^{\tau+2}\int_\Omega |\partial_i F\varrho\psi(t-\tau)\phi(x)\mathcal{A}_i[\phi b(\varrho)]|\,dxdt\\ &\le \operatorname*{ess\,sup}_{t>1}\|A_i[\phi b(\varrho)\|_{L^\infty(\mathbb{R}^3)}\int_{\tau-1}^{\tau+2}\int_\Omega |\partial_i F\varrho|\,dxdt\\ &\le Y_5(b)\operatorname*{ess\,sup}_{t>1}|\nabla F|\int_{\tau-1}^{\tau+2}\int_\Omega \varrho\,dxdt\\ &= m_0 Y_5(b)\operatorname*{ess\,sup}_{t>1}|\nabla F|,\end{aligned}$$

in the same way, we get

$$\begin{aligned}&\int_{\tau-1}^{\tau+2}\int_\Omega |p(\varrho)\psi(t-\tau)(\partial_i\phi)\mathcal{A}_i[\phi b(\varrho)]|\,dxdt\\ &\quad\le \operatorname*{ess\,sup}_{t>1}\|A_i[\phi b(\varrho)\|_{L^\infty(\mathbb{R}^3)}\int_{\tau-1}^{\tau+2}\int_\Omega |p(\varrho)\nabla\phi|\,dxdt\\ &\quad\le Y_5(b)\int_{\tau-1}^{\tau+2}\int_\Omega p(\varrho)|\nabla\phi|\,dxdt.\end{aligned}$$

Hence, from above, we get (3.11). □

Consider a sequence $\tau_n\to\infty$ and denote

$$\varrho_n(t,x)\overset{\text{def}}{=\!=}\varrho(t+\tau_n,x),\quad t\in(0,1),\quad x\in\Omega.$$

Proposition 3.2. *Under the hypotheses of Theorem 1.1, any sequence $\tau_n\to\infty$, contains a subsequence such that*

$$\varrho_n(t,x)=\varrho(t+\tau_n)\to\varrho_s\quad \text{in } L^q((0,1)\times\Omega),\quad q\in[1,\infty) \tag{3.23}$$

where ϱ_s is a solution of the stationary problem in $\mathscr{D}'(\Omega)$, moreover,

$$\int_\Omega \varrho_s\,dx\le\int_\Omega\varrho(t)\,dx=m_0.$$

Proof. From the definition of weak solution, we know $\varrho,\mathbf{u}$ satisfy

$$\begin{aligned}&\varrho\in C([0,T];L^1(\Omega))\cap L^\infty((0,T)\times\Omega),\quad \varrho(0)=\varrho_0,\\ &\sqrt{\varrho}\mathbf{u}\in L^\infty([0,T];(L^2(\Omega))^3),\ \mathbf{u}\in L^r([0,T];(W_0^{1,r}(\Omega))^3).\end{aligned}$$

Hence, choosing a subsequence if necessary, we can obtain that (see [22])

$$\begin{aligned}&\varrho_n(t,x)\to\varrho_s\quad \text{in } C((0,1);L^q_{weak}\Omega),\quad q\in[1,\infty),\\ &\mathbf{u}_n(t,x)\rightharpoonup\mathbf{u}_s\quad \text{weakly in } L^r([0,1];(W_0^{1,r}(\Omega))^3).\end{aligned}$$

Furthermore,

$$\int_\Omega\varrho_s\,dx\le\liminf_{\tau_n\to\infty}\int_\Omega\varrho_n(t)\,dx=m_0.$$

Since, $\varrho,\mathbf{u}$ are solutions to (1.1) in the sense of normalized solution, in particular

$$\varrho_t+\operatorname{div}(\varrho\mathbf{u})=0\quad\text{in}\quad\mathscr{D}'((0,T)\times\Omega). \tag{3.24}$$

Taking test function $\varphi(x,t)=\eta(t)\phi(x)$ in (3.24), where

$$\eta(t)\in\mathscr{D}(0,1),\quad \phi\in\mathscr{D}(\Omega),$$

integrating by parts

$$\int_0^1\left(\int_\Omega\varrho_n\phi\,dx\right)\eta'(t)\,dt+\int_0^1\int_\Omega\varrho_n\mathbf{u}_n\nabla\phi\eta\,dxdt=0.$$

Then, by Lemma 3.1, we get

$$\begin{aligned}\lim_{\tau_n\to\infty}\int_0^1\left(\int_\Omega\varrho_n\phi\,dx\right)\eta'(t)\,dx&=-\lim_{\tau_n\to\infty}\int_0^1\int_\Omega\varrho_n\mathbf{u}_n\nabla\phi\eta\,dxdt\\&\le C\lim_{\tau_n\to\infty}\int_0^1\left(\int_\Omega\varrho_n\,dx\right)^{\frac{1}{2}}\\&\quad\left(\int_\Omega\varrho_n|\mathbf{u}_n|^2\,dx\right)^{\frac{1}{2}}\,dt\\&\le Cm_0\lim_{\tau_n\to\infty}\int_0^1\|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^2(\Omega)}\,dt\to0,\end{aligned}$$

that is

$$\int_0^1\left(\int_\Omega\varrho_s\phi\,dx\right)\eta'(t)\,dt=0.$$

Since the arbitrariness of η, we have, ϱ_s must be independent of t.

By the definition of the pressure $p=p(\varrho)$ and (1.12), passing to the limit in (1.2) and using Lemma 3.1 again, we have

$$\nabla\overline{p}=\nabla F\varrho_s\quad\text{in}\quad\mathscr{D}'(\Omega),$$

where the symbol $\overline{p(\varrho)}$ denotes a weak limit of the sequence $p(\varrho_n)$.

On the other hand, repeat the procedure in [1], we have

$$\int_0^1\int_\Omega\overline{p(\varrho)}\mathrm{div}\,\mathbf{u}_s-\overline{p(\varrho)\mathrm{div}\,\mathbf{u}}\,dxdt\le0. \tag{3.25}$$

Taking $P(\varrho)=\psi(\varrho)\varrho$, from (1.12), we know that, $P(\varrho)$ is a strictly convex function. Making use of the renormalized Eq. (2.2), one has the form

$$\partial_t[P(\varrho_s)]+\mathrm{div}_x[P(\varrho_s)\mathbf{u}_s]+p(\varrho_s)\mathrm{div}_x\mathbf{u}=0.$$

Noting that, we also have

$$\partial_t[\overline{P(\varrho)}]+\mathrm{div}_x[\overline{P(\varrho)}\mathbf{u}_s]+\overline{p(\varrho)\mathrm{div}_x\mathbf{u}}=0,$$

considering (3.25), we conclude that

$$\begin{aligned}\left[\int_\Omega(\overline{P(\varrho)}-P(\varrho_s)\,dx)\right]\Big|_0^1&=-\int_0^1\int_\Omega(\overline{p(\varrho)\mathrm{div}\,\mathbf{u}}-p(\varrho_s)\mathrm{div}\,\mathbf{u}_s)\,dxdt\\&\le-\int_0^1\int_\Omega(\overline{p(\varrho)}-p(\varrho_s))\mathrm{div}\,\mathbf{u}_s\,dxdt.\end{aligned} \tag{3.26}$$

Using the convexity of P, we have

$$\int_\Omega(\overline{P(\varrho)}-P(\varrho_s))\mathrm{div}\,\mathbf{u}\,dx\ge d\limsup_{\tau_n\to\infty}\int_\Omega|\rho_n-\varrho_s|^2\,dx,\quad\text{for a certain }d>0,$$

while

$$\begin{aligned}&\int_0^1\int_\Omega(\overline{P(\varrho)}-P(\varrho_s))\mathrm{div}\,\mathbf{u}\,dxdt\\&\quad=-\lim_{\tau_n\to\infty}\int_0^1\int_\Omega(p(\varrho_n)-p(\varrho_s))\mathrm{div}\,\mathbf{u}\,dxdt\\&\qquad+C\limsup_{\tau_n\to\infty}\int_0^1\int_\Omega|\varrho_n-\varrho_s|^2\,dxdt.\\&\quad\le-\int_0^1\int_\Omega(\overline{p(\varrho)}-p(\varrho_s))\mathrm{div}\,\mathbf{u}\,dxdt.\end{aligned}\tag{3.27}$$

Combining (3.26) and (3.27), and by virtue of Gronwall's inequality, we observe that

$$\overline{p(\varrho)}=p(\varrho_s),$$

where we have used the fact

$$[\overline{p(\varrho)}-p(\varrho_s)](0,\cdot)=0.$$

In particular,

$$\varrho_n\to\varrho_s\quad\text{in }L^2((0,1)\times\Omega),\quad\overline{p(\varrho)}=p(\varrho_s),$$

this yields the strong convergence in (3.23). □

4. Stationary solutions

In this part, we give the well-know results on the stationary problem (1.8). In the work of da Veiga [17], they proved the necessary and sufficient conditions for the solutions existence of the stationary problem for an arbitrary $F\in L^\infty(\Omega)$. Further, Feireisl and Petzeltová [14] showed the optimal condition in terms of F for the problem to possess a unique nonnegative solution ϱ.

Let p be a continuously differentiable real function defined on $\mathbb{R}^+=\{s\in\mathbb{R}:s>0\}$, such that $p'(s)>0$, $\forall s\in\mathbb{R}^+$. Assume,

$$0<\operatorname*{ess\,inf}_{x\in\Omega}\varrho(x),\quad\operatorname*{ess\,sup}_{x\in\Omega}\varrho(x)<+\infty,\tag{4.1}$$

and, for a fixed $m>0$

$$\frac{1}{|\Omega|}\int_\Omega\varrho(x)\,dx=m.\tag{4.2}$$

Define

$$\pi(s)=\int_m^s\frac{p'(t)}{t}\,dt\quad\forall s\in\mathbb{R}^+,\tag{4.3}$$

we denote by $[a,b]$ the range of π, $\pi(\mathbb{R}^+)=[a,b]$. Since $\pi(m)=0$, one has $-\infty\le a<0<b\le+\infty$. We define $\Phi=\pi^{-1}$, clearly, $\Phi([a,b])=\mathbb{R}^+$, we set $\Phi(a)=0$, $\Phi(b)=+\infty$.

Definition 4.1. Let $F\in L^\infty(\Omega)$, a function ϱ is called an equilibrium solution of (1.8), if $\varrho\in L^\infty(\Omega)$, and if

$$\pi(\varrho(x))=F(x)+C,\quad\text{a.e in }\Omega,$$

and (4.1), (4.2) hold.

We set $m_0 = \operatorname{ess\,inf} F$ in Ω, $M_0 = \operatorname{ess\,sup} F$ in Ω, one has the following result.

Theorem 4.1. *Let $F \in L^\infty$ be given. There exists an equilibrium solution $\varrho(x)$ if and only of there exists a constant*

$$C \in [a - m_0, b - M_0], \tag{4.4}$$

such that

$$\frac{1}{|\Omega|}\int_\Omega \Phi(C + F(x))\,dx = m. \tag{4.5}$$

If such a constant exists then the (unique) equilibrium solution is given by

$$\varrho(x) = \Phi(C + F(x)), \quad \forall x \in \Omega. \tag{4.6}$$

Theorem 4.2. *Under the assumption of Theorem 4.1, there exists an equilibrium solution $\varrho(x)$ if and only if*

$$a - m_0 < b - M_0,$$

and

$$\frac{1}{|\Omega|}\int_\Omega \Phi(a - m_0 + F(x))\,dx < m < \frac{1}{|\Omega|}\int_\Omega \Phi(b - M_0 + F(x))\,dx.$$

In this case the equilibrium solution $\varrho(x)$ is given by (4.6), where C is the (unique) solution of (4.4)–(4.5).

The proof of Theorems 4.1 and 4.2 have been given by da Veiga [17].

5. The proof of Theorem 1.1

In this part, we will prove the main result.

We know that for every sequence $\tau_n \to \infty$, the time shifts $\varrho_n = \varrho(t + \tau_n)$ converges to the stationary state ϱ_s, more accurately,

$$\varrho_n \to \varrho_s \quad \text{strongly in } L^q((0,1)\times\Omega), \quad q \in [1,\infty).$$

Energy inequality implies converge of the energy $E(t)$ for $t \to \infty$ to some finite number

$$E_\infty := \operatorname*{ess\,sup}_{t\to\infty} E(t).$$

Further, Lemma 3.1 shows that

$$\lim_{\tau\to\infty}\int_\tau^{\tau+1}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2\; dxdt = 0.$$

Thus,

$$E_\infty = \lim_{\tau_n\to\infty}\int_{\tau_n}^{\tau_n+1}\int_\Omega (P(\varrho) - \varrho F)\; dxdt = \int_\Omega P(\varrho_s) - F\varrho_s\,dx = E[\varrho_s].$$

Moreover, using the continuity Eq. (1.1) and Lemma 3.1, one has easily observe that

$$\varrho(t) \to \varrho_s \text{ weakly in } L^q(\Omega) \text{ as } t \to \infty, \quad q \in [1,\infty).$$

Then, we have

$$\begin{aligned} E_\infty &= \int_\Omega \left(P(\varrho_s) - F\varrho_s\right)\, dx \\ &\le \liminf_{t\to\infty} \int_\Omega \left(P(\varrho(t)) - F\varrho(t)\right)\, dx \\ &\le \limsup_{t\to\infty} \int_\Omega \left(P(\varrho(t)) - F\varrho(t)\right)\, dx \\ &\le \operatorname{ess}\limsup_{t\to\infty} \int_\Omega \left(\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + P(\varrho(t)) - F\varrho(t)\right)\, dx \\ &= \operatorname{ess}\lim_{t\to\infty} E(t) = E_\infty. \end{aligned}$$

Hence,

$$\begin{aligned} &\operatorname{ess}\sup_{\tau>t} \int_\Omega \varrho(\tau)|\mathbf{u}(\tau)|^2\, dx \to 0, \\ &\varrho(t) \to \varrho_s \quad \text{strongly in } L^q(\Omega), \quad q \in [1,\infty). \end{aligned}$$

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