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Decay rates for the quadratic and super-quadratic tilt-excess of integral varifolds

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Abstract. This paper concerns integral varifolds of arbitrary dimension in an open subset of Euclidean space satisfying integrability conditions on their first variation. Firstly, the study of pointwise power decay rates almost everywhere of the quadratic tilt-excess is completed by establishing the precise decay rate for two-dimensional integral varifolds of locally bounded first variation. In order to obtain the exact decay rate, a coercive estimate involving a height-excess quantity measured in Orlicz spaces is established. Moreover, counter-examples to pointwise power decay rates almost everywhere of the super-quadratic tilt-excess are obtained. These examples are optimal in terms of the dimension of the varifold and the exponent of the integrability condition in most cases, for example if the varifold is not two-dimensional. These examples also demonstrate that within the scale of Lebesgue spaces no local higher integrability of the second fundamental form, of an at least two-dimensional curvature varifold, may be deduced from boundedness of its generalised mean curvature vector. Amongst the tools are Cartesian products of curvature varifolds.

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Keywords. Integral varifold, First variation, Generalised mean curvature vector, Quadratic tilt-excess, Super-quadratic tilt-excess, Orlicz space height-excess, Curvature varifold, Second fundamental form, Cartesian product of varifolds.

Introduction

Overview

Integral varifolds constitute an analytically tractable model for singular geometric objects which admit appropriate notions of tangent plane and mean curvature vector, see Almgren [5], Allard [4], and Simon [32]. Due to good

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compactness properties they also arise naturally as weak limits of smooth submanifolds of some ambient space and may be used to represent solutions to geometrical variational problems. Our principal objective is the study of regularity properties entailed by integrability conditions on the first variation of such varifolds by means of decay rates of tilt-excess. For this purpose the classical quadratic tilt-excess and the super-quadratic tilt-excess which arises in the study of area minimising integral currents are employed. The three main results of this study may be informally described as follows.

Firstly, for two-dimensional integral varifolds of locally bounded first variation (i.e. $m = 2$ and $p = 1$ in the general hypotheses below), the optimal decay rate almost everywhere of the quadratic tilt-excess is established, see Theorems A and B for the decay rate and its sharpness respectively. For all other values of (m, p) the best decay rate amongst *powers* was determined in [22]. The present result not only fills this gap but in fact sharply exhibits the best decay rate amongst *all* rates, not just powers. To obtain this precision, a coercive estimate for the quadratic tilt-excess is derived which involves a height-excess quantity measured in an Orlicz space occurring naturally in sharp embeddings of Sobolev spaces. This seems to be the first time that a regularity estimate for varifolds relies on Orlicz spaces; in fact, the only previous usage of Orlicz spaces in the context of varifolds appears to be the Poincaré type embedding results of Hutchinson, see [16, Theorems 2 and 4].

Secondly, for at least two-dimensional integral varifolds with locally bounded mean curvature and "no boundary" ($p = \infty$), negative results concerning decay rates almost everywhere of super-quadratic tilt-excess are shown in Theorem C. The importance of these examples stems from the fact that they imply that—even just almost everywhere and in co-dimension one—there is no analogue in the present situation of the "main analytic estimate" in Almgren's proof of interior almost everywhere regularity of area minimising integral currents in arbitrary codimension, see [6] and [7, § 3]. (Of course, Almgren in fact proved a stronger Hausdorff dimension estimate on the interior singular set.) This provides a serious obstacle to an, otherwise canonical, approach to the question of possible almost everywhere regularity of stationary integral varifolds which is a key open problem in varifold theory.

Thirdly, for one-dimensional integral varifolds of locally bounded first variation ($p = 1$), almost everywhere differentiability of the tangent plane map (restricted to the set of points of approximate continuity) is proven, see Theorem D. This implies in particular that such varifolds near almost every point are representable as graphs of a finite number of Lipschitzian functions with small constant. These results as well as the estimates involved in deriving them should prove useful in the study which parts of the structural description of one-dimensional stationary varifolds with a uniform lower bound on their density, see Allard and Almgren in [1], generalise to locally bounded first variation.

In combination with previous results the preceding three main results in particular yield a nearly complete picture concerning power decay rates almost everywhere of quadratic and super-quadratic tilt-excess.

Known results

In order to more formally describe the results of the present paper in the context of known results, consider the following set of hypotheses; the notation is explained in Sect. 1. Additionally, the terms "[twice] weakly differentiable" are employed with their usual meaning, see e.g. [21, pp. 9–10].

General hypotheses. *Suppose m and n are positive integers, $m < n$, $1 \leq p \leq \infty$, U is an open subset of $\mathbf{R}^n$, V is an m dimensional integral varifold in U whose first variation δV is representable by integration,*[1] *the generalised mean curvature vector $\mathbf{h}(V, \cdot)$ of V belongs to $\mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$, and if $p > 1$ then*

$$(\delta V)(\theta) = -\int \mathbf{h}(V, z) \bullet \theta(z) \, \mathrm{d}\|V\| z \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Let $Z = U \cap \{z : \operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)\}$ and define $\tau : Z \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by $\tau(z) = \operatorname{Tan}^m(\|V\|, z)_\natural$ for $z \in Z$.

The set Z consists of all points such that the closed cone of $(\|V\|, m)$ approximate tangent vectors forms an m dimensional plane; for these points z, $\tau(z)$ denotes the orthogonal projection of $\mathbf{R}^n$ onto this plane.

The study of regularity properties is usually preceded by a more basic study of the density ratio by means of the monotonicity identity, see Allard [4, § 5], Simon [32, § 17] and [23, § 4], and its consequence, the isoperimetric inequality, see Allard [4, § 7] and Michael and Simon [26, § 2]. In particular, if $p > m$ then $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ is an upper semicontinuous real valued function whose domain is U, see Simon [32, 17.8]. To which extent these properties persist if $p = m$ is unclear; the cases $m = 1$ and $m = 2$ are treated in [23, 4.8] and Kuwert and Schätzle [17, Appendix] respectively. From Allard [4, 3.5 (1), 8.3] one is at least assured that $\mathscr{H}^m \llcorner \operatorname{spt} \|V\| \leq \|V\|$ if $p = m$. If $p < m$, one easily constructs examples with $\operatorname{spt} \|V\| = U$, see [23, 14.1]. Yet, there are precise local estimates available on the size of the set of points where the density ratio is small, see [19, § 2].

In order to put the study of regularity properties into perspective, it is instructive to consider as well the behaviour of the Laplace operator.

Model case. *Suppose m and n are positive integers, $m < n$, $1 \leq p \leq \infty$, $u \in \mathbf{L}_1^{\mathrm{loc}}(\mathscr{L}^m, \mathbf{R}^{n-m})$, the distributional Laplacian $T \in \mathscr{D}'(\mathbf{R}^m, \mathbf{R}^{n-m})$ of u, defined by $T(\phi) = \int u \bullet \operatorname{Lap} \phi \, \mathrm{d}\mathscr{L}^m$ for $\phi \in \mathscr{D}(\mathbf{R}^m, \mathbf{R}^{n-m})$, is representable by integration, and if $p > 1$ then there exists $f \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^{n-m})$ satisfying*

$$T(\phi) = \int f(x) \bullet \phi(x) \, \mathrm{d}\mathscr{L}^m x \quad \text{for } \phi \in \mathscr{D}(U, \mathbf{R}^{n-m}).$$

If $1 < p < \infty$ then u is twice weakly differentiable and the distributional partial derivatives up to second order of u correspond to functions in $\mathbf{L}_p^{\mathrm{loc}}(\mathscr{L}^m, \mathbf{R}^{n-m})$; this is a consequence of the usual a priori estimate based on the Calderón Zygmund inequality, see e.g. [22, 3.5], and convolution.

[1] That is, in the terminology of Simon [32, 39.2], V is of locally bounded first variation.

This implies differentiability results in Lebesgue spaces for the weak derivative, see for instance Calderón and Zygmund [10, Theorem 12, p. 204].

For an integral varifold V, the existence of a notion of first order derivative, i.e., that τ is defined $\|V\|$ almost everywhere, is a simple consequence of its rectifiability, see Allard [4, 3.5 (1)]. However, this derivative behaves rather like an approximate derivative than a weak derivative as is exemplified by the fact that a Poincaré inequality only holds under additional hypotheses on the first variation, see [20, p. 372]. More information on the validity of Sobolev Poincaré type inequalities may be retrieved from Hutchinson [16], [20, § 4] and [23, § 10].

As the Grassmann manifold is compact, the map τ belongs to $\mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, \mathbf{R}^n))$ for $1 \le q \le \infty$. Yet, it is important to understand for which q effective coercive estimates are available. Classically, this is the case for $q = 2$, see Allard [4, 8.13] and its refinements Brakke [8, 5.5] and [21, 4.10, 4.14].

Question 1. *Suppose $2 < q < \infty$ and $1 < p < \infty$.*

Do the general hypotheses imply that for $\|V\|$ almost all c there exists $1 \le \gamma < \infty$ such that there holds

$$\left(r^{-m} \int_{\mathbf{B}(c,r/2)} |\tau(z) - \tau(c)|^q \,\mathrm{d}\|V\|z \right)^{1/q} \le \gamma \left(\left(r^{-m} \int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^2 \,\mathrm{d}\|V\|z \right)^{1/2} + \left(r^{p-m} \int_{\mathbf{B}(c,r)} |\mathbf{h}(V,z)|^p \,\mathrm{d}\|V\|z \right)^{1/p} \right)$$

whenever $0 < r \le \gamma^{-1}$? A similar question may be phrased for $q = \infty$ or $p \in \{1, \infty\}$.

In case $m \ge 2$ and $q = \infty$ such estimates are known to fail by Brakke [8, 6.1]. In view of 10.4 the Question 1 is related to the possible size of the exceptional sets occurring in approximations by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions. It is also related to the second “main analytic estimate”—so termed in Almgren’s announcement [6, p. 6]—of Almgren’s regularity proof for area minimising currents of arbitrary codimension, see Almgren [7, 3.29, 3.30] and De Lellis and Spadaro [11, Theorem 7.1].

Passing from first order to second order quantities, the analogous property to weak differentiability of the weak derivative of u would be generalised V weak differentiability of τ, or equivalently, V being a curvature varifold in the sense of Hutchinson, see 1 and 3.3. Considering three half lines emanating from the origin in $\mathbf{R}^2$ at equal angles shows that even a stationary integral varifold need not to be a curvature varifold, see Mantegazza [18, 3.4, 3.11]. In view of [22, 4.8], one may however still define a notion of approximate second fundamental form, $\mathrm{ap}\,\mathbf{b}(V, \cdot)$, of a varifold satisfying the general hypotheses such that

$$\operatorname{ap}\mathbf{b}(V,z) = \mathbf{b}(M,z) \quad \text{for } \|V\| \text{ almost all } z \in U \cap M$$

whenever M is an m dimensional submanifold of $\mathbf{R}^n$ of class 2. Since the corresponding approximate mean curvature vector is $\|V\|$ almost equal to $\mathbf{h}(V,\cdot)$ by [22, 4.8], we assume $m \geq 2$ in the following question.

Question 2. *Suppose $m \geq 2$, $p = \infty$, and $0 < q < \infty$.*
Do the general hypotheses imply that for $\|V\|$ almost all c

$$\int_{\mathbf{B}(c,r)} \|\operatorname{ap}\mathbf{b}(V,z)\|^q \, \mathrm{d}\|V\|z < \infty \quad \textit{for some } r > 0?$$

The existence proof in [22, 4.8] does not provide any integral estimates of the resulting quantity and, in fact, no positive results are known. Considering the scaling behaviour of the above integral, the example in Brakke [8, 6.1] shows that the answer is in the negative whenever $q \geq 2$.

In case V happens to be a curvature varifold, one may deduce differentiability results for τ in Lebesgue spaces from general facts about generalised V weakly differentiable functions, see [23, 11.4, 15.9–15.12]. The next question concerns to which extent these properties persist for non-curvature varifolds.

Question 3. *Suppose $0 < \alpha \leq 1$ and $2 \leq q < \infty$.*
Do the general hypotheses imply that

$$\limsup_{r \to 0+} r^{-\alpha} \left(r^{-m} \int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^q \, \mathrm{d}\|V\|z \right)^{1/q} < \infty$$

for $\|V\|$ almost all c? A similar question may be phrased for $q = \infty$.

This may be seen as a pointwise Hölder condition with exponent α on τ at c measured in a Lebesgue space with exponent q. If $p > m$ and $p \geq 2$, proving uniform estimates for

$$r^{-\alpha} \left(r^{-m} \int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^2 \, \mathrm{d}\|V\|z \right)^{1/2}$$

for all c in some relatively open subset of $\operatorname{spt}\|V\|$ and all $0 < r \leq \varepsilon$ for some $\varepsilon > 0$ is—via Hölder continuity of τ—the key to Allard's regularity theorem, see [4, § 8]. If $m \geq 2$ and $p = \infty$, then τ may be discontinuous at points in a set of positive $\|V\|$ measure and $\operatorname{spt}\|V\|$ may fail to be associated to an $\mathbf{R}^{n-m}$ valued or even $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function near those points by Brakke [8, 6.1]. Therefore, the condition in Question 3 also acts as replacement for more classical notions of regularity which are known to possibly fail on a set of positive $\|V\|$ measure.

Evidently, if $m \geq 2$ and $q = \infty$ the answer is in the negative whenever $1 \leq p \leq \infty$ by Brakke [8, 6.1]. If $p < m$, the answer is in the negative whenever $\alpha q > mp/(m-p)$, see [19, 1.2 (iv)].

Turning to positive results, only the case $q = 2$ has been systematically studied. The sharpest known results were obtained in [22] building partly on [21] and extending methods and results of Brakke [8, 5.5, 5.7] and Schätzle

[29, 5.4], [30, Theorem 5.1], and [31, Theorem 3.1]. Namely, if $m = 1$ or $m = 2$ and $p > 1$ or $m > 2$ and $p \geq 2m/(m+2)$, then the answer is in the affirmative for $\alpha = 1$ and $q = 2$, see [22, 5.2 (2)], and if $m = 2$ and $\alpha < 1$ or $\sup\{2, p\} < m$ and $\alpha = mp/(2(m-p)) < 1$ then the answer is in the affirmative for $q = 2$, see [22, 5.2 (1)].

Therefore, the previously known results for Question 3 may be summarised as follows. If $q = 2$, only the case $(m, p, \alpha) = (2, 1, 1)$ was left open. If $2 < q < \infty$, the gap between positive results and known counterexamples was quite large. And if $q = \infty$, only the case $m = 1$ remained open. The initial motivation for the present work was to solve some of these open cases of Question 3.

Finally, notice that Question 3 could be phrased for $1 \leq q < 2$ as well, see [19, p. 248, Problem (ii)], and Question 2 could include the case $p < \infty$. However, no effort has been made to resolve these additional cases in the present study.

Results of the present article

The results may be summarised as follows. All cases of Question 1 are answered; in the negative if $m \geq 2$ and in the affirmative if $m = 1$, see 11.6 and 12.2. Question 2 is answered in the negative if $q > 1$, see 10.5. All cases of Question 3 except the case $(m, p, \alpha) = (2, 1, 2/q)$ for $2 < q < \infty$ are treated, see 11.1, 11.3–11.5, and 12.4. Finally, for the case $(m, p, q) = (2, 1, 2)$ of Question 3, the precise decay rate is determined, see 5.8 and 9.2.

Beginning with the last item, the following theorem is established.

Theorem A. (see 9.2) *Suppose $m = 2$ and $p = 1$.*

If n, U, V, and τ satisfy the conditions of the general hypotheses, then

$$\lim_{r \to 0+} r^{-4} (\log(1/r))^{-1} \int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^2 \, \mathrm{d}\|V\| z = 0$$

for $\|V\|$ almost all c.

This result is sharply complemented by the following negative result.

Theorem B. (see 5.8) *Suppose $m = 2$, $n = 3$, $p = 1$, $U = \mathbf{R}^3$, and ω is a modulus of continuity.*

Then there exist V and τ satisfying the conditions of the general hypotheses and C with $\|V\|(C) > 0$ satisfying

$$\limsup_{r \to 0+} \omega(r)^{-1} r^{-4} (\log(1/r))^{-1} \int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^2 \, \mathrm{d}\|V\| z > 0$$

whenever $c \in C$.

In fact, the varifold constructed in Theorem B may be chosen to be associated to the graph of a Lipschitzian function with small Lipschitz constant. Theorem B in particular answers the case $(m, p, \alpha) = (2, 1, 1)$ of the part $q = 2$ of Question 3 in the negative.

Theorem C. (see 10.8 and 10.10) *Suppose $m \geq 2$, $n = m+1$, $p = \infty$, $U = \mathbf{R}^n$, and ω is a modulus of continuity.*

Then there exist a curvature varifold $V \in \mathbf{IV}_m(\mathbf{R}^n)$ *and* τ *satisfying the conditions of the general hypotheses, an* m *dimensional submanifold* M *of* $\mathbf{R}^n$ *of class* ∞ *which is relatively open in* $\operatorname{spt}\|V\|$, $\varepsilon > 0$, $B \subset \{r : r > 0\}$ *with* $\inf B = 0$, *and* C *with* $\|V\|(C) > 0$ *such that the following properties hold whenever* $c \in C$.

(1) *If* $r \in B$, *then*
$$\|V\|(\mathbf{B}(c,r) \cap \{z : \|\tau(z) - \tau(c)\| \geq 1/3\}) \geq \omega(r) r^{m+2} (\log(1/r))^{-2}.$$

(2) *If* $r \in B$ *and* $T = \operatorname{im} \tau(c)$, *then*
$$\mathscr{H}^m(H(T,c,r)) \geq \omega(r) r^{m+2} (\log(1/r))^{-2},$$
where $H(T,c,r) = T \cap \mathbf{B}(T_\natural(c), r) \sim T_\natural[\mathbf{C}(T,c,r,r) \cap \operatorname{spt}\|V\|]$.

(3) *If* $r > 0$ *and* $1 < q < \infty$, *then*
$$\textstyle\int_{M \cap \mathbf{B}(c,r)} |\mathbf{b}(M,z)|^q \, \mathrm{d}\mathscr{H}^m z = \infty.$$

(4) *If* $0 < r \leq \varepsilon$, *then*
$$\|V\|(\mathbf{B}(c,r) \cap \{z : \mathbf{\Theta}^m(\|V\|, z) \leq \mathbf{\Theta}^m(\|V\|, c) - 1\}) \geq \omega(r) r^m.$$

If ω satisfies the Dini condition and "$\omega(r)$" is replaced by "$\omega(r)^2$" in parts (1) and (2), then one may take $B = \{r : 0 < r \leq \varepsilon\}$, see 10.3 and 10.5.

Part (1) answers Question 3 in the negative whenever $q > 2$ and $\alpha > 2/q$. Notice that the $\|V\|$ measure of sets of the form considered in part (1) occurs as upper bound on the size of the exceptional sets of the usual approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions, 10.4. In part (2) we provide a lower bound on the size of "holes" of the varifold. As these regions will always be part of one of the exceptional sets of any approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions, this yields a corresponding lower bound on the size of this set. For any $\delta > 0$, this lower bound also rules out an upper bound on the size of the exceptional sets by a suitable multiple of
$$E^{1+\delta}, \quad \text{where } E = r^{-m} \textstyle\int_{\mathbf{B}(c,r)} |\tau(z) - \tau(c)|^2 \, \mathrm{d}\|V\| z.$$
Such upper bound would be the natural analogue for varifolds satisfying the general hypotheses with $m \geq 2$ and $p = \infty$ of the aforementioned second "main analytic estimate" for area minimising currents obtained by Almgren in [7, 3.29 (8)]. Of course, our example does not (obviously) preclude a possible bound involving $E(\log(1/E))^{-2}$, for $E > 0$, in place of $E^{1+\delta}$, or the original bound $E^{1+\delta}$ under the additional hypothesis of stationarity.

Evidently, part (3) provides a negative answer to Question 2 whenever $q > 1$; in fact, even if the integral in question is restricted to the "regular part" of V. Part (4) contains the observation that the regions of significantly smaller density around a given point may be as large as the approximate continuity of the density permits, see 10.11.

If $m \geq 2$, part (1) of Theorem C leaves little room for positive answers to Question 3 for $q > 2$ except of those which follow from the boundedness of τ and the known positive results for $q = 2$, see 11.1 and 11.3–11.5. In fact, only the case $(m, p, \alpha) = (2, 1, 2/q)$ for $2 < q < \infty$ remains open. This case is related to the question of availability of estimates of τ in certain Lorentz spaces, see 11.5.

The combination of positive and negative answers to Question 3 obtained so far in particular implies that the answer to Question 1 is in the negative whenever $m \geq 2$, see 11.6.

Turning to the case $m = 1$, a positive answer to Question 1 follows from 12.2.

Theorem D. (see 12.4 and 12.5) *Suppose $m = 1$ and $p = 1$.*

If n, U, V, Z, and τ satisfy the general conditions, then there exists a subset A of Z with $\|V\|(A \sim Z) = 0$ such that the following two statements hold for $\|V\|$ almost all $z \in A$.

(1) *The map $\tau | A$ is differentiable at z relative to A and*

$$\mathrm{D}(\tau|A)(z) = (\|V\|, 1)\,\mathrm{ap}\,\mathrm{D}\,\tau(z).$$

(2) *If $\varepsilon > 0$ then there exist a positive integer Q, $0 < r < \infty$, and $f_i : T \cap \mathbf{B}(T_\natural(z), r) \to T^\perp \cap \mathbf{B}(T_\natural^\perp(z), r)$ with* $\operatorname{Lip} f_i \leq \varepsilon$ *for $i = 1, \ldots, Q$ and*

$$\boldsymbol{\Theta}^1(\|V\|, \zeta) = \operatorname{card}\left\{i : f_i(T_\natural(\zeta)) = T_\natural^\perp(\zeta)\right\}$$

for $\mathscr{H}^1$ almost all $\zeta \in \mathbf{C}(T, z, r, r)$.

Part (1) includes a positive answer to Question 3 for $m = 1$, $\alpha = 1$, and $q = \infty$. Part (2) expresses the varifold as finite sum of graphs of Lipschitzian functions.

Outline of the proofs

Theorem A. In order to explain the proof of Theorem A, it is instructive to recall the strategy of proof of similar results in [22, 5.2]. The proof rests on an idea of Schätzle underlying [30, Theorem 3.1]. He realised that in the presence of a coercive estimate, see Brakke [8, 5.5], second order rectifiability of a varifold satisfying the general hypotheses with $p = 2$ implies second order behaviour of the quadratic tilt-excess and the quadratic height-excess. Having second order rectifiability at one's disposal from [22, 4.8], this procedure has been employed in [22, 5.2 (2)] with a refined coercive estimate. Specifically, Brakke's estimate [8, 5.5] was improved by introducing height-excess quantities measured in $\mathbf{L}_q(\|V\|)$ with $q = \frac{2m}{m-2}$ if $m > 2$, see [21, 4.10, 4.11]. Employing an approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions and interpolation in [21, 5.7, 6.4] yielded a coercive estimate involving approximate height-excess quantities, that is height-excess that excludes an arbitrary set of small $\|V\|$ measure, see [21, 9.5]. Finally, in all cases in which this method implied a positive answer to Question 3 for $q = 2$ and $\alpha < 1$, the differentiation theory from [19, 3.7] was employed in conjunction again with the second order rectifiability to establish that the limit in question is actually equal zero $\|V\|$ almost everywhere.

The results obtained by the above method in case $m = 2$ were not sharp as the coercive estimate [21, 4.10] did only use Lebesgue spaces so that subsequently the non-sharp embedding of weakly differentiable functions with square-integrable weak derivative on $\mathbf{U}(a, r)$, for $a \in \mathbf{R}^2$ and $0 < r < \infty$,

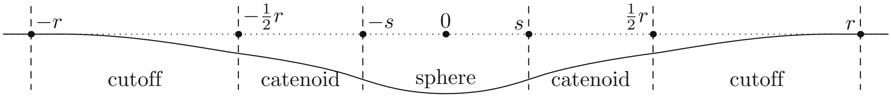

Figure 1. Rotating the solid line around the vertical axis illustrates the support of the varifold constructed in 5.6

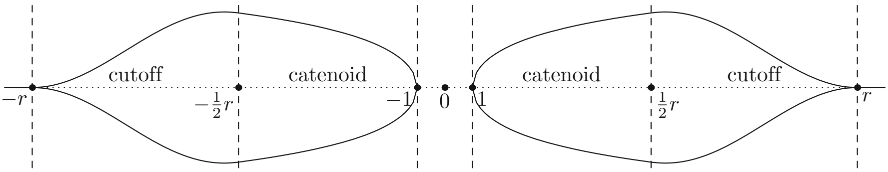

Figure 2. Rotating the solid line around the vertical axis illustrates the support of the varifold constructed in 10.2

into $\mathbf{L}_q(\mathscr{L}^2 \llcorner \mathbf{U}(a,r))$ for $q < \infty$ needed to be employed. To be able to employ the sharp embedding into Orlicz spaces, see [2, 8.27, 8.28], we therefore modify the above procedure. In particular, we obtain a coercive estimate involving the appropriate Orlicz space in 6.6 which takes the role of [21, 4.10]. Moreover, to be able to proceed after obtaining a weaker form of Theorem A which results from replacing "lim" and "$= 0$" by "lim sup" and "$< \infty$" in its statement, the differentiation result [19, 3.7] is adapted in 4.2 to include rates which are not powers. It should also be noted that as the current proof does not aim at decay *estimates*, i.e., estimates for positive radii, such as [21, 10.2] but only at decay *rates*, i.e., the behaviour as the radius approaches zero, we are able to encapsulate the usage of the approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions in the construction of a real valued auxiliary function in 7.7. In comparison to [21, 9.5] where the coercive estimate involving the approximate height-excess quantities was derived as corollary to the more elaborate estimates in [21, 9.4 (9)] aiming at the proof of [21, 10.2], this greatly simplifies our derivation of the corresponding estimates in 9.2.

Theorems B and C. The qualitative construction principle for both theorems is that of Brakke [8, 6.1]. The basis for obtaining quantitative information is provided by the following variant of classical propositions, see 2.5. *If m is a positive integer*, $0 < \lambda < 1$, *and ω is a modulus of continuity, then there exist a countable disjointed family G of cubes contained in the unit cube C of $\mathbf{R}^m$, $B \subset \mathbf{R} \cap \{r : r > 0\}$ with* $\inf B = 0$, *and $\varepsilon > 0$ such that $\mathscr{L}^m$ almost all $a \in C \sim \bigcup G$ have the following property: If $0 < r \leq \varepsilon$, then there exists a subfamily H of G with $\bigcup H \subset \mathbf{B}(a,r)$,*

$$\mathscr{L}^m\left(\bigcup H\right) \geq \omega(r) r^m \quad \text{and} \quad \text{if } r \in B \text{ then } \operatorname{card} H = 1.$$

Moreover, B may be required to equal $\{r : 0 < r \leq \varepsilon\}$ if and only if ω satisfies the Dini condition, see 2.2 and 2.3. Denoting by F the collection of balls with centres and "radii" equal to those of the cubes in G and fixing an isometric injection of $\mathbf{R}^m$ into $\mathbf{R}^{m+1}$, we associate to each member of F an m dimensional submanifold of $\mathbf{R}^{m+1}$ of class 1 involving a piece of a catenoid, see Figs. 1 and 2. The varifolds whose existence is asserted in Theorems B and C then consist of the submanifolds corresponding to the members of F together with the image of $\mathbf{R}^m \sim \bigcup F$ under the injection [taken with multiplicity two in case of Theorem C].

Theorem D. The key to the proof of Theorem D is an a priori estimate for "weak subsolutions to Poisson's equation for the Laplace–Beltrami operator on V", see 12.1 and 12.3. This estimate is adapted from Allard and Almgren, see [1, 5 (6)], and implies in particular a positive answer to Question 1 if $m = 1$ and $q = \infty$, see 12.2. Since a positive answer to Question 3 for $m = 1$ and $q = 2$ is already known from [22, 5.2 (2)], part (1) is now a consequence of suitable differentiation result, see 4.4, which in turn is based on [19, 3.1]. Part (2) then follows by a suitable approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-1})$ valued functions, see [20, 3.15], and a selection theorem for such function, see Almgren [7, 1.10].

Curvature varifolds. Our treatment of curvature varifolds makes use of generalised weakly differentiable functions introduced in [23, 8.3], in particular their differentiability properties, see [23, 11.2, 11.4]. This approach rests on the characterisation of curvature varifolds in terms of such functions, see [23, 15.6]. Otherwise, only some more elementary facts are cited from [23].

An open problem

The cases of Question 3 remaining open are related to the following question concerning the decay behaviour of tilt-excess quantities measures in the Lorentz space with exponent $(2, \infty)$, see 11.5.

Question 4. *Suppose $m = 2$ and $p = 1$.*
Do the general hypotheses imply that

$$\limsup_{r \to 0+} r^{-1} E(c, r) < \infty \quad \textit{for } \|V\| \textit{ almost all } c,$$

where $E(c, r) = \sup\{ t r^{-m/2} \|V\|(\mathbf{B}(c, r) \cap \{z : |\tau(z) - \tau(c)| > t\})^{1/2} : 0 < t < \infty\}$?

If the answer should turn out to be in the affirmative, it would imply Theorem A as a corollary as well as a positive answer to Question 3 in the remaining cases $\alpha = 2/q$ for $2 < q < \infty$ both by interpolation based on the boundedness of τ. For the model case of the Laplace operator the procedure used to prove Theorem A could be adapted since coercive estimates in Lorentz spaces are available for the Laplace operator; such estimates are reviewed for instance in Dolzmann and Müller [12, § 3].

Organisation of the paper

In Sect. 1 the notation is introduced. Sections 2, 3 and 4 are of preparatory nature supplying the density properties of the Lebesgue measure, properties of Cartesian products of varifolds, and some differentiation theory for functions on varifolds respectively. Sections 5, 6, 7, 8 and 9 treat the quadratic tilt-excess whereas Sects. 10 and 11 cover the super-quadratic case. This includes, in Sect. 10, the example concerning the integrability of the second fundamental form of curvature varifolds. Finally, one dimensional varifolds are considered in Sect. 12.

1. Notation

The notation generally follows Federer [13] and Allard [4] with some modifications and additions described in [23, § 1]. Here, concerning the paragraph "Definitions in the text" of [23, § 1], only the notions of *generalised weakly differentiable function*, *generalised weak derivative*, $V\ \mathbf{D}\, f$, and the associated space $\mathbf{T}(V, Y)$, see [23, 8.3], will be employed.

Less common symbols and terminology. A family is disjointed if and only if two distinct members have empty intersection, see [13, 2.1.6]. Whenever f is a function it is a subset of the Cartesian product of its domain, $\operatorname{dmn} f$, and its image, $\operatorname{im} f$, see [13, p. 669]. The Lebesgue measure of the unit ball in $\mathbf{R}^m$ will be denoted by $\boldsymbol{\alpha}(m)$, see [13, 2.7.16]. Whenever μ measures X and Y is a Banach space, $\mathbf{A}(\mu, Y)$ denotes the vectorspace of those Y valued μ measurable functions f such that there exists a separable subspace Z of Y satisfying $\mu(f^{-1}[Y \sim Z]) = 0$, see [13, 2.3.8]. In this case,

$$\mu_{(p)}(f) = \left(\textstyle\int |f|^p \,\mathrm{d}\mu\right)^{1/p} \quad \text{whenever } 1 \le p < \infty,$$
$$\mu_{(\infty)}(f) = \inf\{t : \mu(\{x : |f(x)| > t\}) = 0\}$$

whenever $f \in \mathbf{A}(\mu, Y)$, see [13, 2.4.12]. Notice also that no usage of equivalence classes of functions is made.

Modifications. Extending Allard [4, 2.5 (1)], whenever M is a submanifold of $\mathbf{R}^n$ of class 2 and $a \in M$ the *second fundamental form of M at a* is the unique symmetric bilinear mapping

$$\mathbf{b}(M, a) : \operatorname{Tan}(M, a) \times \operatorname{Tan}(M, a) \to \operatorname{Nor}(M, a)$$

such that whenever $g : M \to \mathbf{R}^n$ is of class 1 and $g(z) \in \operatorname{Nor}(M, z)$ for $z \in M$,

$$u \bullet \langle v, \mathrm{D}\, g(a)\rangle = -\mathbf{b}(M, a)(u, v) \bullet g(a) \quad \text{for } u, v \in \operatorname{Tan}(M, a).$$

Additional notation. By a *modulus of continuity* we mean a function $\omega : \{t : 0 \le t \le 1\} \to \{t : 0 \le t \le 1\}$ such that $\lim_{t\to 0+} \omega(t) = 0$ and

$$\omega(t) = 0 \text{ if and only if } t = 0 \quad \text{whenever } 0 \le t \le 1,$$
$$\omega(s) \le \omega(t) \quad \text{whenever } 0 \le s \le t \le 1.$$

Adapting Morrey [25, p. 54], such ω is said to satisfy the *Dini condition* if and only if $\int_0^1 \omega(t) t^{-1} \,\mathrm{d}\mathscr{L}^1 t < \infty$. Following [21, p. 8], whenever $n, m \in \mathscr{P}$, $m < n$, $T \in \mathbf{G}(n, m)$, $a \in \mathbf{R}^n$, $0 < r < \infty$, and $0 < h < \infty$ we define the *closed cuboid* by

$$\mathbf{C}(T,a,r,h) = \mathbf{R}^n \cap \left\{ z : |T_\natural(z-a)| \le r \text{ and } |T_\natural^\perp(z-a)| \le h \right\}.$$

Definitions in the text. The notions of *curvature varifold* and its *second fundamental form* $\mathbf{b}(V,z)$, are explained in 1. The Orlicz space seminorm $\mu_{(\Phi)}$ is defined in 2. The space $\mathbf{Q}_Q(Y)$ metrised by $\mathscr{G}$ occurs in 3, the notion of *affine* $\mathbf{Q}_Q(Y)$ valued function and the corresponding seminorm $\|f\|$ are explained in 4. Finally, the notions of *affinely approximable* and *approximately affinely approximable* for $\mathbf{Q}_Q(Y)$ valued functions and the corresponding symbols $\mathrm{A}\, f$ and $\mathrm{ap}\,\mathrm{A}\, f$ are defined in 4.

2. Density properties of Lebesgue measure

The purpose of this section is to provide two examples of subsets of positive Lebesgue measure whose complement is as large as possible near each point of the set, see 2.2 and 2.5. In this respect the size of the complement is measured either by the Lebesgue measure or by the behaviour of the distance function on the complement (equivalently by the size of cubes contained in the complement). In the light of known positive results, the examples obtained are sharp, see 2.3 and 2.6. Using these examples, certain varifolds will be constructed in Sects. 5 and 10 to demonstrate that the tilt-excess decays proven in Sects. 9 and 11 are sharp in many cases.

Both sets are constructed by removing a suitable disjointed subcollection of the family of all dyadic subcubes from the unit cube.

2.1. Here, we collect some useful terminology for later reference.

Suppose m is a positive integer. Define *open cubes* by

$$C(a,r) = \mathbf{R}^m \cap \{ x : a_j < x_j < a_j + r \text{ for } j = 1, \ldots, m \}$$

whenever $a \in \mathbf{R}^m$ and $0 < r < \infty$. Note that $a \notin C(a,r)$. We shall work with the following definition of dyadic cubes. Whenever i is a nonnegative integer define $W(i)$ to consist of all open cubes

$$C(2^{-i}a, 2^{-i}) \subset C(0,1)$$

corresponding to $a \in \mathbf{Z}^m$ with $0 \le a_j \le 2^i - 1$ for $j = 1, \ldots, m$. Let

$$Z = \bigcup \{ W(i) : i = 0,1,2,\ldots \}, \quad N = \textstyle\bigcup_{i=1}^\infty \left((\operatorname{Clos} C(0,1)) \sim \bigcup W(i) \right).$$

Note that $W(i)$ is disjointed and $W(i) \cap W(j) = \varnothing$ if $i \neq j$ and $\mathscr{L}^m(N) = 0$. Observe

$$\text{either } Q \subset R \text{ or } R \subset Q \quad \text{whenever } Q, R \in Z \text{ and } Q \cap R \neq \varnothing.$$

Example 2.2. Suppose m, W, Z, and N are as in 2.1, ω is a modulus of continuity satisfying the Dini condition, and $0 \le \lambda < 1$.

Then there exist $0 < \varepsilon \le 1$ and a disjointed subfamily G of Z such that

$$A = C(0,1) \sim \left(N \cup \bigcup G \right)$$

satisfies the following two conditions:

(1) $\mathscr{L}^m(A) \ge \lambda$.

(2) If $a \in A$ and $0 < r \le \varepsilon$, then there exists $Q \in G$ with

$$Q \subset \mathbf{U}(a,r) \quad \text{and} \quad \mathscr{L}^m(Q) \ge \omega(r)r^m.$$

Proof. Choose $0 < s \le 2^{-1}m^{-1/2}$ such that $\omega(2m^{1/2}s) \le 2^{-2m}m^{-m/2}$, define $\phi : \mathbf{R} \cap \{t : 0 \le t \le 1\} \to \mathbf{R}$ by

$$\phi(t) = 2^{2m}m^{m/2}\omega(2m^{1/2}t) \quad \text{for } 0 \le t \le s, \qquad \phi(t) = 1 \quad \text{for } s < t \le 1,$$

and note that ϕ is a modulus of continuity satisfying the Dini condition. Choose a positive integer k such that

$$2^{-k} \le s, \quad (\log 2)^{-1} \int_0^{2^{1-k}} \phi(t)t^{-1}\, \mathrm{d}\mathscr{L}^1 t \le 1 - \lambda.$$

Define a sequence β_i of integers by the requirement

$$2^{m(i-\beta_i+1)} > \phi(2^{-i}) \ge 2^{m(i-\beta_i)}.$$

For $k \le i \in \mathbf{Z}$ observe that $\beta_{i+1} \ge \beta_i \ge i$ and inductively define families F_i to consist of the open cubes $C(b, 2^{-\beta_i})$, see 2.1, corresponding to all $C(b, 2^{-i}) \in W(i)$ satisfying the following condition:

If j is an integer, $k \le j \le i-1$, and $R \in F_j$ then $C(b, 2^{-\beta_i}) \cap R = \varnothing$.

Let $\varepsilon = m^{1/2}2^{1-k}$ and note that $\varepsilon \le 1$. Define

$$G = \bigcup\{F_i : k \le i \in \mathbf{Z}\}, \quad A = C(0,1) \sim (N \cup \bigcup G).$$

Notice that F_i is disjointed for $k \le i \in \mathbf{Z}$ and G is a disjointed subfamily of Z. Estimating

$$\sum_{i=k}^{\infty} 2^{m(i-\beta_i)} \le \sum_{i=k}^{\infty} \phi(2^{-i}) \le \sum_{i=k}^{\infty} (\log 2)^{-1} \int_{2^{-i}}^{2^{1-i}} \phi(t)t^{-1}\, \mathrm{d}\mathscr{L}^1 t \le 1 - \lambda,$$

and $\operatorname{card} F_i \le \operatorname{card} W(i) = 2^{im}$ whenever $k \le i \in \mathbf{Z}$, one obtains

$$\mathscr{L}^m(\bigcup G) \le \sum_{i=k}^{\infty} 2^{m(i-\beta_i)} \le 1 - \lambda, \quad \mathscr{L}^m(A) \ge \lambda.$$

Suppose $a \in A$ and $0 < r \le \varepsilon$.
There exist i and b such that

$$k \le i \in \mathbf{Z}, \quad 2^{-i} \le m^{-1/2}r \le 2^{1-i}, \quad a \in S = C(b, 2^{-i}) \in W(i).$$

The proof will be concluded by showing: *There exists $Q = C(c,t) \in G$ having the property*

$$t \ge 2^{-\beta_i} \quad \text{and} \quad Q \cap S \ne \varnothing$$

and this property implies

$$Q \subset \mathbf{U}(a,r), \quad \mathscr{L}^m(Q) \ge \omega(r)r^m.$$

Concerning the existence of Q, if $C(b, 2^{-\beta_i}) \cap R \ne \varnothing$ for some integer j with $k \le j \le i-1$ and some $R \in F_j$, then one may take $Q = R$, and otherwise one may take $Q = C(b, 2^{-\beta_i}) \in F_i$. Concerning the implication of the property, estimate

$$2^{-i} \le s, \quad \phi(2^{-i}) \ge 2^{2m} m^{m/2} \omega(r), \quad t^m \ge 2^{-m\beta_i} > \phi(2^{-i}) 2^{m(-i-1)} \ge \omega(r) r^m$$

and note that $Q \subset S$, since $a \in S \sim Q$ and $S, Q \in Z$, hence $Q \subset \mathbf{U}(a, r)$. □

Remark 2.3. Whenever ω violates the Dini condition and $\lambda > 0$ there do not exist ε, G and A as in 2.2 as may be verified by applying Topsøe [34, Theorem 3] with $\|x\|$, N, $\mathscr{B}$, $\psi(r)$, and δ_0 replaced by $\sup\{|x_i| : i = 1, \ldots, m\}$, m, G, $\omega(r)^{1/m} r/2$, and ε.

Remark 2.4. The basic construction principle is that of Mejlbro and Topsøe [27, Theorem 2] who established the sharpness of the Dini condition in a similar context.

Example 2.5. Suppose m, W, Z, and N are as in 2.1, ω is a modulus of continuity, and $0 \le \lambda < 1$.

Then there exist $0 < \varepsilon \le 1$, $B \subset \mathbf{R} \cap \{r : r > 0\}$ with $\inf B = 0$, and a disjointed subfamily G of Z such that

$$A = C(0,1) \sim (N \cup \bigcup G) \quad \text{with} \quad \mathscr{L}^m(A) \ge \lambda$$

satisfying the following condition: If $a \in A$ and $0 < r \le \varepsilon$, then there exists a subset H of $G \cap \{Q : Q \subset \mathbf{U}(a, r)\}$ such that

$$\mathscr{L}^m(\bigcup H) \ge \omega(r) r^m \quad \text{and} \quad \operatorname{card} H = 1 \text{ if } r \in B.$$

Proof. Choose s with

$$0 < s \le 2^{-1} m^{-1/2}, \quad \omega(2m^{1/2} s) \le 2^{-2m} m^{-m/2},$$

define $\phi : \mathbf{R} \cap \{t : 0 \le t \le 1\} \to \mathbf{R}$ by

$$\phi(t) = 2^{2m} m^{m/2} \omega(2m^{1/2} t) \quad \text{for } 0 \le t \le s, \qquad \phi(t) = 1 \quad \text{for } s < t \le 1,$$

and note that ϕ is a modulus of continuity. Inductively choose sequences α_i and β_i of nonnegative integers subject to the conditions

$$\begin{aligned} \phi(2^{-\alpha_i}) &\le (1-\lambda) 2^{-i}, \qquad \alpha_{i+1} > \beta_i \ge \alpha_i, \\ 2^{m(\alpha_{i+1} - \beta_{i+1} + 1)} &> \phi(2^{-\alpha_i}) \ge 2^{m(\alpha_{i+1} - \beta_{i+1})} \end{aligned}$$

whenever i is a positive integer and, in case $i > 1$, define families F_i to consist of the open cubes $C(b, 2^{-\beta_i})$, see 2.1, corresponding to all $C(b, 2^{-\alpha_i}) \in W(\alpha_i)$ satisfying the following condition:

If j is an integer, $1 < j \le i - 1$, and $R \in F_j$ then $C(b, 2^{-\beta_i}) \cap R = \varnothing$.

Define $\varepsilon = \inf\{2^{-\alpha_1} m^{1/2}, 2m^{1/2} s\}$ and note that $\varepsilon \le 1$. Let

$$G = \bigcup \{F_i : 1 < i \in \mathbf{Z}\}, \quad A = C(0,1) \sim (N \cup \bigcup G),$$
$$B = \{m^{1/2} 2^{1-\alpha_i} : 1 < i \in \mathbf{Z}\}.$$

Observe that G is disjointed. Since $\operatorname{card} F_i \le 2^{m\alpha_i}$ for any positive integer i, we have

$$\mathscr{L}^m(\bigcup G) \le \sum_{i=2}^{\infty} 2^{m(\alpha_i - \beta_i)} \le \sum_{i=2}^{\infty} \phi(2^{-\alpha_{i-1}}) \le 1 - \lambda.$$

Hence $\mathscr{L}^m(A) \geq \lambda$.

Suppose $a \in A$ and $0 < r \leq \varepsilon$.

There exist k and c satisfying

$$\alpha_1 < k \in \mathbf{Z}, \quad 2^{-k} < m^{-1/2} r \leq 2^{1-k}, \quad a \in S = C(c, 2^{-k}) \in W(k)$$

and i such that

$$1 < i \in \mathbf{Z} \quad \text{and} \quad \alpha_{i-1} < k \leq \alpha_i.$$

Defining $I = W(\alpha_i) \cap \{Q : Q \subset S\}$, one notes that

$$\operatorname{card} I = 2^{m(\alpha_i - k)},$$

in particular $\operatorname{card} I = 1$ if $r \in B$. Moreover, one concludes

$$C(b, 2^{-\alpha_i}) \in I \quad \text{implies} \quad C(b, 2^{-\beta_i}) \subset R \subset S \text{ for some } R \in G;$$

in fact, either $C(b, 2^{-\beta_i}) \cap R \neq \varnothing$ for some integer j with $1 < j \leq i-1$ and some $R \in F_j$, hence

$$R \cap S \neq \varnothing, \quad C(b, 2^{-\beta_i}) \subset R \subset S$$

as $\beta_i \geq \beta_j$ and $a \in S \sim R$, because $a \in A \subset C(0,1) \sim \bigcup G$, or

$$C(b, 2^{-\beta_i}) \in F_i \subset G, \quad C(b, 2^{-\beta_i}) \subset C(b, 2^{-\alpha_i}) \subset S.$$

Consequently, there exists a subset H of $G \cap \{R : R \subset S\}$ such that

$$\bigcup \{C(b, 2^{-\beta_i}) : C(b, 2^{-\alpha_i}) \in I\} \subset \bigcup H, \quad \operatorname{card} H = 1 \text{ if } r \in B.$$

Noting $2^{-1} m^{-1/2} r \leq \inf\{2^{-\alpha_{i-1}}, s\}$, one infers

$$\mathscr{L}^m(\bigcup H) \geq 2^{m(\alpha_i - \beta_i - k)} \geq 2^{-2m} m^{-m/2} \phi(2^{-\alpha_{i-1}}) r^m \geq \omega(r) r^m.$$

Since $S \subset \mathbf{U}(a, r)$, the conclusion follows. □

Remark 2.6. From the classical differentiation theory of Vitali and Lebesgue, see for instance [13, 2.8.17, 2.9.11], it is evident that the lower bound on $\mathscr{L}^m(\mathbf{U}(a,r) \sim A)$ exhibited here is the known optimal one, see Tolstoff [33, Théorème 3, p. 263]. In order to demonstrate the sharpness of our results in 9.2, the additional property "$\operatorname{card} H = 1$ if $r \in B$" will be employed in 5.8. Clearly, B may not be required to equal $\{r : 0 < r \leq \varepsilon\}$ if ω violates the Dini condition and $\lambda > 0$ by 2.3.

3. Cartesian product of varifolds

The purpose of this section is to establish basic properties of the Cartesian product of varifolds. The construction preserves rectifiability, integrality and maps curvature varifolds to curvature varifolds, see 3.6. We also note a version of the coarea formula for rectifiable varifolds, see 3.5 (2).

The proof of the rectifiability of the Cartesian product of rectifiable varifolds needs to take into account that $(\mathscr{H}^m, m)$ *rectifiable sets do not possess a similar stability property*, see 3.7. Our treatment of curvature varifolds is based on the characterisation of such varifolds in terms of generalised weakly differentiable functions obtained in [23, 15.6].

In the present paper only products of varifolds with planes are employed. More general products will be required in the study of the geodesic distance on the support of the weight measure of certain varifolds, see [24, § 6].

Lemma 3.1. *Suppose $m, n \in \mathscr{P}$, $m \le n$, M is an m dimensional submanifold of $\mathbf{R}^n$ of class 2, $\tau : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is defined by $\tau(z) = \operatorname{Tan}(M, z)_\natural$ for $z \in M$, and $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ is normed by $\|\cdot\|$.*

Then the following three statements hold.

(1) *If $z \in M$ and $u, v \in \operatorname{Tan}(M, z)$, then $\mathbf{b}(M, z)(u, v) = \langle u, \langle v, \mathrm{D}\,\tau(z)\rangle\rangle$.*
(2) *If $z \in M$ and $u, v, w \in \mathbf{R}^n$ then*

$$\begin{aligned}&\langle v, \langle u, \mathrm{D}\,\tau(z) \circ \tau(z)\rangle\rangle \bullet w \\ &\quad = \mathbf{b}(M, z)(\langle u, \tau(z)\rangle, \langle v, \tau(z)\rangle) \bullet w + \mathbf{b}(M, z)(\langle u, \tau(z)\rangle, \langle w, \tau(z)\rangle) \bullet v.\end{aligned}$$

(3) *If $z \in M$, then $\|\mathbf{b}(M, z)\| = \|\mathrm{D}\,\tau(z) \circ \tau(z)\|$.*

Proof. Define $\nu : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by $\nu(z) = \mathbf{1}_{\mathbf{R}^n} - \tau(z)$ for $z \in M$. Differentiating the equations $\tau(z) \circ \tau(z) = \tau(z)$ and $\nu(z) \circ \tau(z) = 0$ for $z \in M$, one obtains for $z \in M$ and $u \in \operatorname{Tan}(M, z)$ that

$$\tau(z) \circ \langle u, \mathrm{D}\,\tau(z)\rangle \circ \tau(z) = 0, \quad \nu(z) \circ \langle u, \mathrm{D}\,\tau(z)\rangle \circ \nu(z) = 0.$$

In order to prove (1), suppose $z \in M$ and $u, v \in \operatorname{Tan}(M, z)$, notice that $\langle u, \langle v, \mathrm{D}\,\tau(z)\rangle\rangle \in \operatorname{Nor}(M, z)$, and differentiate the equation $\langle u, \tau(\zeta)\rangle \bullet g(\zeta) = 0$ for $\zeta \in M$, to obtain $\langle u, \langle v, \mathrm{D}\,\tau(z)\rangle\rangle \bullet g(z) = -u \bullet \langle v, \mathrm{D}\, g(z)\rangle$. Expressing

$$\begin{aligned}\langle v, \langle u, \mathrm{D}\,\tau(z) \circ \tau(z)\rangle\rangle \bullet w = \ &\langle\langle v, \tau(z)\rangle, \langle u, \mathrm{D}\,\tau(z) \circ \tau(z)\rangle\rangle \bullet \langle w, \nu(z)\rangle \\ &+ \langle\langle v, \nu(z)\rangle, \langle u, \mathrm{D}\,\tau(z) \circ \tau(z)\rangle\rangle \bullet \langle w, \tau(z)\rangle\end{aligned}$$

for $z \in M$ and $u, v, w \in \mathbf{R}^n$, (2) follows from the symmetry of $\langle u, \mathrm{D}\,\tau(z) \circ \tau(z)\rangle$ and (1). Finally, noting

$$|\langle v, \tau(z)\rangle||\langle w, \nu(z)\rangle| + |\langle v, \nu(z)\rangle||\langle w, \tau(z)\rangle| \le |v||w|$$

for $z \in M$ and $v, w \in \mathbf{R}^n$ by Hölder's inequality, (3) follows from (1) and (2). □

Remark 3.2. Items 3.1 (1) (2) are in analogy with Hutchinson [15, 5.1.1 (i) (ii)].

Definition 1. Suppose $m, n \in \mathscr{P}$, $m \le n$, and U is an open subset of $\mathbf{R}^n$.

Then V is called *m dimensional curvature varifold in U* if and only if the following three conditions are satisfied:

(1) V is an m dimensional integral varifold in U.
(2) $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$.
(3) If $Y = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{\sigma : \sigma = \sigma^*\}$, $Z = U \cap \{z : \operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)\}$, and $\tau : Z \to Y$ is defined by $\tau(z) = \operatorname{Tan}^m(\|V\|, z)_\natural$ for $z \in Z$, then τ is a generalised V weakly differentiable function.

In this case one defines for $z \in Z \cap \operatorname{dmn} V\,\mathbf{D}\,\tau$ the *second fundamental form of V at z* by

$$\begin{aligned}\mathbf{b}(V, z) : \operatorname{Tan}^m(\|V\|, z) \times \operatorname{Tan}^m(\|V\|, z) \to \mathbf{R}^n, \\ \mathbf{b}(V, z)(u, v) = \langle u, \langle v, V\,\mathbf{D}\,\tau(z)\rangle\rangle \quad \text{whenever } u, v \in \operatorname{Tan}^m(\|V\|, z).\end{aligned}$$

Remark 3.3. In view of [23, 15.4–15.6] the preceding definition of curvature varifold is equivalent to Hutchinson's original definition in [15, 5.2.1]. Recalling [22, 4.8] and [23, 11.2], one infers that, for $\|V\|$ almost all z, the second fundamental form of V at z is a symmetric bilinear map with values in $\operatorname{Nor}^m(\|V\|, z)$ which is related to $V\ \mathbf{D}\,\tau(z)$ as $\mathbf{b}(M,z)$ is related to $\mathrm{D}\,\tau(z) \circ \tau(z)$ in 3.1 (2) (3); in fact, if M is an m dimensional submanifold of $\mathbf{R}^n$ of class 2 and $\sigma : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ satisfies $\sigma(z) = \operatorname{Tan}(M,z)_\natural$ for $z \in M$, then

$$\operatorname{Tan}(M,z) = \operatorname{Tan}^m(\|V\|, z) \quad \text{and} \quad \mathrm{D}\,\sigma(z) = (\|V\|, m)\operatorname{ap}\mathrm{D}\,\tau(z)$$

for $\|V\|$ almost all $z \in U \cap M$ by [13, 2.8.18, 2.9.11, 3.2.17] and Allard [4, 3.5 (2)]. Notice also that, for $\|V\|$ almost all z,

$$\mathbf{b}(V,z) \circ (\operatorname{Tan}^m(\|V\|, z)_\natural \times \operatorname{Tan}^m(\|V\|, z)_\natural)$$

corresponds to the generalised second fundamental form at $(z, \operatorname{Tan}^m(\|V\|, z))$ in the sense of Hutchinson [15, 5.2.5].

3.4. Suppose V and W are finite dimensional vectorspaces, $f \in \operatorname{Hom}(V, W)$, and $g \in \operatorname{Hom}(W, V)$. Then $\operatorname{trace}(g \circ f) = \operatorname{trace}(f \circ g)$; in fact, the argument of [13, 1.4.5] for the case $V = W$ applies to the present case as well.

3.5. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, and $V \in \mathbf{RV}_m(U)$. Then the following two statements hold.

(1) There exist sequences of compact subset C_j of m dimensional submanifolds of U of class 1 and $0 < d_j < \infty$ such that

$$V(k) = \sum_{j=1}^{\infty} d_j \int_{C_j} k(z, \operatorname{Tan}^m(\mathscr{H}^m \llcorner C_j, z))\, \mathrm{d}\mathscr{H}^m z$$

for $k \in \mathscr{K}(U \times \mathbf{G}(n,m))$.

(2) If $m \ge \mu \in \mathscr{P}$, $M = \{z : 0 < \boldsymbol{\Theta}^m(\|V\|, z) < \infty\}$, and $f : U \to \mathbf{R}^\mu$ is Lipschitzian, then f is $(\|V\|, m)$ differentiable at $\|V\|$ almost all z and there holds

$$\begin{aligned}\int g(z) \big\| {\textstyle\bigwedge_\mu} (\|V\|, m)\operatorname{ap}\mathrm{D}\, f(z) \big\|\, \mathrm{d}\|V\| z \\ = \int \int_{M \cap f^{-1}[\{y\}]} g(z) \boldsymbol{\Theta}^m(\|V\|, z)\, \mathrm{d}\mathscr{H}^{m-\mu} z\, \mathrm{d}\mathscr{L}^\mu y\end{aligned}$$

whenever g is a $\|V\|$ integrable $\overline{\mathbf{R}}$ valued function, where $\infty \cdot 0 = 0$.

(1) may be verified by means of [13, 2.8.18, 2.9.11, 3.2.17, 3.2.29]. The first half of (2) is implied by [21, 4.5 (2)]. Concerning the second half of (2), one notices that $g|C_j$ is $\mathscr{H}^m \llcorner C_j$ integrable and

$$\boldsymbol{\Theta}^m(\|V\|, z) = \sum_{j \in J(z)} d_j \quad \text{for } \mathscr{H}^m \text{ almost all } z \in U,$$

$$(\|V\|, m)\operatorname{ap}\mathrm{D}\, f(z) = (\mathscr{H}^m \llcorner C_j, m)\operatorname{ap}\mathrm{D}\, f(z) \quad \text{for } \mathscr{H}^m \text{ almost all } z \in C_j,$$

where $J(z) = \mathscr{P} \cap \{j : z \in C_j\}$, by Allard [4, 2.8 (4a), 3.5 (2)] and obtains

$$\int_{C_j} g(z) \big\| \textstyle\bigwedge_\mu (\mathscr{H}^m \llcorner C_j, m) \operatorname{ap} \mathrm{D} f(z) \big\| \,\mathrm{d}\mathscr{H}^m z = \int \int_{C_j \cap f^{-1}[\{y\}]} g(z) \,\mathrm{d}\mathscr{H}^{m-\mu} z \,\mathrm{d}\mathscr{L}^\mu y$$

from [13, 2.10.35, 3.2.22 (3)], hence multiplying by d_j and summing over j by means of [13, 2.10.25] yields the conclusion since M and $\bigcup_{j=1}^\infty C_j$ are $\mathscr{H}^m$ almost equal by Allard [4, 3.5 (1b)] and the equation for $\boldsymbol{\Theta}^m(\|V\|, z)$.

Theorem 3.6. *Suppose for $i \in \{1, 2\}$, $m_i, n_i \in \mathscr{P}$, $m_i \le n_i$, $p_i : \mathbf{R}^{n_1} \times \mathbf{R}^{n_2} \to \mathbf{R}^{n_i}$ satisfy $p_i(z_1, z_2) = z_i$ for $(z_1, z_2) \in \mathbf{R}^{n_1} \times \mathbf{R}^{n_2}$, U_i are open subsets of $\mathbf{R}^{n_i}$, $V_i \in \mathbf{V}_{m_i}(U_i)$, and $W \in \mathbf{V}_{m_1+m_2}(U_1 \times U_2)$ satisfies*

$$W(k) = \int k((z_1, z_2), S_1 \times S_2) \,\mathrm{d}(V_1 \times V_2)((z_1, S_1), (z_2, S_2))$$

for $k \in \mathscr{K}((U_1 \times U_2) \times \mathbf{G}(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, m_1 + m_2))$.

Then the following eight statements hold.

(1) *There holds $\|W\| = \|V_1\| \times \|V_2\|$.*

(2) *If V_1 and V_2 are rectifiable, so is W and, for $\|W\|$ almost all (z_1, z_2),*

$$\operatorname{Tan}^{m_1+m_2}(\|W\|, (z_1, z_2)) = \operatorname{Tan}^{m_1}(\|V_1\|, z_1) \times \operatorname{Tan}^{m_2}(\|V_2\|, z_2),$$
$$\boldsymbol{\Theta}^{m_1+m_2}(\|W\|, (z_1, z_2)) = \boldsymbol{\Theta}^{m_1}(\|V_1\|, z_1)\boldsymbol{\Theta}^{m_2}(\|V_2\|, z_2).$$

(3) *If V_1 and V_2 are integral, so is W.*

(4) *If $S_i \in \mathbf{G}(n_i, m_i)$ for $i \in \{1, 2\}$ and $h \in \operatorname{Hom}(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, \mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, then*

$$(S_1 \times S_2)_\natural = p_1^* \circ (S_1)_\natural \circ p_1 + p_2^* \circ (S_2)_\natural \circ p_2,$$
$$(S_1 \times S_2)_\natural \bullet h = (S_1)_\natural \bullet (p_1 \circ h \circ p_1^*) + (S_2)_\natural \bullet (p_2 \circ h \circ p_2^*).$$

(5) *If $\theta \in \mathscr{D}(U_1 \times U_2, \mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, then*

$$(\delta W)(\theta) = \int (\delta V_1)_{z_1}(p_1(\theta(z_1, z_2))) \,\mathrm{d}\|V_2\| z_2 + \int (\delta V_2)_{z_2}(p_2(\theta(z_1, z_2))) \,\mathrm{d}\|V_1\| z_1.$$

(6) *If $\|\delta V_i\|$ are Radon measures for $i \in \{1, 2\}$, then*

$$\|\delta W\| \le \|\delta V_1\| \times \|V_2\| + \|V_1\| \times \|\delta V_2\|$$

and, for $\theta \in \mathbf{L}_1(\|\delta V_1\| \times \|V_2\| + \|V_1\| \times \|\delta V_2\|, \mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$, the equation in (5) holds.

(7) *If, for $i \in \{1, 2\}$, V_i are rectifiable, $\|\delta V_i\|$ are Radon measures, Y_i are finite dimensional normed vectorspaces and $f_i \in \mathbf{T}(V_i, Y_i)$, and $f : \operatorname{dmn} f_1 \times \operatorname{dmn} f_2 \to Y_1 \times Y_2$ satisfies*

$$f(z_1, z_2) = (f_1(z_1), f_2(z_2)) \quad \text{for } z_1 \in \operatorname{dmn} f_1 \text{ and } z_2 \in \operatorname{dmn} f_2,$$

then $f \in \mathbf{T}(W, Y_1 \times Y_2)$ and, for $\|W\|$ almost all (z_1, z_2),

$$W \mathbf{D} f(z_1, z_2)(u_1, u_2) = (V_1 \mathbf{D} f_1(z_1)(u_1), V_2 \mathbf{D} f_2(z_2)(u_2))$$

whenever $u_1 \in \mathbf{R}^{n_1}$ *and* $u_2 \in \mathbf{R}^{n_2}$*.*

(8) *If* V_1 *and* V_2 *are curvature varifolds, then so is* W *and, for* $\|W\|$ *almost all* (z_1, z_2)*,*

$$\mathbf{b}(W,(z_1,z_2))((u_1,u_2),(v_1,v_2)) = (\mathbf{b}(V_1,z_1)(u_1,u_2), \mathbf{b}(V_2,z_2)(u_2,v_2))$$

whenever $u_1, v_1 \in \mathbf{R}^{n_1}$ *and* $u_2, v_2 \in \mathbf{R}^{n_2}$*.*

Proof of (1) follows from Fubini's theorem. □

Proof of (2) If C_i are compact subsets of m_i dimensional submanifolds of U_i of class 1 for $i \in \{1, 2\}$, then $C_1 \times C_2$ is $(\mathscr{H}^{m_1+m_2}, m_1 + m_2)$ rectifiable with

$$(\mathscr{H}^{m_1} \llcorner C_1) \times (\mathscr{H}^{m_2} \llcorner C_2) = \mathscr{H}^{m_1+m_2} \llcorner (C_1 \times C_2),$$

by [13, 3.2.23] and, for $\mathscr{H}^{m_1+m_2}$ almost all $(z_1, z_2) \in C_1 \times C_2$,

$$\begin{aligned}\operatorname{Tan}^{m_1}(\mathscr{H}^{m_1} \llcorner C_1, z_1) \times \operatorname{Tan}^{m_2}(\mathscr{H}^{m_2} \llcorner C_2, z_2)\\ = \operatorname{Tan}^{m_1+m_2}(\mathscr{H}^{m_1+m_2} \llcorner (C_1 \times C_2), (z_1, z_2))\end{aligned}$$

by [13, 2.8.18, 2.9.11]. Therefore the assertion may be verified by means of 3.5 (1), Allard [4, 2.8 (4a), 3.5 (2)], [13, 2.1.1 (11)], and (1). □

Proof of (3) This is a consequence of (2) and Allard [4, 3.5 (1c)]. □

Proof of (4) (5) (6) The first equation of (4) is obvious and implies the second equation by 3.4. (5) is a consequence of (4) and Fubini's theorem. (5) implies (6). □

Proof of (7) Assume $\dim(Y_1 \times Y_2) \geq 1$. First, consider the case $\dim(Y_1 \times Y_2) \geq 2$. Define $F : p_1^{-1}[\operatorname{dmn} V_1 \mathbf{D} f_1] \cap p_2^{-1}[\operatorname{dmn} V_2 \mathbf{D} f_2] \to \operatorname{Hom}(\mathbf{R}^{n_1} \times \mathbf{R}^{n_2}, Y_1 \times Y_2)$ by

$$F(z_1,z_2)(u_1,u_2) = (V_1 \mathbf{D} f_1(z_1)(u_1), V_2 \mathbf{D} f_2(z_2)(u_2))$$

whenever $z_1 \in \operatorname{dmn} V_i \mathbf{D} f_i$ and $u_i \in \mathbf{R}^{n_i}$ for $i \in \{1, 2\}$. By [23, 8.4] it sufficient to prove

$$\begin{aligned}&(\delta W)((\gamma \circ f)\theta)\\ &\quad = \int \gamma(f(z)) T_\natural \bullet \mathrm{D}\,\theta(z)\,\mathrm{d}W(z,T) + \int \langle \theta(z), \mathrm{D}\,\gamma(f(z)) \circ F(z)\rangle\,\mathrm{d}\|W\|z\end{aligned}$$

whenever $\theta \in \mathscr{D}(U_1 \times U_2, \mathbf{R}^{n_1} \times \mathbf{R}^{n_2})$ and $\gamma \in \mathscr{D}(Y_1 \times Y_2, \mathbf{R})$. Recalling [23, 2.15, 3.1],[2] one may assume additionally that for some $\gamma_i \in \mathscr{D}(Y_i, \mathbf{R})$ for $i \in \{1, 2\}$, there holds

$$\gamma(y_1,y_2) = \gamma_1(y_1)\gamma_2(y_2) \quad \text{for } (y_1,y_2) \in Y_1 \times Y_2.$$

[2] The topologies on $\mathscr{D}(Y_1 \times Y_2, \mathbf{R})$ considered in [23, 2.13] and [13, 4.1.1] differ but possess the same convergent sequences, see [23, 2.15, 2.17 (3)].

In this case one computes, noting

$$\begin{aligned}&(\delta V_i)_{z_i}(\gamma_i(f_i(z_i))p_i(\theta(z_1,z_2)))\\&\quad=\int \gamma_i(f_i(z_i))(S_i)_\natural \bullet (p_i\circ \mathrm{D}\,\theta(z_1,z_2)\circ p_i^*)\,\mathrm{d}V_i(z_i,S_i)\\&\qquad+\int \langle p_i(\theta(z_1,z_2)), \mathrm{D}\,\gamma_i(f_i(z_i))\circ V_i\ \mathbf{D}\,f_i(z_i)\rangle\,\mathrm{d}\|V_i\|z_i\end{aligned}$$

whenever $i,j\in\{1,2\}$, $i\neq j$ and $z_j\in\operatorname{dmn} f_j$, and using (4) and (6),

$$\begin{aligned}&(\delta W)((\gamma\circ f)\theta)\\&\quad=\int \gamma_2(f_2(z_2))(\delta V_1)_{z_1}(\gamma_1(f_1(z_1))p_1(\theta(z_1,z_2)))\,\mathrm{d}\|V_2\|z_2\\&\qquad+\int \gamma_1(f_1(z_1))(\delta V_2)_{z_2}(\gamma_2(f_2(z_2))p_2(\theta(z_1,z_2)))\,\mathrm{d}\|V_1\|z_1\\&\quad=\int \gamma(f(z))T_\natural\bullet \mathrm{D}\,\theta(z)\,\mathrm{d}W(z,T)+\int\langle\theta(z),\mathrm{D}\,\gamma(f(z))\circ F(z)\rangle\,\mathrm{d}\|W\|z.\end{aligned}$$

If $\dim(Y_1\times Y_2)=1$, one may assume $\dim Y_1=1$, hence $\dim Y_2=0$, and similarly consider $\gamma_1\in\mathscr{E}(Y_1,\mathbf{R})$ such that $\operatorname{spt}\mathrm{D}\,\gamma_1$ is compact and $\gamma_2=1$. □

Proof of (8) Define $Y_i=\operatorname{Hom}(\mathbf{R}^{n_i},\mathbf{R}^{n_i})\cap\{\sigma:\sigma=\sigma^*\}$ and functions $f_i\in\mathbf{T}(V_i,Y_i)$ by $f_i(z_i)=\operatorname{Tan}^{m_i}(\|V_i\|,z_i)_\natural$ whenever $z_i\in U_i$ and $\operatorname{Tan}^{m_i}(\|V_i\|,z_i)\in\mathbf{G}(n_i,m_i)$ for $i\in\{1,2\}$. Associate $f\in\mathbf{T}(W,Y_1\times Y_2)$ to f_1 and f_2 as in (7). Define Y to be the vectorspace of symmetric endomorphisms of $\mathbf{R}^{n_1}\times\mathbf{R}^{n_2}$ and let the linear map $L:Y_1\times Y_2\to Y$ be defined by

$$L(\sigma_1,\sigma_2)=p_1^*\circ\sigma_1\circ p_1+p_2^*\circ\sigma_2\circ p_2\quad\text{for }\sigma_1\in Y_1\text{ and }\sigma_2\in Y_2.$$

In view of (2) and (4), one infers

$$\operatorname{Tan}^{m_1+m_2}(\|W\|,(z_1,z_2))_\natural=L(f(z_1,z_2))\quad\text{for }\|W\|\text{ almost all }(z_1,z_2).$$

Therefore W is a curvature varifold by (1), (3), (6), and [23, 8.6] and, by (7), there holds, for $\|W\|$ almost all (z_1,z_2),

$$W\ \mathbf{D}\,f(z_1,z_2)(u_1,u_2)=L(V_1\ \mathbf{D}\,f_1(z_1)(u_1),V_2\ \mathbf{D}\,f_2(z_2)(u_2))$$

for $u_1\in\mathbf{R}^{n_1}$ and $u_2\in\mathbf{R}^{n_2}$. Recalling (2), the equation for the second fundamental form of W now follows. □

Remark 3.7. The behaviour of rectifiable varifolds described in (2) is in contrast with the more subtle behaviour of $(\mathscr{H}^m,m)$ rectifiable sets; in fact, if $m,n\in\mathscr{P}$ and $m<n$ there exist compact subsets C_1 and C_2 of $\mathbf{R}^n$ such that

$$\mathscr{H}^m(C_1)=\mathscr{H}^m(C_2)=0,\quad \mathscr{H}^l(C_1\times C_2)=\infty$$

whenever $0\le l<m+n$, see [13, 2.10.29, 3.2.24].

Remark 3.8. Concerning (7) and (8), notice that, for general W, neither is membership in $\mathbf{T}(W,Y_1\times Y_2)$ implied by membership of the component functions in $\mathbf{T}(W,Y_i)$ nor is $\mathbf{T}(W,Y_1\times Y_2)$ closed under addition, see [23, 8.25].

4. Differentiation results

In the present section a differentiation theorem for measures relative to varifolds is provided. It slightly generalises [19, 3.1] so as to become applicable in the proof of the sharp decay rate almost everywhere for the quadratic tilt-excess of certain two-dimensional varifolds in Sect. 9. Additionally, a corollary for use in the study of one-dimensional varifolds in Sect. 12 is noted.

Results of the type considered here occur for instance in Calderón and Zygmund [10, Theorem 10, p. 189] in which paper a differentiation theory of higher order in Lebesgue spaces is developed. The classical Rademacher theorem is contained in that theory as special case, see Calderón and Zygmund [10, Theorem 12, p. 204].

4.1. Suppose $m, n \in \mathscr{P}$, $m \le n$, $1 \le p \le \infty$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, z) \ge 1$ for $\|V\|$ almost all z. If $p > 1$, then suppose additionally that $\mathbf{h}(V, \cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ and

$$\delta V(\theta) = -\int \mathbf{h}(V, z) \bullet \theta(z)\, \mathrm{d}\|V\| z \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Therefore $V \in \mathbf{RV}_m(U)$ by Allard [4, 5.5 (1)]. If $p = 1$ let $\psi = \|\delta V\|$. If $1 < p < \infty$ define a Radon measure ψ over U by the requirement $\psi(B) = \int_B |\mathbf{h}(V, z)|^p \, \mathrm{d}\|V\| z$ whenever B is Borel subset of U.

Theorem 4.2. *Suppose m, n, p, U and V are as in 4.1, $1 \le p \le m$, ω is a modulus of continuity, μ measures U with $\mu(U \sim \operatorname{spt} \|V\|) = 0$, and Z is a $\|V\|$ measurable set with $\mu(Z) = 0$. In case $p < m$ suppose additionally that there exist $1 < q \le \infty$ and $f \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|)$ such that*

$$\liminf_{r \to 0+} r^{(1-1/q)mp/(p-m)} \omega(r) > 0,$$

$$\mu(B) = \int_B f \, \mathrm{d}\|V\| \quad \textit{whenever } B \textit{ is Borel subset of } U.$$

Then for $\mathscr{H}^m$ almost all $z \in Z$

$$\limsup_{r \to 0+} r^{-m} \omega(r)^{-1} \mu\, \mathbf{B}(z, r) \quad \textit{equals either } 0 \textit{ or } \infty.$$

Proof. For $i \in \mathscr{P}$ let G_i denote the set of all $z \in \operatorname{spt} \|V\|$ such that either $\mathbf{U}(z, 1/i) \not\subset U$ or

$$\|\delta V\| \, \mathbf{B}(z, r) > (2\boldsymbol{\gamma}(m))^{-1} \|V\|(\mathbf{B}(z, r))^{1-1/m} \quad \text{for some } 0 < r < 1/i.$$

Notice that $G_{i+1} \subset G_i$ and that G_i is relatively open in $\operatorname{spt} \|V\|$ by an argument analogous to [13, 2.9.14].

We start with some preliminary observations. Consider the *case* $p < m$. If $q < \infty$, define ν to be the Radon measure over U characterised by the requirement $\nu(B) = \int_B f^q \, \mathrm{d}\|V\|$ whenever B is a Borel subset of U. Using Hölder's inequality, one may employ [19, 2.9, 2.10] with m, n, μ, s, ε, and Γ replaced by $n - m$, m, $\|V\|$, m, $(2\boldsymbol{\gamma}(m))^{p/(p-m)}$, and $5m\gamma(m)$ to see that for $\mathscr{H}^m$ almost all $z \in U$ there exists an $i \in \mathscr{P}$ such that

$$\boldsymbol{\Theta}^{m^2/(m-p)}(\|V\| \llcorner G_i, z) = 0.$$

This implies, by another use of Hölder's inequality, that for $\mathscr{H}^m$ almost all $z \in U$ there exists an $i \in \mathscr{P}$ such that

$$\boldsymbol{\Theta}^{m(1+(1-1/q)p/(m-p))}(\mu \llcorner G_i, z) = 0,$$

since, if $q < \infty$, then $\boldsymbol{\Theta}^{*m}(\nu, z) < \infty$ for $\mathscr{H}^m$ almost all $z \in U$ due to [13, 2.10.19 (3)]. Observe that if $z \in \operatorname{spt} \|V\| \sim \bigcup_{i=1}^\infty G_i$, then, according to [19, 2.5], $\boldsymbol{\Theta}_*^m(\|V\|, z) > 0$; hence, $\operatorname{spt} \|V\| \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = 0\} \subset \bigcap_{i=1}^\infty G_i$. Recalling that G_i are relatively open in $\operatorname{spt} \|V\|$ and $\mu(U \sim \operatorname{spt} \|V\|) = 0$, one infers that

$$\boldsymbol{\Theta}^m(\|V\|, z) = 0 \quad \text{implies} \quad \boldsymbol{\Theta}^{m(1+(1-1/q)p/(m-p))}(\mu, z) = 0$$

for $\mathscr{H}^m$ almost all $z \in U$. In *case* $p = m$, one notices that

$$\boldsymbol{\Theta}_*^m(\|V\|, z) \geq 1/2 \quad \text{whenever } z \in \operatorname{spt} \|V\|,$$
$$\operatorname{spt} \|V\| \cap \left\{z : \|\delta V\|(\{z\}) < (2\boldsymbol{\gamma}(m))^{-1}\right\} \subset U \cap \textstyle\bigcup_{i=1}^\infty \{c : G_i \cap \mathbf{B}(c, 1/i) = \varnothing\},$$
$$\operatorname{spt} \|V\| \cap \{z : \|\delta V\|(\{z\}) \geq (2\boldsymbol{\gamma}(m))^{-1}\} \text{ is countable}$$

by [23, 4.8 (4), 7.6] and [19, 2.5]

Therefore, one may assume in both cases that $Z \subset \{z : \boldsymbol{\Theta}^{*m}(\|V\|, z) > 0\}$. As this implies that Z is a union of countably many $\mathscr{H}^m$ measurable sets of finite $\mathscr{H}^m$ measure by [13, 2.10.19 (3)], one may also assume that Z is compact.

Define sets

$$Z_j = Z \cap \{z : \mu\, \mathbf{B}(z, r) \leq j r^m \omega(r+) \text{ for all } 0 < r < 1/j\}$$

whenever $j \in \mathscr{P}$ and $1/j < \operatorname{dist}(Z, \mathbf{R}^n \sim U)$ whose union contains

$$Z \cap \left\{ z : \limsup_{r \to 0+} r^{-m} \omega(r)^{-1} \mu\, \mathbf{B}(z, r) < \infty \right\}$$

and observe that the sets Z_j are compact, see [13, 2.9.14]. Hence, it is sufficient to prove for each $j \in \mathscr{P}$ with $1/j < \operatorname{dist}(Z, \mathbf{R}^n \sim U)$ that

$$\lim_{r \to 0+} r^{-m} \omega(r)^{-1} \mu\, \mathbf{B}(c, r) = 0 \quad \text{for } \mathscr{H}^m \text{ almost all } c \in Z_j.$$

In fact, this equality will be proven whenever $c \in Z_j$ and $j \in \mathscr{P}$ with $1/j < \operatorname{dist}(Z, \mathbf{R}^n \sim U)$ satisfy for some $i \in \mathscr{P}$ with $i \geq 2j$ that $\boldsymbol{\Theta}^m(\|V\| \llcorner U \sim Z_j, c) = 0$ and

$$\boldsymbol{\Theta}^{m^2/(m-p)}(\|V\| \llcorner G_i, c) = 0 \quad \text{if } p < m, \qquad G_i \cap \mathbf{B}(c, 1/i) = \varnothing \quad \text{if } p = m$$

as $\mathscr{H}^m$ almost all $c \in Z_j$ do according to [13, 2.10.19 (4)] and the second paragraph of this proof.

For such c, j, and i, suppose $0 < \varepsilon \leq (6m\boldsymbol{\gamma}(m))^{-1}$, $0 < r < 1/i$, and

$$\|V\|(\mathbf{B}(c, 2r) \sim Z_j) < \varepsilon^m r^m.$$

Whenever $\zeta \in \mathbf{B}(c, r) \cap (\operatorname{spt} \|V\|) \sim (G_i \cup Z_j)$ and $s = \operatorname{dist}(\zeta, Z_j)$ there exists $z \in Z_j$ with $s = |\zeta - z|$ and one infers, using [19, 2.5] with n, m, U, μ, ε, and ϱ replaced by m, $n - m$, $\mathbf{U}(\zeta, 1/i)$, $\|V\|$, $(2\boldsymbol{\gamma}(m))^{-1}$, and $s/2$, that

$$s \leq |\zeta - c| \leq r < 1/i \leq 1/(2j), \quad \mathbf{B}(\zeta, s/2) \subset \mathbf{B}(z, 3s/2) \cap \mathbf{B}(c, 2r) \sim Z_j,$$
$$(4m\boldsymbol{\gamma}(m))^{-m} s^m \leq \|V\|\, \mathbf{B}(\zeta, s/2) \leq \|V\|(\mathbf{B}(c, 2r) \sim Z_j) < \varepsilon^m r^m, \quad 3s/2 < r,$$
$$\mu\, \mathbf{B}(\zeta, s/2) \leq \mu\, \mathbf{B}(z, 3s/2) \leq j(3s/2)^m \omega((3s/2)+) \leq \gamma \omega(r) \|V\|\, \mathbf{B}(\zeta, s/2),$$

where $\gamma = j(6m\boldsymbol{\gamma}(m))^m$. Therefore the Besicovitch–Federer covering theorem yields the existence of countable disjointed families $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ of closed balls such that

$$\begin{aligned}\mathbf{B}(c,r) \cap (\operatorname{spt} \|V\|) \sim (G_i \cup Z_j) \subset \textstyle\bigcup\bigcup\{F_k : k = 1, \ldots, \boldsymbol{\beta}(n)\} \subset \mathbf{B}(c, 2r) \sim Z_j,\\ \mu(S) \leq \gamma\omega(r)\|V\|(S) \quad \text{whenever } S \in F_k \text{ and } k = 1, \ldots, \boldsymbol{\beta}(n).\end{aligned}$$

Recalling $\mu(Z) = 0$, the conclusion follows from

$$\begin{aligned}\mu(\mathbf{B}(c,r) \sim G_i) &= \mu(\mathbf{B}(c,r) \cap (\operatorname{spt} \|V\|) \sim (G_i \cup Z_j))\\ &\leq \sum_{k=1}^{\boldsymbol{\beta}(n)} \sum_{S \in F_k} \mu(S) \leq \sum_{k=1}^{\boldsymbol{\beta}(n)} \gamma\omega(r) \sum_{S \in F_k} \|V\|(S)\\ &\leq \boldsymbol{\beta}(n)\gamma\omega(r)\|V\|(\mathbf{B}(c, 2r) \sim Z_j) \leq \boldsymbol{\beta}(n)\gamma\varepsilon^m r^m \omega(r),\end{aligned}$$

since ε can be chosen arbitrary small. □

Remark 4.3. The preceding theorem is a slight generalisation of [19, 3.1] which treated the case that $\omega(r)$ equals a positive power of r; see [19, 3.2–3.4] for comments on earlier developments and the sharpness of certain hypotheses. The case $q = 1$ is excluded since in this case there is no modulus of continuity satisfying the condition on the limit inferior.

Corollary 4.4. *Suppose m, n, p, U, and V are as in 4.1, $p = m$, g maps $\|V\|$ almost all of U into $\{t : 0 \leq t \leq \infty\}$, $\operatorname{dmn} g \subset \operatorname{spt} \|V\|$, and $0 < q < \infty$.*

Then, for $\|V\|$ almost all z, either

(1) $\limsup_{\zeta \to z} |\zeta - z|^{-q} g(\zeta) = \infty$, *or*
(2) $g(z) = 0$ *and* $\lim_{\zeta \to z} |\zeta - z|^{-q} g(\zeta) = 0$.[3]

Proof. Define $C = \{z : g(z) = 0\}$ and $Z = U \cap \{\limsup_{\zeta \to z} |\zeta - z|^{-q} g(\zeta) < \infty\}$. Then Z is a Borel subset of U and $Z \cap (\operatorname{dmn} g) \sim C$ is countable; in fact, Z is the union of the relatively closed subsets of U defined by

$$Z_i = U \cap \{z : g(\zeta) \leq i|\zeta - z|^q \text{ whenever } \zeta \in \operatorname{dmn} g \text{ and } 0 < |\zeta - z| < 1/i\}$$

for $i \in \mathscr{P}$ and $Z_i \cap (\operatorname{dmn} g) \sim C$ is contained in the set of isolated points of Z_i. Therefore $\|V\|(Z \sim C) = 0$ and $C \cap Z$ is $\|V\|$ measurable. Hence, applying 4.2 with $\omega(r)$, $\mu(B)$, and Z replaced by r, $(\sup(\{0\} \cup g[B]))^{(m+1)/q}$, and $C \cap Z$, one derives

$$\limsup_{\zeta \to z} |\zeta - z|^{-q} g(\zeta) = 0 \quad \text{for } \|V\| \text{ almost all } z \in Z.$$

As $\operatorname{dmn} g$ does not contain isolated points, the conclusion follows. □

Remark 4.5. Clearly, the special case [19, 3.1] of 4.2 would also be sufficient for the proof of the corollary.

[3] Our usage of limits is affected by the requirement

$$\limsup_{z \to c} g(z) = \lim_{\varepsilon \to 0+} \sup\{g(z) : z \in \mathbf{B}(c, \varepsilon) \cap \operatorname{dmn} g\},$$

in particular $\limsup_{z \to c} g(z) = -\infty$ if $c \notin \operatorname{Clos} \operatorname{dmn} g$.

5. Quadratic tilt-excess, an example

The purpose of this section is to exhibit a two dimensional integral varifold whose first variation is representable by integration in order to render the decay estimates of Sect. 9 sharp. The varifold constructed for that purpose in 5.8 will in fact be associated to the graph of a Lipschitzian function with small Lipschitz constant.

The basic building block of this example (see Fig. 1) is a varifold consisting of a piece of a sphere, a piece of a catenoid and a smooth join to a plane, see 5.6. Suitably rescaled copies of this varifold will then be used to fill the holes of a set previously constructed in 2.5.

5.1. (see [13, 5.1.9]) Suppose $1 < n \in \mathscr{P}$ and $\mathbf{p} : \mathbf{R}^n \to \mathbf{R}^{n-1}$ and $\mathbf{q} : \mathbf{R}^n \to \mathbf{R}$ satisfy

$$\mathbf{p}(z) = (z_1, \ldots, z_{n-1}) \quad \text{and} \quad \mathbf{q}(z) = z_n \quad \text{whenever } z = (z_1, \ldots, z_n) \in \mathbf{R}^n.$$

Then the statements of Allard [4, 8.9] may be supplemented as follows.

(1) If $S, T \in \mathbf{G}(n, n-1)$, then $|S_\natural - T_\natural| = 2^{1/2} \|S_\natural - T_\natural\|$.
(2) If $L \in \operatorname{Hom}(\mathbf{R}^{n-1}, \mathbf{R})$, $S = \mathbf{R}^n \cap \{z : L(\mathbf{p}(z)) = \mathbf{q}(z)\}$, and $T = \operatorname{im} \mathbf{p}^*$, then $|L| = \|L\|$ and $\|S_\natural - T_\natural\| = (1 + \|L\|^2)^{-1/2} \|L\|$.

In fact, if $v \in \mathbf{S}^{n-1}$ and with $S = \{z : z \bullet v = 0\}$ then (1) is implied by

$$2^{-1} |S_\natural - T_\natural|^2 = T_\natural \bullet S_\natural^\perp = |T_\natural(v)|^2 = \|T_\natural \circ S_\natural^\perp\|^2 = \|S_\natural - T_\natural\|^2$$

and in case of (2) one may take $v = (1 + \|L\|^2)^{-1/2} (\mathbf{p}^*(L^*(1)) - \mathbf{q}^*(1))$.

5.2. If $1 < n \in \mathscr{P}$, I is an open subset of $\{t : 0 < t < \infty\}$, and $g : I \to \mathbf{R}$ is of class 2, then $N = \mathbf{R}^n \cap \{z : \mathbf{q}(z) = g(|\mathbf{p}(z)|)\}$, see 5.1, is an $n-1$ dimensional submanifold of $\mathbf{R}^n$ of class 2 and if $z \in N$ and $t = |\mathbf{p}(z)|$ then

$$\begin{aligned}
|\mathbf{h}(N, z)| &= (1 + g'(t)^2)^{-1/2} \left| (n-2) t^{-1} g'(t) + (1 + g'(t)^2)^{-1} g''(t) \right| \\
&\le (n-2) t^{-1} |g'(t)| + |g''(t)|, \\
\|\mathbf{b}(N, z)\| &= (1 + g'(t)^2)^{-1/2} \sup\{t^{-1} |g'(t)|, (1 + g'(t)^2)^{-1} |g''(t)|\} \\
&\le \sup\{t^{-1} |g'(t)|, |g''(t)|\}
\end{aligned}$$

as may be verified using the formulae occurring in [14, pp. 356–357, 388–391].

5.3. We will employ the area cosinus hyperbolicus, $\operatorname{arcosh} : \{t : 1 \le t < \infty\} \to \mathbf{R}$, given by $\operatorname{arcosh}(t) = \log \big(t + (t^2 - 1)^{1/2} \big)$ for $1 \le t < \infty$. Notice that

$$\operatorname{arcosh}'(t) = (t^2 - 1)^{-1/2}, \quad \operatorname{arcosh}''(t) = -t (t^2 - 1)^{-3/2}$$

for $1 < t < \infty$, hence for $3/2 \le t < \infty$ also that

$$\log t \le \operatorname{arcosh}(t) \le 3 \log t, \quad 1/t \le \operatorname{arcosh}'(t) \le 3/t, \quad -3/t^2 \le \operatorname{arcosh}''(t) < 0.$$

Moreover, $\operatorname{arcosh}(t) - \operatorname{arcosh}(s) \ge \log(t/s)$ for $1 \le s \le t < \infty$.

Lemma 5.4. *Suppose* $N = \mathbf{R}^3 \cap \{z : |\mathbf{q}(z)| = \operatorname{arcosh}(|\mathbf{p}(z)|)\}$, *see* 5.1 *and* 5.3.

Then N is a 2 dimensional submanifold of $\mathbf{R}^3$ of class ∞, $\mathbf{h}(N,z)=0$ for $z\in N$, and whenever $1\le r<\infty$ there holds

$$\mathscr{H}^2\left(N\cap\mathbf{p}^{-1}[\mathbf{B}(0,r)]\right)=2\boldsymbol{\alpha}(2)\left(\operatorname{arcosh}(r)+r(r^2-1)^{1/2}\right),$$
$$\int_{N\cap\mathbf{p}^{-1}[\mathbf{B}(0,r)]}|\operatorname{Tan}(N,z)_\natural-\mathbf{p}^*\circ\mathbf{p}|^2\,\mathrm{d}\mathscr{H}^2z=8\boldsymbol{\alpha}(2)\operatorname{arcosh}(r).$$

Proof. The asserted equations are readily verified by means of 5.1–5.3. □

Remark 5.5. The surface N is known as the catenoid, see e.g. [28, p. 18].

Lemma 5.6. *Suppose $n=3$, $\mathbf{p}$ and $\mathbf{q}$ are related to n as in 5.1, and $2\le s\le r/2<\infty$.*

Then there exist $h:\mathbf{R}^2\to\mathbf{R}$ of class 1 and $V\in\mathbf{IV}_2(\mathbf{R}^3)$ satisfying

$$\begin{aligned}&\operatorname{Lip}h\le\Gamma/s,\quad \operatorname{Lip}\mathrm{D}\,h<\infty,\quad \|V\|=\mathscr{H}^2\llcorner\operatorname{im}(\mathbf{p}^*+\mathbf{q}^*\circ h),\\ &\|V\|\llcorner\mathbf{p}^{-1}[\mathbf{R}^2\sim\mathbf{U}(0,r)]=\mathscr{H}^2\llcorner\mathbf{p}^*[\mathbf{R}^2\sim\mathbf{U}(0,r)],\\ &1\le\boldsymbol{\alpha}(2)^{-1}r^{-2}\|V\|\left(\mathbf{p}^{-1}[\mathbf{U}(0,r)]\right)\le 1+\Gamma r^{-2}\log r,\quad \|\delta V\|(\mathbf{R}^3)\le\Gamma,\\ &\|\delta V\|\text{ is absolutely continuous with respect to }\|V\|,\\ &\boldsymbol{\Theta}^2(\|V\|,z)=1\text{ and }|\mathbf{q}(z)|\le 3\log r\text{ whenever }z\in\operatorname{spt}\|V\|,\\ &\int_{\mathbf{p}^{-1}[\mathbf{U}(0,r)]\times\mathbf{G}(3,2)}|S_\natural-\mathbf{p}^*\circ\mathbf{p}|^2\,\mathrm{d}V(z,S)\ge\log(r/(2s)),\end{aligned}$$

where Γ is a universal, positive, finite number.

Proof. Abbreviate $f_1=\operatorname{arcosh}$, see 5.3. Define $d:\{\sigma:1<\sigma<\infty\}\to\mathbf{R}$ by

$$d(\sigma)=f_1(\sigma)+\sigma/f_1'(\sigma)$$

for $1<\sigma<\infty$ and note that $d(\sigma)-\sigma^2$ is a nondecreasing as a function of σ with $d(\sigma)-\sigma^2\to-1$ as $\sigma\to 1+$, hence

$$d(\sigma)-\sigma^2\ge-1\quad\text{for }1<\sigma<\infty.$$

Define $f_2:\{t:-s^2<t<s^2\}\to\mathbf{R}$ by

$$f_2(t)=a(s)-\left(s^4-t^2\right)^{1/2}\quad\text{for }-s^2<t<s^2,$$

hence $f_2(t)\ge-1$ for $-s^2<t<s^2$. Choose $\gamma\in\mathscr{E}(\mathbf{R},\mathbf{R})$ with $0\le\gamma\le 1$ and

$$\gamma(t)=1\quad\text{if }t\le 1/2,\qquad\gamma(t)=0\quad\text{if }t\ge 1,\qquad -3\le\gamma'(t)\le 0$$

whenever $t\in\mathbf{R}$. Noting

$$s^4-s^2=s^2/f_1'(s)^2>0,\quad f_2(s)=f_1(s),\quad f_2'(s)=f_1'(s),$$

one defines a function $g:\mathbf{R}\to\mathbf{R}$ of class 1 with $\operatorname{Lip}g'<\infty$ by

$$g(t)=f_2(|t|)-f_1(r)\quad\text{if }|t|\le s,\qquad g(t)=(f_1(|t|)-f_1(r))\gamma(|t|/r)\quad\text{else}$$

whenever $t\in\mathbf{R}$.

One computes

$$\begin{aligned}g'(t)&=f_1'(t)\gamma(t/r)+(f_1(t)-f_1(r))r^{-1}\gamma'(t/r),\\ g''(t)&=f_1''(t)\gamma(t/r)+2f_1'(t)r^{-1}\gamma'(t/r)+(f_1(t)-f_1(r))r^{-2}\gamma''(t/r)\end{aligned}$$

for $s < t < \infty$, hence, taking into account 5.3, there exists a positive, finite number Δ_1 determined by γ such that

$$\begin{aligned}
&-\Delta_1 \log r \le g(t) \le 0 \quad \text{for } 0 \le t < \infty,\\
&0 \le g'(t) \le \Delta_1 \inf\{1/s, 1/t\} \quad \text{for } 0 \le t < r,\\
&g'(t) \ge 1/t \quad \text{for } s \le t \le r/2, \quad |g''(t)| \le \Delta_1 r^{-2} \quad \text{for } r/2 < t < r.
\end{aligned}$$

Define $h : \mathbf{R}^2 \to \mathbf{R}$ by

$$h(x) = g(|x|) \quad \text{for } x \in \mathbf{R}^2$$

and note that h is of class 1 with

$$\operatorname{Lip} h \le \Delta_1/s, \quad \operatorname{Lip} \mathrm{D}\, h < \infty.$$

Let $M = \mathbf{R}^3 \cap \{z : \mathbf{q}(z) = h(\mathbf{p}(z))\}$, define $V \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$V(k) = \int_M k(z, \operatorname{Tan}(M, z))\, \mathrm{d}\mathscr{H}^2 z \quad \text{for } k \in \mathscr{K}(\mathbf{R}^3 \times \mathbf{G}(3,2))$$

and observe

$$\delta V(\theta) = -\int_M \mathbf{h}(M, z) \bullet \theta(z)\, \mathrm{d}\mathscr{H}^2 z \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^3, \mathbf{R}^3).$$

Since $M \cap \mathbf{p}^{-1}[\mathbf{U}(0,s)]$ is a piece of a sphere of radius s^2 one computes

$$\begin{aligned}
&|\mathbf{h}(M, z)| = 2s^{-2} \quad \text{whenever } z \in M \cap \mathbf{p}^{-1}[\mathbf{U}(0,s)],\\
&\int_{M \cap \mathbf{p}^{-1}[\mathbf{U}(0,s)]} |\mathbf{h}(M,z)|\, \mathrm{d}\mathscr{H}^2 z = 4\boldsymbol{\alpha}(2) s/(s + (s^2-1)^{1/2}) \le 4\boldsymbol{\alpha}(2)
\end{aligned}$$

and since $M \cap \mathbf{p}^{-1}[\mathbf{B}(0,r/2) \sim \mathbf{B}(0,s)]$ is a piece of a catenoid one obtains

$$\mathbf{h}(M, z) = 0 \quad \text{whenever } z \in M \cap \mathbf{p}^{-1}[\mathbf{B}(0,r/2) \sim \mathbf{B}(0,s)]$$

from 5.4. Moreover, recalling $2 \le s \le r/2$, one estimates

$$\begin{aligned}
0 &\le \mathscr{H}^2\left(M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r)]\right) - \boldsymbol{\alpha}(2) r^2\\
&= 2\boldsymbol{\alpha}(2) \int_0^r \left((1 + g'(t)^2)^{1/2} - 1\right) t\, \mathrm{d}\mathscr{L}^1 t\\
&\le \boldsymbol{\alpha}(2) \int_0^r g'(t)^2 t\, \mathrm{d}\mathscr{L}^1 t \le 2\boldsymbol{\alpha}(2) \Delta_1^2 \log r.
\end{aligned}$$

From 5.2 and the estimates for g' and g'', one obtains a positive, finite number Δ_2 determined by γ such that

$$|\mathbf{h}(M,z)| \le \Delta_2 r^{-2} \quad \text{whenever } z \in M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r) \sim \mathbf{B}(0,r/2)].$$

Combining the preceding estimates, we obtain, recalling $r \ge 2$, that

$$\|\delta V\|(\mathbf{R}^3) \le 4\boldsymbol{\alpha}(2) + \boldsymbol{\alpha}(2)\Delta_2(1 + \Delta_1^2).$$

Since $M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r/2) \sim \mathbf{B}(0,s)]$ is a piece of a catenoid, 5.4 implies

$$\int_{\mathbf{p}^{-1}[\mathbf{U}(0,r)] \times \mathbf{G}(3,2)} |S_\natural - \mathbf{p}^* \circ \mathbf{p}|^2\, \mathrm{d}V(z,S) \ge 8\boldsymbol{\alpha}(2)(\operatorname{arcosh}(r/2) - \operatorname{arcosh}(s)),$$

hence 5.3 implies the conclusion. □

5.7. Occasionally, we denote the open cube with centre a and side length $2r$ by

$$O(a,r) = \mathbf{R}^m \cap \{(x_1,\ldots,x_m) : |x_i - a_i| < r \text{ for } i = 1,\ldots,m\}$$

for $m \in \mathscr{P}$, $a = (a_1,\ldots,a_m) \in \mathbf{R}^m$, and $0 < r < \infty$.

Example 5.8. Suppose $n = 3$, $\mathbf{p}$ and $\mathbf{q}$ are related to n as in 5.1, $T = \operatorname{im}\mathbf{p}^*$, $\varepsilon > 0$, and ω is a modulus of continuity.

Then there exist $f : \mathbf{R}^2 \to \mathbf{R}$, $C \subset T$, and $V \in \mathbf{IV}_2(\mathbf{R}^3)$ satisfying

$$\begin{aligned}
&\operatorname{Lip} f \le \varepsilon, \quad \|V\| = \mathscr{H}^2 \llcorner \operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f), \quad \|V\|(C) > 0,\\
&\|\delta V\|(\mathbf{R}^3) < \infty, \quad \|\delta V\| \text{ is absolutely continuous with respect to } \|V\|,\\
&\limsup_{r\to 0+} r^{-1}(\log(1/r))^{-1/2}\omega(r)^{-1} \left(\fint_{\mathbf{B}(c,r)\times\mathbf{G}(3,2)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z,S) \right)^{1/2} > 0
\end{aligned}$$

whenever $c \in C$, here $0^{-1} = \infty$.

Proof. Take G and A as furnished by 2.5 with m and λ replaced by 2 and $1/2$, abbreviate $\Delta = \Gamma_{5.6}$, $\lambda = \inf\{1/4, \varepsilon/(2\Delta)\}$, and let $C = \mathbf{p}^*[A]$. Define $W \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$W(k) = \int_{T \sim \mathbf{p}^*[\bigcup G]} k(z,T) \, \mathrm{d}\mathscr{H}^2 z \quad \text{for } k \in \mathscr{K}(\mathbf{R}^3 \times \mathbf{G}(3,2)).$$

Whenever $Q = O(a,t) \in G$ and $t > \lambda$ let $f_Q : \mathbf{R}^2 \to \mathbf{R}$ and $X_Q \in \mathbf{IV}_2(\mathbf{R}^3)$ be defined by

$$f_Q(x) = 0 \quad \text{for } x \in \mathbf{R}^2, \qquad \|X_Q\| = \mathscr{H}^2 \llcorner \operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f_Q).$$

Whenever $Q = O(a,t) \in G$ and $t \le \lambda$ apply 5.6 with s and r replaced by $(2\lambda)^{-1}$ and $1/t$ to construct $f_Q : \mathbf{R}^2 \to \mathbf{R}$ and $X_Q \in \mathbf{IV}_2(\mathbf{R}^3)$ such that $t^{-2} f_Q \circ \boldsymbol{\tau}_a \circ \boldsymbol{\mu}_{t^2}$ and $(\boldsymbol{\mu}_{t^{-2}} \circ \boldsymbol{\tau}_{-a})_\# X_Q$ satisfy the conditions of 5.6 in place of h and V implying

$$\begin{aligned}
&\operatorname{spt} f_Q \subset \operatorname{Clos} Q, \quad \operatorname{Lip} f_Q \le \varepsilon, \quad \|X_Q\| = \mathscr{H}^2 \llcorner \operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f_Q),\\
&\|X_Q\| \llcorner \mathbf{p}^{-1}[\mathbf{R}^2 \sim Q] = \mathscr{H}^2 \llcorner \mathbf{p}^*[\mathbf{R}^2 \sim Q],\\
&\|\delta X_Q\|(\mathbf{R}^3) \le \Delta t^2, \quad \|\delta X_Q\| \text{ is absolutely continuous with respect to } \|X_Q\|,\\
&|\mathbf{q}(z)| \le 3t^2 \log(1/t) \quad \text{whenever } z \in \operatorname{spt}\|X_Q\|,\\
&\int_{\mathbf{p}^{-1}[Q]\times\mathbf{G}(3,2)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(z,S) \ge t^4 \left(\log(1/t) - \log(1/\lambda)\right).
\end{aligned}$$

Recall that G is disjointed and define $f : \mathbf{R}^2 \to \mathbf{R}$ and $V \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$f(x) = \sum_{Q \in G} f_Q(x) \quad \text{for } x \in \mathbf{R}^2, \qquad \|V\| = \mathscr{H}^2 \llcorner \operatorname{im}(\mathbf{p}^* + \mathbf{q}^* \circ f).$$

Note that $V = W + \sum_{Q\in G} X_Q \llcorner (\mathbf{p}^{-1}[Q] \times \mathbf{G}(3,2))$ and $\|V\|(C) \ge 1/2$. Observe

$$\|\delta V\|(\mathbf{R}^3) < \infty, \quad \|\delta V\| \text{ is absolutely continuous with respect to } \|V\|.$$

Suppose $c \in C$ and $\delta > 0$.

Then there exist r and $O(a,t) = Q$ such that

$$0 < r \le \delta, \quad Q \in G, \quad Q \subset \mathbf{U}(a,r), \quad \mathscr{L}^2(Q) \ge \omega(r) r^2,$$

hence $t \le r$. Since $(2t)^2 \ge \omega(r) r^2$ and

$$t^4 \left(\log(1/t) - \log(1/\lambda)\right) \ge 2^{-4} \omega(r)^2 r^4 \left(\log(1/r) - \log(1/\lambda)\right),$$

the estimates for X_Q imply the assertion. □

6. A coercive estimate

In this section we provide, in 6.6, the first main ingredient for the proof of the decay rates almost everywhere of the quadratic tilt-excess of two dimensional integral varifolds whose first variation is representable by integration, namely a coercive estimate. In this estimate the quadratic tilt-excess is controlled by the variation measure of the first variation and the height-excess. In order to be effective for the present purpose, two aspects are crucial. Firstly, in the height-excess only the set of points where the density ratio is bounded from below are taken into account. Secondly, the height-excess term which is multiplied by a first variation term is measured in the Orlicz space seminorm naturally corresponding to square summable weak derivatives in two dimensions.

In the basic form of such coercive estimate all quantities are measured as square integrals, see Allard [4, 8.13]. In [8, 5.5] Brakke devised an interpolation procedure to obtain estimates in which the variation is measured by its variation measure. This was further refined in [21, 4.14] by allowing the height-excess to be measured in different Lebesgue spaces and in [21, 4.10] by restricting the height-excess to the set of points where the density ratio is bounded from below using a possibly discontinuous "cut-off" function, see 6.7. In 6.6 we additionally refine Brakke's interpolation procedure to include the relevant Orlicz space norm.

Definition 2. If $\Phi : \{t : 0 \le t < \infty\} \to \{t : 0 \le t < \infty\}$ is a nondecreasing convex function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$, μ measures X, and Z is a Banach space, then one defines the seminorm $\mu_{(\Phi)}$ on $\mathbf{A}(\mu, Z)$ by

$$\mu_{(\Phi)}(f) = \inf \left\{ \lambda : 0 < \lambda \le \infty, \int \Phi \circ |\lambda^{-1} f| \, \mathrm{d}\mu \le 1 \right\} \quad \text{for } f \in \mathbf{A}(\mu, Z).$$

Remark 6.1. Notice that $\mu_{(\Phi)}(f) = 0$ if and only if $f(x) = 0$ for μ almost all x. Moreover, if $\mu_{(\Phi)}(f) > 0$ then $\int \Phi \circ |\lambda^{-1} f| \, \mathrm{d}\mu \le 1$ for $\lambda = \mu_{(\Phi)}(f)$ and equality holds if $\int \Phi \circ |s^{-1} f| \, \mathrm{d}\mu < \infty$ for some $0 < s < \mu_{(\Phi)}(f)$.

Remark 6.2. The functions Φ and $\mu_{(\Phi)}$ are a "Young's function" and its corresponding "Luxemburg norm" in the terminology of [9, Chapter 4, 8.1, 8.6].

Remark 6.3. Suppose Φ, μ, X and Z are as in 2. Then the following basic properties hold.

(1) If $0 < c < \infty$ and $f \in \mathbf{A}(\mu, Z)$, then $(c\mu)_{(\Phi)}(f) = \mu_{(c\Phi)}(f)$.
(2) If $0 < \varepsilon \le 1$ and $f \in \mathbf{A}(\mu, Z)$, then $\varepsilon \mu_{(\Phi)}(f) \le \mu_{(\varepsilon\Phi)}(f)$.

(3) If $u : X \to Y$, $f : Y \to Z$, and $f \circ u \in \mathbf{A}(\mu, Z)$, then $f \in \mathbf{A}(u_\# \mu, Z)$ and $(u_\# \mu)_{(\Phi)}(f) = \mu_{(\Phi)}(f \circ u)$, see [13, 2.1.2, 2.4.18 (1)].

6.4. Suppose $2 \leq m \in \mathscr{P}$ and $\kappa : \{t : 0 \leq t < \infty\} \to \{t : 0 \leq t < \infty\}$ satisfies

$$\kappa(0) = 0, \qquad \kappa(t) = t\left(1 + (\log(1 + 1/t))^{1-1/m}\right) \quad \text{for } 0 < t < \infty.$$

Then one verifies that κ is continuous increasing and concave, in particular $\kappa(\tau t) \leq \tau \kappa(t)$ for $1 \leq \tau < \infty$ and $0 \leq t < \infty$.

6.5. Suppose $2 \leq m \in \mathscr{P}$ and $\Phi : \{t : 0 \leq t < \infty\} \to \{0 \leq t < \infty\}$ is defined by

$$\Phi(t) = \exp\left(t^{m/(m-1)}\right) - 1 \quad \text{for } 0 \leq t < \infty.$$

Then Φ satisfies the conditions of 2, and Φ maps $\{t : 0 \leq t < \infty\}$ univalently onto $\{t : 0 \leq t < \infty\}$ with

$$\Phi^{-1}(t) = (\log(1 + t))^{1-1/m} \quad \text{for } 0 \leq t < \infty.$$

Therefore $\mu_{(\Phi)}(1) = 1/\Phi^{-1}(1/\mu(X))$ whenever μ measures X and $0 < \mu(X) < \infty$ by 6.1. Notice, if $0 \leq \alpha < \infty$ then

$$\inf\{\alpha t + 1/\Phi(t) : 0 < t < \infty\} \leq \kappa(\alpha),$$

where κ is as in 6.4; in fact, consider $t = \Phi^{-1}(1/\alpha)$ if $\alpha > 0$.

Theorem 6.6. *Suppose m, n, p, U, and V are as in 4.1, $p = 1 < m$, κ and Φ are related to m as in 6.4 and 6.5, C and K are compact subsets of U, $C \subset K$, $0 < r < \infty$, H is the set of all $z \in \operatorname{spt} \|V\|$ such that*

$$\|V\| \mathbf{B}(z, s) \geq (40 \boldsymbol{\gamma}(m) m)^{-m} s^m \quad \textit{whenever } 0 < s < \infty,\ \mathbf{B}(z, s) \subset K,$$

$c \in \mathbf{R}^n$, $T \in \mathbf{G}(n, m)$, and $h : U \to \mathbf{R}$ satisfies $h(z) = \operatorname{dist}(z - c, T)$ for $z \in U$.

Then there holds

$$r^{-m} \int_{\{z : \mathbf{U}(z,r) \subset C\} \times \mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z, S) \leq \Gamma \Big(r^{-m} \|\delta V\|(K)^{m/(m-1)}$$
$$+ \kappa(r^{-m} \|\delta V\|(K) (\|V\| \llcorner C \cap H)_{(r^{-m}\Phi)}(h)) + r^{-m-2} \textstyle\int_{C \cap H} |h|^2 \, \mathrm{d}\|V\| \Big),$$

where Γ is a positive, finite number depending only on m.

Proof. Assume $c = 0$, hence $h = \left|T_\natural^\perp | U\right|$, and notice that in view of 6.3, Allard [4, 3.2 (2), 4.12 (1)], one may employ homotheties to reduce the problem to the case $r = 1$. Abbreviate $\mu = (\|V\| \llcorner C \cap H)_{(\Phi)}(h)$ and denote $(\|V\|, m)$ approximate differentials by "ap D".

Select $\phi \in \mathscr{D}(U, \mathbf{R})$ with

$$0 \leq \phi \leq 1, \quad \operatorname{spt} \phi \subset C, \quad \{z : \mathbf{U}(z, 1) \subset C\} \subset \{z : \phi(z) = 1\}, \quad |\mathrm{D}\, \phi| \leq 2.$$

Using [21, 4.7] with $\delta = \frac{1}{40}$, one obtains a Borel function $f : U \to \{t : 0 \leq t \leq 1\}$ with $f | U \sim K = 0$ such that the varifolds $V_1, V_2 \in \mathbf{RV}_m(U)$ defined by

$$V_1(A) = \int_A^* f(z) \, \mathrm{d}V(z, S) \quad \text{for } A \subset U \times \mathbf{G}(n, m)$$

and $V_2 = V - V_1$ satisfy

$$f(z) = 1 \text{ and } \operatorname{ap} \mathrm{D} f(z) = 0 \text{ for } \|V\| \text{ almost all } z \in U \sim H,$$

$$\int \phi(z)^2 \|S_\natural - T_\natural\|^2 \, \mathrm{d} V_1(z,S) \le 4 \|V_1\|(K) \le \Delta \|\delta V\|(K)^{m/(m-1)},$$

$$\|\delta V_2\| \le (1-f)\|\delta V\| + |\operatorname{ap} \mathrm{D} f| \|V\|, \quad \|V\|(|\operatorname{ap} \mathrm{D} f|) \le (400)^m \|\delta V\|(K),$$

where $\Delta = 4(400)^{m^2/(m-1)}(\boldsymbol{\gamma}(m)m)^{m/(m-1)}$; compare [21, p. 24, l. 14–20]. In particular, one infers

$$\|V_2\| \le \|V\| \llcorner H, \quad \|\delta V_2\|(K) \le (800)^m \|\delta V\|(K).$$

Defining $g = \phi^2 (T_\natural^\perp | U)$, one derives

$$\int \phi(z)^2 \|S_\natural - T_\natural\|^2 \, \mathrm{d} V_2(z,S) \le \sup \left\{ 16 \int |\mathrm{D} \phi|^2 |h|^2 \, \mathrm{d} \|V_2\|, 2|(\delta V_2)(g)| \right\}$$

as in Brakke [8, 5.5, p. 139, l. 1–14], hence

$$\int_{\{z : \mathbf{U}(z,1) \subset C\} \times \mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2 \, \mathrm{d} V(z,S)$$
$$\le \Delta \|\delta V\|(K)^{m/(m-1)} + 2|(\delta V_2)(g)| + 64 \textstyle\int_{C \cap H} |h|^2 \, \mathrm{d} \|V\|.$$

If $\mu = 0$, then $g(z) = 0$ for $\|V\|$ almost all $z \in H$, hence $\mathrm{D}\, g(z) | \operatorname{Tan}^m(\|V\|, z) = 0$ for $\|V\|$ almost all $z \in H$ by [13, 2.10.19 (4), 3.2.16] and $(\delta V_2)(g) = 0$. Therefore one may assume $\mu > 0$.

In order to estimate $|(\delta V_2)(g)|$, suppose $0 < t < \infty$, define $\eta : \{s : 0 \le s < \infty\} \to \mathbf{R}$ by

$$\eta(0) = 1, \qquad \eta(s) = \inf\{1, t/s\} \quad \text{for } 0 < s < \infty.$$

Moreover, let $Z = \{z : t < h(z)\}$ and define Lipschitzian maps by

$$g_1 = \phi^2 (\eta \circ h) T_\natural^\perp | U, \quad g_2 = g - g_1.$$

Since $g_2 | U \sim Z = 0$, one notices that

$$\operatorname{ap} \mathrm{D}\, g_2(z) = 0 \quad \text{for } \|V\| \text{ almost all } z \in U \sim Z$$

by [13, 2.10.19 (4)]. Additionally, one computes

$$|g_1(z)| \le t \quad \text{for } z \in U, \qquad \|\mathrm{D}\, g_2(z)\| \le 2\phi(z)^2 + |\mathrm{D} \phi(z)|^2 h(z)^2 \quad \text{for } z \in Z,$$

see the case $r = 1$ of [21, 4.10, p. 24, l. 26–p. 25, l. 12]. It follows that

$$\|\mathrm{D}\, g_2(z)\| \le 2\phi(z)^2 \Phi(t/\mu)^{-1} \Phi(h(z)/\mu) + |\mathrm{D} \phi(z)|^2 h(z)^2 \quad \text{for } z \in Z.$$

Therefore one estimates, using [21, 4.5 (4)] and 6.1,

$$|(\delta V_2)(g)| \le t \|\delta V_2\|(K) + 2\Phi(t/\mu)^{-1} \int_C \Phi \circ |\mu^{-1} h| \, \mathrm{d} \|V_2\| + 4 \int_C |h|^2 \, \mathrm{d} \|V_2\|$$
$$\le (800)^m t \|\delta V\|(K) + 2\Phi(t/\mu)^{-1} + 4 \int_{C \cap H} |h|^2 \, \mathrm{d} \|V\|,$$

hence, noting 6.5 with $\alpha = \|\delta V\|(K)\mu$, one may take $\Gamma = \Delta + (1600)^m$. □

Remark 6.7. Notice that the function f furnished by [21, 4.7] will necessarily be discontinuous in some cases, see [21, 4.8]. However, inspecting the proof of [21, 4.7] and using [23, 8.7], one is at least assured that f is generalised weakly differentiable in the sense of [23, 8.3] with

$$V\ \mathbf{D}\, f(z) = (\|V\|, m)\, \mathrm{ap}\, \mathrm{D}\, f(z) \circ \mathrm{Tan}^m(\|V\|, z)_\natural \quad \text{for } \|V\| \text{ almost all } z.$$

7. Approximation

In this section we construct a real valued Lipschitzian auxiliary function from an integral varifold in a cylinder whose first variation is representable by integration, see 7.7. This auxiliary function captures information on the height-excess measured in Lebesgue and Orlicz spaces and the quadratic tilt-excess of the varifold. In conjunction with the basic coercive estimate in 6.6 and the interpolation inequalities of Sect. 8 it will be used in 9.1 to obtain a coercive estimate involving an approximate height quantity.

The auxiliary function is constructed as "upper envelope of the modulus" of an approximating Lipschitzian $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued function constructed in [20, 3.15]. Approximations by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions are a powerful tool, originating from Almgren [7, 3.1–3.12], whose handling is at times complex. The fact that in the present setting we are able to encapsulate their usage in the construction of the real valued auxiliary function considerably simplifies our proof of decay rates for the quadratic tilt-excess in 9.2.

Definition 3. (see Almgren [7, 1.1 (1) (2)]) Suppose Q is a positive integer and Y is a finite dimensional inner product space. Then

$$\mathbf{Q}_Q(Y) = \left\{ \sum_{i=1}^{Q} [y_i] : y_1, \ldots, y_Q \in Y \right\}$$

is metrised by $\mathscr{G}$ such that, whenever $y_1, \ldots, y_Q \in Y$ and $\upsilon_1, \ldots, \upsilon_Q \in Y$, $\mathscr{G}\big(\sum_{i=1}^{Q} [y_i], \sum_{i=1}^{Q} [\upsilon_i]\big)$ equals the infimum of the set of numbers

$$\left(\sum_{i=1}^{Q} \left| y_i - \upsilon_{\pi(i)} \right|^2 \right)^{1/2}$$

corresponding to all permutations π of $\{1, \ldots, Q\}$.

Definition 4. (see Almgren [7, 1.1 (9) (10)]) Suppose m and Q are positive integers and Y is a finite dimensional inner product space.

A function $f : \mathbf{R}^m \to \mathbf{Q}_Q(Y)$ is called *affine* if and only if there exist affine functions $f_i : \mathbf{R}^m \to Y$ corresponding to $i = 1, \ldots, Q$ such that

$$f(x) = \sum_{i=1}^{Q} [f_i(x)] \quad \text{whenever } x \in \mathbf{R}^m$$

and in this case $\|f\| = \mathrm{Lip}\, f$. Moreover, if $a \in A \subset \mathbf{R}^m$ and $f : A \to \mathbf{Q}_Q(Y)$ then f is *affinely approximable at* a if and only if $a \in \mathrm{Int}\, A$ and there exists an affine function $g : \mathbf{R}^m \to \mathbf{Q}_Q(Y)$ such that

$$g(a) = f(a) \quad \text{and} \quad \lim_{x \to a} \mathscr{G}(f(x), g(x))/|x-a| = 0.$$

The function g is unique and denoted by $\mathrm{A}\, f(a)$. The concept of *approximate affine approximability* is obtained through replacement of the condition $a \in \operatorname{Int} A$ by $a \in A$ and replacement of lim by ap lim. The corresponding affine function is denoted by $\mathrm{ap}\,\mathrm{A}\, f(a)$.

Remark 7.1. In comparison to Almgren [7, 1.1 (10)], the requirement "$g(a) = f(a)$" has been added. Consequently, [approximate] affine approximability implies [approximate] continuity. Moreover, supposing $Q = 1$ and denoting by $i : \mathbf{Q}_1(Y) \to Y$ the canonical isometry, the function f is [approximately] affinely approximable at a if and only if $i \circ f$ is [approximately] differentiable at a, see [13, 3.1.1, 3.1.2], and in this case

$$i \circ \mathrm{ap}\,\mathrm{A}\, f(a) = i(f(a)) + \mathrm{ap}\,\mathrm{D}(i \circ f)(a).$$

7.2. Suppose Q is a positive integer, Y is a finite dimensional inner product space, $a \in \mathbf{R}^m$, and $f : \mathbf{R}^m \to \mathbf{Q}_Q(Y)$ is affine. Then

$$\operatorname{Lip} f = \limsup_{x \to a} |x-a|^{-1}\mathscr{G}(f(x), f(a)) = \mathrm{ap}\,\limsup_{x \to a} |x-a|^{-1}\mathscr{G}(f(x), f(a));$$

in fact, in view of [7, 1.1 (9)] only the last equation needs to be proven. For this purpose denote the approximate limit superior by λ and define the Lipschitzian function $g : \mathbf{R}^m \to \mathbf{R}$ by $g(x) = \mathscr{G}(f(x), f(a)) - \lambda|x-a|$ for $x \in \mathbf{R}^m$. One infers $\mathrm{ap}\,\mathrm{D}\, g^+(a) = 0$, whence it follows $\mathrm{D}\, g^+(a) = 0$ by [13, 3.1.5] with C, B, f, η, and M replaced by $\mathbf{R}^m$, $\mathbf{R}^m$, g^+, 1, and $\operatorname{Lip} g$; therefore $\limsup_{x \to a} |x-a|^{-1}\mathscr{G}(f(x), f(a)) \le \lambda$. The reverse inequality follows since $a \in \operatorname{Int} \operatorname{dmn} f$.

7.3. Suppose Q is a positive integer, Y is a finite dimensional inner product space, $a \in \mathbf{R}^m$, f maps a subset of $\mathbf{R}^m$ into $\mathbf{Q}_Q(Y)$, and f is approximately affine approximable at a. Then 7.2 implies

$$\|\mathrm{ap}\,\mathrm{A}\, f(a)\| = \mathrm{ap}\,\limsup_{x \to a} |x-a|^{-1}\mathscr{G}(f(x), f(a)).$$

Lemma 7.4. *Suppose Q is a positive integer, Y is a finite dimensional inner product space, and $\sigma : \mathbf{Q}_Q(Y) \to \mathbf{R}$ satisfies*

$$\sigma(S) = \sup\{|y| : y \in \operatorname{spt} S\} \quad \text{for } S \in \mathbf{Q}_Q(Y).$$

Then $\operatorname{Lip} \sigma \le 1$.

Proof. One may express $\sigma = p \circ \xi \circ g$, where $g : \mathbf{Q}_Q(Y) \to \mathbf{Q}_Q(\mathbf{R})$ denotes the push forward induced by the norm on Y mapping Y into $\mathbf{R}$, and $\xi : \mathbf{Q}_Q(\mathbf{R}) \to \mathbf{R}^Q$ and $p : \mathbf{R}^Q \to \mathbf{R}$ are characterised by

$$\xi\left(\sum_{i=1}^{Q} [y_i]\right) = (y_1, \ldots, y_Q) \quad \text{if } y_i \le y_{i+1} \text{ for } i = 1, \ldots, Q-1,$$

$$p(y_1, \ldots, y_Q) = y_Q$$

whenever $(y_1, \ldots, y_Q) \in \mathbf{R}^Q$. Clearly, $\operatorname{Lip} p \le 1$. Moreover, one readily verifies $\operatorname{Lip} g \le 1$. Finally, $\operatorname{Lip} \xi \le 1$ by Almgren [7, 1.1 (4)]. □

Lemma 7.5. *Suppose Q is a positive integer, Y is a finite dimensional inner product space, $a \in \mathbf{R}^m$, f maps a subset of $\mathbf{R}^m$ into $\mathbf{Q}_Q(Y)$, and $\sigma : \mathbf{Q}_Q(Y) \to \mathbf{R}$ is Lipschitzian.*

Then the following two statements hold.

(1) *If f is affinely approximable at a and $\sigma \circ f$ is differentiable at a, then $|\operatorname{D}(\sigma \circ f)(a)| \le \operatorname{Lip}(\sigma) \| \operatorname{A} f(a) \|$.*
(2) *If f is approximately affinely approximable at a and $\sigma \circ f$ is approximately differentiable at a, then $|\operatorname{ap} \operatorname{D}(\sigma \circ f)(a)| \le \operatorname{Lip}(\sigma) \| \operatorname{ap} \operatorname{A} f(a) \|$.*

Proof. (2) is a consequence of 7.3 together with 7.1 and implies (1). □

7.6. Notice that

$$\sup\{\boldsymbol{\alpha}(m) : m \in \mathscr{P}\} < 6;$$

in fact, using $3 < \boldsymbol{\Gamma}(1/2)^2 < 3.2$ and $(m+2)\boldsymbol{\alpha}(m+2) = 2\boldsymbol{\Gamma}(\frac{1}{2})^2 \boldsymbol{\alpha}(m)$ for $m \in \mathscr{P}$ by [13, 3.2.13], one obtains $\boldsymbol{\alpha}(6) = \frac{1}{6}\boldsymbol{\Gamma}(\frac{1}{2})^6 < \frac{8}{15}\boldsymbol{\Gamma}(\frac{1}{2})^4 = \boldsymbol{\alpha}(5) < 6$ and the supremum does not exceed $\boldsymbol{\alpha}(5)$.

Theorem 7.7. *Suppose $m, n, Q \in \mathscr{P}$ and $1 < m < n$.*

Then there exists a positive, finite number Γ with the following property. If $0 < r < \infty$, $T = \operatorname{im} \mathbf{p}^*$, $V \in \mathbf{IV}_m(\mathbf{R}^n \cap \mathbf{U}(0, 4r))$,

$$(Q - 1/2)\boldsymbol{\alpha}(m) r^m \le \|V\|(\mathbf{C}(T, 0, r, r)) \le (Q + 1/2)\boldsymbol{\alpha}(m) r^m,$$
$$\|V\|(\mathbf{C}(T, 0, r, 2r) \sim \mathbf{C}(T, 0, r, r/2)) \le (1/2)\boldsymbol{\alpha}(m) r^m,$$
$$\|V\| \, \mathbf{U}(0, 4r) \le Q \boldsymbol{\alpha}(m)(5r)^m,$$
$$\eta = \|\delta V\|(\mathbf{U}(0, 4r))^{m/(m-1)} + \int \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z, S),$$

H consists of all $z \in \mathbf{C}(T, 0, r, r)$ such that

$$\|V\| \, \mathbf{B}(z, s) \ge (40 \boldsymbol{\gamma}(m) m)^{-m} s^m \quad \textit{whenever } 0 < s < 2r,$$

and Φ is as in 6.5, then there exists a Borel subset X of $\mathbf{R}^m \cap \mathbf{B}(0, r)$ and a function $f : X \to \mathbf{R}$ with $\operatorname{Lip} f \le 1$ satisfying the following five conditions whenever $1 \le q \le \infty$ and A is a subset of X:

(1) $\mathscr{L}^m(\mathbf{B}(0, r) \sim X) \le \Gamma \eta$.
(2) $(\|V\| \llcorner H)_{(q)}(T_\natural^\perp) \le \Gamma \left((\mathscr{L}^m \llcorner X)_{(q)}(f) + \eta^{1/q + 1/m} \right)$.
(3) $(\|V\| \llcorner H)_{(r^{-m}\Phi)}(T_\natural^\perp) \le \Gamma \left((\mathscr{L}^m \llcorner X)_{(r^{-m}\Phi)}(f) + \eta^{1/m} \right)$.
(4) $(\mathscr{L}^m \llcorner A)_{(2)}(f) \le (\|V\| \llcorner H \cap \mathbf{p}^{-1}[A])_{(2)}(T_\natural^\perp)$.
(5) $(\mathscr{L}^m \llcorner X)_{(2)}(\operatorname{ap} \operatorname{D} f) \le (2Q\eta)^{1/2}$.

Proof. Notice that $(\boldsymbol{\gamma}(m) m)^{-m} \le \boldsymbol{\alpha}(m)$, see for instance [19, 2.4]. Define $\beta = m/(m-1)$, and

$$\Delta_1 = \varepsilon_{[20,\ 3.15]} \left(n - m, m, Q, 1, 5^m Q, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, (40 \boldsymbol{\gamma}(m) m)^{-m} / \boldsymbol{\alpha}(m) \right),$$
$$\Delta_2 = (\log 2)^{1/\beta}, \quad \Delta_3 = 1/\Phi^{-1}(1/(6(Q+1))), \quad \Delta_4 = (20 \boldsymbol{\gamma}(m) m)^{-m} \Delta_1^\beta,$$
$$\Delta_5 = \Delta_1^2 (Q \boldsymbol{\alpha}(m) n)^{-1} (60 \boldsymbol{\gamma}(m) m)^{-2m},$$
$$\Delta_6 = \sup\{3 + 2Q + (12Q + 6)5^m, 8(Q+2)\},$$

$$\Delta_7 = (1/2)\boldsymbol{\alpha}(m)\lambda_{[20,\ 3.15\,(4)]}(m,1/2,1/4)^m 6^{-m},$$
$$\Delta_8 = \sup\{\Gamma_{[20,\ 3.15\,(6)]}(m), 2\boldsymbol{\alpha}(m)^{-1/m}\}, \quad \Delta_9 = \Delta_6 n\boldsymbol{\beta}(n)\sup\{1,\Delta_1^{-2}\},$$
$$\Delta_{10} = (12)^{m+1}Q\sup\{Q,\Delta_8\Delta_9^{1+1/m}\}, \quad \Delta_{11} = \inf\{1,\Delta_4,\Delta_5,\Delta_9^{-1}\Delta_7\},$$
$$\Delta_{12} = 2\Delta_{10}\Delta_2^{-1}, \quad \Delta_{13} = \sup\{6(Q+1)\Delta_{11}^{-1-1/m}, \Delta_3\Delta_{11}^{-1/m}\},$$
$$\Gamma = \sup\{\Delta_9,\Delta_{10},\Delta_{12},\Delta_{13}\}.$$

Notice that $\Delta_2 < 1 \le \Delta_9$.

Suppose r, T, V, η, H, and Φ are related to m, n, and Q as in the body of the lemma. Since the statement of the lemma is invariant by replacing V, f with $(\boldsymbol{\mu}_{(1/r)})_\# V$, $r^{-1}f\circ\boldsymbol{\mu}_r$, we can assume $r=1$.

One may also assume $\eta \le \Delta_{11}$ since otherwise

$$\mathscr{L}^m\,\mathbf{B}(0,1) \le 6 \le \Delta_{13}\Delta_{11} \le \Gamma\eta,$$
$$(\|V\| \llcorner \mathbf{C}(T,0,1,1))_{(q)}(T_\natural^\perp) \le 6(Q+1) \le \Delta_{13}\Delta_{11}^{1+1/m} \le \Gamma\eta^{1/q+1/m},$$
$$(\|V\| \llcorner \mathbf{C}(T,0,1,1))_{(\Phi)}(T_\natural^\perp) \le \Delta_3 \le \Delta_{13}\Delta_{11}^{1/m} \le \Gamma\eta^{1/m}$$

by 7.6 and 6.5, hence one may take $X=\varnothing$ and $f=\varnothing$.

One applies [20, 3.15] with

$$m,n,L,M,\delta_1,\delta_2,\delta_3,\delta_4,\delta_5,a,h,\mu \text{ and } \varepsilon_1 \quad \text{replaced by}$$
$$n-m, m, 1, 5^m Q, \tfrac12, \tfrac12, \tfrac12, \tfrac14, (40\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m), 0, r, \|V\| \text{ and } \Delta_1$$

to obtain $\bar B$, $\bar f$ and $\bar H$ named B, f and H there.

First, *it will be shown that* $H=\bar H$; in fact, noting $\eta \le \inf\{\Delta_4,\Delta_5\}$, one estimates

$$\|\delta V\|\,\mathbf{U}(z,2) \le \eta^{1/\beta} \le \Delta_4^{1/\beta} \le \Delta_1\|V\|(\mathbf{U}(z,2))^{1/\beta},$$
$$\int_{\mathbf{U}(z,2)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|\,\mathrm{d}V(z,S) \le \|V\|(\mathbf{U}(0,4))^{1/2}n^{1/2}\eta^{1/2}$$
$$\le (Q\boldsymbol{\alpha}(m))^{1/2}3^m n^{1/2}\Delta_5^{1/2} \le \Delta_1\|V\|\,\mathbf{U}(z,2).$$

whenever $z\in\mathbf{C}(T,0,1,1)$ and $\|V\|\,\mathbf{U}(z,2) \ge (20\boldsymbol{\gamma}(m)m)^{-m}$.

Choose a Borel subset X of $\operatorname{dmn}\bar f$ with $\mathscr{L}^m((\operatorname{dmn}\bar f)\sim X)=0$ and define $f: X\to\mathbf{R}$ by

$$f(x) = \sup\{|y| : y\in\operatorname{spt}\bar f(x)\} \quad \text{whenever } x\in X.$$

Clearly, $\operatorname{Lip} f \le \operatorname{Lip}\bar f \le 1$ by 7.4 and one infers

$$|\operatorname{ap}\mathrm{D}\,f(x)| \le \|\operatorname{ap}\mathrm{A}\,\bar f(x)\| \quad \text{for } \mathscr{L}^m \text{ almost all } x\in X;$$

in fact, $\bar f$ is approximately affinely approximable at $\mathscr{L}^m$ almost all $x\in X$ by [20, 3.15 (7a)] and f is approximately differentiable by [13, 2.8.18, 2.9.11, 3.1.8] at $\mathscr{L}^m$ almost all $x\in X$ so the assertion follows from 7.4 and 7.5 (2).

Next, *it will be proven that* $\mathscr{L}^m(\mathbf{B}(0,1)\sim X) \le \Delta_9\eta$. For this purpose define sets B_1 and B_2 consisting of those $z\in\mathbf{C}(T,0,1,1)$ satisfying

$$\|\delta V\|\,\mathbf{B}(z,s) > \Delta_1\|V\|(\mathbf{B}(z,s))^{1/\beta} \text{ for some } 0<s<2,$$
$$\int_{\mathbf{B}(z,s)\times\mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2\,\mathrm{d}V(z,S) > \Delta_1^2 n^{-1}\|V\|\,\mathbf{B}(z,s) \text{ for some } 0<s<2$$

respectively. To estimate $\|V\|(B_1)$ we employ the Besicovitch–Federer covering theorem which provides disjointed families $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ of closed balls such that

$$B_1 \subset \bigcup \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\} \subset \mathbf{U}(0,4),$$
$$\|V\|(C) < \Delta_1^{-\beta} \|\delta V\|(C)^\beta \quad \text{whenever } C \in F_i \text{ and } i = 1, \ldots, \boldsymbol{\beta}(n),$$

and we obtain

$$\begin{aligned} \|V\|(B_1) &\le \Delta_1^{-\beta} \sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{C \in F_i} \|\delta V\|(C)^\beta \\ &\le \Delta_1^{-\beta} \sum_{i=1}^{\boldsymbol{\beta}(n)} \Big(\sum_{C \in F_i} \|\delta V\|(C) \Big)^\beta \le \Delta_1^{-\beta} \boldsymbol{\beta}(n) \|\delta V\|(\mathbf{U}(0,4))^\beta. \end{aligned}$$

In a similar fashion we find another disjointed families $F_1, \ldots, F_{\boldsymbol{\beta}(n)}$ of closed balls such that

$$B_2 \subset \bigcup \bigcup \{F_i : i = 1, \ldots, \boldsymbol{\beta}(n)\} \subset \mathbf{U}(0,4),$$
$$\|V\|(C) < \Delta_1^{-2} n \int_{C \times \mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(z,S) \quad \text{for } C \in F_i,\ i{=}1, \ldots, \boldsymbol{\beta}(n),$$

and in consequence

$$\begin{aligned} \|V\|(B_2) &\le \Delta_1^{-2} n \sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{C \in F_i} \int_{C \times \mathbf{G}(n,m)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(z,S) \\ &\le \Delta_1^{-2} n \boldsymbol{\beta}(n) \int \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(z,S). \end{aligned}$$

Verifying $\bar{B} \subset B_1 \cup B_2$ by means of Hölder's inequality, the asserted estimate follows from [20, 3.15 (3)].

Since in particular $\mathscr{L}^m(\mathbf{B}(0,1) \sim X) \le \Delta_9 \Delta_{11} \le \Delta_7$, one applies [20, 3.15 (6)] with S replaced by $Q[0]$ to estimate, concerning (2),

$$\begin{aligned} &(\|V\| \llcorner H)_{(q)}(T_\natural^\perp) \\ &\quad \le (12)^{m+1} Q \left(Q^{1/2} (\mathscr{L}^m \llcorner X)_{(q)}(f) + \Delta_8 \mathscr{L}^m (\mathbf{B}(0,1) \sim X)^{1/q+1/m} \right) \\ &\quad \le \Delta_{10} \left((\mathscr{L}^m \llcorner X)_{(q)}(f) + \eta^{1/q+1/m} \right) \quad \text{for } 1 \le q \le \infty. \end{aligned}$$

Consequently, concerning (3), one notes that $\eta \le 1$ and $\Phi(\Delta_2) = 1$ and estimates

$$\begin{aligned} \int_H \Phi \circ |\gamma^{-1} T_\natural^\perp| \,\mathrm{d}\|V\| &= \sum_{i=1}^\infty i!^{-1} \gamma^{-i\beta} \textstyle\int_H |T_\natural^\perp|^{\beta i} \,\mathrm{d}\|V\| \\ &\le \frac{1}{2} \sum_{i=1}^\infty i!^{-1} (2\Delta_{10}/\gamma)^{\beta i} \left(\int_X |f|^{\beta i} \,\mathrm{d}\mathscr{L}^m + \eta^{1+\beta i/m} \right) \\ &= \frac{1}{2} \int_X \Phi \circ |2\Delta_{10} \gamma^{-1} f| \,\mathrm{d}\mathscr{L}^m + \tfrac{1}{2} \eta \Phi(2\Delta_{10} \eta^{1/m} \gamma^{-1}) \le 1 \end{aligned}$$

whenever $2\Delta_{10}\Delta_2^{-1}\big((\mathscr{L}^m \llcorner X)_{(\Phi)}(f) + \eta^{1/m}\big) < \gamma < \infty$, hence

$$(\|V\| \llcorner H)_{(\Phi)}(T_\natural^\perp) \leq \Delta_{12}\left((\mathscr{L}^m \llcorner X)_{(\Phi)}(f) + \eta^{1/m}\right).$$

To prove (4) and (5), recall

$$H \cap \mathbf{p}^{-1}[\operatorname{dmn} \bar{f}] = \{z : \mathbf{q}(z) \in \operatorname{spt} \bar{f}(\mathbf{p}(z))\} \subset \{z : \boldsymbol{\Theta}^m(\|V\|, z) \in \mathscr{P}\}$$

from [20, 3.15 (2) (4)] and observe: *If A is a subset of X, g is an $\mathscr{L}^m \llcorner A$ measurable real valued function and h is an $\|V\| \llcorner H \cap \mathbf{p}^{-1}[A]$ measurable real valued function such that*

$$\mathscr{L}^m(A \cap \{x : g(x) > t\} \sim \mathbf{p}[H \cap \{z : h(z) > t\}]) = 0 \quad \textit{for } 0 < t < \infty,$$

then $(\mathscr{L}^m \llcorner A)_{(q)}(g) \leq (\|V\| \llcorner H \cap \mathbf{p}^{-1}[A])_{(q)}(h)$*;* in fact

$$\begin{aligned}\mathscr{L}^m(A \cap \{x : g(x) > t\}) &\leq \mathscr{H}^m(H \cap \mathbf{p}^{-1}[A] \cap \{z : h(z) > t\}) \\ &\leq \|V\|(H \cap \mathbf{p}^{-1}[A] \cap \{z : h(z) > t\})\end{aligned}$$

by [13, 2.10.35] and Allard [4, 3.5 (1b)].

One applies this observation with g and h replaced by f and $\left|T_\natural^\perp|U\right|$ to deduce (4). Recalling $|\operatorname{ap} \mathrm{D} f(x)| \leq \|\operatorname{ap} \mathrm{A} f(x)\|$ for $\mathscr{L}^m$ almost all $x \in X$ together with [20, 3.15 (7d)], one applies the observation once more, with

$$g(x) \text{ and } h(z) \quad \text{replaced by} \quad |\operatorname{ap} \mathrm{D} f(x)| \text{ and } (2Q)^{1/2}\|\operatorname{Tan}^m(\|V\|, z)_\natural - T_\natural\|$$

to infer (5). □

8. Embedding results

In the present section we formulate for convenient reference two embedding results for Sobolev functions in Euclidean space which measure the lower order term only on a set of suitably large Lebesgue measure.

Lemma 8.1. *Suppose* $2 \leq m \in \mathscr{P}$, $a \in \mathbf{R}^m$, $0 < r < \infty$, $0 < \varepsilon \leq (\boldsymbol{\alpha}(m)/2)^{1/m}$, *$A$ is an $\mathscr{L}^m$ measurable subset of* $\mathbf{U}(a, r)$, $\mathscr{L}^m(\mathbf{U}(a, r) \sim A) \leq (\varepsilon r)^m$, *and* $f \in \mathbf{W}^{1,1}(\mathbf{U}(a, r))$.

Then there holds

$$\begin{aligned}&r^{-1}(\mathscr{L}^m \llcorner \mathbf{U}(a, r))_{(m)}(f) \\ &\qquad \leq \Gamma\left(\varepsilon^{1/2}(\mathscr{L}^m \llcorner \mathbf{U}(a, r))_{(m)}(\mathbf{D} f) + \varepsilon^{-1/2} r^{-1}(\mathscr{L}^m \llcorner A)_{(m)}(f)\right),\end{aligned}$$

where Γ is a positive, finite number depending only on m.

Proof. By Hölder's inequality it is sufficient to prove the statement that results from replacing $r^{-1}(\mathscr{L}^m \llcorner \mathbf{U}(a, r))_{(m)}(f)$ by $r^{-1/2}(\mathscr{L}^m \llcorner \mathbf{U}(a, r))_{(2m)}(f)$. The latter is a special case of [21, 6.3] taking $\zeta = \frac{2}{3}m$, $\xi = m$, $s = m$ and $\lambda = (\varepsilon r)^m$. □

Theorem 8.2. *Suppose* $2 \leq m \in \mathscr{P}$, *Φ is related to m as in 6.5,* $a \in \mathbf{R}^m$, $0 < r < \infty$, *A is an $\mathscr{L}^m$ measurable subset of $\mathbf{U}(a, r)$ with* $\mathscr{L}^m(A) \geq \frac{1}{2}\boldsymbol{\alpha}(m) r^m$, *and* $f \in \mathbf{W}^{1,1}(\mathbf{U}(a, r))$,

Then there holds

$$(\mathscr{L}^m \llcorner \mathbf{U}(a,r))_{(r^{-m}\Phi)}(f) \leq \Gamma \left((\mathscr{L}^m \llcorner \mathbf{U}(a,r))_{(m)}(\mathbf{D} f) + r^{-1}(\mathscr{L}^m \llcorner A)_{(m)}(f) \right),$$

where Γ is a positive, finite number depending only on m.

Proof. The problem may be reduced firstly to the case $A = \mathbf{U}(a,r)$ by 8.1 and secondly to the case $a = 0$ and $r = 1$ using translations and homotheties. The remaining case is a special case of [2, 8.27]. □

9. Quadratic tilt-excess, decay rates

In this section we prove sharp decay rates of the quadratic tilt-excess for two dimensional integral varifolds whose first variation is a Radon measure, see 9.2. This result rests on two pillars. Firstly, on the second order rectifiability of such varifolds obtained in [22, 4.8]. Secondly, on an approximate coercive estimate by which we mean an estimate of the tilt-excess in terms of the first variation, the height-excess measured on a set of suitably large weight measure and small contributions from the tilt-excess, see 9.1.

Accordingly, in order to derive the approximate coercive estimate 9.1 from the coercive estimate 6.6, one needs to estimate the height-excess occurring in 6.6 by approximate height-quantities together with the variation measure of the first variation and quadratic tilt-excess. The approximation 7.7 reduces such an estimate to the case of a real valued Lipschitzian functions which has been treated in 8.1 and 8.2.

Since currently no analogous estimates to 8.1 and 8.2 are available for real valued Lipschitzian function over varifolds, the authors have chosen the path using the approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions leading to 7.7. Yet, it would be of interest to investigate whether the embedding theory for Lipschitzian functions on varifolds can be extended so as to yield a proof without such approximation. In a somewhat different vein much of that theory has been extended to generalised weakly differentiable functions in [23, § 10].

Lemma 9.1. *Suppose $2 < n \in \mathscr{P}$, $Q \in \mathscr{P}$, $c \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{IV}_2(\mathbf{U}(c, 8r))$,*

$$\|V\| \, \mathbf{B}(c, 2r) \geq (Q - 1/2)\boldsymbol{\alpha}(2)(2r)^2, \quad \|V\| \, \mathbf{U}(c, 8r) \leq (Q + 1/4)\boldsymbol{\alpha}(2)(8r)^2,$$

$T \in \mathbf{G}(n, 2)$, $0 < \varepsilon \leq 1$, Z is $\|V\|$ measurable, and

$$\alpha = r^{-1}\|\delta V\| \, \mathbf{U}(c, 8r), \quad \beta = r^{-1} \left(\int \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z, S) \right)^{1/2}, \quad \alpha + \beta \leq \varepsilon,$$

$$\gamma = r^{-2} \left(\int Z \operatorname{dist}(z - c, T)^2 \, \mathrm{d}\|V\| z \right)^{1/2}, \quad \|V\|(\mathbf{U}(c, 8r) \sim Z) \leq (\varepsilon r)^2,$$

and $\kappa : \{t : 0 \leq t < \infty\} \to \mathbf{R}$ satisfies (see 6.4)

$$\kappa(0) = 0, \qquad \kappa(t) = t \left(1 + (\log(1 + 1/t))^{1/2} \right) \quad \text{for } 0 < t < \infty.$$

Then there holds

$$r^{-2}\int_{\mathbf{U}(c,r)\times\mathbf{G}(n,2)}\|S_\natural - T_\natural\|^2\,\mathrm{d}V(z,S) \le \Gamma\left(\kappa(\alpha(\alpha+\beta+\gamma)) + \varepsilon\beta^2 + \varepsilon^{-1}\gamma^2\right),$$

where Γ *is a positive, finite number depending only on* n *and* Q*.*

Proof. Considering $(\boldsymbol{\mu}_{1/r})_\# V$ in place of V one may assume $r=1$ and, using isometries, one may assume $c=0$ and $T=\operatorname{im}\mathbf{p}^*$, see 5.1. Moreover, one may assume Z to be a Borel set.

Define

$$\begin{aligned}
&\Delta_1 = 1+\Gamma_{7.7}(2,n,Q), \quad \Delta_2 = \inf\left\{1/3, 2\Delta_1^{-1/2}\right\},\\
&\Delta_3 = 2(Q+1/2)^{1/2}(Q+3/8)^{-1/2}, \quad \Delta_4 = (2Q)^{1/2}+\Delta_1^{1/2},\\
&\Delta_5 = \Delta_1^{5/4}\Gamma_{8.1}(2)2^{1/2}, \quad \Delta_6 = \inf\left\{1, (\Delta_3^2-4)^{1/2}/4\right\}, \quad \Delta_7 = \Delta_5(1+\Delta_4),\\
&\Delta_8 = 4\Delta_1(\Gamma_{8.2}(2)+1)(\Delta_4+1),\\
&\Delta_9 = \sup\{\Delta_2^{-1}, \Delta_6^{-2}\}, \quad \Gamma = \sup\{\Delta_9, \Gamma_{6.6}(2)(1+3\Delta_7^2+\Delta_8)\}.
\end{aligned}$$

If $\varepsilon > \Delta_2$ then $\beta^2 \le \Delta_2^{-1}\varepsilon\beta^2 \le \Delta_9\varepsilon\beta^2$ and if $\gamma > \Delta_6$, then $\beta^2 \le \varepsilon^2 \le 1 \le \Delta_6^{-2}\varepsilon^{-1}\gamma^2 \le \Delta_9\varepsilon^{-1}\gamma^2$. Therefore one may assume $\varepsilon \le \Delta_2$ and $\gamma \le \Delta_6$.

Abbreviate $C = \mathbf{C}(T,0,2,2)$, $K = \mathbf{R}^n \cap \{z : \operatorname{dist}(z,C)\le 4\}$, and

$$H = C \cap \{z : \|V\|\,\mathbf{B}(z,s) \ge (80\boldsymbol{\gamma}(2))^{-2}s^2 \text{ for } 0<s<4\}.$$

Notice that

$$\begin{aligned}
&\mathbf{U}(0,1)\subset\{z : \mathbf{U}(z,1)\subset C\}, \quad K\subset\mathbf{U}(0,8),\\
&C\cap H_{6.6}\subset H, \quad \text{where } H_{6.6} \text{ denotes the set named “}H\text{” in 6.6.}
\end{aligned}$$

In order to apply 7.7 with m and r replaced by 2 and 2, one estimates

$$\begin{aligned}
\|V\|\,\mathbf{U}(0,8) &\le 100Q\boldsymbol{\alpha}(2),\\
\|V\|(\mathbf{C}(T,0,2,4)\sim\mathbf{C}(T,0,2,1)) &\le \|V\|(\mathbf{U}(0,8)\cap\{z : \operatorname{dist}(z,T)\ge 1\})\\
&\le \|V\|(\mathbf{U}(0,8)\sim Z)+\gamma^2 \le \varepsilon^2+1 \le 2\boldsymbol{\alpha}(2),
\end{aligned}$$

and, noting $2<\Delta_3<8$ and

$$C\subset\mathbf{U}(0,\Delta_3)\cup\left(\mathbf{U}(0,8)\cap\{z : \operatorname{dist}(z,T)^2\ge\Delta_3^2-4\}\right),$$

one employs the monotonicity identity (see for instance [23, 4.5, 4.6]) and the bounds on ε and γ to infer

$$\begin{aligned}
\|V\|(C) &\le \|V\|\,\mathbf{U}(0,\Delta_3) + \|V\|(\mathbf{U}(0,8)\cap\{z : \operatorname{dist}(z,T)^2\ge\Delta_3^2-4\})\\
&\le \Delta_3^2\left(8^{-2}\|V\|\,\mathbf{U}(0,8) + \int_2^8 t^{-2}\|\delta V\|\,\mathbf{U}(0,t)\,\mathrm{d}\mathscr{L}^1 t\right)\\
&\quad + \|V\|(\mathbf{U}(0,8)\sim Z) + (\Delta_3^2-4)^{-1}\gamma^2\\
&\le \Delta_3^2\left((Q+1/4)\boldsymbol{\alpha}(2) + \tfrac{3}{8}\alpha + \varepsilon^2 + (\Delta_3^2-4)^{-1}\gamma^2\right)\\
&\le \Delta_3^2(Q+3/8)\boldsymbol{\alpha}(2) = 4(Q+1/2)\boldsymbol{\alpha}(2).
\end{aligned}$$

Using the hypotheses one also gets $\|V\|(C)\ge 4(Q-1/2)\boldsymbol{\alpha}(2)$. Therefore, applying 7.7 with m and r replaced by 2 and 2 in conjunction with Kirszbraun's theorem, see [13, 2.10.43], one obtains a Borel set X and a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $f|X$ satisfies the conditions of 7.7 and $\operatorname{Lip} f \le 1$, in particular f is weakly differentiable with $\mathbf{D}\, f(x) = \mathrm{D}\, f(x)$ for $\mathscr{L}^2$ almost all x by [3, 2.13, 2.14]. Define

$$A = X \sim \mathbf{p}[C \cap \{z : \boldsymbol{\Theta}^2(\|V\|, z) \in \mathscr{P}\} \sim Z]$$

and notice that A is $\mathscr{L}^2$ measurable by [3, 2.55] and [13, 2.2.13]. Since

$$\begin{aligned}\mathscr{L}^2(\mathbf{p}[C \cap \{z : \boldsymbol{\Theta}^2(\|V\|, z) \in \mathscr{P}\} \sim Z]) \\ \le \mathscr{H}^2(C \cap \{z : \boldsymbol{\Theta}^2(\|V\|, z) \in \mathscr{P}\} \sim Z) \le \|V\|(C \sim Z) \le \varepsilon^2\end{aligned}$$

by [13, 2.10.35] and Allard [4, 3.5 (1b)], one infers from 7.7 (1) that

$$\mathscr{L}^2(\mathbf{U}(0,2) \sim A) \le \mathscr{L}^2(\mathbf{B}(0,2) \sim X) + \mathscr{L}^2(X \sim A) \le \Delta_1 \varepsilon^2 \le 2\boldsymbol{\alpha}(2).$$

Noting Allard [4, 3.5 (1c)] and [13, 2.8.17, 2.9.11, 3.1.2] and observing that from the definition of A it follows that $H \cap \mathbf{p}^{-1}[A] \cap \{z : \boldsymbol{\Theta}^2(\|V\|, z) \in \mathscr{P}\} \subset Z$, one applies 7.7 (4) and 7.7 (1) (5) to obtain the following auxiliary estimates

$$(\mathscr{L}^2 \llcorner A)_{(2)}(f) \le \gamma, \quad (\mathscr{L}^2 \llcorner \mathbf{U}(0,2))_{(2)}(\mathrm{D}\, f) \le \Delta_4(\alpha + \beta).$$

Next, defining Φ as in 6.5, it will be shown that

$$\begin{aligned}(\|V\| \llcorner H)_{(2)}(T_\natural^\perp) &\le \Delta_7(\alpha + \varepsilon^{1/2}\beta + \varepsilon^{-1/2}\gamma), \\ (\|V\| \llcorner H)_{(\Phi)}(T_\natural^\perp) &\le \Delta_8(\alpha + \beta + \gamma).\end{aligned}$$

To prove the first estimate, one notes $\alpha^2 \le \alpha$, $\beta^2 \le \varepsilon\beta \le \varepsilon^{1/2}\beta$ and applies 7.7 (2) and 8.1 with r and ε replaced by 2 and $2^{-1}\Delta_1^{1/2}\varepsilon$ to deduce

$$\begin{aligned}(\|V\| \llcorner H)_{(2)}(T_\natural^\perp) &\le \Delta_1 \left((\mathscr{L}^2 \llcorner \mathbf{U}(0,2))_{(2)}(f) + \alpha^2 + \beta^2 \right) \\ &\le \Delta_5 \left(\varepsilon^{1/2}(\mathscr{L}^2 \llcorner \mathbf{U}(0,2))_{(2)}(\mathrm{D}\, f) + \varepsilon^{-1/2}(\mathscr{L}^2 \llcorner A)_{(2)}(f) + \alpha + \varepsilon^{1/2}\beta \right),\end{aligned}$$

hence the estimate follows using the auxiliary estimates. To prove the second estimate, one employs 6.3 (2), 7.7 (3), and 8.2 to infer

$$\begin{aligned}(\|V\| \llcorner H)_{(\Phi)}(T_\natural^\perp) &\le 4\Delta_1 \left((\mathscr{L}^2 \llcorner \mathbf{U}(0,2))_{(\Phi/4)}(f) + \alpha + \beta \right) \\ &\le 4\Delta_1(\Gamma_{8.2}(2) + 1) \left((\mathscr{L}^2 \llcorner \mathbf{U}(0,2))_{(2)}(\mathrm{D}\, f) + (\mathscr{L}^2 \llcorner A)_{(2)}(f) + \alpha + \beta \right),\end{aligned}$$

hence the estimate follows from the auxiliary estimates.

To conclude the proof, one employs 6.6 to obtain

$$\begin{aligned}&\int_{\mathbf{U}(0,1) \times \mathbf{G}(n,2)} \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z,S) \\ &\quad \le \Gamma_{6.6}(2) \left(\alpha^2 + \kappa(\Delta_8 \alpha(\alpha + \beta + \gamma)) + 3\Delta_7^2(\alpha^2 + \varepsilon\beta^2 + \varepsilon^{-1}\gamma^2) \right),\end{aligned}$$

hence, noting $\alpha^2 \le \kappa(\alpha^2)$ and $\kappa(\Delta_8\alpha(\alpha+\beta+\gamma)) \le \Delta_8 \kappa(\alpha(\alpha+\beta+\gamma))$ by 6.4, the conclusion is now readily derived. □

Theorem 9.2. *Suppose* $2 < n \in \mathscr{P}$, U *is an open subset of* $\mathbf{R}^n$, $V \in \mathbf{IV}_2(U)$, *and* $\|\delta V\|$ *is a Radon measure.*

Then, for V *almost all* (z, T), *there holds*

$$\lim_{r\to 0+} r^{-4}(\log(1/r))^{-1} \int_{\mathbf{B}(z,r)\times\mathbf{G}(n,2)} \|S_\natural - T_\natural\|^2 \,\mathrm{d}V(\zeta, S) = 0.$$

Proof. Define $Z = U \cap \{z : \operatorname{Tan}^2(\|V\|, z) \in \mathbf{G}(n,2)\}$ and $\tau : Z \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by $\tau(z) = \operatorname{Tan}^2(\|V\|, z)_\natural$ for $z \in Z$. Recall that

$$V(k) = \int_Z k(z, \tau(z))\boldsymbol{\Theta}^2(\|V\|, z)\,\mathrm{d}\mathscr{H}^2 z \quad \text{for } k \in \mathscr{K}(U \times \mathbf{G}(n,2))$$

from Allard [4, 3.5 (1b)] and that there exists a countable collection C of 2 dimensional submanifolds of $\mathbf{R}^n$ of class 2 such that $\|V\|(U \sim \bigcup C) = 0$ from [22, 4.8]. Notice that

$$\operatorname{Tan}(M, z) = \operatorname{Tan}^2(\|V\|, z) \quad \text{for } \|V\| \text{ almost all } z \in U \cap M$$

for $M \in C$ by [13, 2.8.18, 2.9.11, 3.2.17] and Allard [4, 3.5 (2)]. In particular, one may construct a sequence of functions $\tau_i : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ of class 1 such that the sets $Z_i = U \cap \{z : \tau(z) = \tau_i(z)\}$ cover $\|V\|$ almost all of U. For $i \in \mathscr{P}$, applying 4.2 with m, p, $\omega(r)$, Z, f, and q replaced by 2, 1, $r^2(1+\log(1/r))$, Z_i, $\|\tau - \tau_i\|^2$, and ∞ one infers, for $\|V\|$ almost all $z \in Z_i$, that

$$\begin{aligned}
&\limsup_{r\to 0+} r^{-4}(\log(1/r))^{-1} \int_{\mathbf{B}(z,r)} \|\tau(\zeta) - \tau(z)\|^2 \,\mathrm{d}\|V\|\zeta \\
&\quad = \limsup_{r\to 0+} r^{-4}(\log(1/r))^{-1} \int_{\mathbf{B}(z,r)} \|\tau - \tau_i\|^2 \,\mathrm{d}\|V\| \in \{0, \infty\}.
\end{aligned}$$

Therefore it is sufficient to prove for $\|V\|$ almost all c that

$$\limsup_{r\to 0+} r^{-4}(\log(1/r))^{-1} \int_{\mathbf{B}(c,r)} \|\tau(z) - \tau(c)\|^2 \,\mathrm{d}\|V\|z < \infty.$$

For $\|V\|$ almost all $c \in U$ there exist $Q \in \mathscr{P}$ and $M \in C$ such that

$$\boldsymbol{\Theta}^2(\|V\|, c) = Q, \quad \boldsymbol{\Theta}^{*2}(\|\delta V\|, c) < \infty, \quad \boldsymbol{\Theta}^2(\|V\| \llcorner U \sim M, c) = 0,$$
$$\lim_{r\to 0+} r^{-2} \int_{\mathbf{B}(c,r)} \|\tau(z) - \tau(c)\| \,\mathrm{d}\|V\|z = 0$$

by Allard [4, 3.5 (1c)] and [13, 2.8.18, 2.9.5, 2.9.9, 2.9.11]. Considering such c, Q and M and abbreviating $T = \operatorname{Tan}(M, c)_\natural$, it follows

$$\tau(c) = T_\natural, \quad \limsup_{s\to 0+} s^{-6} \int_{\mathbf{B}(c,s)\cap M} \operatorname{dist}(z - c, T)^2 \,\mathrm{d}\|V\|z < \infty$$

since M is a submanifold of class 2. Defining

$$\eta = \sup\{1, \Gamma_{9.1}(n, Q)\}, \quad \varepsilon = 2^{-14}\eta^{-1},$$

one obtains the existence of $0 < r \le 1/4$ and $1 \le \xi < \infty$ such that $\mathbf{U}(c, 8r) \subset U$ and for $0 < s \le r$

$$s^{-1} \|\delta V\| \, \mathbf{U}(c, 8s) + s^{-2} \left(\int_{\mathbf{U}(c,8s) \cap M} \operatorname{dist}(z - c, T)^2 \, \mathrm{d}\|V\| z \right)^{1/2} \le \xi s,$$

and V satisfies the hypotheses of 9.1

with r and Z replaced by s and $\mathbf{U}(c, 8s) \cap M$.

Abbreviating

$$f(s) = s^{-2} \int_{\mathbf{U}(c,s) \times \mathbf{G}(n,2)} \|S_\natural - T_\natural\|^2 \, \mathrm{d}V(z, S) \quad \text{for } 0 < s \le 8r,$$

$$\Delta = \sup \left\{ 2^{20} \eta^2 \varepsilon^{-1} \xi^2, 2^6 f(8r) r^{-2} (\log(1/(8r)))^{-1} \right\},$$

one inductively proves that

$$f(s) \le \Delta s^2 \log(1/s) \quad \text{whenever } 0 < s \le 8r;$$

in fact, the inequality is evident if $r \le s \le 8r$ and if it holds with s replaced by $8s$ for some $0 < s \le r$, then, recalling $r \le 1/4$, one notes that

$$s^2 \le \xi s \left(\xi s + 2^6 \Delta^{1/2} s (\log(1/s))^{1/2} \right) \le 8^{-1} \eta^{-1} \Delta s^2 (\log(1/s))^{1/2},$$

$$1 + (\log(1 + 1/s^2))^{1/2} \le 4 (\log(1/s))^{1/2}, \quad \eta \varepsilon^{-1} \xi^2 s^2 \le 4^{-1} \Delta s^2 \log(1/s),$$

to infer from 9.1 that

$$\begin{aligned} f(s) &\le \eta \left(\kappa \left(\xi s (\xi s + 2^6 \Delta^{1/2} s (\log(1/s))^{1/2}) \right) + 2^{12} \varepsilon \Delta s^2 \log(1/s) + \varepsilon^{-1} \xi^2 s^2 \right) \\ &\le 8^{-1} \Delta s^2 (\log(1/s))^{1/2} \left(1 + (\log(1 + 1/s^2))^{1/2} \right) + 2^{-1} \Delta s^2 \log(1/s) \\ &\le \Delta s^2 \log(1/s), \end{aligned}$$

where κ is as in 9.1. □

Remark 9.3. In view of 5.8 the decay rate is sharp for integral varifolds. For curvature varifolds a stronger conclusion is attainable, see [23, 15.9].

Remark 9.4. It is an open problem whether the integrality hypothesis on V could be replaced by the requirement "$\boldsymbol{\Theta}^m(\|V\|, z) \ge 1$ for $\|V\|$ almost all z".

10. Super-quadratic tilt-excess, an example

In this section we provide examples of curvature varifolds satisfying the conditions of 4.1 with $p = \infty$, hence in particular having bounded generalised mean curvature vector, for which there is a set of positive weight measure such that in arbitrarily small balls around the points of that set there is a portion of relatively large measure where the tilt is greater than $1/3$. In fact, the Hausdorff measure of the regions in the affine tangent planes which are not covered by the varifold, i.e., the size of "holes", is large at these scales which is essentially a stronger statement, see 10.4. The power of the decay of the super-quadratic tilt-excess exhibited by these varifolds is the smallest possible, see 10.5 and

11.1. In 10.8, the example is modified so as to yield the largest possible size of points of small density permitted by the approximate lower semicontinuity of the density, see 10.11.

The qualitative construction principle was described by Brakke in [8, 6.1] for two dimensional integral varifolds. Our implementation employs additionally the estimates obtained in 10.2 for certain varifolds, see Fig. 2, and the sets constructed in 2.2 and 2.5.

10.1. If ϕ is a measure, A is ϕ measurable, $\phi(A) < \infty$, $f \in \mathbf{L}_\infty(\phi)$, $\varepsilon > 0$, and $\varepsilon\phi(A) \le \int_A f \,\mathrm{d}\phi$, then

$$(\varepsilon/2)\phi(A) \le \phi(A \cap \{x : f(x) \ge \varepsilon/2\})\phi_{(\infty)}(f).$$

Lemma 10.2. *Suppose $n = 3$, $\mathbf{p}$ and $\mathbf{q}$ are related to n as in 5.1, and $4 \le r < \infty$.*

Then there exists a curvature varifold $V \in \mathbf{IV}_2(\mathbf{R}^3)$ satisfying

$$\begin{aligned}
&\operatorname{spt}\|V\| \subset \operatorname{im}\mathbf{p}^* \cup \mathbf{p}^{-1}[\mathbf{U}(0,r)],\\
&\|V\| \llcorner \mathbf{p}^{-1}[\mathbf{R}^2 \sim \mathbf{U}(0,r)] = 2\mathscr{H}^2 \llcorner \operatorname{im}\mathbf{p}^* \sim \mathbf{U}(0,r),\\
&\mathbf{p}^{-1}[\mathbf{U}(0,r)] \cap \operatorname{spt}\|V\| \textit{ is a two dimensional submanifold of } \mathbf{R}^3 \textit{ of class } \infty,\\
&\boldsymbol{\Theta}^2(\|V\|, z) = 1 \quad \textit{for } z \in \mathbf{p}^{-1}[\mathbf{U}(0,r)] \cap \operatorname{spt}\|V\|,\\
&0 \le \|V\|\left(\mathbf{p}^{-1}[\mathbf{U}(0,r)]\right) - 2\mathscr{L}^2\,\mathbf{U}(0,r) \le \Gamma(\log r)^2,\\
&\|\delta V\| \le \Gamma(r^{-2}\log r)\|V\|, \quad \textstyle\int \|\mathbf{b}(V,z)\|\,\mathrm{d}\|V\|z \le \Gamma\log r,\\
&|\mathbf{p}(z)| \ge 1 \textit{ and } |\mathbf{q}(z)| \le 3\log r \quad \textit{whenever } z \in \operatorname{spt}\|V\|,\\
&V\left((\mathbf{R}^3 \times \mathbf{G}(3,2)) \cap \{(z,S) : \|S_\natural - \mathbf{p}^* \circ \mathbf{p}\| \ge 1/3\}\right) \ge 1,
\end{aligned}$$

where Γ is a universal positive, finite number.

Proof. Choose $\gamma \in \mathscr{D}(\mathbf{R},\mathbf{R})$ with $\{t : \gamma(t) > 0\} = \mathbf{U}(0,1)$ and

$$0 \le \gamma(t) \le 1 \quad \text{for } t \in \mathbf{R}, \quad \gamma(t) = 1 \quad \text{for } -1/2 \le t \le 1/2.$$

Recalling 5.3, define $g : \mathbf{R} \cap \{t : 1 < t < \infty\} \to \mathbf{R}$ by

$$g(t) = \operatorname{arcosh}(t)\gamma(t/r) \quad \text{for } 1 < t < \infty,$$

hence there exists a positive, finite number Δ_1 determined by γ such that

$$|g'(t)| \le \Delta_1 r^{-1}\log r \quad \text{and} \quad |g''(t)| \le \Delta_1 r^{-2}\log r \qquad \text{for } r/2 \le t \le r.$$

Defining $h : \mathbf{R}^2 \sim \mathbf{B}(0,1) \to \mathbf{R}$ by $h(x) = g(|x|)$ for $x \in \mathbf{R}^2 \sim \mathbf{B}(0,1)$, let

$$M = \mathbf{R}^3 \cap \{z : \mathbf{q}(z) = h(\mathbf{p}(z))\}.$$

Notice that $\mathbf{h}(M,z) = 0$ for $z \in M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r/2)]$ by 5.4 and

$$\begin{aligned}
&M \cap \mathbf{p}^{-1}[\mathbf{R}^2 \sim \mathbf{U}(0,r)] = \operatorname{im}\mathbf{p}^* \sim \mathbf{U}(0,r),\\
&\mathbf{p}[M] \subset \mathbf{R}^2 \sim \mathbf{U}(0,1), \quad \mathbf{q}[M] \subset \mathbf{B}(0, 3\log r),\\
&M \cap \mathbf{p}^{-1}[\mathbf{B}(0,2)] \subset \{z : \|\operatorname{Tan}(M,z)_\natural - \mathbf{p}^* \circ \mathbf{p}\| \ge 1/3\}.
\end{aligned}$$

by 5.1 (2) and 5.3. Recalling $r \ge 4$, one may deduce from 5.3 and 5.4 that

$$0 \le \mathscr{H}^2\left(M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r/2)]\right) - \mathscr{L}^2(\mathbf{U}(0,r/2)) \le 12\log r.$$

Noting that $(1+s)^{1/2} \le 1 + s/2$ for $-1 \le s < \infty$, one estimates

$$0 \le \mathscr{H}^2\left(M \cap \mathbf{p}^{-1}[\mathbf{U}(0,r) \sim \mathbf{U}(0,r/2)]\right) - \mathscr{L}^2(\mathbf{U}(0,r) \sim \mathbf{U}(0,r/2))$$

$$\le 8 \int_{r/2}^{r} \left((1+g'(t)^2)^{1/2} - 1\right) t \,\mathrm{d}\mathscr{L}^1 t \le 4 \int_{r/2}^{r} g'(t)^2 t \,\mathrm{d}\mathscr{L}^1 t \le 2\Delta_1^2 (\log r)^2.$$

In view of the estimates for g' and g'' and 5.2, one notes

$$\|\mathbf{b}(M,z)\| \le 2\Delta_1 r^{-2} \log r \quad \text{whenever } z \in M \text{ and } r/2 \le |\mathbf{p}(z)| \le r,$$

hence, using 5.2 and 5.3, one obtains

$$\int_M \|\mathbf{b}(M,z)\| \,\mathrm{d}\mathscr{H}^2 z \le \Delta_2 \log r,$$

where Δ_2 is a positive, finite number determined by γ.

Employing the reflection $L : \mathbf{R}^3 \to \mathbf{R}^3$ given by $L(z) = \mathbf{p}^*(\mathbf{p}(z)) - \mathbf{q}^*(\mathbf{q}(z))$ for $z \in \mathbf{R}^3$, one defines a curvature varifold $V \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$V(k) = \textstyle\int_M k(z, \operatorname{Tan}(M,z)) + k(L(z), L[\operatorname{Tan}(M,z)]) \,\mathrm{d}\mathscr{H}^2 z$$

for $k \in \mathscr{K}(\mathbf{R}^3 \times \mathbf{G}(3,2))$. Hence one may take $\Gamma = 4 \sup\{(3+\Delta_1)^2, \Delta_2\}$. □

Example 10.3. Suppose m is an integer with $m \ge 2$ and ω is a modulus of continuity satisfying the Dini condition.

Then there exist ε, R, T, C, M, and V satisfying

$\varepsilon > 0$, $\quad R \in \mathbf{G}(m+1, m-2)$, $\quad T \in \mathbf{G}(m+1, m)$, $\quad C$ is a Borel subset of T,

M is an m dimensional submanifold of $\mathbf{R}^{m+1}$ of class ∞,

$V \in \mathbf{IV}_m(\mathbf{R}^{m+1})$ is a curvature varifold with $\boldsymbol{\Theta}^m(\|V\|, z) = 1$ for $z \in M$,

C, M, and T, are invariant under translations in directions belonging to R,

$\operatorname{spt}\|V\| \subset M \cup T$, $\quad \|\delta V\| \le \|V\|$, $\quad \|V\|(C) > 0$, $\quad \boldsymbol{\Theta}^m(\|V\|, c) = 2$,

$\|V\|(\mathbf{B}(c,r) \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = 1\}) \ge \omega(r) r^m$,

$$\inf\big\{V((\mathbf{B}(c,r) \times \mathbf{G}(m+1,m)) \cap \{(z,S) : \|S_\natural - T_\natural\| \ge 1/3\}),$$
$$\mathscr{H}^m(T \cap \mathbf{B}(c,r) \sim T_\natural[\operatorname{spt}\|V\|])\big\} \ge \omega(r)^2 r^{m+2} (\log(1/r))^{-2}$$

whenever $c \in C$ and $0 < r \le \varepsilon$ and, if $m > 2$, then there also exists a curvature varifold $V' \in \mathbf{IV}_2(\ker R_\natural)$ such that

$$V(k) = \textstyle\int_{\mathbf{R}^{m+1} \times R} k(x+y, \operatorname{im}(P_\natural + R_\natural)) \,\mathrm{d}V' \times \mathscr{H}^{m-2}((x,P),y)$$

whenever $k \in \mathscr{K}\left(\mathbf{R}^{m+1}, \mathbf{G}(m+1,m)\right)$.

Proof. Suppose $\mathbf{p}$ and $\mathbf{q}$ as related to $n = m+1$ as in 5.1 define $T = \operatorname{im}\mathbf{p}^*$. Notice that it is sufficient to prove the assertion obtained by replacing closed balls "$\mathbf{B}(c,r)$" by open cubes "$O(c,r)$", see 5.7. Hence, in view of 3.6, the construction may be reduced to the case $m = 2$ by considering suitable products with $m-2$ dimensional planes if $m > 2$. Let $\Delta = 3\sup\{\Gamma_{10.2}, 3\}$ and choose $0 < \eta \le 1$ such that $\omega(\eta) \le 2^{-6}\Delta^{-2}$.

Define a modulus of continuity ψ satisfying the Dini condition such that $\psi(r) = \sup\{8\Delta\omega(r), 4r^2\}$ for $0 \le r \le \eta$. Apply 2.2 with m, ω, and λ replaced by 2, ψ, and 1/2 to obtain a number δ, named "ε" there, as well as G and A. Let

$$\varepsilon = \inf\left\{\delta, \Delta^{-1}, \eta\right\}, \quad B = \{r : 0 < r \le \varepsilon\}.$$

Define $W \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$W(k) = 2\int_{T \sim \mathbf{p}^*[\bigcup G]} k(z,T)\,\mathrm{d}\mathscr{H}^2 z \quad \text{for } k \in \mathscr{K}(\mathbf{R}^3 \times \mathbf{G}(3,2)).$$

Whenever $Q = O(a,s) \in G$ and $s > \Delta^{-1}$ let $X_Q \in \mathbf{IV}_2(\mathbf{R}^3)$ be defined by

$$X_Q(k) = 2\int_T k(z,T)\,\mathrm{d}\mathscr{H}^2 z \quad \text{for } k \in \mathscr{K}(\mathbf{R}^3 \times \mathbf{G}(3,2))$$

and set $M_Q = \varnothing$. Whenever $Q = O(a,s) \in G$ and $s \le \Delta^{-1}$ apply 10.2 with r replaced by $\Delta s^{-1}\log(1/s)$ to construct a curvature varifold $X_Q \in \mathbf{IV}_2(\mathbf{R}^3)$ such that $(\boldsymbol{\mu}_{\Delta s^{-2}\log(1/s)} \circ \boldsymbol{\tau}_{-a})_\# X_Q$ satisfies the conditions of 10.2 in place of V implying

$$\|X_Q\| \llcorner \mathbf{p}^{-1}[\mathbf{R}^2 \sim \mathbf{U}(a,s)] = 2\mathscr{H}^2 \llcorner T \sim \mathbf{U}(\mathbf{p}^*(a),s),$$

M_Q is a two dimensional submanifold of $\mathbf{R}^3$ of class ∞,

$$\begin{aligned}
&\boldsymbol{\Theta}^2(\|X_Q\|,z) = 1 \quad \text{for } z \in M_Q,\\
&\|\delta X_Q\| \le \|X_Q\|, \qquad |\mathbf{q}(z)| \le s^2 \quad \text{for } z \in \operatorname{spt}\|X_Q\|,\\
&\|X_Q\|\left(O(\mathbf{p}^*(a),s) \cap \{z : \boldsymbol{\Theta}^2(\|X_Q\|,z) = 1\}\right) \ge 2^{-1}\mathscr{L}^2(Q),\\
&\|X_Q\|\left(\mathbf{p}^{-1}[Q]\right) \le 2\mathscr{L}^2(Q) + s^4, \quad \int \|\mathbf{b}(X_Q,z)\|\,\mathrm{d}\|X_Q\|z \le s^2,\\
&\inf\Big\{X_Q((\mathbf{p}^{-1}[Q] \times \mathbf{G}(3,2)) \cap \{(z,S) : \|S_\natural - T_\natural\| \ge 1/3\}),\\
&\qquad \mathscr{H}^2(\mathbf{p}^*[Q] \sim T_\natural[\operatorname{spt}\|X_Q\|])\Big\} \ge \Delta^{-2}s^4(\log(1/s))^{-2},
\end{aligned}$$

where $M_Q = \mathbf{p}^{-1}[\mathbf{U}(a,s)] \cap \operatorname{spt}\|X_Q\|$. Now, let $M = \bigcup\{M_Q : Q \in G\}$ and define $V \in \mathbf{IV}_2(\mathbf{R}^3)$ by

$$V = W + \sum_{Q \in G} X_Q \llcorner (\mathbf{p}^{-1}[Q] \times \mathbf{G}(3,2)).$$

Note that $\|V\|(\mathbf{p}^*[A]) \ge 1$ and $\boldsymbol{\Theta}^2(\|V\|,c) = 2$ for $\|V\|$ almost all $c \in \mathbf{p}^*[A]$ by Allard [4, 2.8 (4a), 3.5 (2)]. Let $C = \mathbf{p}^*[A] \cap \{c : \boldsymbol{\Theta}^2(\|V\|,c) = 2\}$. Moreover, observe that V is a curvature varifold with $\|\delta V\| \le \|V\|$. Finally, if $c \in \mathbf{p}^*[A]$ and $r \in B$, then there exists $O(a,s) = Q \in G$ with $Q \subset \mathbf{U}(\mathbf{p}(c),r)$ and $\mathscr{L}^2(Q) = 4s^2 \ge \psi(r)r^2$, in particular $O(\mathbf{p}^*(a),s) \subset O(c,r)$, $s \le \Delta^{-1}$, and

$$\Delta^{-2}s^4(\log(1/s))^{-2} \ge 2^{-6}\Delta^{-2}\psi(r)^2r^4(\log(1/r))^{-2} \ge \omega(r)^2r^4(\log(1/r))^{-2}$$

since $r \ge s \ge \psi(r)^{1/2}r/2 \ge r^2$. □

Remark 10.4. Concerning the relation of the two terms occurring in the infimum, the following observation is particularly appropriate. *If* n, Q, L, M, δ_1,

δ_2, δ_3, δ_4, ε, m, s, S, U, V, δ, *and* B *are as in* [22, 4.1], $p = m$, ψ *is related to* m, n, p, U *and* V *as in* 4.1, *and* $\psi(U)^{1/m} \le \delta$, *then*

$$\|V\|(B) \le 2\delta^{-1} n^{1/2} \boldsymbol{\beta}(n) V((U \times \mathbf{G}(n,m)) \cap \{(z,R) : |R_\natural - S_\natural| \ge \delta/2\});$$

in fact, this follows from 10.1, Allard [4, 8.9 (3)], and the Besicovitch–Federer covering theorem.

Remark 10.5. Taking ω in 10.3 such that $\omega(t) = (\log(1/t))^{-1}(\log(\log(1/t)))^{-2}$ for $0 < t \le e^{-e}$, where e denotes the Euler's number, one obtains

$$\lim_{r \to 0+} r^{-m-2-\delta} \int_{\mathbf{B}(c,r) \times \mathbf{G}(n,m)} \|S_\natural - T_\natural\|^\iota \, \mathrm{d}V(z,S) = \infty$$

whenever $c \in C$, $\delta > 0$, and $1 \le \iota < \infty$. Taking $\iota > 2$ and $\delta = \iota - 2$, one infers

$$\int_{M \cap \mathbf{U}(c,r)} \|\mathbf{b}(M,z)\|^q \, \mathrm{d}\mathscr{H}^m z = \infty$$

whenever $c \in \operatorname{spt}(\|V\| \llcorner C)$, $0 < r < \infty$, and $1 < q < \infty$; in fact, the Cartesian product structure of M and V reduces the problem to the case $m = 2$ in which, in view of 3.1 (3) and 3.3, one may apply [23, 11.4 (3)] with $f(z)$ replaced by $\operatorname{Tan}^2(\|V\|, z)_\natural$.

Remark 10.6. For comparison note the following well known proposition: *If* m *and* n *are positive integers,* $m \le n$, $0 \le K < \infty$, $V \in \mathbf{IV}_m(\mathbf{R}^n)$ *with* $\|\delta V\| \le K\|V\|$ *then there exists a relatively open, dense subset* A *of* $\operatorname{spt}\|V\|$ *such that for any* $1 \le q < \infty$ *there holds*

$$\limsup_{r \to 0+} r^{-m-q} \int_{\mathbf{B}(a,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^q \, \mathrm{d}V(z,S) < \infty$$

for V *almost all* $(a,T) \in A \times \mathbf{G}(n,m)$; in fact, one may combine Allard [4, 8.1 (1)] with elliptic regularity theory as provided, e.g., in [22, 3.6, 3.21] and properties of Sobolev functions, see Calderón and Zygmund [10, Theorem 12, p. 204] or [35, Theorem 3.4.2]. In particular, the tangent plane behaviour exhibited in the preceding example may not occur at V almost all points.

Remark 10.7. Example 10.3 is a refinement of the example described by Brakke in [8, 6.1].

Example 10.8. Suppose m is an integer with $m \ge 2$ and ω is a modulus of continuity.

Then there exist ε, B, R, T, C, M, and V satisfying

$$\varepsilon > 0, \quad B \subset \mathbf{R} \cap \{t : t > 0\}, \quad R \in \mathbf{G}(m+1, m-2), \quad T \in \mathbf{G}(m+1, m),$$
$$\inf B = 0, \quad C \text{ is a Borel subset of } T,$$
$$M \text{ is an } m \text{ dimensional submanifold of } \mathbf{R}^{m+1} \text{ of class } \infty,$$
$$V \in \mathbf{IV}_m(\mathbf{R}^{m+1}) \text{ is a curvature varifold with } \boldsymbol{\Theta}^m(\|V\|, z) = 1 \text{ for } z \in M,$$
$$C,\ M, \text{ and } T \text{ are invariant under translations in directions belonging to } R,$$
$$\operatorname{spt}\|V\| \subset M \cup T, \quad \|\delta V\| \le \|V\|, \quad \|V\|(C) > 0, \quad \boldsymbol{\Theta}^m(\|V\|, c) = 2,$$
$$\|V\|(\mathbf{B}(c,r) \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = 1\}) \ge \omega(r) r^m \quad \text{for } 0 < r \le \varepsilon,$$
$$\inf \big\{V((\mathbf{B}(c,r) \times \mathbf{G}(m+1,m)) \cap \{(z,S) : \|S_\natural - T_\natural\| \ge 1/3\}),$$
$$\mathscr{H}^m(T \cap \mathbf{B}(c,r) \sim T_\natural[\operatorname{spt}\|V\|])\big\} \ge \omega(r) r^{m+2} (\log(1/r))^{-2} \quad \text{for } r \in B$$

whenever $c \in C$ and, if $m > 2$, then there also exists a curvature varifold $V' \in \mathbf{IV}_2(\ker R_\natural)$ such that

$$V(k) = \int_{\mathbf{R}^{m+1} \times R} k(x+y, \operatorname{im}(P_\natural + R_\natural)) \, \mathrm{d}V' \times \mathscr{H}^{m-2}((x,P), y)$$

whenever $k \in \mathscr{K}\left(\mathbf{R}^{m+1}, \mathbf{G}(m+1,m)\right)$.

Proof. Modify the construction of 10.3 by replacing its second paragraph by "Define a modulus of continuity ψ such that $\psi(r) = \sup\{8\Delta\omega(r)^{1/2}, 4r^2\}$ for $0 \le r \le \eta$. Apply 2.5 with m, ω, and λ replaced by 2, ψ, and 1/2 to obtain a number δ, named 'ε' there, as well as B, G and A. Let $\varepsilon = \inf\left\{\delta, \Delta^{-1}, \eta\right\}$." and "$\omega(r)^2$" in the last displayed inequality by "$\omega(r)$", and adding "If $c \in \mathbf{p}^*[A]$ and $0 < r \le \varepsilon$ there exists H such that $H \subset G \cap \{Q : Q \subset \mathbf{U}(\mathbf{p}(c), r)\}$ and $\mathscr{L}^2(\bigcup H) \ge \psi(r) r^2$, in particular $O(a,s) \in H$ implies $O(\mathbf{p}^*(a), s) \subset O(c,r)$ and $s \le \Delta^{-1}$." at the end, to obtain a construction for the present assertion. □

Remark 10.9. The main modification of the construction of 10.8 in comparison to 10.3 is the usage of 2.5 in place of 2.2 and that B is a (countable) set constructed in 2.5 rather than an interval.

Remark 10.10. As in 10.5, one obtains

$$\int_{M \cap \mathbf{U}(c,r)} \|\mathbf{b}(M,z)\|^q \, \mathrm{d}\mathscr{H}^m z = \infty$$

whenever $c \in \operatorname{spt}(\|V\| \llcorner C)$, $0 < r < \infty$, and $1 < q < \infty$.

Remark 10.11. Since $\boldsymbol{\Theta}^m(\|V\|, c) = 2$ for $c \in C$, the lower bound on

$$\|V\|(\mathbf{B}(c,r) \cap \{z : \boldsymbol{\Theta}^m(\|V\|, z) = 1\})$$

is the largest one permitted by the approximate continuity of $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ with respect to $\|V\|$ and the standard Vitali relation, see [13, 2.8.18, 2.9.13].

11. Super-quadratic tilt-excess, decay rates

The present section concerns integral varifolds of at least two dimensions, deferring the one dimensional case to Sect. 12. Its purpose is to complement the examples concerning the decay rates of the super-quadratic tilt-excess constructed in 10.3, 10.8 and [19, § 1] by positive results, see 11.1. This yields a sharp dividing line in most cases, see 11.3–11.5. Additionally, we prove that the examples constructed in 10.3 and 10.8 are essentially sharp also with respect to the size of holes the varifolds contain, see 11.7.

The positive results follow readily from the existing theory. For the super-quadratic tilt-excess, these are the second order rectifiability and its consequences for the decay of the quadratic tilt-excess in conjunction with the differentiation theory both obtained in [22, 4.8, 5.2] and [19, § 3] respectively. Concerning the estimate for the size of the holes, we additionally employ an approximation by $\mathbf{Q}_Q(\mathbf{R}^{n-m})$ valued functions, see [20, 3.15], and more basic results on the size of the set where the first variation is large from [19, § 2].

Theorem 11.1. *Suppose m, n, p, U, and V satisfy the hypotheses of 4.1, $V \in \mathbf{IV}_m(U)$, $2 < q < \infty$, and either*

1. *$m = 2$ and $p > 1$, or*
2. *$m > 2$ and $p \geq 2m/(m+2)$.*

Then for V almost all (z, T) there holds

$$\lim_{r \to 0+} r^{-m-2} \int_{\mathbf{B}(z,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^q \,\mathrm{d}V(\zeta, S) = 0.$$

Proof. Assume $m < n$. First, note that since the function mapping $S \in \mathbf{G}(n,m)$ to $|S_\natural - T_\natural|$ is bounded for any $T \in \mathbf{G}(n,m)$, we have

$$\limsup_{r \to 0+} r^{-m-2} \int_{\mathbf{B}(z,r) \times \mathbf{G}(n,m)} |S_\natural - T_\natural|^q \,\mathrm{d}V(\zeta, S) < \infty$$

for V almost all (z, T) by [22, 5.2 (2)] and Hölder's inequality. Second, note that [22, 4.8] implies the existence of a sequence of functions $\tau_i : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ of class 1 such that

$$\|V\| \left(U \sim \bigcup_{i=1}^{\infty} Z_i \right) = 0,$$

where $Z_i = U \cap \{z : \tau_i(z) = \operatorname{Tan}^m(\|V\|, z)_\natural\}$, hence

$$\lim_{r \to 0+} r^{-m-2} \int_{\mathbf{B}(z,r)} |\tau_i(\zeta) - \operatorname{Tan}^m(\|V\|, \zeta)_\natural|^q \,\mathrm{d}\|V\|\zeta = 0$$

for $\|V\|$ almost all $z \in Z_i$ by [19, 3.7 (i)] with Z, f, α, r, and g replaced by $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$, $\operatorname{Tan}^m(\|V\|, \cdot)_\natural$, $2/q$, ∞, and τ_i. The conclusion then follows, since the functions τ_i are of class 1. □

Remark 11.2. The concept of proof is the same as in [22, 5.2 (1)].

Remark 11.3. Note that the number 2 in r^{-m-2} cannot be replaced by any larger number by 10.3 even if $n = m + 1$ and "lim" is replaced by "lim inf".

Remark 11.4. Note the following proposition: *If m, n, p, U, and V are as in 4.1, $V \in \mathbf{IV}_m(U)$, $m > 2$, $p < 2m/(m+2)$, and $2 \le q < \infty$, then*

$$\lim_{r \to 0+} r^{-m-mp/(m-p)} \int_{\mathbf{B}(z,r)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|^q \, \mathrm{d}V(z,S) = 0$$

for V almost all (z,T); in fact, it suffices to combine [22, 5.2 (1)] with Hölder's inequality. Taking $\alpha_1 = \alpha_2$ slightly larger than $q^{-1}mp(m-p)^{-1}$ in [19, 1.2], one infers that $mp/(m-p)$ cannot be replaced by any larger number in the preceding statement even if $n = m+1$ and "lim" is replaced by "lim inf".

Remark 11.5. Note the following proposition: *If m, n, p, U, and V are as in 4.1, $V \in \mathbf{IV}_m(U)$, $m = 2$, $p = 1$, $0 < s < 2$, and $2 \le q < \infty$, then*

$$\limsup_{r \to 0+} r^{-2-s} \int_{\mathbf{B}(z,r)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|^q \, \mathrm{d}V(\zeta,S) < \infty$$

for V almost all (z,T); in fact, again, it suffices to combine [22, 5.2 (1)] with Hölder's inequality. Taking $\alpha_1 = \alpha_2$ slightly larger than $2q^{-1}$ in [19, 1.2, 1.3], one infers that s cannot be replaced by any number larger than 2 in the preceding statement and s cannot be replaced by 2 in case $q = 2$ by 5.8 both even if $n = m+1$ and "lim sup" is replaced by "lim inf". This leaves open the case $s = 2$ and $q > 2$. An affirmative answer to the latter case would be implied by interpolation if one would know

$$\limsup_{r \to 0+} r^{-1}\phi(z,r,T) < \infty \quad \text{for } V \text{ almost all } (z,T),$$

where $\phi(z,r,T)$ abbreviates

$$r^{-1} \sup \left\{ tV((\mathbf{B}(z,r)\times\mathbf{G}(n,m)) \cap \{(\zeta,S) : |S_\natural - T_\natural| > t\})^{1/2} : 0 < t < \infty \right\}.$$

Remark 11.6. *If $2 \le m \in \mathscr{P}$, $1 \le p < \infty$, and $2 < q < \infty$, then there exist $V \in \mathbf{IV}_m(U)$ related to m, $n = m+1$, p, and $U = \mathbf{R}^n$ as in 4.1 and A with $V(A) > 0$ satisfying*

$$\lim_{r\to 0+} \frac{\left(r^{-m} \int_{\mathbf{B}(z,r)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|^q \, \mathrm{d}V(\zeta,S) \right)^{1/q}}{\left(r^{-m} \int_{\mathbf{B}(z,r)\times\mathbf{G}(n,m)} |S_\natural - T_\natural|^2 \, \mathrm{d}V(\zeta,S) \right)^{1/2} + r^{1-m/p}\psi(\mathbf{B}(z,r))^{1/p}} = \infty$$

whenever $(z,T) \in A$, where ψ is as in 4.1; in fact, choosing α such that $q^{-1}mp(m-p)^{-1} < \alpha < 2^{-1}mp(m-p)^{-1}$ if $m > 2$ and $p < 2m/(m+2)$ and $2/q < \alpha < 1$ otherwise, 11.3–11.5 yield $V \in \mathbf{IV}_m(U)$ related to m, $n = m+1$, p, and $U = \mathbf{R}^n$ as in 4.1 and A with $V(A) > 0$ such that

$$\liminf_{r \to 0+} r^{-\alpha} \left(r^{-m} \int \mathbf{B}(z,r) \times \mathbf{G}(n,m) |S_\natural - T_\natural|^q \, \mathrm{d}V(\zeta,S) \right)^{1/q} > 0$$

whenever $(z,T) \in A$, hence [22, 5.2] and [13, 2.8.18, 2.9.5, 2.9.8] imply the assertion. *The same statement holds for $p = \infty$ if $\psi(\mathbf{B}(z,r))^{1/p}$ is replaced by $(\|V\| \llcorner \mathbf{B}(z,r))_{(\infty)}(\mathbf{h}(V,\cdot))$.*

Theorem 11.7. *Suppose m, n, p, U, and V satisfy the hypotheses of 4.1, $V \in \mathbf{IV}_m(U)$, and either*

(1) *$m = 2$ and $p > 1$, or*
(2) *$m > 2$ and $p \geq 2m/(m+2)$.*

Then for V almost all (c, T) there holds

$$\lim_{r \to 0+} r^{-m-2} \mathscr{H}^m(H(T, c, r)) = 0,$$

where $H(T, c, r) = T \cap \mathbf{B}(T_\natural(c), r) \sim T_\natural[\mathbf{C}(T, c, r, r) \cap \{z : \mathbf{\Theta}^{*m}(\|V\|, z) > 0\}]$.

Proof. Assume $1 < p < m$. If $m = n$, then $\delta V = 0$ by [22, 4.8], hence the conclusion follows from Allard [4, 4.6 (3)]. Therefore assume $m < n$.

Suppose Q is a positive integer. Recalling [19, 2.4], define

$$\lambda = \varepsilon_{[20,\ 3.15]}(n-m, m, Q, 1, 5^m Q, 1/4, 1/4, 1/4, 1/4, (2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)),$$
$$Z = U \cap \{z : \operatorname{Tan}^m(\|V\|, z) \in \mathbf{G}(n, m)\}$$

and $\tau : Z \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ by

$$\tau(z) = \operatorname{Tan}^m(\|V\|, z)_\natural \quad \text{whenever } z \in Z.$$

Let B_i consist of all $z \in \operatorname{spt} \|V\|$ such that either $\mathbf{B}(z, 1/i) \not\subset U$ or

$$\|\delta V\| \, \mathbf{B}(z, s) > \lambda \, \|V\|(\mathbf{B}(z, s))^{1-1/m} \quad \text{for some } 0 < s < 1/i$$

whenever i is a positive integer. Note that $B_{i+1} \subset B_i$. Moreover, let $D_i(c)$ denote the set of all $z \in U$ such that either $\mathbf{B}(z, 1/i) \not\subset U$ or

$$\int_{\mathbf{B}(z,s)} |\tau(\zeta) - \tau(c)| \, \mathrm{d}\|V\|\zeta > \lambda \, \|V\| \, \mathbf{B}(z, s) \quad \text{for some } 0 < s < 1/i$$

whenever $c \in Z$ and i is a positive integer. Note that $D_{i+1}(c) \subset D_i(c)$.

Next, the following assertion will be proven. *For $\|V\|$ almost all c there exists i such that*

$$\lim_{r \to 0+} r^{-m-2} \|V\|(B_i \cap \mathbf{B}(c, r)) = 0, \qquad \lim_{r \to 0+} r^{-m-2} \|V\|(D_i(c) \cap \mathbf{B}(c, r)) = 0.$$

Noting $mp/(m-p) \geq 2$ and applying [19, 2.9, 2.10] with m, n, μ, s, ε, and Γ replaced by $n-m$, m, $\|V\|$, m, $\inf\left\{(2\boldsymbol{\gamma}(m))^{-p/(m-p)}, \lambda^{p/(m-p)}\right\}$, and $8\boldsymbol{\gamma}(m)m$ yields the first equality. In view of [22, 5.2 (2)], applying [19, 3.7 (ii)] with n, m, μ, Z, f, α, q, and r replaced by m, $n-m$, $\|V\|$, $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$, τ, 1, 2, and ∞ one obtains the second equality.

Note that for V almost all (c, T) with density $\mathbf{\Theta}^m(\|V\|, c) = Q$ the hypotheses of [20, 3.15] (Lipschitz approximation theorem) with m, n, L, M, δ_1, δ_2, δ_3, δ_4, δ_5, a, h, and μ replaced by $n-m$, n, 1, $5^m Q$, $1/4$, $1/4$, $1/4$, $1/4$, $(2\boldsymbol{\gamma}(m)m)^{-m}/\boldsymbol{\alpha}(m)$, c, r, and $\|V\|$ are satisfied for all sufficiently small $r > 0$. Therefore the conclusions (1)–(3) of [20, 3.15] with ε_1 replaced by λ in conjunction with the assertion of the preceding paragraph yield the conclusion. □

Remark 11.8. Possibly up to logarithmic factors, the estimate obtained is sharp even in case $n - m = 1$ and $p = \infty$ by 10.3 and 10.8.

12. The one dimensional case

For completeness, we consider in this section one dimensional integral varifolds of locally bounded first variation. In that case we prove that there is a set, almost equal to the support of the weight measure of the varifold, such that the tangent map of the varifold is differentiable relative to this set almost everywhere, see 12.4. This implies that near almost all points the varifold may be expressed by a finite sum of graphs of Lipschitzian functions, see 12.5.

The differentiability result for the tangent map mainly relies on an adaptation of a coercive estimate of Allard and Almgren in [1, § 5] in conjunction with differentiability results of approximate and integral nature obtained for that map in [22, § 5]. The corollary then follows from a suitable approximation by Lipschitzian $\mathbf{Q}_Q(\mathbf{R}^{n-1})$ valued functions, see [20, 3.15], in combination with a structural result for such function, see Almgren [7, 1.10].

Theorem 12.1. *Suppose U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_1(U)$, $M = \{z : 0 < \boldsymbol{\Theta}^m(\|V\|, z) < \infty\}$, μ is a Radon measure over U, $f : U \to \mathbf{R}$ is a Lipschitzian function,*

$$\int \operatorname{ap} \mathrm{D} f \bullet \operatorname{ap} \mathrm{D} \theta \, \mathrm{d}\|V\| \le \mu(\theta) \quad \text{whenever } \theta \in \mathscr{D}(U, \mathbf{R}) \text{ and } \theta \ge 0,$$

where "ap" denotes approximate differentiation with respect to $(\|V\|, 1)$, and $\phi \in \mathscr{D}(U, \mathbf{R})$ with $\phi \ge 0$.

Then there holds

$$(\mathscr{H}^1 \llcorner M)_{(\infty)}(\phi |\operatorname{ap} \mathrm{D} f| \boldsymbol{\Theta}^1(\|V\|, \cdot)) \le \operatorname{Lip}(\phi) \int_{\operatorname{spt} \mathrm{D} \phi} |\operatorname{ap} \mathrm{D} f| \, \mathrm{d}\|V\| + \mu(\phi).$$

Proof. Abbreviate $\gamma = \operatorname{Lip}(\phi) \int_{\operatorname{spt} \mathrm{D} \phi} |\operatorname{ap} \mathrm{D} f| \, \mathrm{d}\|V\| + \mu(\phi)$ and define

$$h(t) = \int_{M \cap \{z : f(z) = t\}} \phi(z) |\operatorname{ap} \mathrm{D} f(z)| \boldsymbol{\Theta}^1(\|V\|, z) \, \mathrm{d}\mathscr{H}^0 z \quad \text{whenever } t \in \mathbf{R}$$

and $T = \{t : h(t) \le \gamma\}$. Since 3.5 (2) and Allard [4, 3.5 (1b)] imply

$$\operatorname{ap} \mathrm{D} f(z) = 0 \quad \text{for } \mathscr{H}^1 \text{ almost all } z \in M \cap f^{-1}[N]$$

whenever $\mathscr{L}^1(N) = 0$, it is sufficient to prove $\mathscr{L}^1(\mathbf{R} \sim T) = 0$.

Approximating θ by convolution and using [21, 4.5 (3)], one obtains

$$\int \operatorname{ap} \mathrm{D} f \bullet \operatorname{ap} \mathrm{D} \theta \, \mathrm{d}\|V\| \le \mu(\theta)$$

whenever $\theta : U \to \mathbf{R}$ is a nonnegative Lipschitzian function with compact support. Employing 3.5 (2) with $g(z)$ replaced by $\phi(z)\psi'(f(z))|\operatorname{ap} \mathrm{D} f(z)|$ and taking $\theta = \phi \cdot (\psi \circ f)$ yields

$$\begin{aligned} \int \psi' h \, \mathrm{d}\mathscr{L}^1 &= \int \phi(\psi' \circ f) |\operatorname{ap} \mathrm{D} f|^2 \, \mathrm{d}\|V\| \\ &= \int \operatorname{ap} \mathrm{D} f \bullet \operatorname{ap} \mathrm{D} \theta \, \mathrm{d}\|V\| - \int (\psi \circ f) \operatorname{ap} \mathrm{D} f \bullet \operatorname{ap} \mathrm{D} \phi \, \mathrm{d}\|V\| \le \gamma \end{aligned}$$

whenever $\psi \in \mathscr{E}(\mathbf{R}, \mathbf{R})$ and $0 \leq \psi \leq 1$. Letting ψ approach the characteristic function of $\{u : t < u\}$ shows that $t \in T$ whenever t is a Lebesgue point of h and the conclusion follows from [13, 2.8.18, 2.9.8]. □

Remark 12.2. If $\theta \in \mathscr{D}(U, \mathbf{R})$ and $f \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$, then $(\delta V)(\theta \operatorname{grad} f) = \int \operatorname{ap} \mathrm{D} f \bullet \operatorname{ap} \mathrm{D} \theta \, \mathrm{d}\|V\|$. Consequently, if $\|\delta V\|$ is a Radon measure, then

$$\|V\|_{(\infty)}(\phi | \operatorname{ap} \mathrm{D} L|) \leq n \left(\|L\| \|\delta V\|(\phi) + \operatorname{Lip}(\phi) \textstyle\int_{\operatorname{spt} \phi} |\operatorname{ap} \mathrm{D} L| \, \mathrm{d}\|V\| \right)$$

whenever $L \in \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$.

Remark 12.3. The method of proof originates from Allard and Almgren [1, 5 (6)]. Adapting the terminology of [14, p. 41, p. 188, p. 391] to varifolds, our presentation views f as a "weak subsolution to Poisson's equation for the Laplace–Beltrami operator on V", see also Allard [4, 7.5].

Theorem 12.4. *Suppose $1 < n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_1(U)$, $\|\delta V\|$ is a Radon measure,*

$$C = \{(z, \mathbf{B}(z, r)) : z \in U,\ 0 < r < \infty\}, \quad Z = U \cap \{z : \operatorname{Tan}^1(\|V\|, z) \in \mathbf{G}(n, 1)\},$$

$\tau : Z \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ satisfies $\tau(z) = \operatorname{Tan}^1(\|V\|, z)_\natural$ for $z \in Z$, and A is the set of points in $\operatorname{spt} \|V\|$ at which τ is $(\|V\|, C)$ approximately continuous.

Then $\|V\|(U \sim A) = 0$ and, for $\|V\|$ almost all $z \in A$, $\tau | A$ is differentiable relative to A at z with

$$\mathrm{D}(\tau | A)(z) = (\|V\|, 1) \operatorname{ap} \mathrm{D} \tau(z).$$

Proof. First, notice that $\|V\|(U \sim A) = 0$ by [13, 2.8.18, 2.9.13] and that, for $\|V\|$ almost all z, τ is $(\|V\|, 1)$ approximately differentiable at z and

$$\limsup_{r \to 0+} r^{-2} \int_{\mathbf{B}(z,r)} |\tau(\zeta) - \tau(z)| \, \mathrm{d}\|V\| z < \infty$$

by [22, 5.2 (2), 5.5]. If z additionally satisfies $\boldsymbol{\Theta}^{*1}(\|\delta V\|, z) < \infty$, as $\|V\|$ almost all z do by [13, 2.10.19 (3)], then

$$\limsup_{A \ni \zeta \to z} |\tau(\zeta) - \tau(z)| / |\zeta - z| < \infty;$$

in fact, it is sufficient to take $L = \mathbf{1}_{\mathbf{R}^n} - \tau(z)$ and a suitable ϕ in 12.2 since $|\tau(\zeta) - \tau(z)|^2 = 2 |L \circ \tau(\zeta)|^2$ for $\zeta \in Z$ by Allard [4, 8.9 (1) (2)],

Next, one obtains a sequence of functions $\tau_i : U \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ of class 1 such that the sets $Z_i = Z \cap \{z : \tau(z) = \tau_i(z)\}$ cover $\|V\|$ almost all of U and

$$(\|V\|, 1) \operatorname{ap} \mathrm{D} \tau(z) = \mathrm{D} \tau_i(z) | \operatorname{Tan}^1(\|V\|, z) \quad \text{for } \|V\| \text{ almost all } z \in Z_i$$

by [23, 11.1 (2) (4)] and [13, 3.2.16]. Defining $f_i = (\tau - \tau_i) | A$, the preceding paragraph yields

$$\limsup_{\zeta \to z} |f_i(\zeta)| / |\zeta - z| < \infty \quad \text{for } \|V\| \text{ almost all } z \in Z_i,$$

hence, in view of 4.4 and [13, 3.1.22], it follows

$$\lim_{\zeta \to z} |f_i(\zeta)| / |\zeta - z| = 0, \quad \mathrm{D}(\tau | A)(z) = \mathrm{D} \tau_i(z) | \operatorname{Tan}(A, z)$$

for $\|V\|$ almost all $z \in Z_i$. Noting

$$\operatorname{Tan}^1(\|V\|, z) \subset \operatorname{Tan}(A, z) \subset \operatorname{Tan}(\operatorname{spt}\|V\|, z) \quad \text{for } z \in U$$

by [13, 3.2.16], the conclusion now follows from [23, 11.3]. □

Corollary 12.5. *Suppose $1 < n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_1(U)$, $\|\delta V\|$ is a Radon measure, and $\varepsilon > 0$.*

Then, for V almost all (z, T), there exist $Q \in \mathscr{P}$, $0 < r < \infty$, and $f_i : T \cap \mathbf{B}(T_\natural(z), r) \to T^\perp \cap \mathbf{B}(T_\natural^\perp(z), r)$ with $\operatorname{Lip} f_i \le \varepsilon$ for $i = 1, \ldots, Q$ such that

$$\boldsymbol{\Theta}^1(\|V\|, \zeta) = \operatorname{card}\{i : f_i(T_\natural(\zeta)) = T_\natural^\perp(\zeta)\}$$

for $\mathscr{H}^1$ almost all $\zeta \in \mathbf{C}(T, z, r, r)$.

Proof. Let Z, τ and A be defined as in 12.4, in particular $\|V\|(U \sim A) = 0$. In view of 12.4 and Allard [4, 3.5 (1)] it is sufficient to prove the conclusion at a point (z, T) such that $z \in A$, $T = \operatorname{im} \tau(z)$, $\|\delta V\|(\{z\}) = 0$, $\tau | A$ is continuous at z, and, for some $Q \in \mathscr{P}$, also

$$r^{-1} \int k(r^{-1}(\zeta - z), S) \, \mathrm{d}V(\zeta, S) \to Q \int_T k(\zeta, T) \, \mathrm{d}\mathscr{H}^1 \zeta \quad \text{as } r \to 0+$$

for $k \in \mathscr{K}(\mathbf{R}^n \times \mathbf{G}(n, 1))$. Define $\delta_1 = \delta_2 = \delta_3 = 1/2$, $\delta_4 = 1/4$, and $\delta_5 = (2\boldsymbol{\alpha}(1)\boldsymbol{\gamma}(1))^{-1}$ and recall $\delta_5 \le 1$, see e.g. [19, 2.4]. From [20, 3.15] one obtains

$$\lambda = \varepsilon_{[20,\ 3.15]}(n - 1, 1, Q, \varepsilon, 5Q, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5).$$

Choose $0 < r < \infty$ such that

$$\begin{aligned}
&(Q - 1/2)\boldsymbol{\alpha}(1) r \le \|V\|(\mathbf{C}(T, z, r, r)) \le (Q + 1/2)\boldsymbol{\alpha}(1) r, \\
&\|V\|(\mathbf{C}(T, z, r, 5r/4) \sim \mathbf{C}(T, z, r, r/2)) \le (1/2)\boldsymbol{\alpha}(1) r, \\
&\mathbf{U}(z, 4r) \subset U, \quad \|V\| \, \mathbf{U}(z, 4r) \le 5Q\boldsymbol{\alpha}(1) r, \quad \|\delta V\| \, \mathbf{U}(z, 4r) \le \lambda, \\
&|\tau(\zeta) - \tau(z)| \le \lambda \quad \text{whenever } \zeta \in A \cap \mathbf{U}(z, 4r).
\end{aligned}$$

Applying [20, 3.15 (1)–(3)] with m, n, L, M, a, h, and ε_1 by $n - 1$, 1, ε, $5Q$, z, r, and λ and noting that the set B occurring there is empty, one infers the existence of a function f with values in $\mathbf{Q}_Q(T^\perp)$ with $\operatorname{dmn} f \subset T \cap \mathbf{B}(T_\natural(z), r)$, $\operatorname{Lip} f \le \varepsilon$, and

$$\begin{aligned}
&\operatorname{spt} f(x) \subset \mathbf{B}(T_\natural^\perp(z), r) \quad \text{for } x \in \operatorname{dmn} f, \\
&\boldsymbol{\Theta}^1(\|V\|, \zeta) = \boldsymbol{\Theta}^0(\|f(T_\natural(\zeta))\|, T_\natural^\perp(\zeta)) \quad \text{for } \zeta \in \mathbf{C}(T, z, r, r) \cap T_\natural^{-1}[\operatorname{dmn} f], \\
&\mathscr{H}^1(T \cap \mathbf{B}(T_\natural(z), r) \sim \operatorname{dmn} f) + \|V\| \left(\mathbf{C}(T, z, r, r) \sim T_\natural^{-1}[\operatorname{dmn} f] \right) = 0.
\end{aligned}$$

Consequently, [13, 2.10.19 (4)] implies

$$\boldsymbol{\Theta}^1(\|V\|, \zeta) = 0 \quad \text{for } \mathscr{H}^1 \text{ almost all } \zeta \in \mathbf{C}(T, z, r, r) \sim T_\natural^{-1}[\operatorname{dmn} f].$$

Defining $g = \operatorname{Clos} f$, one infers that g is a function, $\operatorname{dmn} g = T \cap \mathbf{B}(T_\natural(z), r)$, $\operatorname{Lip} g \le \varepsilon$, and $\operatorname{spt} g(x) \subset \mathbf{B}(T_\natural^\perp(z), r)$ for $x \in \operatorname{dmn} g$. Now one readily verifies

$$\boldsymbol{\Theta}^0(\|g(T_\natural(\zeta))\|, T_\natural^\perp(\zeta)) = 0 \quad \text{for } \mathscr{H}^1 \text{ almost all } \zeta \in \mathbf{C}(T, z, r, r) \sim T_\natural^{-1}[\operatorname{dmn} f]$$

and the conclusion, both by means of Almgren [7, 1.10 (2), 1.10 (1) (iii)]. □

Remark 12.6. The fact $\mathscr{H}^m(\{(x,y) : x \in X \text{ and } y \in \operatorname{spt} g(x)\}) = 0$ whenever $\mathscr{L}^m(X) = 0$ and $g : X \to \mathbf{Q}_Q(\mathbf{R}^{n-m})$ is Lipschitzian, deduced from Almgren [7, 1.10 (2), 1.10 (1) (iii)] for $m = 1$ in the preceding proof, clearly holds for arbitrary $n > m \in \mathscr{P}$ by Almgren [7, 1.5 (11) (iii) (c)] or [20, 2.5 (1)].

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