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Singular Adams inequality for biharmonic operator on Heisenberg Group and its applications

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Abstract. The goal of this paper is to establish singular Adams type inequality for biharmonic operator on Heisenberg group. As an application, we establish the existence of a solution to

$$\Delta^2_{\mathbb{H}^n} u = \frac{f(\xi, u)}{\rho(\xi)^a} \text{ in } \Omega, \ u|_{\partial\Omega} = 0 = \frac{\partial u}{\partial \nu}\bigg|_{\partial\Omega},$$

where $0 \in \Omega \subseteq \mathbb{H}^4$ is a bounded domain, $0 \leq a \leq Q$, $(Q = 10)$. The special feature of this problem is that it contains an exponential nonlinearity and singular potential.

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1. Introduction

In this article, we are interested to establish Adams type inequality for biharmonic operator on Heisenberg group. We also establish Adams type inequality with singular potential. As an application of Adams type inequality, we prove the existence of a solution to the following biharmonic equation with Dirichlet's boundary condition on Heisenberg group:

$$\begin{aligned} \Delta^2_{\mathbb{H}^n} u &= \frac{f(\xi,u)}{\rho(\xi)^a} \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0 = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}, \end{aligned} \tag{1.1}$$

where $0 \in \Omega \subseteq \mathbb{H}^4$ is a bounded domain, $0 \leq a < Q$, $Q = 10$ is the homogeneous dimension of $\mathbb{H}^4$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies either subcritical or critical exponential growth condition. It is interesting to observe that in case of $\Omega \subseteq \mathbb{H}^n$, $n \geq 5$, by the Sobolev embedding theorem, the nonlinearity cannot exceed the degree $\frac{2Q}{Q-4}$, while the Adams' inequality allows the nonlinearities to have exponential growth when $n = 4$. Therefore Adams' inequality motivates us to discuss the above problem with exponential growth in $\Omega \subseteq \mathbb{H}^4$.

Problem (1.1), in bounded domains of $\mathbb{R}^4$ has been discussed by Macedo [40]. Macedo established the existence of a solution to the following problem with the aid of singular version of Adams' inequality and by variational arguments:

$$\begin{aligned} \Delta^2 u &= \frac{f(x,u)}{|x|^a} \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0 = \left.\frac{\partial u}{\partial \nu}\right|_{\partial\Omega}, \end{aligned} \tag{1.2}$$

where $0 \in \Omega \subseteq \mathbb{R}^4$ is a bounded domain, $0 \leq a < 4$. de Souza [19] established the existence of solution for the critical problem with singular potential $\dfrac{1}{|x|^a}$ in the case of n-Laplace operator in whole $\mathbb{R}^n$, using variational techniques. do Ó et. al. [20] established the existence of a critical point to the following functional

$$J(u) = \frac{1}{n}\int_{\mathbb{R}^n} (|\nabla u|^n + |u|^n)dx - \int_{\mathbb{R}^n} \frac{F(u)}{|x|^a}, \tag{1.3}$$

where $n \geq 2$, $F : \mathbb{R}^n \to \mathbb{R}$ is of class C^1 and $0 \leq a < n$. For the related works, see the references cited in [19, 20, 40].

For the Trudinger–Moser type inequality in unbounded domains of $\mathbb{R}^2$, and further generalizations in unbounded domains in $\mathbb{R}^n$, we refer to [39, 48]. For more details about Moser–Trudinger inequality, we refer to a survey by Chang and Wang [12]. Several existence results have been proved for problems involving Laplace and n-Laplace operator with exponential nonlinearities, see for instance [3, 4, 7, 16–18, 21, 22, 47] and references cited therein.

Let us recall the developments on Trudinger–Moser inequality. Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$ be a bounded domain. The Sobolev embedding theorem says that for $p < n$, $W^{1,p}_0(\Omega) \hookrightarrow L^q(\Omega)$, $1 \leq q \leq \frac{np}{n-p}$. For the limiting case $p = n$, we have

$$W_0^{1,n} \hookrightarrow L^q(\Omega),\ 1 \leq q < \infty$$

but it is well known (see, Example 4.43 [2]) that

$$W_0^{1,n}(\Omega) \not\hookrightarrow L^\infty(\Omega).$$

Then there is a natural question that what is the smallest possible space in which, we have embedding of $W_0^{1,n}(\Omega)$? This question was answered by Trudinger [49]. Trudinger proved that $W_0^{1,n}(\Omega)$ is embedded into Orlicz space $L_A(\Omega)$, where

$$A(t) = \exp\left(t^{\frac{p}{p-1}}\right) - 1$$

is an N function. Inequality by Trudinger [49], which was later sharpened by Moser [43] is as follows:

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, $u \in W_0^{1,n}(\Omega)$, $n \geq 2$ and*

$$\int_\Omega |\nabla u(x)|^n dx \leq 1,$$

then there exists a constant C, which depends on n only such that

$$\int_\Omega \exp(\alpha u^p) dx \leq Cm(\Omega),$$

where

$$p = \frac{n}{n-1},\ \alpha \leq \alpha_n = n\omega_n^{\frac{1}{n-1}},\ m(\Omega) = \int_\Omega dx$$

and ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere.

The integral on the left actually is finite for any positive α, but if $\alpha > \alpha_n$ it can be made arbitrarily large by an appropriate choice of u.

In order to deal with problems involving higher order elliptic operators with exponential type nonlinearities, Adams [1] extended the sharp inequality by J. Moser to higher order Sobolev spaces. Adams proved the following:

Theorem 1.2. *Let Ω be a bounded and open subset of $\mathbb{R}^n$. If m is a positive integer less than n, then there exists a constant $C_0 = C(m,n)$ such that for all $u \in C^m(\mathbb{R}^n)$ with support contained in Ω and $\|\nabla^m u\|_p \leq 1$, $p = \frac{n}{m}$, we have*

$$\frac{1}{|\Omega|}\int_\Omega \exp(\beta |u(x)|^{\frac{n}{n-m}}) dx \leq C_0$$

for all $\beta \leq \beta(n,m)$ where

$$\beta(n,\ m) = \begin{cases} \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})}\right]^{p'}, & \text{when } m \text{ is odd,} \\ \frac{n}{w_{n-1}}\left[\frac{\pi^{n/2}2^m\Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})}\right]^{p'}, & \text{when } m \text{ is even,} \end{cases}$$

$p' = \frac{p}{p-1}$. *Furthermore, for any* $\beta > \beta(n,m)$, *the integral can be made as large as desired, where*

$$\nabla^m u = \begin{cases} \triangle^{\frac{m}{2}} u, & \text{for } m \text{ even}, \\ \nabla \triangle^{\frac{m-1}{2}} u, & \text{for } m \text{ odd}. \end{cases}$$

For applications of Adams' inequality to polyharmonic equations involving exponential type nonlinearities, we refer to [23,31,33,42]. A version of Moser–Trudinger inequality with singular potential was established by Adimurthi and Sandeep [5]. They proved the following:

Theorem 1.3. *Let* Ω *be an open and bounded subset of* $\mathbb{R}^n$. *Let* $n \geq 2$ *and* $u \in W_0^{1,n}(\Omega)$. *Then for every* $\alpha > 0$ *and* $\beta \in [0, n)$,

$$\int_\Omega \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx < \infty.$$

Moreover,

$$\sup_{\|u\| \leq 1} \int_\Omega \frac{\exp\left(\alpha |u|^{\frac{n}{n-1}}\right)}{|x|^\beta} dx < \infty$$

if and only if $\dfrac{\alpha}{\alpha_n} + \dfrac{\beta}{n} \leq 1$, *where* $\|u\| = \left(\int_\Omega |\Delta u|^n\right)^{\frac{1}{n}}$.

Motivated by this singular version of Moser–Trudinger inequality several authors studied the following problem

$$\begin{aligned} &-\Delta_n u + \lambda u|u|^{n-2} = \gamma \tfrac{f(x,u)}{|x|^\beta} + kh(x,u) \text{ in } \Omega \subseteq \mathbb{R}^n, \\ &u = 0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

in bounded as well as unbounded domains. See for instance, [5,6,19,34] and references cited therein. Lam and Lu [32] established a version of singular Adams' inequality on bounded domains. More precisely, they proved that:

Theorem 1.4. *Let* $0 \leq \alpha < n$ *and* Ω *be a bounded domain in* $\mathbb{R}^n$. *Then for all* $0 \leq \beta \leq \beta_{\alpha,n,m} = \left(1 - \frac{\alpha}{n}\right) \beta(n,m)$, *we have*

$$\sup_{u \in W_0^{m,\frac{n}{m}}(\Omega),\ \|\nabla^m u\|_{\frac{n}{m}} \leq 1} \int_\Omega \frac{e^{\beta |u|^{\frac{n}{n-m}}}}{|x|^\alpha} dx < \infty. \tag{1.5}$$

When $\beta > \beta_{\alpha,n,m}$, *the supremum is infinite. Moreover, when* m *is an even number, the Sobolev space* $W_0^{m,\frac{n}{m}}(\Omega)$ *in the above supremum can be replaced by a larger Sobolev space* $W_N^{m,\frac{n}{m}}(\Omega)$.

In case of Heisenberg group $\mathbb{H}^n$, Cohn and Lu [15] established a Moser–Trudinger type inequality on bounded domains of $\mathbb{H}^n$. They proved the following result:

Theorem 1.5. *Let $\mathbb{H}^n$ be a n-dimensional Heisenberg group, $Q = 2n+2$, $Q' = \frac{Q}{Q-1}$, and $\alpha_Q = (2\pi^n \Gamma(\frac{1}{2})\Gamma(\frac{Q-1}{2})\Gamma(\frac{Q}{2})^{-1}\Gamma(n)^{-1})^{Q'-1}$. Then there exists a constant C_0 depending only on Q such that for all $\Omega \subseteq \mathbb{H}^n$, $|\Omega| < \infty$,*

$$\sup_{u \in W_0^{1,Q}(\Omega), \|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)} \leq 1} \frac{1}{|\Omega|} \int_\Omega \exp\left(\alpha_Q |u(\xi)|^{Q'}\right) d\xi \leq C_0 < \infty. \tag{1.6}$$

If α_Q is replaced by any larger number, the integral in (1.6) is still finite for any $u \in W^{1,Q}(\mathbb{H}^n)$, but the supremum is infinite.

Lam et. al. [35] established the Moser–Trudinger type inequality with a singular potential. Their result reads as follows:

Theorem 1.6. *Let $\mathbb{H}^n$ be a n-dimensional Heisenberg group, $\Omega \subseteq \mathbb{H}^n$, $|\Omega| < \infty$, $Q = 2n+2$, $Q' = \frac{Q}{Q-1}$, $0 \leq \beta < Q$, and $\alpha_Q = Q\sigma_Q^{\frac{1}{Q-1}}$, $\sigma_Q = \int_{\rho(z,t)=1} |z|^Q d\mu$. Then there exists a constant C_0 depending only on Q and β such that*

$$\sup_{u \in W_0^{1,Q}(\Omega), \|\nabla_{\mathbb{H}^n} u\|_{L^Q(\Omega)} \leq 1} \frac{1}{|\Omega|^{1-\frac{\beta}{Q}}} \int_\Omega \exp\left(\alpha_Q (1 - \frac{\beta}{Q}) |u(\xi)|^{Q'}\right) d\xi \leq C_0 < \infty. \tag{1.7}$$

If $\alpha_Q\left(1 - \frac{\beta}{Q}\right)$ is replaced by any larger number, then the supremum is infinite.

Motivated by the above research works, in order to obtain the existence of a solution to (1.1) on Heisenberg group which involves exponential and singular nonlinearity, it is natural to establish singular Adams type inequality on Heisenberg group. In fact, in this article, we first establish Adams type inequality for biharmonic operator on Heisenberg group and also establish the singular Adams type inequality. We, then prove existence of a solution to (1.1) as an application to Adams type inequality, where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function satisfying either subcritical or critical exponential growth condition.

We point out that very little research works are available for the existence of solution to singular elliptic equations on Heisenberg group even for the Laplacian, see for instance [13,41,50]. For existence results related to Laplace equation without singularity, we refer to [8–11,14,27–30,36–38,45,51,52]. For existence result concerning biharmonic operator on Heisenberg group, we refer to [53] and for qualitative questions related to biharmonic operator on Heisenberg group, we refer to [24].

Next, we define subcritical and critical growth for $f(\xi, u)$.

We say that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ has subcritical growth on $\Omega \subseteq \mathbb{H}^4$ if

$$\lim_{|u|\to\infty} \frac{|f(\xi,u)|}{\exp(\alpha u^2)} = 0, \text{ uniformly on } \Omega, \forall\, \alpha > 0. \tag{1.8}$$

We say that f has critical exponential growth if there exists $\alpha_0 >$ such that

$$\lim_{|u|\to\infty} \frac{|f(\xi,u)|}{\exp(\alpha u^2)} = 0, \text{ uniformly on } \Omega, \forall\, \alpha > \alpha_0 \tag{1.9}$$

and

$$\lim_{|u|\to\infty} \frac{|f(\xi,u)|}{\exp(\alpha u^2)} = +\infty, \text{ uniformly on } \Omega, \forall\, \alpha < \alpha_0. \tag{1.10}$$

We define

$$\Lambda = \inf_{0\neq u\in W_0^{2,2}(\Omega)} \frac{\|u\|^2}{\int_\Omega \frac{|u|^2}{\rho(\xi)^a}} > 0, \tag{1.11}$$

where $\xi = (z,t)$ and $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$, $0 \le a < Q = 10$.

We assume the following conditions on the nonlinearity f :

(H1) $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous, $f(\xi,u) \ge 0$ on $\Omega \times [0,\infty)$, $f(\xi,u) \le 0$ when $u \le 0$.

(H2) There exists $R_0 > 0$, $M > 0$ such that, $\forall\, u \ge R_0$, $\forall \xi \in \Omega$

$$0 < F(\xi,u) \le M f(\xi,u),$$

where $F(\xi,u) = \int_0^u f(\xi,s)ds$.

(H3) There exist $R_0 > 0$, $\theta > 2$ such that $\forall\, |u| \ge R_0$, $\forall \xi \in \Omega$,

$$\theta F(\xi,u) \le u f(\xi,u).$$

(H4) $\limsup_{u\to 0^+} \frac{2F(\xi,u)}{|u|^2} < \Lambda$, where Λ is defined by (1.11).

(H5) $\lim_{u\to\infty} u f(\xi,u) \exp(-\alpha_0 |u|^{p'}) \ge \beta_1 > \frac{(Q-a)A_Q}{Q\alpha_0 R^{Q-a}\mathcal{M}}$, where $p' = \frac{Q}{Q-2}$, $\mathcal{M}$ and A_Q are defined in Sect. 2.

We remark that Problem (1.1) has the following special features, which makes it challenging to study:

(i) It contains the nonlinearity f, which is of exponential growth and potential $\frac{1}{\rho(\xi)^a}$, $0 \le a \le Q$, which has singularity at $\xi = 0$. This problem is handled by the use of singular version of Adams type inequality.

(ii) The case $a = Q$, is critical in the potential. Since we do not have the singular Adams type inequality in case of $a = Q$, therefore, we use the approximation method. More precisely, we approximate the Problem (1.1) with a sequence of problems which are subcritical in potential, i.e. $a < Q$ and then, we pass the limit to conclude that Problem (1.1) has a nontrivial solution in case $a = Q$.

Next, we state our main results, which we will prove in next sections.

Theorem 1.7. *Let Ω be a bounded domain in $\mathbb{H}^n, n = 4$, $p = \frac{Q}{2}$, $p' = \frac{Q}{Q-2}$, where $Q = 2n+2$ is homogeneous dimension of $\mathbb{H}^n$. Then there exists a constant $C(\Omega)$ such that for all $u \in C_0^\infty(\Omega)$ and $\|\Delta_{\mathbb{H}^n} u\|_p \le 1$,*

$$\int_\Omega \exp\left(A_Q |u(\xi)|^{p'}\right) \le C_0 < \infty, \tag{1.12}$$

where

$$A_Q = \frac{Q}{c_0 \gamma_n^{p'}},$$

$$c_0 = \int_{\Sigma} d\mu,\, \Sigma = \{\xi \in \mathbb{H}^n : |\xi| = 1\} \tag{1.13}$$

and

$$\gamma_n = \left(n(n+1) \int_{\mathbb{H}^n} |z|^2 (|z|^4 + t^2 + 1)^{-\frac{n+4}{2}} d\xi \right)^{-1}. \tag{1.14}$$

Furthermore, if we choose any number greater than A_Q *then inequality fails to hold.*

Theorem 1.8. *Let* Ω *be a bounded domain in* $\mathbb{H}^n, n = 4$ *and* $0 \le a < Q$, *where* $Q = 2n + 2$ *is homogeneous dimension of* $\mathbb{H}^n$. *Then there exists a constant* $C_0(\Omega)$ *such that for all* $u \in C_0^\infty(\Omega)$ *and* $\|\Delta_{\mathbb{H}^n} u\|_{Q/2} \le 1$,

$$\int_\Omega \frac{\exp\left(A_Q \left(1 - \frac{a}{Q}\right) |u(\xi)|^{Q/(Q-2)}\right)}{\rho(\xi)^a} \le C_0 < \infty,$$

where $\rho(\xi) = \sqrt{(|z|^4 + t^2)^{\frac{1}{4}}}$, $A_Q = \dfrac{Q}{c_0 \gamma_n^{p'}}$ *and* γ_n *and* c_0 *are as defined by* (1.13) *and* (1.14), *respectively. Furthermore, if we choose any number greater than* $A_Q(1 - \frac{a}{Q})$ *then inequality fails to hold.*

Theorem 1.9. *Assume that* f *satisfies the subcritical growth condition* (1.8) *and* (H1)–(H5) *hold, then Problem* (1.1) *has a weak solution for* $0 < a < Q$.

Theorem 1.10. *Assume that* f *satisfies the critical growth condition* (1.9), (1.10) *and* (H1)–(H5) *hold, then Problem* (1.1) *has a weak solution for* $0 < a < Q$.

We say (1.1) has a critical potential case when $a = Q$. In this case there is no singular adams type inequality. In critical potential case, we establish the following:

Theorem 1.11. *Assume that* f *satisfies the subcritical growth condition* (1.8) *and* (H1)–(H5) *hold, then Problem* (1.1) *has a weak solution for* $a = Q$.

Theorem 1.12. *Assume that* f *satisfies the critical growth condition* (1.9),(1.10) *and* (H1)–(H5) *hold, then Problem* (1.1) *has a weak solution for* $a = Q$.

The plan of the article is as follows. In Sect. 2, we give important preliminaries on Heisenberg group and auxiliary results, which are used to prove the main theorems. In Sect. 3, we prove Theorem 1.7 and 1.8. In Sect. 4, we prove Theorems 1.9–1.12.

2. Preliminaries and auxiliary results

First, let us recall the briefs on the Heisenberg group $\mathbb{H}^n$. The Heisenberg group $\mathbb{H}^n = (\mathbb{R}^{2n+1}, \cdot)$, is the space $\mathbb{R}^{2n+1}$ with the non-commutative law of product

$$(x, y, t) \cdot (x', y', t') = (x + x',\, y + y', t + t' + 2(\langle y, x'\rangle - \langle x,\, y'\rangle)),$$

where $x,\, y,\, x',\, y' \in \mathbb{R}^n$, $t,\, t' \in \mathbb{R}$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$. This operation endows $\mathbb{H}^n$ with the structure of a Lie group. The Lie algebra of $\mathbb{H}^n$ is generated by the left-invariant vector fields

$$T = \frac{\partial}{\partial t},\;\; X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t},\;\; Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t},\;\; i = 1,\, 2, .3,\, \ldots,\, n.$$

These generators satisfy the non-commutative formula

$$[X_i, Y_j] = -4\delta_{ij} T,\;\; [X_i, X_j] = [Y_i, Y_j] = [X_i, T] = [Y_i, T] = 0.$$

Let $z = (x,\, y) \in \mathbb{R}^{2n},\;\; \xi = (z,\, t) \in \mathbb{H}^n$. The parabolic dilation

$$\delta_\lambda \xi = (\lambda x,\, \lambda y,\, \lambda^2 t)$$

satisfies

$$\delta_\lambda(\xi_0 . \xi) = \delta_\lambda \xi . \delta_\lambda \xi_0$$

and

$$|\xi| = (|z|^4 + t^2)^{\frac{1}{4}} = ((x^2 + y^2)^2 + t^2)^{\frac{1}{4}}$$

is a norm with respect to the parabolic dilation which is known as Korányi gauge norm $N(z,\, t)$. In other words, $\rho(\xi) = (|z|^4 + t^2)^{\frac{1}{4}}$ denotes the Heisenberg distance between ξ and the origin. Similarly, one can define the distance between $(z,\, t)$ and $(z',\, t')$ on $\mathbb{H}^n$ as follows:

$$\rho(z,\, t; z', t') = \rho((z',\, t')^{-1} . (z,\, t)).$$

It is clear that the vector fields $X_i,\, Y_i,\, i = 1,\, 2,\, \ldots,\, n$ are homogeneous of degree 1 under the norm $|\cdot|$ and T is homogeneous of degree 2. The Lie algebra of Heisenberg group has the stratification $\mathbb{H}^n = V_1 \oplus V_2$, where the $2n$-dimensional horizontal space V_1 is spanned by $\{X_i,\, Y_i\},\; i = 1,\, 2,\, \ldots,\, n$, while V_2 is spanned by T. The Korányi ball of center ξ_0 and radius r is defined by

$$B_{\mathbb{H}^n}(\xi_0,\, r) = \{\xi : |\xi^{-1} . \xi_0| \le r\}$$

and it satisfies

$$|B_{\mathbb{H}^n}(\xi_0,\, r)| = |B_{\mathbb{H}^n}(0,\, r)| = r^d |B_{\mathbb{H}^n}(\mathbf{0},\, 1)|,$$

where $|.|$ is the $(2n+1)$-dimensional Lebesgue measure on $\mathbb{H}^n$ and $d = 2n+2$ is the so called the homogeneous dimension of Heisenberg group $\mathbb{H}^n$. The Heisenberg gradient and Heisenberg Laplacian or the Laplacian–Kohn operator on $\mathbb{H}^n$ are given by

$$\nabla_{\mathbb{H}^n} = (X_1,\, X_2,\, \ldots,\, X_n,\, Y_1,\, Y_2,\, \ldots,\, Y_n)$$

and

$$\begin{aligned} \Delta_{\mathbb{H}^n} &= \sum_{i=1}^n X_i^2 + Y_i^2 \\ &= \sum_{i=1}^n \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial t} - 4x_i \frac{\partial^2}{\partial y_i \partial t} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial t^2} \right). \end{aligned}$$

Folland [25] proved the existence of fundamental solution for the sublaplacian $-\Delta_{\mathbb{H}^n}$ on the Heisenberg group $\mathbb{H}^n$. Using Corollary 1 [25], we have the following representation formula for each $u \in C_0^\infty(\Omega)$,

$$u(\xi) = -\gamma_n \int_{\mathbb{H}^n} \Delta_{\mathbb{H}^n} u(\eta) |\xi \cdot \eta^{-1}|^{2-Q} d\eta, \tag{2.1}$$

where $Q = 2n + 2$ is the homogeneous dimension of the Heisenberg group $\mathbb{H}^n$ and

$$\gamma_n = \left(n(n+1) \int_{\mathbb{H}^n} |z|^2 (|z|^4 + t^2 + 1)^{-\frac{n+4}{2}} d\xi \right)^{-1}, \; \xi = (z, t). \tag{2.2}$$

Next, we define convolution on $\mathbb{H}^n$, see [26] for details.

Definition 2.1. (*Convolution*) If f and g are measurable functions on $\mathbb{H}^n$, then their convolution $f * g$ is defined as

$$(f * g)(\xi) = \int_{\mathbb{H}^n} f(\eta) g(\eta^{-1} \cdot \xi) d\eta = \int_{\mathbb{H}^n} f(\xi \cdot \eta^{-1}) g(\eta) d\eta,$$

provided the integrals converge.

Definition 2.2. ($D^{1,p}(\Omega)$ *and* $D_0^{1,p}(\Omega)$ *Space*) Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{1,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ such that } \; u, |\nabla_{\mathbb{H}^n} u| \in L^p(\Omega)\}.$$

$D^{1,p}(\Omega)$ is equipped with the norm

$$\|u\|_{D^{1,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

$D_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{1,p}(\Omega)} = \left(\int_\Omega |\nabla_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$

Definition 2.3. ($D^{2,p}(\Omega)$ *and* $D_0^{2,p}(\Omega)$ *Space*) Let $\Omega \subseteq \mathbb{H}^n$ be open and $1 < p < \infty$. Then we define

$$D^{2,p}(\Omega) = \{u : \Omega \to \mathbb{R} \text{ such that } \; u, |\nabla_{\mathbb{H}^n} u|, |\Delta_{\mathbb{H}^n} u| \in L^p(\Omega)\}.$$

$D^{2,p}(\Omega)$ is equipped with the norm

$$\|u\|_{D^{2,p}(\Omega)} = \left(\|u\|_{L^p(\Omega)} + \|\nabla_{\mathbb{H}^n} u\|_{L^p(\Omega)} + \|\Delta_{\mathbb{H}^n} u\|^p \right)^{\frac{1}{p}}.$$

$D_0^{2,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{D_0^{2,p}(\Omega)} = \left(\int_\Omega |\Delta_{\mathbb{H}^n} u|^p dz dt \right)^{\frac{1}{p}}.$$

Theorem 2.4. (Embedding Theorem) *Let* $k \in \mathbb{N}$ *and* $p \in [1, \infty)$.

(i) *If* $k < \frac{Q}{p}$, *then* $D_0^{k,p}(\Omega)$ *is continuously embedded into* $L^{p^*}(\Omega)$, *for* $\dfrac{1}{p^*} = \dfrac{1}{p} - \dfrac{k}{Q}$.

(ii) *If* $k = \frac{Q}{p}$, *then* $D_0^{k,p}(\Omega)$ *is continuously embedded into* $L^r(\Omega)$, *for* $r \in [1, \infty)$.

(iii) *If* $k > \frac{Q}{p}$, *then* $D_0^{k,p}(\Omega)$ *is continuously embedded into* $C^{0,\gamma}(\bar{\Omega})$, *for all* $0 \le \gamma < k - \frac{Q}{p}$.

Now, we define the Adams functions. Let $B := B(0,1)$ denote the unit ball in $\mathbb{H}^4$ and $B_\ell = B(0,\ell)$ denotes the ball with center 0 and radius ℓ. We have the following result.

Lemma 2.5. [31] *For all* $\ell \in (0,1)$ *there exists* $U_\ell \in D := \{u \in D_0^{2,2}(B) : u\,|_{B_\ell} = 1\}$, *such that*

$$\|U_\ell\| = C(B_\ell, B) \le \frac{A_Q}{Q \log\left(\frac{1}{\ell}\right)},$$

where $C(K,E)$ *denote the conductor capacity of* K *in* E, *whenever* E *is an open set and* K *a relatively compact subset of* E, *which is defined as follows:*

$$C(K,E) := \inf\{\|\Delta_{\mathbb{H}^n} u\|_2^2 : u \in C_0^\infty(E),\ u\,|_K = 1\}.$$

Let $0 \in \Omega$ and $R \le \operatorname{dist}(0, \partial\Omega)$, the Adams function is defined as follows:

$$\tilde{A}_r(\xi) = \begin{cases} \sqrt{\frac{Q\log\left(\frac{R}{r}\right)}{A_Q}}\, U_{r/R}\left(\frac{\xi}{R}\right), & |\xi| < R; \\ 0, & |\xi| \ge R, \end{cases} \tag{2.3}$$

where $0 < r < R$. It is easy to check that $\left\|\tilde{A}_r\right\| \le 1$ and we denote

$$\mathcal{M} = \lim_{k\to\infty} \int_{\frac{1}{k} \le |\xi| \le 1} \exp\left(Q \log k |U_{R/k}(\xi)|\right) d\xi.$$

We have $\mathcal{M} > 0$, for the details, we refer to [31].

Next, we recall decreasing rearrangement of functions on Heisenberg group. For the details about rearrangement on Heisenberg group, we refer to [26]. Let Ω be a bounded and measurable subset of $\mathbb{H}^n$. Let $f : \Omega \to \mathbb{R}$ be a measurable function. For $t \in \mathbb{R}$, the level set $\{f > t\}$ is defined as

$$\{f > t\} = \{\xi \in \Omega : f(\xi) > t\}.$$

Sets $\{f < t\}$, $\{f \ge t\}$ and $\{f = t\}$ can be defined in an analogous way.

Definition 2.6. (*Distribution Function*) Let $f : \Omega \to \mathbb{R}$ be a measurable function then distribution function of f is given by

$$\lambda_f(t) = |\{f > t\}|,$$

where $|A|$ denotes the Lebesgue measure of the set A.

It is easy to see that distribution function is a monotonically decreasing function of t and

$$\lambda_f(t) = \begin{cases} 0, & t \ge ess\sup(f), \\ |\Omega|, & t \le ess\inf(f). \end{cases}$$

Thus the range of λ_f is the interval $[0, |\Omega|]$.

Definition 2.7. (*Decreasing Rearrangement*) Let $\Omega \subset \mathbb{H}^n$ be bounded and let $f : \Omega \to \mathbb{R}$ be a measurable function. Then the decreasing rearrangement of f is defined as

$$f^*(0) = ess\sup(f),$$

$$f^*(s) = \inf\{t : \lambda_f(t) < s\},\ s > 0.$$

Lemma 2.8. *Let $\Omega \subset \mathbb{H}^n$ be bounded and let $f : \Omega \to \mathbb{R}$ be a measurable function. Then for $0 < p < \infty$,*

$$\int_\Omega |f(\xi)|^p d\xi = \int_0^{|\Omega|} |f^*(t)|^p dt.$$

Proof. For a proof, we refer to Chapter 1 [26]. □

Lemma 2.9. (Hardy–Littlewood inequality) *Let $\Omega \subset \mathbb{H}^n$ be bounded and let $f, g : \Omega \to \mathbb{R}$ be a measurable functions. Then*

$$\int_\Omega |f(\xi)g(\xi)|d\xi \le \int_0^{|\Omega|} f^*(t)g^*(t)dt.$$

Proof. For a proof, we refer to Chapter 1 [26]. □

The function f^{**} on $(0, \infty)$ is defined as

$$f^{**}(t) = \frac{1}{t}\int_0^\infty f^*(s)ds.$$

Next, we state Vitali's convergence theorem. We refer to [46] for the proof.

Theorem 2.10. (Vitali's convergence theorem) *Let $(X, \mathcal{F}, \mu)$ be a measure space such that $\mu(X) < \infty$. Suppose*

(i) *$\{f_n\}$ is uniformly integrable,*
(ii) *$f_n(x) \to f(x)$ a.e. as $n \to \infty$,*
(iii) *$|f(x)| < \infty$, a.e. in X,*

then $f \in L^1(X, \mu)$ and

$$\lim_{n\to\infty} \int_X |f_n - f|d\mu = 0.$$

Theorem 2.11. (Converse of Vitali's theorem) *Let $(X, \mathcal{F}, \mu)$ be a measure space such that $\mu(X) < \infty$. Let $f_n \in L^1(X, \mu)$ and*

$$\lim_{n\to\infty} \int_E f_n d\mu$$

exists for every $E \in \mathcal{F}$, then $\{f_n\}$ is uniformly integrable.

Let

$$J : D_0^{2,2}(\Omega) \longrightarrow \mathbb{R}$$

be a functional defined by

$$J(u) = \frac{1}{2}\int_\Omega |\Delta_{\mathbb{H}^n} u|^2 dx - \int_\Omega \frac{F(\xi, u)}{\rho(\xi)^a} dx, \tag{2.4}$$

where $F(\xi,u)=\int_0^u f(\xi,s)ds$. Throughout this article, we denote $\|\cdot\|_{D_0^{2,2}(\Omega)}$ by $\|\cdot\|$ and $||\cdot||_p$ denotes the standard L^p-norm.

3. Proof of Theorem 1.7 and Theorem 1.8

In order to prove Theorems 1.7 and 1.8, we need the following results. In this paper C is some generic constant which may vary from line to line. Kohn and Lu [15] proved the following theorem:

Theorem 3.1. *Let* $0<\alpha<Q$, $Q-\alpha p=0$, $p'=\dfrac{Q}{Q-\alpha}$ *and let*

$$(I_\alpha * f)(\xi)=\int_{\mathbb{H}^n}|\xi\cdot\eta^{-1}|^{\alpha-Q}f(\eta)d\eta. \tag{3.1}$$

Then there exists a constant C *such that for all* $\Omega\subseteq\mathbb{H}^n$, $|\Omega|<\infty$, *and for all* $f\in L^p(\mathbb{H}^n)$ *with support in* Ω,

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\frac{Q}{c_0}\left|\frac{(I_\alpha * f)(\xi)}{\|f\|_{L^p(\mathbb{H}^n)}}\right|^{p'}\right)d\xi\le C,$$

where $c_0=\int_\Sigma d\mu$, $\Sigma=\{\xi\in\mathbb{H}^n:|\xi|=1\}$. *Furthermore, if* Q/c_0 *is replaced by a greater number, then the statement is false.*

In particular, for $\alpha=2$, we get the following corollary:

Corollary 3.2. *Let* $p=\frac{Q}{2}$, $p'=\dfrac{Q}{Q-2}$ *and* $(I_2*f)(\xi)=\int_{\mathbb{H}^n}|\xi\cdot\eta^{-1}|^{2-Q}f(\eta)d\eta$. *Then there exists a constant* C *such that for all* $\Omega\subseteq\mathbb{H}^n$, $|\Omega|<\infty$, *and for all* $f\in L^p(\mathbb{H}^n)$ *with support in* Ω,

$$\frac{1}{|\Omega|}\int_\Omega \exp\left(\frac{Q}{c_0}\left|\frac{(I_2 * f)(\xi)}{\|f\|_{L^p(\mathbb{H}^n)}}\right|^{p'}\right)d\xi\le C,$$

where $c_0=\int_\Sigma d\mu$, $\Sigma=\{\xi\in\mathbb{H}^n:|\xi|=1\}$. *Furthermore, if* Q/c_0 *is replaced by a greater number, then the statement is false.*

Lemma 3.3. *Let* $0<\alpha<1$, $1<p<\infty$ *and* $b(s,t)$ *be a non-negative measurable function on* $(-\infty,\infty)\times[0,\infty)$. *such that almost everywhere,*

$$b(s,t)\le 1,\ \text{when } 0<s<t,$$

$$\sup_{t>0}\left(\int_{-\infty}^0+\int_t^\infty b(s,t)^{p'}ds\right)^{\frac{1}{p'}}=b<\infty.$$

Then there is a constant $C(p,\alpha)$ *such that if for* $\phi\ge 0$

$$\int_{-\infty}^\infty \phi(s)^p ds\le 1,$$

then

$$\int_0^\infty \exp(-F_\alpha(t))dt\le C,$$

where

$$F_\alpha(t) = \alpha t - \alpha \left(\int_{-\infty}^{\infty} b(s,t)\phi(s) ds \right)^{p'}.$$

Proof. In case of $\alpha = 1$, this lemma was proved by Adams [1], which was later modified for the case $0 < \alpha \leq 1$ by Lam and Lu [32]. We refer to [1,32] for the details. □

Let $U = f * g$ denote the convolution on $\mathbb{H}^n$. Then O'Neil [44] proved the following lemma:

Lemma 3.4.

$$U^*(t) \leq U^{**}(t) \leq t f^{**}(t) g^{**}(t) + \int_t^{\infty} f^*(s) g^*(s) ds.$$

Now, we are ready to prove Theorem 1.7.

Proof of Theorem 1.7 Using (2.1), we get

$$\begin{aligned} |u(\xi)| &\leq \gamma_n \int_\Omega \Delta_{\mathbb{H}^n} u(\eta) |\xi \cdot \eta^{-1}|^{2-Q} d\eta \\ &\leq \gamma_n |(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)| \text{ (by (3.1) with } \alpha = 2) \\ |u(\xi)|^{p'} &\leq \gamma_n^{p'} |(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)|^{p'}. \end{aligned} \tag{3.2}$$

Using Corollary 3.2 and Eq. (3.2), we get

$$\int_\Omega \exp\left(A_Q |u(\xi)|^{p'} \right) d\xi \leq \int_\Omega \exp\left(A_Q \gamma_n^{p'} |(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)|^{p'} \right) \leq C_0,$$

provided $A\gamma_n^{p'} \leq \dfrac{Q}{c_0}$, i.e. , $A \leq \dfrac{Q}{c_0 \gamma_n^{p'}}$. This completes the first part of the proof.

The proof of sharpness of the constant has similar lines as pp. 393 [1], so we omit the details.

In order to prove Theorem 1.8, first we prove auxiliary lemmas, which are used in the proof.

Lemma 3.5. *Let* $g(\xi) = \rho(\xi)^{2-Q}$, *then*

$$g^*(t) = \left(\frac{c_0}{Qt} \right)^{\frac{1}{p'}},$$

and

$$g^{**}(t) = p g^*(t),$$

where $\rho(\xi) = |\xi| = (|z|^4 + t^2)^{\frac{1}{4}}$, $p = \frac{Q}{2}$, $p' = \frac{Q}{Q-2}$ *and* c_0 *is defined in* (1.13).

Proof. We have

$$g^*(t) = \inf\{s > 0 : \lambda_g(s) \leq t\},$$

where

$$\lambda_g(s) = |\{\xi \in \Omega : g(\xi) > s\}|.$$

Now,

$$\begin{aligned}|\{\xi \in \Omega : g(\xi) > s\}| &= |\{\xi \in \Omega : |\xi|^{2-Q} > s\}| \\ &= |\{\xi \in \Omega : |\xi| < s^{-\frac{1}{Q-2}}\}|. \end{aligned} \tag{3.3}$$

By using polar coordinates (Proposition 1.15 [26]), from (3.3), we obtain

$$\begin{aligned}\lambda_g(s) &= \int_\Sigma \int_0^{s^{-\frac{1}{Q-2}}} r^{Q-1} dr d\mu, \text{ where } \Sigma \text{ is defined in (1.13)} \\ &= \frac{c_0}{Q} s^{-\frac{Q}{Q-2}}. \end{aligned} \tag{3.4}$$

From (3.4), we see that, for any $t > 0$,

$$\begin{aligned}\lambda_g(s) < t &\Rightarrow \frac{c_0}{Q} s^{-\frac{Q}{Q-2}} < t \\ &\Rightarrow s^{-\frac{Q}{Q-2}} < \frac{Q}{c_0} t \\ &\Rightarrow s > \left(\frac{c_0}{Qt}\right)^{\frac{Q-2}{Q}} = \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}. \end{aligned} \tag{3.5}$$

From (3.5), we obtain

$$g^*(t) \geq \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}. \tag{3.6}$$

Now, for $s = \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}$,

$$\lambda_g(s) = t. \tag{3.7}$$

From (3.7), we obtain

$$g^*(t) \leq \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}. \tag{3.8}$$

From (3.6) and (3.8), we conclude that

$$g^*(t) = \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}.$$

Next, we compute $g^{**}(t)$.

$$\begin{aligned}g^{**}(t) &= \frac{1}{t}\int_0^t g^*(s) ds \\ &= \frac{1}{t}\int_0^t \left(\frac{c_0}{Qs}\right)^{\frac{1}{p'}} ds \\ &= \frac{1}{t}\left(\frac{c_0}{Q}\right)^{\frac{1}{p'}} \int_0^t s^{-\frac{1}{p'}} ds \end{aligned}$$

$$
\begin{aligned}
&= p\frac{1}{t}\left(\frac{c_0}{Q}\right)^{\frac{1}{p'}} t^{\frac{1}{p}} \\
&= pg^*(t).
\end{aligned}
$$

This completes the proof. □

Lemma 3.6. *Let $\Omega \subseteq \mathbb{H}^n, n = 4$ be a bounded domain, $p = \frac{Q}{2}$, $p' = \dfrac{Q}{Q-2}$, $0 \leq a < Q$ and $(I_2 * f)(\xi) = \int_{\mathbb{H}^n} |\xi \cdot \eta^{-1}|^{2-Q} f(\eta) d\eta$. Then there exists a constant $C > 0$ such that for all $f \in L^p(\mathbb{H}^n)$ with support in Ω,*

$$
\frac{1}{|\Omega|} \int_\Omega \frac{\exp\left(\frac{Q}{c_0}\left(1 - \frac{a}{Q}\right) \left|\frac{(I_2 * f)(\xi)}{\|f\|_{L^p(\mathbb{H}^n)}}\right|^{p'}\right)}{\rho(\xi)^a} \leq C,
$$

where $c_0 = \int_\Sigma d\mu$, $\Sigma = \{\xi \in \mathbb{H}^n : |\xi| = 1\}$. Furthermore, if $\frac{Q}{c_0}\left(1 - \frac{a}{Q}\right)$ is replaced by a greater number, then the statement no longer holds.

Proof. Let

$$
u(\xi) = (g * f)(\xi), \text{ where}
$$

$$
g(\xi) = \rho(\xi)^{2-Q}.
$$

Then by definition

$$
u(\xi) = (I_2 * f)(\xi)
$$

and by Lemma 3.5, we get

$$
g^*(t) = \left(\frac{c_0}{Qt}\right)^{\frac{1}{p'}}, \; g^{**}(t) = pg^*(t). \tag{3.9}
$$

By Lemma 3.4, we get

$$
\begin{aligned}
u^*(t) \leq u^{**}(t) &\leq tf^{**}(t)g^{**}(t) + \int_t^{|\Omega|} f^*(s)g^*(s)ds \\
&= t.\frac{1}{t}pg^*(t)\int_0^t f^*(s)ds + \int_t^{|\Omega|} f^*(s)\left(\frac{c_0}{Q}\right)^{\frac{1}{p'}} s^{-\frac{1}{p'}} ds \text{ (by (3.9))} \\
&= \left(\frac{c_0}{Q}\right)^{\frac{1}{p'}} \left(pt^{-\frac{1}{p'}} \int_0^t f^*(s)ds + \int_t^{|\Omega|} s^{-\frac{1}{p'}} f^*(s)ds\right). \qquad (3.10)
\end{aligned}
$$

Now, using the change of variables,

$$
\phi(s) = |\Omega|^{\frac{1}{p}} f^*(|\Omega| e^{-s}) e^{-\frac{s}{p}}, \tag{3.11}
$$

we get

$$
\begin{aligned}
\int_\Omega (f(x))^p dx &= \int_0^{|\Omega|} (f^*(t))^p dt \\
&= \int_0^\infty (\phi(s))^p ds. \qquad (3.12)
\end{aligned}
$$

Let $h(\xi) = \frac{1}{\rho(\xi)}$, then $h^*(\xi) = \frac{C_Q}{t}^{\frac{a}{Q}}$, where C_Q is volume of unit ball in $\mathbb{H}^n$. By the Hardy–Littlewood inequality (Lemma 2.9), we obtain

$$\int_\Omega \frac{\exp\left(\left(1-\frac{a}{Q}\right)\frac{Q}{c_0}|u(\xi)|^{p'}\right)}{\rho(\xi)^a}d\xi \leq (C_Q)^{\frac{a}{Q}}\int_0^{|\Omega|}\frac{\exp\left(\left(1-\frac{a}{Q}\right)\frac{Q}{c_0}(u^*(t))^{p'}\right)}{t^{\frac{a}{Q}}}. \tag{3.13}$$

Let us introduce the change of variable

$$t = |\Omega|e^{-s}, \text{ then } dt = -|\Omega|e^{-s}ds$$

and using this change of variable, we get

$$\begin{aligned}
&(C_Q)^{\frac{a}{Q}}\int_0^{|\Omega|}\frac{\exp\left(\left(1-\frac{a}{Q}\right)\frac{Q}{c_0}(u^*(t))^{p'}\right)}{t^{\frac{a}{Q}}}dt\\
&= (C_Q)^{\frac{a}{Q}}\int_0^{\infty}\frac{\exp\left(1-\frac{a}{Q}\right)\frac{Q}{c_0}(u^*(|\Omega|e^{-s}))^{p'}}{(|\Omega|e^{-s})^{\frac{a}{Q}}}|\Omega|e^{-s}ds\\
&\leq (C_Q)^{\frac{a}{Q}}|\Omega|^{1-\frac{a}{Q}}\int_0^\infty \exp\left[\left(1-\frac{a}{Q}\right)\left\{p(|\Omega|e^{-s})^{-\frac{1}{p'}}\int_0^{|\Omega|e^{-s}} f^*(z)dz\right.\right.\\
&\quad \left.\left.+\int_{|\Omega|e^{-s}}^{|\Omega|} f^*(z)z^{-\frac{1}{p'}}dz\right\}^{p'} - \left(1-\frac{a}{Q}\right)s\right]ds \text{ (by (3.10))}\\
&= (C_Q)^{\frac{a}{Q}}|\Omega|^{1-\frac{a}{Q}}\int_0^\infty \exp\left[\left(1-\frac{a}{Q}\right)\left\{pe^{\frac{s}{p'}}\int_s^\infty \phi(w)e^{-\frac{w}{p'}}dw+\int_0^s\phi(w)dw\right\}^{p'}\right.\\
&\quad \left.-\left(1-\frac{q}{Q}\right)s\right]ds \text{ (by using the value of } f^*(z) \text{ from (3.11))}\\
&= (C_Q)^{\frac{a}{Q}}|\Omega|^{1-\frac{a}{Q}}\int_0^\infty \exp\left[-F_{\left(1-\frac{a}{Q}\right)}(s)\right]ds,
\end{aligned} \tag{3.14}$$

where $F_{1-\frac{a}{Q}}(s)$ is as in Lemma 3.3 with

$$b(s,t) = \begin{cases} 0 & -\infty < s \leq 0,\\ 1 & 0 < s < t,\\ pe^{\frac{t-s}{p'}} & t < s < \infty. \end{cases}$$

Since $u(\xi) = (I_2 * f)(\xi)$, therefore in view of (3.13), it is enough to show that

$$\int_0^\infty \phi(s)^p ds \leq 1 \text{ implies } \int_0^\infty \exp\left(-F_{\left(1-\frac{a}{Q}\right)}(s)\right)ds \leq C. \tag{3.15}$$

(3.15) follows by using Lemma 3.3. This completes the proof. □

Proof of Theorem 1.8 Using the Formula (2.1), we get

$$\begin{aligned}|u(\xi)| &\leq \gamma_n \int_{\Omega} \Delta_{\mathbb{H}^n} u(\xi) |\xi \cdot \eta^{-1}|^{2-Q} d\eta \\ &\leq \gamma_n |(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)| \text{ (by (3.1) with } \alpha = 2) \\ |u(\xi)|^{p'} &\leq \gamma_n^{p'} |(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)|^{p'}. \end{aligned} \tag{3.16}$$

Using Lemma 3.6 and (3.16), we get

$$\begin{aligned}&\int_{\Omega} \frac{\exp\left(A_Q\left(1-\frac{a}{Q}\right)|u(\xi)|^{p'}\right)}{\rho(\xi)^a} \\ &\leq \int_{\Omega} \frac{\exp\left(A_Q\left(1-\frac{a}{Q}\right)\gamma_n^{p'}|(I_2 * \Delta_{\mathbb{H}^n} u)(\xi)|^{p'}\right)}{\rho(\xi)^a} \leq C_0,\end{aligned}$$

provided $A_Q\left(1-\dfrac{a}{Q}\right)\gamma_n^{p'} \leq \dfrac{Q}{c_0}$.

For the sharpness of the constant, we refer to [1]. This completes the proof.

4. Proof of Theorems 1.9–1.12

In order to prove Theorems 1.9–1.12, we obtain mountain pass geometry of the associated functional. The following lemmas deal with the geometric requirements of mountain pass theorem. We have $p = \frac{Q}{2}$ and $p' = \frac{Q}{Q-2}$ throughout the section.

Lemma 4.1. *Assume that f satisfies* (1.8) *and suppose* (H1)–(H5) *hold. Then there exists $\rho > 0$ such that*

$$J(u) > 0, \text{ if } \|u\| = \rho.$$

Proof. By (H4), we have that

$$\limsup_{s \to 0^+} \frac{2F(\xi, s)}{|s|^2} < \Lambda,$$

which by definition is same as

$$\inf_{\beta > 0} \sup \left\{ \frac{2F(\xi, s)}{|s|^2} : 0 < s < \beta \right\} < \Lambda. \tag{4.1}$$

Since (4.1) is strict inequality, therefore, we can choose a number $\tau > 0$ such that

$$\inf_{\beta > 0} \sup \left\{ \frac{2F(\xi, s)}{|s|^2} : 0 < s < \beta \right\} < \Lambda - \tau. \tag{4.2}$$

Since in (4.2) infimum is strictly less than $\Lambda - \tau$, therefore there exists $\delta > 0$ such that

$$\sup \left\{ \frac{2F(\xi, s)}{|s|^2} : 0 < s < \delta \right\} < \Lambda - \tau. \tag{4.3}$$

Thus for $|s| < \delta$

$$\frac{2F(\xi, s)}{|s|^2} < \Lambda - \tau,$$

or

$$F(\xi, s) < \frac{1}{2}(\Lambda - \tau)|s|^2. \tag{4.4}$$

Since f has subcritical exponential growth therefore there exist constants $C > 0$ and $\gamma > 0$ such that

$$|f(\xi, t)| \leq C \exp(\gamma t^{p'}),\ \forall \xi \in \Omega,\ \forall t \in \mathbb{R}. \tag{4.5}$$

Thus we have

$$\begin{aligned} |F(\xi, s)| &= \left| \int_0^s f(\xi, t) dt \right| \\ &\leq \int_0^s |f(\xi, t)| dt \\ &\leq C \int_0^s \exp(\gamma t^{p'}) dt \ \ \text{(by (4.5))} \\ &\leq C \exp(\gamma s^{p'}). \end{aligned} \tag{4.6}$$

Now for $|s| \geq \delta$ and $q > 2$, there exists a constant $K(\delta, q)$ such that

$$|F(\xi, s)| \leq K|s|^q \exp(\gamma s^{p'}), \quad \forall\, |s| \geq \delta. \tag{4.7}$$

On using (4.4) and (4.6), we get

$$F(\xi, s) \leq \frac{1}{2}(\Lambda - \tau)|s|^2 + K \exp(\gamma |s|^{p'})|s|^q, \tag{4.8}$$

for all $\xi \in \Omega$, $s \in \mathbb{R}$ and for some $\gamma, \tau > 0$ and $q > 2$.

Now consider r and r' such that $\frac{1}{r} + \frac{1}{r'} = 1$, then by Hölder's inequality, we have

$$\begin{aligned} \int_\Omega \frac{\exp(\gamma |u|^{p'})|u|^q}{\rho(\xi)^a} d\xi &\leq \left(\int_\Omega \frac{\exp(\gamma r |u|^{p'})}{\rho(\xi)^{ar}} dx \right)^{\frac{1}{r}} \left(\int_\Omega |u|^{qr'} d\xi \right)^{\frac{1}{r'}} \\ &\leq \left(\int_\Omega \frac{\exp\left(\gamma r \, \|u\|^{p'} \left(\frac{|u|}{\|u\|} \right)^{p'} \right)}{\rho(\xi)^{ar}} \right)^{\frac{1}{r}} \left(\int_\Omega |u|^{qr'} d\xi \right)^{\frac{1}{r'}}. \end{aligned} \tag{4.9}$$

Now, if we choose $r > 1$ sufficiently close to 1, so that $ar < Q$ and $\|u\| \leq \sigma$ such that $\gamma r \sigma^2 < A_Q \left(1 - \frac{a}{Q}\right)$. Then by Theorem 1.8 and (4.9), we get

$$\int_\Omega \frac{\exp(\gamma |u|^{p'})|u|^q}{\rho(\xi)^a} dx \leq C \left(\int_\Omega |u|^{qr'} dx \right)^{\frac{1}{r'}}. \tag{4.10}$$

Therefore, we get

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{\Lambda-\tau}{2}\int_\Omega \frac{|u|^2}{\rho(\xi)^a}dx - C\left(\int_\Omega |u|^{r'q}\right)^{\frac{1}{r'}}. \tag{4.11}$$

Now, we have

$$\Lambda = \inf_{0\neq u\in D_0^{2,2}(\Omega)} \frac{\|u\|^2}{\int_\Omega \frac{|u|^2}{\rho(\xi)^a}}. \tag{4.12}$$

(4.12) implies that

$$\Lambda \leq \frac{\|u\|^2}{\int_\Omega \frac{|u|^2}{\rho(\xi)^a}} \ \ \forall 0\neq u\in D_0^{2,2}(\Omega)$$

or

$$\int_\Omega \frac{|u|^2}{\rho(\xi)^a} \leq \frac{1}{\Lambda}\|u\|^2. \tag{4.13}$$

On using (4.13) in (4.11), we get

$$J(u) \geq \frac{1}{2}\|u\|^2 - \frac{\Lambda-\tau}{2\Lambda}\|u\|^2 - C\|u\|_{r'q}^q. \tag{4.14}$$

Since by Theorem 2.4, $D_0^{2,2}(\Omega)$ is continuously embedded into $L^s(\Omega)$, for all $1\leq s<\infty$. Therefore, in particular, for $s=r'q$, we get

$$\|u\|_{r'q} \leq C\|u\|. \tag{4.15}$$

On using (4.15) in (4.14), we get

$$J(u) \geq \frac{1}{2}\left(1-\frac{\Lambda-\tau}{\Lambda}\right)\|u\|^2 - C\|u\|^q.$$

Since $\tau>0$ and $q>2$, choose $\rho>0$ such that

$$\frac{1}{2}\left(1-\frac{\Lambda-\tau}{\Lambda}\right)\rho - C\rho^{q-1} > 0,$$

then, we have

$$J(u) \geq \|u\|\left(\frac{1}{2}\left(1-\frac{\Lambda-\tau}{\Lambda}\right)\|u\| - C\|u\|^{q-1}\right) > 0,$$

whenever $\|u\|=\rho$. This completes the proof. □

Lemma 4.2. *There exists $e\in D_0^{2,2}(\Omega)$ with $\|e\|>\rho$ such that*

$$J(e) < \int_{\|u\|=\rho} J(u).$$

Proof. Let $0\neq u\in D_0^{2,2}(\Omega)$ and $u\geq 0$. By (H2) and (H3), there exist $c>0$ and $d>0$ such that

$$F(\xi,s) \geq cs^\theta - d,\ \forall(\xi,s)\in\Omega\times\mathbb{R}^+,\ \text{where } \theta>2. \tag{4.16}$$

For $t>0$, we have

$$J(tu) \leq \frac{t^2}{2}\int_\Omega |\Delta_{\mathbb{H}^n}u|^2 d\xi - ct^\theta\int_\Omega \frac{u^\theta}{\rho(\xi)^a}dx + d\int_\Omega \frac{1}{\rho(\xi)^a}d\xi. \tag{4.17}$$

Since $\theta > 2$, (4.17) implies that $J(tu) \to -\infty$ as $t \to \infty$. By setting $e = tu$ with t large enough, we get $\|e\| > \rho$ and

$$J(e) < \inf_{\|u\|=\rho} J(u).$$

This completes the proof. □

Lemma 4.3. *Assume that f satisfies subcritical growth condition (1.8). Then the functional J satisfies Palais–Smale condition at level c, for all $c \in \mathbb{R}$.*

Proof. Let $\{u_k\} \subseteq D_0^{2,2}(\Omega)$ ba a PS sequence at level c, that is,

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega \frac{F(\xi, u_k)}{\rho(\xi)^a} d\xi \to c, \text{ as } k \to \infty \tag{4.18}$$

and

$$|DJ(u_k)v| = \left| \int_\Omega \Delta_{\mathbb{H}^n} u_k \Delta_{\mathbb{H}^n} v d\xi - \int_\Omega \frac{f(\xi, u_k)v}{\rho(\xi)^a} \right| \le \epsilon_k \|v\|, \tag{4.19}$$

where $\epsilon_k \to 0$ as $k \to \infty$. On taking $v = u_k$ in (4.19), we get

$$|DJ(u_k)u_k| = \left| \int_\Omega |\Delta_{\mathbb{H}^n} u_k|^2 d\xi - \int_\Omega \frac{f(\xi, u_k)u_k}{\rho(\xi)^a} \right| \le \epsilon_k \|u_k\|, \tag{4.20}$$

On multiplying (4.18) with θ and subtracting (4.20) from it, we get

$$\left(\frac{\theta}{2} - 1\right)\|u_k\|^2 + \int_\Omega \frac{1}{\rho(\xi)^a}(f(\xi, u_k)u_k - \theta F(\xi, u_k))dx \le O(1) + \epsilon_k \|u_k\|. \tag{4.21}$$

By (H6), there exist $R_0 > 0$ and $\theta > 2$ such that, for $\|u\| \ge R_0$,

$$\theta F(\xi, u) \le u f(\xi, u). \tag{4.22}$$

On using (4.22), in (4.21), we get

$$\left(\frac{\theta}{2} - 1\right)\|u_k\|^2 \le O(1) + \epsilon_k \|u_k\|. \tag{4.23}$$

Since $\theta > 2$, (4.23) shows that $\{u_k\}$ is bounded, therefore, up to a subsequence

$$\begin{aligned} u_k &\rightharpoonup u_0 \text{ in } D_0^{2,2}(\Omega) \\ u_k &\to u_0 \text{ in } L^p(\Omega), \forall p \ge 1 \\ u_k(\xi) &\to u_0(\xi) \text{ a.e. in } \Omega. \end{aligned}$$

Since f has subcritical growth on Ω, therefore there exists a constant $C_k > 0$ such that

$$f(\xi, s) \le C_k \exp\left(\frac{A_Q}{2k^2}|s|^{p'}\right), \ \forall (\xi, s) \in \Omega \times \mathbb{R}. \tag{4.24}$$

Thus

$$\begin{aligned}
&\left|\int_\Omega \frac{f(\xi,u_k)}{\rho(\xi)^a}(u_k-u)d\xi\right| \leq \int_\Omega \frac{|f(\xi,u_k)|}{\rho(\xi)^a}|(u_k-u)|dx\\
&\leq \int_\Omega C_k \frac{\exp\left(\frac{A_Q}{2k^2}|u_k|^{p'}\right)}{\rho(\xi)^a}|u_k-u|d\xi\\
&\leq C\left(\int_\Omega \frac{\exp\left(\frac{rA_Q\|u_k\|^{p'}}{k^2}\frac{|u_k|^{p'}}{\|u_k\|^{p'}}\right)}{\rho(\xi)^{ar}}\right)^{\frac{1}{r}}\left(\int_\Omega |u_k-u|^{r'}\right)^{\frac{1}{r'}},\\
&\text{(where } r>1 \text{ and such that } ar<Q \text{ and } \frac{1}{r}+\frac{1}{r'}=1)\\
&\leq C\,\|u_k-u\|_{r'} \quad \text{(by Theorem 1.8)}\\
&\to 0 \text{ as } k\to\infty.
\end{aligned} \tag{4.25}$$

Similarly, we can show that

$$\int_\Omega \frac{f(\xi,u)}{\rho(\xi)^a}(u_k-u)d\xi \to 0 \ \text{ as } k\to\infty. \tag{4.26}$$

Also, we have

$$\langle DJ(u_k)-DJ(u),u_k-u\rangle \to 0,\ \text{ as } k\to\infty.$$

Thus $u_k \to u$ in $D_0^{2,2}(\Omega)$. This completes the proof. □

4.1. Subcritical growth. Proof of Theorem 1.9

Using Lemmas 4.1, 4.2 one can show that J satisfies the geometric requirements of mountain pass theorem. Also Lemma 4.3, shows that J satisfies Palais–Smale conditions. Therefore, we conclude the proof of Theorem 1.9 by applying mountain pass theorem.

4.2. The critical growth

In this case, we need the following lemma to establish the existence of solution.

Lemma 4.4. *Assume that f satisfies critical exponential growth condition* (1.9) *and* (1.10) *and suppose* (H1)–(H5) *hold. Then there exists $k>0$ such that*

$$\max\{J(tA_k): t\geq 0\} < \left(\frac{Q-a}{2Q}\right)\frac{A_Q}{\alpha_0},$$

where $A_k=\tilde{A}_{R/k}$ is defined by (2.3).

Proof. We shall prove this result by method of contradiction. Suppose that for all k, we have

$$\max\{J(tA_k): t\geq 0\} \geq \left(\frac{Q-a}{2Q}\right)\frac{A_Q}{\alpha_0}. \tag{4.27}$$

Therefore for all k there exists a $t_k>0$ at which maximum is attained and

$$J(t_kA_k)=\frac{t_k^2\,\|A_k\|^2}{2}-\int_\Omega \frac{F(\xi,t_kA_k)}{\rho(\xi)^a}dx \geq \left(\frac{Q-a}{2Q}\right)\frac{A_Q}{\alpha_0} \tag{4.28}$$

and

$$t_k^2 \|A_k\|^2 = \int_\Omega \frac{t_k A_k f(\xi, t_k A_k}{\rho(\xi)^a} d\xi. \tag{4.29}$$

Since $F(\xi, s) \geq 0$ and $\|A_k\|^2 \leq 1$, therefore from (4.28), we get

$$t_k^2 \geq \left(\frac{Q-a}{Q}\right) \frac{A_Q}{\alpha_0}. \tag{4.30}$$

Also for a given $\tau > 0$, there exists $R_\tau > 0$ such that for all $u \geq R_\tau$, we have

$$uf(\xi, u) \geq (\beta_1 - \tau) \exp(\alpha_0 |u|^{p'}). \tag{4.31}$$

On using (4.31) in (4.29), we get

$$\begin{aligned}
t_k^2 &\geq (\beta_1 - \tau) \int_{B_{R/k}} \frac{\exp(\alpha_0 |t_k A_k|^{p'})}{\rho(\xi)^a} dx \\
&= (\beta_1 - \tau) \frac{w_{Q-1}}{Q-a} \left(\frac{R}{k}\right)^{Q-a} \exp\left(\alpha_0 t_k^{p'} \left(\frac{Q \log k}{A_Q}\right)^{p'}\right) \\
&= (\beta_1 - \tau) \frac{w_{Q-1} R^{Q-a}}{Q-a} \exp\left[\alpha_0 t_k^{p'} \left(\frac{Q \log k}{A_Q}\right)^{p'} - (Q-a)\log(k)\right] \\
1 &\geq (\beta_1 - \tau) \frac{w_3 R^{Q-a}}{Q-a} \exp\left[\alpha_0 t_k^{p'} \left(\frac{Q \log k}{A_Q}\right)^{p'} - (Q-a)\log(k) - 2\log(t_k)\right].
\end{aligned} \tag{4.32}$$

(4.32) shows that $\{t_k\}$ is a bounded sequence, otherwise up to a subsequence right hand side of (4.32) tends to ∞ as $k \to \infty$. Also, we have

$$t_k^2 \to \left(\frac{Q-a}{Q}\right) \frac{A_Q}{\alpha_0} \quad \text{as } k \to \infty \tag{4.33}$$

and

$$\|A_k\| \to 1 \quad \text{as } k \to \infty.$$

Also observe that, by definition of A_k, as $k \to \infty$, we have,

$$A_k(\xi) \to 0, \quad \text{a.e. } \xi \in \Omega.$$

Let

$$X_k = \{\xi \in \Omega : t_k A_k \geq R_\tau\}$$

and

$$Y_k = \Omega \backslash X_k,$$

then the characteristic function of Y_k, $\chi_{Y_k} \to 1$, a.e. $\xi \in \Omega$. By Lebesgue dominated convergence theorem, we get

$$\int_{Y_k} t_k A_k \frac{f(\xi, t_k A_k)}{\rho(\xi)^a} d\xi \to 0 \tag{4.34}$$

and

$$\int_{Y_k} \frac{\exp(\alpha_0 |t_k A_k|^2)}{\rho(\xi)^a} dx \to \frac{w_3 R^{Q-a}}{Q-a}, \text{ as } k \to \infty. \tag{4.35}$$

Since $t_k^2 \geq \frac{Q-a}{Q}\frac{A_Q}{\alpha_0}$, therefore

$$\begin{aligned}\int_{B_R} \frac{\exp(\alpha_0|t_kA_k|^2)}{\rho(\xi)^a}d\xi &\geq \int_{B_R} \frac{\exp\left(\frac{Q-a}{Q}A_Q|A_k|^2\right)}{\rho(\xi)^a}d\xi \\ &= \int_{|x|\leq \frac{R}{k}} \frac{\exp\left(\frac{Q-a}{Q}A_Q|A_k|^2\right)}{\rho(\xi)^a}d\xi + \int_{\frac{R}{k}\leq |x|\leq R} \frac{\exp\left(\frac{Q-a}{Q}A_Q|A_k|^2\right)}{\rho(\xi)^a}d\xi \\ &= \int_{|x|\leq \frac{R}{k}} \frac{\exp \frac{Q-a}{Q}(A_Q|A_k|^2)}{\rho(\xi)^a}d\xi + \int_{\frac{R}{k}\leq |x|\leq R} \frac{\exp\left(\frac{Q-a}{Q}A_Q|A_k|^2\right)}{\rho(\xi)^a}d\xi \\ &= \frac{w_3R^{Q-a}}{Q-a} + R^{Q-a}\mathcal{M} \end{aligned} \tag{4.36}$$

Since

$$\begin{aligned} t_k^2 \geq (\beta_1-\tau)\int_{|x|\leq R} \frac{\exp(\alpha_0|t_kA_k|^2)}{\rho(\xi)^a}d\xi + \int_{Y_k} \frac{t_kA_kf(\xi,t_kA_k)}{\rho(\xi)^a}d\xi \\ -(\beta_1-\tau)\int_{Y_k} \frac{\exp(\alpha_0|t_kA_k|^2)}{\rho(\xi)^a}d\xi, \end{aligned}$$

therefore

$$\frac{Q-a}{Q}\frac{A_Q}{\alpha_0} \geq (\beta_1-\tau)R^{Q-a}\mathcal{M}$$

or

$$\beta_1 \leq \frac{A_Q}{R^{Q-a}\mathcal{M}\alpha_0}\frac{Q-a}{Q},$$

which is a contradiction to (H5). This completes the proof. □

Lemma 4.5. *Assume that f satisfies critical exponential growth condition (1.9) and (1.10). Let $\{u_k\} \subseteq D_0^{2,2}(\Omega)$ be a Palais–Smale sequence. Then $\{u_k\}$ has a subsequence, still denoted by $\{u_k\}$, and $u \in D_0^{2,2}(\Omega)$ such that*

(i) $u_k \rightharpoonup u$ *in* $D_0^{2,2}(\Omega)$

(ii) $\dfrac{f(\xi,u_k)}{\rho(\xi)^a} \to \dfrac{f(\xi,u)}{\rho(\xi)^a}$ *in* $L^1(\Omega)$.

Proof. Let $\{u_k\}$ be a Palais–Smale sequence, then

$$J(u_k) = \frac{1}{2}\|u_k\|^2 - \int_\Omega F(\xi,u_k)d\xi \to c, \text{ as } k\to\infty \tag{4.37}$$

and

$$|J'(u_k)v| = \left|\int_\Omega \Delta_{\mathbb{H}^n}u_k\Delta_{\mathbb{H}^n}v d\xi - \int_\Omega f(\xi,u_k)vd\xi\right| \leq \tau_k\|v\|. \tag{4.38}$$

Also by Lemma 4.4,

$$c < \frac{Q-a}{2Q}\frac{A_Q}{\alpha_0}.$$

From (4.37) and (4.38), we get

$$\begin{aligned} C + \tau_n \|u_k\| &\geq \left(\frac{\theta}{2} - 1\right) \|u_k\|^2 - \int_\Omega \frac{(\theta F(\xi, u_k) - f(\xi, u_k)u_k)}{\rho(\xi)^a} d\xi \\ &\geq \left(\frac{\theta}{2} - 1\right) \|u_k\|^2 , \end{aligned} \tag{4.39}$$

which implies that

$$\begin{cases} \|u_k\| \leq C, \\ \displaystyle\int_\Omega \frac{f(\xi, u_k)u_k}{\rho(\xi)^a} d\xi \leq C, \\ \displaystyle\int_\Omega \frac{F(\xi, u_k)}{\rho(\xi)^a} d\xi \leq C. \end{cases} \tag{4.40}$$

Since $D_0^{2,2}(\Omega)$ is a reflexive Banach space, therefore by (4.40), up to a subsequence

$$\begin{cases} u_k \rightharpoonup u \text{ in } D_0^{2,2}(\Omega), \\ u_k \longrightarrow u \text{ in } L^q(\Omega), \, \forall \, 1 \leq q < \infty, \\ u_k(\xi) \longrightarrow u(\xi), \text{ a.e. } \xi \in \Omega. \end{cases}$$

Furthermore, using the arguments similar to Lemma 2.1 [16], we get

$$\frac{f(\xi, u_n)}{\rho(\xi)^a} \to \frac{f(\xi, u)}{\rho(\xi)^a} \text{ in } L^1(\Omega). \tag{4.41}$$

This completes the proof. □

4.3. Proof of Theorem 1.10

By Lemmas 4.1, 4.2, we can find a Palais–Smale sequence $\{u_k\}$ at the level c and by Lemma 4.4, $0 < c < \frac{Q-a}{2Q}\frac{A_Q}{\alpha_0}$. Thus, we have

$$J(u_k) = \frac{1}{2} \|u_k\|^2 - \int_\Omega F(x, u_k) d\xi \longrightarrow c \tag{4.42}$$

and

$$|J'(u_k)v| = \left| \int_\Omega \Delta_{\mathbb{H}^n} u_k \Delta_{\mathbb{H}^n} v dx - \int_\Omega \frac{f(\xi, u_k)v}{\rho(\xi)^a} d\xi \right| \leq \epsilon_k \|v\| . \tag{4.43}$$

By Lemma 4.5, there exists $u \in D_0^{2,2}(\Omega)$ such that

(i) $u_k \rightharpoonup u$ in $D_0^{2,2}(\Omega)$.
(ii) $\frac{f(\xi,u_k)}{\rho(\xi)^a} \to \frac{f(\xi,u)}{\rho(\xi)^a}$ strongly in $L^1(\Omega)$.

Therefore by (4.43), with the aid of Lebesgue dominated convergence theorem, one can pass the limit and get

$$J'(u)v = 0$$

for all $v \in C_c^\infty(\Omega)$. Since $C_c^\infty(\Omega)$ is dense in $D_0^{2,2}(\Omega)$, therefore u is a weak solution to (1.1).

Now, we show that u is non trivial. On the contrary, let if possible $u \equiv 0$, then by (H2) and Lebesgue dominated convergence theorem,

$$\int_\Omega \frac{F(\xi,u_k)}{\rho(\xi)^a} d\xi \to 0 \text{ in } L^1(\Omega) \text{ as } k \to \infty. \tag{4.44}$$

From (4.42), we get

$$\|u_k\|^2 \to 2c < \frac{Q-a}{Q}\frac{A_Q}{\alpha_0}. \tag{4.45}$$

Choose $q > 1$, sufficiently close to 1 such that

$$\frac{Q}{Q-a} q\alpha_0 \|u_k\|^{Q/(Q-2)} < A_Q$$

for k large. Now, since f has critical exponential growth, therefore by Theorem 1.8,

$$\begin{aligned}\int_\Omega \frac{|f(\xi,u_k)}{\rho(\xi)^a} d\xi &\le C \int_\Omega \exp\left(q\alpha_0 \|u_k\|^{Q/(Q-2)} \left| \frac{u_k}{\|u_k\|} \right|^{Q/(Q-2)} \right) d\xi \\ &\le O(1), \quad \text{as } k \to \infty.\end{aligned}$$

Thus, by taking $v = u_k$ in (4.42), we obtain

$$\|u_k\|^2 \to 0 \text{ as } k \to \infty,$$

which is a contradiction. This completes the proof.

4.4. The critical potential case $a = Q$

In this section, we consider the borderline problem with respect to potential, i.e., $a = Q$

$$\begin{aligned} \Delta^2_{\mathbb{H}^n} u &= \frac{f(\xi,u)}{\rho(\xi)^Q} \quad \text{in } \Omega, \\ u|_{\partial\Omega} &= 0 = \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega}, \end{aligned} \tag{4.46}$$

where $0 \in \Omega \subseteq \mathbb{H}^n$, $n = 4$ is a bounded domain and f satisfies the exponential growth condition at subcritical and critical level. This case is delicate in the sense that Theorems 1.9 and 1.10 fail when $a = Q$.

In order to establish the existence of solution to the problem (4.46), we consider the approximate problem which has subcritical potential

$$\begin{aligned} \Delta^2_{\mathbb{H}^n} u_n &= \frac{f(\xi,u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} \quad \text{in } \Omega, \\ u_n|_{\partial\Omega} &= 0 = \left.\frac{\partial u_n}{\partial n}\right|_{\partial\Omega}, \end{aligned} \tag{4.47}$$

The solutions to (4.47) are the critical points of the functional

$$J_n : D^{2,2}_0(\Omega) \to \mathbb{R}$$

defined as

$$J_n(u_n) = \frac{1}{2}\int_\Omega |\Delta_{\mathbb{H}^n} u_n|^2 d\xi - \int_\Omega \frac{F(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi, \tag{4.48}$$

where $F(\xi, u_n) = \displaystyle\int_0^{u_n} f(\xi, s)ds$.

Lemma 4.6. *Suppose* (H1)–(H4) *hold. Then there exists* $\rho > 0$ *such that*

$$J_n(u_n) > 0, \ \textit{if} \ \|u_n\| = \rho.$$

Proof. The proof has the similar lines as the proof of Lemma 4.1, for the sake of brevity, we omit the details. □

Lemma 4.7. *There exists* $e_n \in D_0^{2,2}(\Omega)$ *with* $\|e_n\| > \rho$ *such that*

$$J_n(e_n) < \int_{\|u_n\|=\rho} J_n(u_n).$$

Proof. The proof has similar lines as the proof of Lemma 4.2 and therefore we omit the details for the sake of brevity. □

Lemma 4.8. *The functional* J_n *satisfies Palais–Smale condition at level* c, *for all* $c \in \mathbb{R}$.

Proof. Let $\{u_n^{(m)}\} \subseteq D_0^{2,2}(\Omega)$ ba a (PS) sequence at level c, that is,

$$J_n(u_n^{(m)}) = \frac{1}{2}\left\|u_n^{(m)}\right\|^2 - \frac{F(\xi, u_n^{(m)})}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi \to c, \text{ as } m \to \infty \tag{4.49}$$

and

$$|DJ_n(u_n^{(m)})v| = \left|\int_\Omega \Delta_{\mathbb{H}^n} u_n^{(m)} \Delta_{\mathbb{H}^n} v d\xi - \int_\Omega \frac{f(\xi, u_n^{(m)})v}{\rho(\xi)^{Q-\frac{1}{n}}}\right| \le \epsilon_m \|v\|, \tag{4.50}$$

where $0 < \epsilon_m < 1$ and $\epsilon_m \to 0$ as $m \to \infty$. On taking $v = u_n^{(m)}$ in (4.50), we get

$$|DJ_n(u_n^{(m)})u_n^{(m)}| = \left|\int_\Omega |\Delta_{\mathbb{H}^n} u_n^{(m)}|^2 d\xi - \int_\Omega \frac{f(\xi, u_n^{(m)})u_n^{(m)}}{\rho(\xi)^{Q-\frac{1}{n}}}\right| \le \epsilon_m \left\|u_n^{(m)}\right\|, \tag{4.51}$$

On multiplying (4.49) with θ and subtracting (4.51) from it, we get

$$\begin{aligned} &\left(\frac{\theta}{2} - 1\right)\left\|u_n^{(m)}\right\|^2 + \int_\Omega \frac{1}{\rho(\xi)^{Q-\frac{1}{n}}} \\ &\quad \times (f(\xi, u_n^{(m)})u_n^{(m)} - \theta F(\xi, u_n^{(m)}))d\xi \le O(1) + \epsilon_m \left\|u_n^{(m)}\right\|. \end{aligned} \tag{4.52}$$

By (H6), there exist $R_0 > 0$ and $\theta > 2$ such that, for $\|u_n\| \ge R_0$,

$$\theta F(\xi, u_n) \le u_n f(\xi, u_n). \tag{4.53}$$

On using (4.53), in (4.52), we get

$$\left(\frac{\theta}{2} - 1\right)\left\|u_n^{(m)}\right\|^2 \le O(1) + \epsilon_m \left\|u_n^{(m)}\right\|. \tag{4.54}$$

Since $\theta > 2$, (4.54) shows that $\{u_n^{(m)}\}$ is bounded for each fixed $n \in \mathbb{N}$, that is, $\left\|u_n^{(m)}\right\| \leq K_n$, for some $K_n > 0$ and therefore, up to a subsequence, we have

$$\begin{aligned} u_n^{(m)} &\rightharpoonup w_n \text{ in } D_0^{2,2}(\Omega) \text{ as } m \to \infty. \\ u_n^{(m)} &\longrightarrow w_n \text{ in } L^p(\Omega), \text{ as } m \to \infty \text{ for all } p \geq 1. \\ u_n^{(m)}(\xi) &\longrightarrow w_n(\xi) \text{ a.e. in } \Omega, \text{ as } m \to \infty. \end{aligned}$$

Since f has subcritical growth on Ω, therefore there exists a constant $C_{K_n} > 0$ such that

$$f(\xi, s) \leq C_{K_n} \exp\left(\frac{\beta_n}{2K_n^2}|s|^{Q/(Q-2)}\right), \quad \forall (\xi, s) \in \Omega \times \mathbb{R}, \tag{4.55}$$

where $\beta_n = A_Q\left(Q - a + \frac{1}{n}\right)$. Thus

$$\begin{aligned} &\left|\int_\Omega \frac{f(\xi, u_n^{(m)})}{\rho(\xi)^{Q-\frac{1}{n}}}(u_n^{(m)} - w_n)d\xi\right| \leq \int_\Omega \frac{|f(\xi, u_n^{(m)})|}{\rho(\xi)^{Q-\frac{1}{n}}}|(u_n^{(m)} - w_n)|d\xi \\ &\leq \int_\Omega C_{K_n} \frac{\exp\left(\frac{\beta_n}{2K_n^2}|u_n^{(m)}|^2\right)}{\rho(\xi)^{Q-\frac{1}{n}}}|u_n^{(m)} - w_n|d\xi \\ &\leq C\left(\int_\Omega \frac{\exp\left(\frac{r\beta_n\left\|u_n^{(m)}\right\|^{p'}}{K_n^2}\frac{|u_n^{(m)}|^{p'}}{\left\|u_n^{(m)}\right\|^{p'}}\right)}{\rho(\xi)^{(Q-\frac{1}{n})r}}\right)^{\frac{1}{r}}\left(\int_\Omega |u_n^{(m)} - w_n|^{r'}\right)^{\frac{1}{r'}} \\ &\left(\text{where } r > 1 \text{ and such that } \left(a - \frac{1}{n}\right) r > Q\right. \\ &\left.\text{ and } \frac{1}{r} + \frac{1}{r'} = 1\right) \\ &\leq C\left\|u_n^{(m)} - w_n\right\|_{r'} \\ &\to 0 \text{ as } m \to \infty. \end{aligned} \tag{4.56}$$

Similarly, we can show that

$$\int_\Omega \frac{f(\xi, u_n^{(m)})}{\rho(\xi)^{Q-\frac{1}{n}}}(u_n^{(m)} - w_n)d\xi \to 0 \text{ as } m \to \infty. \tag{4.57}$$

Also, we have

$$\langle DJ(u_n^{(m)}) - DJ(w_n), u_n^{(m)} - w_n\rangle \to 0, \text{ as } m \to \infty.$$

Thus $u_n^{(m)} \to w_n$ in $D_0^{2,2}(\Omega)$. This completes the proof. □

4.5. Proof of Theorem 1.11

Lemmas 4.6, 4.7 show that the functional J_n satisfies the geometric conditions required in mountain pass theorem. Lemma 4.8 shows that J_n satisfies Palais–Smale condition and therefore by mountain pass theorem, we conclude that

Problem (4.47) has a weak solution u_n, for each n, that is,

$$\int_\Omega \Delta_{\mathbb{H}^n} u_n \Delta_{\mathbb{H}^n} v d\xi = \int_\Omega \frac{f(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} v dx, \text{ for all } v \in D_0^{2,2}(\Omega). \tag{4.58}$$

Since $0 < \epsilon_m < 1$ therefore from Eq. (4.54), we have $\|u_n\| \leq C$, for some constant C independent of n. Since $D_0^{2,2}(\Omega)$ is reflexive Banach space therefore, up to a subsequence

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ in } D_0^{2,2}(\Omega) \\ u_n &\to u_0 \text{ in } L^p(\Omega), \forall p \geq 1 \\ u_n(\xi) &\to u_0(\xi) \text{ a.e. in } \Omega. \end{aligned}$$

From (4.54) and the arguments used in Lemma 4.5, we also have the following

$$\int_\Omega \frac{f(\xi, u_n) u_n}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi \leq C \tag{4.59}$$

and

$$\int_\Omega \frac{F(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi \leq C. \tag{4.60}$$

Observe that

$$\frac{f(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} \to \frac{f(\xi, u_0)}{\rho(\xi)^{Q}}, \text{ a.e. in } \Omega. \tag{4.61}$$

Using (4.61) and Vitali's convergence theorem in (4.58), we get that u_0 is a weak solution of (4.46). This completes the proof in the subcritical case.

Now, we establish the existence of solution to (4.46), when f satisfies critical exponential growth condition (1.9) and (1.10).

4.6. Proof of Theorem 1.12

Since for each $n \in \mathbb{N}$, $Q - \frac{1}{n} < Q$, therefore by Theorem 1.10, (4.47) has a weak solution u_n. Moreover, since $0 < \epsilon_m < 1$ therefore by (4.40), there exists $C > 0$ independent of n such that $\|u_n\| \leq C$, therefore, up to a subsequence

$$\begin{aligned} u_n &\rightharpoonup u_0 \text{ in } D_0^{2,2}(\Omega). \\ u_n &\to u_0 \text{ in } L^p(\Omega), \forall p \geq 1. \\ u_n(\xi) &\to u_0(\xi) \text{ a.e. in } \Omega. \end{aligned}$$

From (4.54) and the arguments used in Lemma 4.5, we also have the following

$$\int_\Omega \frac{f(\xi, u_n) u_n}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi \leq C \tag{4.62}$$

and

$$\int_\Omega \frac{F(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} d\xi \leq C. \tag{4.63}$$

Observe that

$$\frac{f(\xi, u_n)}{\rho(\xi)^{Q-\frac{1}{n}}} \to \frac{f(\xi, u_0)}{\rho(\xi)^{Q}}, \text{ a.e. in } \Omega. \tag{4.64}$$

Using (4.64) and Vitali's convergence theorem in (4.58), we get that u_0 is a weak solution of (4.46). This completes the proof in the critical case.

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