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Compactness and existence results in weighted Sobolev spaces of radial functions. Part II: existence

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Abstract. We apply the compactness results obtained in the first part of this work, to prove existence and multiplicity results for finite energy solutions to the nonlinear elliptic equation

$$-\triangle u + V\left(|x|\right) u = g\left(|x|, u\right) \quad \text{in } \Omega \subseteq \mathbb{R}^N, \; N \geq 3,$$

where Ω is a radial domain (bounded or unbounded) and u satisfies $u = 0$ on $\partial\Omega$ if $\Omega \neq \mathbb{R}^N$ and $u \to 0$ as $|x| \to \infty$ if Ω is unbounded. The potential V may be vanishing or unbounded at zero or at infinity and the nonlinearity g may be superlinear or sublinear. If g is sublinear, the case with a forcing term $g\left(|\cdot|, 0\right) \neq 0$ is also considered. Our results allow to deal with V and g exhibiting behaviours at zero or at infinity which are new in the literature and, when $\Omega = \mathbb{R}^N$, do not need to be compatible with each other.

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1. Introduction and main results

In this paper we study the existence and multiplicity of radial solutions to the following problem:

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$$\begin{cases} -\triangle u + V(|x|)\, u = g(|x|, u) & \text{in } \Omega \\ u \in D_0^{1,2}(\Omega), \quad \int_\Omega V(|x|)\, u^2 dx < \infty \end{cases} \tag{P}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, is a spherically symmetric domain (bounded or unbounded), $D_0^{1,2}(\Omega)$ is the usual Sobolev space given by the completion of $C_c^\infty(\Omega)$ with respect to the L^2 norm of the gradient and the potential V satisfies the following basic assumption, where $\Omega_r := \{|x| > 0 : x \in \Omega\}$:

(**V**) $V : \Omega_r \to [0, +\infty)$ is a measurable function such that $V \in L^1(r_1, r_2)$ for some interval $(r_1, r_2) \subseteq \Omega_r$.

More precisely, we define the space

$$H_{0,V}^1(\Omega) := \left\{ u \in D_0^{1,2}(\Omega) : \int_\Omega V(|x|)\, u^2 dx < \infty \right\} \tag{1}$$

(which is nonempty and nontrivial by assumption (**V**)) and look for solutions in the following weak sense: we name *solution* to problem (P) any $u \in H_{0,V}^1(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla h\, dx + \int_\Omega V(|x|)\, uh\, dx = \int_\Omega g(|x|, u)\, h\, dx \qquad \text{for all } h \in H_{0,V}^1(\Omega). \tag{2}$$

Of course, we will say that a solution is *radial* if it is invariant under the action on $H_{0,V}^1(\Omega)$ of the orthogonal group of $\mathbb{R}^N$.

By well known arguments, problem (P) is a model for the stationary states of reaction diffusion equations in population dynamics (see e.g. [18]). Moreover, its nonnegative weak solutions lead to special solutions (*solitary waves* and *solitons*) for several nonlinear field theories, such as nonlinear Schrödinger (or Gross-Pitaevskii) and Klein-Gordon equations, which arise in many branches of mathematical physics, such as nonlinear optics, plasma physics, condensed matter physics and cosmology (see e.g. [9,31]). In this respect, since the early studies of [12,19,25,26], problem (P) has been massively addressed in the mathematical literature, recently focusing on the case with $\Omega = \mathbb{R}^N$ and V possibly vanishing at infinity, that is, $\liminf_{|x|\to\infty} V(|x|) = 0$ (some first results on such a case can be found in [3,7,10,11]; for more recent bibliography, see e.g. [2,5,8,13–15,17,27–29,32,33] and the references therein).

Here we study problem (P) under assumptions that, together with (**V**), allow $V(r)$ to be singular at some points (including the origin if Ω is a ball), or vanishing as $r \to +\infty$ (if Ω is unbounded), or both. Also the case of $V = 0$, or V compactly supported, or V vanishing in a neighbourhood of the origin, will be encompassed by our results. As concerns the nonlinearity, we will mainly focus on the following model case (see Sect. 3 for more general results):

$$g(|x|, u) = K(|x|)\, f(u) \tag{3}$$

where f and the potential K satisfy the following basic assumptions:

(**K**) $K:\Omega_{\mathrm{r}}\to(0,+\infty)$ is a measurable function such that $K\in L^s_{\mathrm{loc}}(\Omega_{\mathrm{r}})$ for some $s>\frac{2N}{N+2}$;

(**f**) $f:\mathbb{R}\to\mathbb{R}$ is continuous and such that $f(0)=0$.

Both the cases of f superlinear and sublinear will be studied. For sublinear f, we will also deal with an additional forcing term, i.e., with nonlinearities of the form:

$$g(|x|,u)=K(|x|)f(u)+Q(|x|). \tag{4}$$

Problem (P) with such g's will be denoted by (P_Q), so that, accordingly, (P_0) will indicate problem (P) with g given by (3).

Besides hypotheses (**V**), (**K**) and (**f**), which will be always tacitly assumed in this section, the potentials V and K will satisfy suitable combinations of the following conditions:

($\mathbf{VK}_0$) $\exists\alpha_0\in\mathbb{R}$ and $\exists\beta_0\in[0,1]$ such that

$$\operatorname*{ess\,sup}_{r\in(0,R_0)}\frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}}<+\infty\quad\text{for some } R_0>0;$$

($\mathbf{VK}_\infty$) $\exists\alpha_\infty\in\mathbb{R}$ and $\exists\beta_\infty\in[0,1]$ such that

$$\operatorname*{ess\,sup}_{r>R_\infty}\frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}}<+\infty\quad\text{for some } R_\infty>0;$$

($\mathbf{V}_0$) $\exists\gamma_0>2$ such that $\operatorname*{ess\,inf}_{r\in(0,R_0)} r^{\gamma_0}V(r)>0$ for some $R_0>0$;

($\mathbf{V}_\infty$) $\exists\gamma_\infty<2$ such that $\operatorname*{ess\,inf}_{r>R_\infty} r^{\gamma_\infty}V(r)>0$ for some $R_\infty>0$.

We mean that $V(r)^0=1$ for every r, so that conditions ($\mathbf{VK}_0$) and ($\mathbf{VK}_\infty$) will also make sense if $V(r)=0$ for $r<R_0$ or $r>R_\infty$, with $\beta_0=0$ or $\beta_\infty=0$ respectively.

Concerning the nonlinearity, our existence results rely on suitable combinations of the following assumptions:

($\mathbf{f}_1$) $\exists q_1,q_2>1$ such that

$$\sup_{t>0}\frac{|f(t)|}{\min\{t^{q_1-1},t^{q_2-1}\}}<+\infty;$$

($\mathbf{F}_1$) $\exists\theta>2$ and $\exists t_0>0$ such that $0\le\theta F(t)\le f(t)t$ for all $t\ge0$ and $F(t_0)>0$;

($\mathbf{F}_2$) $\exists\theta>2$ and $\exists t_0>0$ such that $0<\theta F(t)\le f(t)t$ for all $t\ge t_0$;

($\mathbf{F}_3$) $\exists\theta<2$ such that $\liminf_{t\to0^+}\dfrac{F(t)}{t^\theta}>0$.

Here and in the following, we denote $F(t):=\int_0^t f(s)\,ds$. Observe that the double-power growth condition ($\mathbf{f}_1$) with $q_1\ne q_2$ is more stringent than the following and more usual single-power one:

($\mathbf{f}_2$) $\exists q>1$ such that

$$\sup_{t>0}\frac{|f(t)|}{t^{q-1}}<+\infty$$

(the former implies the latter for $q = q_1$, $q = q_2$ and every q in between). On the other hand, $(\mathbf{f}_1)$ does not require $q_1 \neq q_2$, so that it is actually equivalent to $(\mathbf{f}_2)$ as long as one can take $q_1 = q_2$ (cf. Remarks 2.3 and 7.5).

In order to state our existence results for superlinear nonlinearities, we introduce the following notation. For $\alpha, \gamma \in \mathbb{R}$ and $\beta \in [0,1]$, we define the function

$$q_{**}(\alpha,\beta,\gamma) := 2\frac{2\alpha + (1-2\beta)\gamma + 2(N-1)}{2(N-1)-\gamma} \qquad \text{if } \gamma \neq 2N-2. \tag{5}$$

Then, for $\gamma \geq 2$, we set

$$\underline{\alpha}(\beta,\gamma) := \begin{cases} -(1-\beta)\gamma & \text{if } 2 \leq \gamma \leq 2N-2 \\ -\infty & \text{if } \gamma > 2N-2, \end{cases}$$

$$\underline{q}(\alpha,\beta,\gamma) := \begin{cases} 2 & \text{if } 2 \leq \gamma \leq 2N-2 \text{ and } \alpha > \underline{\alpha}(\beta,\gamma) \\ \max\{2, q_{**}(\alpha,\beta,\gamma)\} & \text{if } \gamma > 2N-2 \text{ and } \alpha > \underline{\alpha}(\beta,\gamma), \end{cases}$$

and

$$\overline{q}(\alpha,\beta,\gamma) := \begin{cases} q_{**}(\alpha,\beta,\gamma) & \text{if } 2 \leq \gamma < 2N-2 \text{ and } \alpha > \underline{\alpha}(\beta,\gamma) \\ +\infty & \text{if } \gamma \geq 2N-2 \text{ and } \alpha > \underline{\alpha}(\beta,\gamma). \end{cases}$$

Theorem 1. *Let $\Omega = \mathbb{R}^N$. Assume that f satisfies $(\mathbf{F}_1)$, or that $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and f satisfies $(\mathbf{F}_2)$. Assume furthermore that $(\mathbf{VK}_0)$, $(\mathbf{VK}_\infty)$ and $(\mathbf{f}_1)$ hold with*

$$\alpha_0 > \underline{\alpha}, \qquad \underline{q} < q_1 < \overline{q}, \qquad q_2 > \max\{2, q_{**}\}, \tag{6}$$

where

$$\underline{\alpha} = \underline{\alpha}(\beta_0, 2), \ \underline{q} = \underline{q}(\alpha_0, \beta_0, 2), \ \overline{q} = \overline{q}(\alpha_0, \beta_0, 2) \ \text{ and } \ q_{**} = q_{**}(\alpha_\infty, \beta_\infty, 2).$$

Then problem (P_0) has a nonnegative radial solution $u \neq 0$. If V also satisfies $(\mathbf{V}_0)$, then we can take $\underline{\alpha} = \underline{\alpha}(\beta_0, \gamma_0)$, $\underline{q} = \underline{q}(\alpha_0, \beta_0, \gamma_0)$ and $\overline{q} = \overline{q}(\alpha_0, \beta_0, \gamma_0)$. If V also satisfies $(\mathbf{V}_\infty)$, then we can take $q_{**} = q_{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

Remark 2. 1. *The inequality $\underline{q} < \overline{q}$ is not an assumption in (6) (even in the cases with assumptions $(\mathbf{V}_0)$, $(\mathbf{V}_\infty)$), since it is ensured by the condition $\alpha_0 > \underline{\alpha}$.*

2. *For $\beta \in [0,1]$ fixed, $\underline{\alpha}(\beta,\gamma)$ is left-continuous and decreasing in $\gamma \geq 2$ (as a real extended function) and one can check that $\underline{q}(\alpha,\beta,\gamma)$ and $\overline{q}(\alpha,\beta,\gamma)$, defined on the set $\{(\alpha,\gamma) : \gamma \geq 2,\ \alpha > \underline{\alpha}(\beta,\gamma)\}$, are continuous and respectively decreasing and increasing, both in γ for α fixed and in α for γ fixed ($\overline{q}$ is continuous and increasing as a real extended valued function). Similarly, $\max\{2, q_{**}(\alpha,\beta,\gamma)\}$ is increasing and continuous both in $\gamma \leq 2$ for $\alpha \in \mathbb{R}$ fixed and in $\alpha \in \mathbb{R}$ for $\gamma \leq 2$ fixed. Therefore, thanks to such monotonicities in γ, Theorem 1 actually improves under assumption $(\mathbf{V}_0)$, or $(\mathbf{V}_\infty)$, or both. Moreover, by both monotonicity and*

continuity (or left-continuity) in α and γ, the theorem is also true if, in $\underline{\alpha}, \underline{q}, \overline{q}$ and q_{**}, we replace $\alpha_0, \alpha_\infty, \gamma_0, \gamma_\infty$ with $\overline{\alpha}_0, \underline{\alpha}_\infty, \overline{\gamma}_0, \underline{\gamma}_\infty$, where

$$\begin{aligned}\overline{\alpha}_0 &:= \sup\{\alpha_0 : \operatorname{ess\,sup}_{r\in(0,R_0)} K(r)\, r^{-\alpha_0} V(r)^{-\beta_0} < +\infty\},\\ \underline{\alpha}_\infty &:= \inf\{\alpha_\infty : \operatorname{ess\,sup}_{r>R_\infty} K(r)\, r^{-\alpha_\infty} V(r)^{-\beta_\infty} < +\infty\},\\ \overline{\gamma}_0 &:= \sup\left\{\gamma_0 : \operatorname{ess\,inf}_{r\in(0,R_0)} r^{\gamma_0} V(r) > 0\right\},\\ \underline{\gamma}_\infty &:= \inf\left\{\gamma_\infty : \operatorname{ess\,inf}_{r>R_\infty} r^{\gamma_\infty} V(r) > 0\right\}.\end{aligned}$$

This is consistent with the fact that $(\mathbf{VK}_0)$, $(\mathbf{V}_0)$ and $(\mathbf{VK}_\infty)$, $(\mathbf{V}_\infty)$ still hold true if we respectively lower α_0, γ_0 and raise $\alpha_\infty, \gamma_\infty$.

3. *Theorem 1 also concerns the case of power-like nonlinearities, since the exponents q_1 and q_2 need not to be different in $(\mathbf{f}_1)$ and one can take $q_1 = q_2$ as soon as $\max\{2, q_{**}\} < \overline{q}$. For example, this is always the case when $(\mathbf{V}_0)$ holds with $\gamma_0 \geq 2N-2$ (which gives $\overline{q} = +\infty$), or when $\alpha_\infty \leq 2(\beta_\infty - 1)$ (which implies $\max\{2, q_{**}(\alpha_\infty, \beta_\infty, 2)\} = 2$).*

The Dirichlet problem in bounded ball domains or exterior spherically symmetric domains can be reduced to the problem in $\Omega = \mathbb{R}^N$ by suitably modifying the potentials V and K (see Sect. 5 below). Hence, by the same arguments yielding Theorem 1, we will also get the following results.

Theorem 3. *Let Ω be a bounded ball. Assume that f satisfies $(\mathbf{F}_1)$, or that $K(|\cdot|) \in L^1(\Omega)$ and f satisfies $(\mathbf{F}_2)$. Assume furthermore that $(\mathbf{VK}_0)$ and $(\mathbf{f}_2)$ hold with*

$$\alpha_0 > \underline{\alpha} \quad \text{and} \quad \underline{q} < q < \overline{q}, \tag{7}$$

where

$$\underline{\alpha} = \underline{\alpha}(\beta_0, 2), \quad \underline{q} = \underline{q}(\alpha_0, \beta_0, 2) \quad \text{and} \quad \overline{q} = \overline{q}(\alpha_0, \beta_0, 2).$$

Then problem (P_0) has a nonnegative radial solution $u \neq 0$. If V also satisfies $(\mathbf{V}_0)$, then we can take $\underline{\alpha} = \underline{\alpha}(\beta_0, \gamma_0)$, $\underline{q} = \underline{q}(\alpha_0, \beta_0, \gamma_0)$ and $\overline{q} = \overline{q}(\alpha_0, \beta_0, \gamma_0)$.

Theorem 4. *Let Ω be an exterior radial domain. Assume that f satisfies $(\mathbf{F}_1)$, or that $K(|\cdot|) \in L^1(\Omega)$ and f satisfies $(\mathbf{F}_2)$. Assume furthermore that $(\mathbf{VK}_\infty)$ and $(\mathbf{f}_2)$ hold with*

$$q > \max\{2, q_{**}\}, \quad \text{where} \quad q_{**} = q_{**}(\alpha_\infty, \beta_\infty, 2).$$

Then problem (P_0) has a nonnegative radial solution $u \neq 0$. If V also satisfies $(\mathbf{V}_\infty)$, then we can take $q_{**} = q_{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

For dealing with the sublinear case, we need some more notation. For $\alpha, \gamma \in \mathbb{R}$ and $\beta \in [0, 1)$, we define the following functions:

$$\alpha_1(\beta, \gamma) := -(1-\beta)\gamma, \quad \alpha_2(\beta) := -(1-\beta)N, \tag{8}$$

$$\alpha_3(\beta, \gamma) := -\frac{(1-2\beta)\gamma + N}{2} \tag{9}$$

and

$$q_*(\alpha, \beta, \gamma) := 2\frac{\alpha - \gamma\beta + N}{N - \gamma} \qquad \text{if } \gamma \neq N. \tag{10}$$

Then, for $\gamma \geq 2$, we set

$$q_0(\alpha,\beta,\gamma) := \begin{cases} \max\{1,2\beta\} & \text{if } 2 \leq \gamma \leq N \text{ and } \alpha \geq \alpha_1(\beta,\gamma) \\ \max\{1,2\beta,q_*(\alpha,\beta,\gamma)\} & \text{if } \gamma > N \text{ and } \alpha \geq \alpha_1(\beta,\gamma). \end{cases}$$

In contrast with the superlinear case, we divide our existence results into two theorems, essentially according as assumption $(\mathbf{VK}_0)$ holds with α_0 large enough with respect to β_0 (and γ_0, if $(\mathbf{V}_0)$ holds), or not (cf. Remark 7.2): in the first case, we need only require that f grows as a single power; in the second case, we assume the double-power growth condition $(\mathbf{f}_1)$, which, however, may still reduce to a single-power one in particular cases of exponents $\alpha_0, \beta_0, \gamma_0, \alpha_\infty, \beta_\infty, \gamma_\infty$ (cf. Remark 7.5).

Theorem 5. *Let $\Omega = \mathbb{R}^N$ and let $Q \in L^2(\mathbb{R}_+, r^{N+1}dr)$, $Q \geq 0$. Assume that f satisfies $(\mathbf{F}_3)$, or that Q does not vanish almost everywhere in (r_1, r_2). Assume furthermore that $(\mathbf{VK}_0)$, $(\mathbf{VK}_\infty)$ and $(\mathbf{f}_2)$ hold with*

$$\beta_0, \beta_\infty < 1, \quad \alpha_0 \geq \alpha_1^{(0)}, \quad \alpha_\infty < \alpha_1^{(\infty)}, \quad \max\{2\beta_\infty, q_0, q_*\} < q < 2, \tag{11}$$

where

$$\alpha_1^{(0)} = \alpha_1(\beta_0, 2), \quad \alpha_1^{(\infty)} = \alpha_1(\beta_\infty, 2), \quad q_0 = q_0(\alpha_0, \beta_0, 2)$$

$$\text{and} \quad q_* = q_*(\alpha_\infty, \beta_\infty, 2).$$

Then problem (P_Q) has a nonnegative radial solution $u \neq 0$. If V also satisfies $(\mathbf{V}_0)$, then the same result holds with $\alpha_1^{(0)} = \alpha_1(\beta_0, \gamma_0)$ and $q_0 = q_0(\alpha_0, \beta_0, \gamma_0)$, provided that $\alpha_0 > \alpha_1^{(0)}$ if $\gamma_0 \geq N$. If V also satisfies $(\mathbf{V}_\infty)$, then we can take $\alpha_1^{(\infty)} = \alpha_1(\beta_\infty, \gamma_\infty)$ and $q_* = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

Theorem 6. *Let $\Omega = \mathbb{R}^N$ and let $Q \in L^2(\mathbb{R}_+, r^{N+1}dr)$, $Q \geq 0$. Assume that f satisfies $(\mathbf{F}_3)$, or that Q does not vanish almost everywhere in (r_1, r_2). Assume furthermore that $(\mathbf{VK}_0)$, $(\mathbf{VK}_\infty)$ and $(\mathbf{f}_1)$ hold with*

$$\beta_0, \beta_\infty < 1, \qquad \max\{\alpha_2, \alpha_3\} < \alpha_0 < \alpha_1^{(0)}, \qquad \alpha_\infty < \alpha_1^{(\infty)}, \tag{12}$$

$$\max\{1, 2\beta_0\} < q_1 < q_*^{(0)}, \qquad \max\left\{1, 2\beta_\infty, q_*^{(\infty)}\right\} < q_2 < 2, \tag{13}$$

where

$$\alpha_2 = \alpha_2(\beta_0), \quad \alpha_3 = \alpha_3(\beta_0, 2), \quad \alpha_1^{(0)} = \alpha_1(\beta_0, 2), \quad \alpha_1^{(\infty)} = \alpha_1(\beta_\infty, 2),$$

$$q_*^{(0)} = q_*(\alpha_0, \beta_0, 2) \quad \text{and} \quad q_*^{(\infty)} = q_*(\alpha_\infty, \beta_\infty, 2).$$

If Q does not vanish almost everywhere in (r_1, r_2), $q_2 = 2$ is also allowed in (13). Then problem (P_Q) has a nonnegative radial solution $u \neq 0$. If V also satisfies $(\mathbf{V}_0)$ with $2 < \gamma_0 < N$, then we can take $\alpha_1^{(0)} = \alpha_1(\beta_0, \gamma_0)$,

$\alpha_3 = \alpha_3(\beta_0, \gamma_0)$ and $q_*^{(0)} = q_*(\alpha_0, \beta_0, \gamma_0)$. If V also satisfies $(\mathbf{V}_\infty)$, then we can take $\alpha_1^{(\infty)} = \alpha_1(\beta_\infty, \gamma_\infty)$ and $q_*^{(\infty)} = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

Remark 7.

1. *The inequalities $\max\{2\beta_\infty, q_0, q_*\} < 2$ in (11) and $\max\{\alpha_2, \alpha_3\} < \alpha_1^{(0)}$, $\max\{1, 2\beta_0\} < q_*^{(0)}$ and $\max\{1, 2\beta_\infty, q_*^{(\infty)}\} < 2$ in (12), (13) are ensured by the other hypotheses of Theorems 5 and 6, so that they are not further assumptions.*
2. *As $(\mathbf{VK}_0)$ remains true if we lower α_0, the assumption $\alpha_0 < \alpha_1^{(0)}$ $(= \alpha_1(\beta_0, \gamma_0),\ 2 \leq \gamma_0 < N)$ in (12) is not a restriction. Nevertheless, if the hypotheses of Theorem 6 are satisfied and $(\mathbf{VK}_0)$ holds with $\alpha_0 \geq \alpha_1^{(0)}$, it is never convenient to reduce α_0 and apply Theorem 6, since one can always apply Theorem 5 and get a better result because $(\mathbf{f}_1)$ with q_1, q_2 satisfying (13) implies $(\mathbf{f}_2)$ for some q satisfying (11). In other words, Theorem 6 is useful with respect to Theorem 5 only when $(\mathbf{VK}_0)$ does not hold for some $\alpha_0 \geq \alpha_1^{(0)}$.*
3. *For $\beta \in [0,1)$ fixed, $\alpha_1(\beta, \gamma)$ is continuous and strictly decreasing in $\gamma \in \mathbb{R}$ and one can check that $q_0(\alpha, \beta, \gamma)$, defined on the set $\{(\alpha, \gamma) : \gamma \geq 2,\ \alpha \geq \alpha_1(\beta, \gamma)\}$, is continuous and decreasing both in γ for α fixed and in α for γ fixed. Similarly, the function defined on $\{(\alpha, \gamma) : \gamma \leq 2,\ \alpha < \alpha_1(\beta, \gamma)\}$ by $\max\{2\beta, q_*(\alpha, \beta, \gamma)\}$ is increasing and continuous both in γ for α fixed and in α for γ fixed. This shows that Theorem 5 improves under assumption $(\mathbf{V}_0)$, or $(\mathbf{V}_\infty)$, or both. Moreover, as in Remark 2.2, the theorem is still true if we replace $\alpha_0, \alpha_\infty, \gamma_0, \gamma_\infty$ with $\overline{\alpha}_0, \underline{\alpha}_\infty, \overline{\gamma}_0, \underline{\gamma}_\infty$ in $\alpha_1^{(0)}, \alpha_1^{(\infty)}, q_0, q_*$, and $\alpha_0 \geq \alpha_1^{(0)}$ with $\alpha_0 > \alpha_1^{(0)}$ in (11).*
4. *The same monotonicities in γ of Remark 7.3, together with the fact that $\max\{\alpha_2(\beta), \alpha_3(\beta, \gamma)\}$ and $q_*(\alpha, \beta, \gamma)$ are respectively decreasing and strictly increasing in $\gamma \in [2, N)$ for $\beta \in [0,1)$ and $\alpha > \alpha_2(\beta)$ fixed, show that Theorem 6 improves under assumption $(\mathbf{V}_0)$, or $(\mathbf{V}_\infty)$, or both. Moreover, as in Remarks 2.2 and 7.3, a version of the theorem with $\alpha_0, \alpha_\infty, \gamma_0, \gamma_\infty$ replaced by $\overline{\alpha}_0, \underline{\alpha}_\infty, \overline{\gamma}_0, \underline{\gamma}_\infty$ also holds, the details of which we leave to the interested reader.*
5. *In Theorem 6, it may happen that $\max\{1, 2\beta_\infty, q_*^{(\infty)}\} < q_*^{(0)}$. In this case, one can take $q_1 = q_2$ and a single-power growth condition on f is thus enough to apply the theorem and get existence. By the way, $\alpha_0 < \alpha_1^{(0)}$ and $\gamma_0 < N$ imply $q_*^{(0)} < 2$ (and therefore $q_1 < 2$), so that the linear case $q_1 = q_2 = 2$ is always excluded.*
6. *In Theorems 5 and 6, the requirement $Q \in L^2(\mathbb{R}_+, r^{N+1}dr)$ just plays the role of ensuring that the linear operator $u \mapsto \int_{\mathbb{R}^N} Q(|x|)\, u\, dx$ is continuous on $H^1_{0,V}(\mathbb{R}^N)$ (see Sect. 5 below). Therefore it can be replaced by any other condition giving the same property, e.g., $Q \in L^{2N/(N+2)}(\mathbb{R}_+, r^{N-1}dr)$ or $QV^{-1/2} \in L^2(\mathbb{R}_+, r^{N-1}dr)$.*
7. *According to the proofs, the solution u of Theorems 5 and 6 satisfies $I(u) = \min\limits_{v \in H^1_{0,V,\mathrm{r}}(\mathbb{R}^N),\, v \geq 0} I(v)$, where*

$$I(v) := \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla v|^2 + V(|x|)\,v^2\right)dx + \\ -\int_{\mathbb{R}^N}\left(K(|x|)\,F(v) + Q(|x|)\,v\right)dx. \tag{14}$$

Moreover, if Q does not vanish almost everywhere in (r_1, r_2), both the theorems still work without assuming $Q \geq 0$ (use Theorem 16 instead of Corollary 17 in the proof), but we cannot ensure anymore that u is nonnegative. In this case, the solution satisfies $I(u) = \min_{v \in H^1_{0,V,\mathrm{r}}(\mathbb{R}^N)} I(v)$.

Exactly as in the superlinear case, the same arguments leading to Theorems 5 and 6 also yield existence results for the Dirichlet problem in bounded balls or exterior radial domains, where, respectively, only assumptions on V and K near the origin or at infinity are needed. In both cases, a single-power growth condition on the nonlinearity is sufficient. The precise statements are left to the interested reader.

We conclude with a multiplicity result for problem (P_0), which, in the superlinear case, requires the following assumption, complementary to $(\mathbf{f}_1)$:

$(\mathbf{f}_1')$ $\exists q_1, q_2 > 1$ such that

$$\inf_{t>0} \frac{f(t)}{\min\{t^{q_1-1}, t^{q_2-1}\}} > 0.$$

Theorem 8. (i) *Under the same assumptions of each of Theorems 1, 3 and 4, if f is also odd and satisfies $(\mathbf{f}_1')$ (with the same exponents q_1, q_2 of $(\mathbf{f}_1)$), then problem (P_0) has infinitely many radial solutions.* (ii) *Under the same assumptions of each of Theorems 5 and 6 with $Q = 0$, if f is also odd, then problem (P_0) has infinitely many radial solutions.*

Remark 9. *The infinitely many solutions of Theorem 8 form a sequence $\{u_n\}$ such that $I(u_n) \to +\infty$ in the superlinear case and $I(u_n) \to 0$ in the sublinear one, where I is the functional defined in (14).*

In [4, Section 3] and [20], we have discussed many examples of pairs of potentials V, K and nonlinearities f satifying our hypotheses. In the same papers, we have also compared such hypotheses with the assumptions of some of the main related results in the previous literature, showing essentially that conditions $(\mathbf{VK}_0)$, $(\mathbf{VK}_\infty)$ and $(\mathbf{f}_1)$ allow to deal with potentials exhibiting behaviours at zero and at infinity which are new in the literature (see [4] and Sect. 2 below) and do not need to be compatible with each other (see both [4,20]). The same examples, and the same discussion, can be repetead here, covering many cases not included in previous papers. In particular, our results for problem (P_0) in $\mathbb{R}^N$ contain and extend in different directions the results of [28,29] (for $p = 2$) and are complementary to the ones of [13,27,32]. To the best of our knowledge, assumptions $(\mathbf{VK}_0)$ and $(\mathbf{VK}_\infty)$ are also new in the study of problem (P_Q) in bounded or exterior domains.

The paper is organized as follows. In Sect. 2 we give an example, complementary to the ones of [4, Section 3], of potentials satisfying our hypotheses but not included in the results known in the literature up to now. In Sect. 3, we introduce our variational approach to problem (P) and give some existence and multiplicity results (Theorems 12-19 and Corollary 17), which are more general than the ones stated in this introduction but rely on a less explicit assumption on the potentials (condition $\left(\mathcal{S}''_{q_1,q_2}\right)$). We also give a suitable symmetric criticality type principle (Proposition 11), since the Palais' classical one [23] does not apply in this case. Sections 4 and 5 are respectively devoted to the proof of the results stated in Sect. 3 and in the Introduction. In particular, the latter follow from the former, by applying the compactness theorems of [4] (see Lemma 26 below). In the Appendix, we prove some pointwise estimates for radial Sobolev functions, already used in [4].

2. An example

A particular feature of our results is that we do not necessarily assume hypotheses on V and K separately, but rather on their ratio: the ratio must have a power-like behaviour at zero and at infinity, but V and K are not obliged to have such a behaviour. This allows us to deal with potentials V, K that grow (or vanish) very fast at zero or infinity, in such a way that they escape the results of the previous literature but the ratio K/V satisfies our hypotheses.

In particular we can also treat some examples of potentials not included among those considered in [13], a paper that deals with a very general class of potentials, the so called *Hardy-Dieudonné class*, which also includes the potentials treated in [27–29,32]. In this class, the functions with the fastest growth at infinity are the n times compositions of the exponential map with itself. Accordingly, let us denote by $e_n : [0,+\infty) \to \mathbb{R}$ the function obtained by composing the exponential map with itself $n \geq 0$ times. The Hardy-Dieudonné class contains e_n for all n and these are the mappings with the fastest growth in the class, so that any function growing faster than every e_n is not in that class. Let us then define $\alpha : [0,+\infty) \to \mathbb{R}$ by setting $\alpha(r) := e_n(n)$ if $n \leq r < n+1$. It is clear that

$$\lim_{r\to+\infty} \frac{\alpha(r)}{e_n(r)} = +\infty \quad \text{for all } n$$

and therefore α does not belong to the Hardy-Dieudonné class. Hence, defining

$$K(r) := \alpha(r) \quad \text{and} \quad V(r) := \alpha(r)\, V_1(r)$$

where V_1 is any potential satisfying $(\mathbf{V})$ and having suitable power-like behavior at zero and infinity, we get a pair of potentials V, K which satisfy our hypotheses but not those of [13]. Of course, one can build similar examples of potential pairs which vanish so fast at infinity (or grow or vanish so fast at zero) that they fall out of the Hardy-Dieudonné class, yet their ratio exhibits a power-like behaviour.

3. Variational approach and general results

Let $N \geq 3$ and let $V : \mathbb{R}_+ \to [0, +\infty]$ be a measurable function satisfying the following hypothesis:

(h_0) $V \in L^1((r_1, r_2))$ for some $r_2 > r_1 > 0$.

Define the Hilbert spaces

$$H_V^1 := H_V^1\left(\mathbb{R}^N\right) := \left\{u \in D^{1,2}\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(|x|)\, u^2 dx < \infty\right\}, \quad (15)$$

$$H_{V,\mathrm{r}}^1 := H_{V,\mathrm{r}}^1\left(\mathbb{R}^N\right) := \left\{u \in H_V^1\left(\mathbb{R}^N\right) : u(x) = u(|x|)\right\} \quad (16)$$

where $D^{1,2}\left(\mathbb{R}^N\right)$ is the usual Sobolev space. H_V^1 and $H_{V,\mathrm{r}}^1$ are endowed with the following inner product and related norm:

$$(u \mid v) := \int_{\mathbb{R}^N} \nabla u \cdot \nabla v\, dx + \int_{\mathbb{R}^N} V(|x|)\, uv\, dx, \quad (17)$$

$$\|u\| := \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(|x|)\, u^2 dx\right)^{1/2}.$$

Of course, $u(x) = u(|x|)$ means that u is invariant under the action on H_V^1 of the orthogonal group of $\mathbb{R}^N$. Note that H_V^1 and $H_{V,\mathrm{r}}^1$ are nonzero by (h_0) and H_V^1 is the space $H_{0,V}^1(\mathbb{R}^N)$ defined in the Introduction.

Let $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function and assume once and for all that there exist $f \in C(\mathbb{R}; \mathbb{R})$ and a measurable function $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that:

(h_1) $|g(r,t) - g(r,0)| \leq K(r)\, |f(t)|$ for almost every $r > 0$ and all $t \in \mathbb{R}$;
(h_2) $K \in L_{\mathrm{loc}}^s((0, +\infty))$ for some $s > \frac{2N}{N+2}$.

Assume furthermore that:

(h_3) the linear operator $u \mapsto \int_{\mathbb{R}^N} g(|x|, 0)\, u\, dx$ is continuous on H_V^1

(see also Remark 20). Of course (h_3) will be relevant only if $g(\cdot, 0) \neq 0$ (meaning that $g(\cdot, 0)$ does not vanish almost everywhere).

Define the following functions of $R > 0$ and $q > 1$:

$$\mathcal{S}_0(q, R) := \sup_{u \in H_{V,\mathrm{r}}^1,\, \|u\|=1} \int_{B_R} K(|x|)\, |u|^q\, dx,$$

$$\mathcal{S}_\infty(q, R) := \sup_{u \in H_{V,\mathrm{r}}^1,\, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)\, |u|^q\, dx,$$

$$\mathcal{R}_0(q, R) := \sup_{u \in H_{V,\mathrm{r}}^1,\, h \in H_V^1,\, \|u\|=\|h\|=1} \int_{B_R} K(|x|)\, |u|^{q-1}\, |h|\, dx,$$

$$\mathcal{R}_\infty(q, R) := \sup_{u \in H_{V,\mathrm{r}}^1,\, h \in H_V^1,\, \|u\|=\|h\|=1} \int_{\mathbb{R}^N \setminus B_R} K(|x|)\, |u|^{q-1}\, |h|\, dx.$$

Note that $\mathcal{S}_0(q,\cdot)$ and $\mathcal{R}_0(q,\cdot)$ are increasing, $\mathcal{S}_\infty(q,\cdot)$ and $\mathcal{R}_\infty(q,\cdot)$ are decreasing and all can be infinite at some R. Moreover, for every (q,R) one has $\mathcal{S}_0(q,R) \leq \mathcal{R}_0(q,R)$ and $\mathcal{S}_\infty(q,R) \leq \mathcal{R}_\infty(q,R)$.

On the functions $\mathcal{S},\mathcal{R}$ and f, we will require suitable combinations of the following conditions (see also Remarks 13.2 and 18), where q_1,q_2 will be specified each time:

(f_{q_1,q_2}) $\exists M>0$ such that $|f(t)| \leq M \min\left\{|t|^{q_1-1},|t|^{q_2-1}\right\}$ for all $t\in\mathbb{R}$;
$\left(\mathcal{S}'_{q_1,q_2}\right)$ $\exists R_1,R_2>0$ such that $\mathcal{S}_0(q_1,R_1)<\infty$ and $\mathcal{S}_\infty(q_2,R_2)<\infty$;
$\left(\mathcal{S}''_{q_1,q_2}\right)$ $\lim_{R\to 0^+}\mathcal{S}_0(q_1,R)=\lim_{R\to+\infty}\mathcal{S}_\infty(q_2,R)=0$;
$(\mathcal{R}_{q_1,q_2})$ $\exists R_1,R_2>0$ such that $\mathcal{R}_0(q_1,R_1)<\infty$ and $\mathcal{R}_\infty(q_2,R_2)<\infty$.

We set $G(r,t):=\int_0^t g(r,s)\,ds$ and

$$I(u):=\frac{1}{2}\|u\|^2-\int_{\mathbb{R}^N} G(|x|,u)\,dx. \tag{18}$$

From the embedding results of [4] and the results of [6] about Nemytskiĭ operators on the sum of Lebesgue spaces (see Sect. 4 below for some recallings on such spaces), we get the following differentiability result.

Proposition 10. *Assume that there exist $q_1,q_2>1$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}'_{q_1,q_2}\right)$ hold. Then (18) defines a C^1 functional on $H^1_{V,\mathrm{r}}$, with Fréchet derivative at any $u\in H^1_{V,\mathrm{r}}$ given by*

$$I'(u)\,h=\int_{\mathbb{R}^N}(\nabla u\cdot\nabla h+V(|x|)\,uh)\,dx-\int_{\mathbb{R}^N} g(|x|,u)\,h\,dx,\quad \forall h\in H^1_{V,\mathrm{r}}. \tag{19}$$

Proposition 10 ensures that the critical points of $I:H^1_{V,\mathrm{r}}\to\mathbb{R}$ satisfy (2) (with $\Omega=\mathbb{R}^N$) for all $h\in H^1_{V,\mathrm{r}}$. The next result shows that such critical points are actually weak solutions to problem (P) (with $\Omega=\mathbb{R}^N$), provided that the slightly stronger version $(\mathcal{R}_{q_1,q_2})$ of condition $\left(\mathcal{S}'_{q_1,q_2}\right)$ holds. Observe that the classical Palais' Principle of Symmetric Criticality [23] does not apply in this case, because we do not know whether or not I is differentiable, not even well defined, on the whole space H^1_V.

Proposition 11. *Assume that there exist $q_1,q_2>1$ such that (f_{q_1,q_2}) and $(\mathcal{R}_{q_1,q_2})$ hold. Then every critical point u of $I:H^1_{V,\mathrm{r}}\to\mathbb{R}$ satisfies*

$$\int_{\mathbb{R}^N}\nabla u\cdot\nabla h\,dx+\int_{\mathbb{R}^N}V(|x|)\,uh\,dx=\int_{\mathbb{R}^N} g(|x|,u)\,h\,dx,\quad\forall h\in H^1_V \tag{20}$$

(i.e., u is a weak solution to problem (P) with $\Omega=\mathbb{R}^N$).

By Proposition 11, the problem of radial weak solutions to (P) (with $\Omega=\mathbb{R}^N$) reduces to the study of critical points of $I:H^1_{V,\mathrm{r}}\to\mathbb{R}$. Concerning the case of superlinear nonlinearities, we have the following existence and multiplicity results.

Theorem 12. *Assume $g(\cdot,0)=0$ and assume that there exist $q_1,q_2>2$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}''_{q_1,q_2}\right)$ hold. Assume furthermore that g satisfies:*

(g_1) *$\exists\theta>2$ such that $0\leq\theta G(r,t)\leq g(r,t)\,t$ for almost every $r>0$ and all $t\geq 0$;*

(g_2) *$\exists t_0>0$ such that $G(r,t_0)>0$ for almost every $r>0$.*

If $K(|\cdot|)\in L^1(\mathbb{R}^N)$, we can replace assumptions (g_1)-(g_2) with:

(g_3) *$\exists\theta>2$ and $\exists t_0>0$ such that $0<\theta G(r,t)\leq g(r,t)\,t$ for almost every $r>0$ and all $t\geq t_0$.*

Then the functional $I:H^1_{V,\mathrm{r}}\to\mathbb{R}$ has a nonnegative critical point $u\neq 0$.

Remark 13. 1. *Assumptions (g_1) and (g_2) imply (g_3), so that, in Theorem 12, the information $K(|\cdot|)\in L^1(\mathbb{R}^N)$ actually allows weaker hypotheses on the nonlinearity.*

2. *In Theorem 12, assumptions (h_1) and (f_{q_1,q_2}) need only to hold for $t\geq 0$. Indeed, all the hypotheses of the theorem still hold true if we replace $g(r,t)$ with $\chi_{\mathbb{R}_+}(t)\,g(r,t)$ ($\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$) and this can be done without restriction since the theorem concerns* nonnegative *critical points.*

Theorem 14. *Assume that there exist $q_1,q_2>2$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}''_{q_1,q_2}\right)$ hold. Assume furthermore that:*

(g_4) *$\exists m>0$ such that $G(r,t)\geq mK(r)\min\{t^{q_1},t^{q_2}\}$ for almost every $r>0$ and all $t\geq 0$;*

(g_5) *$g(r,t)=-g(r,-t)$ for almost every $r>0$ and all $t\geq 0$.*

Finally, assume that g satisfies (g_1), or that $K(|\cdot|)\in L^1(\mathbb{R}^N)$ and g satisfies (g_3). Then the functional $I:H^1_{V,\mathrm{r}}\to\mathbb{R}$ has a sequence of critical points $\{u_n\}$ such that $I(u_n)\to+\infty$.

Remark 15. *The condition $g(\cdot,0)=0$ is implicit in Theorem 14 (and in Theorem 19 below), as it follows from assumption (g_5).*

As to sublinear nonlinearities, we will prove the following results.

Theorem 16. *Assume that there exist $q_1,q_2\in(1,2)$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}''_{q_1,q_2}\right)$ hold. Assume furthermore that g satisfies at least one of the following conditions:*

(g_6) *$\exists\theta<2$ and $\exists t_0,m>0$ such that $G(r,t)\geq mK(r)\,t^{\theta}$ for almost every $r>0$ and all $0\leq t\leq t_0$;*

(g_7) *$g(\cdot,0)$ does not vanish almost everywhere in (r_1,r_2).*

If (g_7) holds, we also allow the case $\max\{q_1,q_2\}=2>\min\{q_1,q_2\}>1$. Then there exists $u\neq 0$ such that

$$I(u)=\min_{v\in H^1_{V,\mathrm{r}}}I(v).$$

If $g(\cdot,t)\geq 0$ almost everywhere for all $t<0$, the minimizer u of Theorem 16 is nonnegative, since a standard argument shows that all the critical points of I are nonnegative (test $I'(u)$ with the negative part u_- and get $I'(u)\,u_-=$

$-\left\| u_- \right\|^2 = 0$). The next corollary gives a nonnegative critical point just asking $g(\cdot, 0) \geq 0$.

Corollary 17. *Assume the same hypotheses of Theorem 16. If $g(\cdot, 0) \geq 0$ almost everywhere, then $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ has a nonnegative critical point $\widetilde{u} \neq 0$ satisfying*

$$I(\widetilde{u}) = \min_{u \in H^1_{V,\mathrm{r}},\, u \geq 0} I(u). \tag{21}$$

Remark 18. 1. *In Theorem 16 and Corollary 17, if assumption (g_6) holds then the case* $\max\{q_1, q_2\} = 2 > \min\{q_1, q_2\} > 1$ *cannot be considered, since (g_6) and (f_{q_1,q_2}) imply* $\max\{q_1, q_2\} \leq \theta < 2$.
2. *Checking the proof, one sees that Corollary 17 actually requires that assumptions (h_1) and (f_{q_1,q_2}) hold only for $t \geq 0$, which is consistent with the concern of the result about* nonnegative *critical points.*

Theorem 19. *Assume that there exist $q_1, q_2 \in (1, 2)$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}''_{q_1,q_2}\right)$ hold. Assume furthermore that g satisfies (g_5) and (g_6). Then the functional $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ has a sequence of critical points $\{u_n\}$ such that $I(u_n) < 0$ and $I(u_n) \to 0$.*

Remark 20. *If the linear operator of assumption (h_3) is just continuous on $H^1_{V,\mathrm{r}}$, then Proposition 11 fails, but all the other results of this section remain valid (as can be easily seen by checking the proofs). This is especially relevant in connection with the radial estimates satisfied by the $H^1_{V,\mathrm{r}}$ mappings (see Appendix), which ensure that (h_3) holds on $H^1_{V,\mathrm{r}}$ provided that $g(|\cdot|, 0)$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and satisfies suitable decay (or growth) conditions at zero and at infinity.*

4. Proof of the general results

In this section we keep the notation and assumptions of the preceding section. Denoting $L^p_K(E) := L^p(E, K(|x|)\,dx)$ for any measurable set $E \subseteq \mathbb{R}^N$, we will make frequent use of the sum space

$$L^{p_1}_K + L^{p_2}_K := \left\{ u_1 + u_2 : u_1 \in L^{p_1}_K\left(\mathbb{R}^N\right),\, u_2 \in L^{p_2}_K\left(\mathbb{R}^N\right) \right\}, \quad 1 < p_i < \infty.$$

We recall from [6] that such a space can be characterized as the set of measurable mappings $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L^{p_1}_K(E) \cap L^{p_2}_K(E^c)$. It is a Banach space with respect to the norm

$$\|u\|_{L^{p_1}_K + L^{p_2}_K} := \inf_{u_1 + u_2 = u} \max\left\{ \|u_1\|_{L^{p_1}_K(\mathbb{R}^N)}, \|u_2\|_{L^{p_2}_K(\mathbb{R}^N)} \right\}$$

and for all $p \in [\min\{p_1, p_2\}, \max\{p_1, p_2\}]$ the continuous embedding $L^p_K \hookrightarrow L^{p_1}_K + L^{p_2}_K$ holds. Moreover, for every $u \in L^{p_1}_K + L^{p_2}_K$ one has

$$\|u\|_{L^{p_1}_K + L^{p_2}_K} \leq \|u\|_{L^{\min\{p_1,p_2\}}_K(\Lambda_u)} + \|u\|_{L^{\max\{p_1,p_2\}}_K(\Lambda^c_u)}, \tag{22}$$

where $\Lambda_u := \left\{x \in \mathbb{R}^N : |u(x)| > 1\right\}$ (see [6, Corollary 2.19]).

Proof of Proposition 10 On the one hand, by $\left(\mathcal{S}'_{q_1,q_2}\right)$ and [4, Theorem 1], the embedding $H^1_{V,\mathrm{r}} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is continuous. On the other hand, by (h_1), (f_{q_1,q_2}) and [6, Proposition 3.8], the functional

$$u \mapsto \int_{\mathbb{R}^N} \left(G(|x|, u) - g(|x|, 0)\, u\right) dx \tag{23}$$

is of class C^1 on $L^{q_1}_K + L^{q_2}_K$ and its Fréchet derivative at any u is given by

$$h \in L^{q_1}_K + L^{q_2}_K \mapsto \int_{\mathbb{R}^N} \left(g(|x|, u) - g(|x|, 0)\right) h\, dx.$$

Hence, by (h_3), we conclude that $I \in C^1(H^1_{V,\mathrm{r}})$ and that (19) holds. □

Proof of Proposition 11 Let $u \in H^1_{V,\mathrm{r}}$. By the monotonicity of $\mathcal{R}_0$ and $\mathcal{R}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $(\mathcal{R}_{q_1,q_2})$. So, by [4, Lemma 1], there exists a constant $C > 0$ (dependent on u) such that for all $h \in H^1_V$ we have

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_1 - 1} |h|\, dx \leq C \|h\|$$

and therefore, by (h_1) and (f_{q_1,q_2}),

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\, g(|x|, u) - g(|x|, 0)\, |\, |h|\, dx \\
&\leq \int_{\mathbb{R}^N} K(|x|)\, |f(u)|\, |h|\, dx \leq M \int_{\mathbb{R}^N} K(|x|) \min\{|u|^{q_1-1}, |u|^{q_2-1}\}\, |h|\, dx \\
&\leq M \left(\int_{B_{R_1}} K(|x|)\, |u|^{q_1-1} |h|\, dx + \int_{B^c_{R_2}} K(|x|)\, |u|^{q_2-1} |h|\, dx \right. \\
&\quad \left. + \int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_1-1} |h|\, dx \right) \\
&\leq M \left(\|u\|^{q_1-1} \|h\| \int_{B_{R_1}} K(|x|) \frac{|u|^{q_1-1}}{\|u\|^{q_1-1}} \frac{|h|}{\|h\|} dx \right. \\
&\quad \left. + \|u\|^{q_2-1} \|h\| \int_{B^c_{R_2}} K(|x|) \frac{|u|^{q_2-1}}{\|u\|^{q_2-1}} \frac{|h|}{\|h\|} dx + C\|h\| \right) \\
&\leq M \left(\|u\|^{q_1-1} \mathcal{R}_0(q_1, R_1) + \|u\|^{q_2-1} \mathcal{R}_\infty(q_2, R_2) + C \right) \|h\|.
\end{aligned}$$

Together with (h_3), this gives that the linear operator

$$T(u)\, h := \int_{\mathbb{R}^N} \left(\nabla u \cdot \nabla h + V(|x|)\, uh\right) dx - \int_{\mathbb{R}^N} g(|x|, u)\, h\, dx$$

is well defined and continuous on H^1_V. Hence, by Riesz representation theorem, there exists a unique $\tilde{u} \in H^1_V$ such that $T(u)\, h = (\tilde{u} \mid h)$ for all $h \in H^1_V$, where $(\cdot \mid \cdot)$ is the inner product of H^1_V defined in (17). Denoting by $O(N)$ the orthogonal group of $\mathbb{R}^N$, by means of obvious changes of variables it is easy to see that for every $h \in H_V^1$ and $g \in O(N)$ one has $(\tilde{u} \mid h(g\cdot)) = \left(\tilde{u}\left(g^{-1}\cdot\right) \mid h\right)$ and $T(u)\, h(g\cdot) = T(u)\, h$, so that $\left(\tilde{u}\left(g^{-1}\cdot\right) \mid h\right) = (\tilde{u} \mid h)$. This means $\tilde{u}\left(g^{-1}\cdot\right) = \tilde{u}$ for all $g \in O(N)$, i.e., $\tilde{u} \in H^1_{V,\mathrm{r}}$. Now assume $I'(u) = 0$ in the dual space of $H^1_{V,\mathrm{r}}$. Then we have $(\tilde{u} \mid h) = T(u)\, h = I'(u)\, h = 0$ for all $h \in H^1_{V,\mathrm{r}}$, which implies $\tilde{u} = 0$. This gives $T(u)\, h = 0$ for all $h \in H^1_V$, which is the thesis (20). □

For future reference, we point out here that, by assumption (h_1), if (f_{q_1,q_2}) holds then $\exists \tilde{M} > 0$ such that for almost every $r > 0$ and all $t \in \mathbb{R}$ one has

$$|G(r,t) - g(r,0)\, t| \leq \tilde{M} K(r) \min\left\{|t|^{q_1}, |t|^{q_2}\right\}. \tag{24}$$

Lemma 21. *Let L_0 be the norm of the operator of assumption (h_3). If there exist $q_1, q_2 > 1$ such that (f_{q_1,q_2}) and $\left(\mathcal{S}'_{q_1,q_2}\right)$ hold, then there exist two constants $c_1, c_2 > 0$ such that*

$$I(u) \geq \frac{1}{2}\|u\|^2 - c_1 \|u\|^{q_1} - c_2 \|u\|^{q_2} - L_0 \|u\| \qquad \text{for all } u \in H^1_{V,\mathrm{r}}. \tag{25}$$

If $\left(\mathcal{S}''_{q_1,q_2}\right)$ also holds, then $\forall \varepsilon > 0$ there exist two constants $c_1(\varepsilon), c_2(\varepsilon) > 0$ such that (25) holds both with $c_1 = \varepsilon$, $c_2 = c_2(\varepsilon)$ and with $c_1 = c_1(\varepsilon)$, $c_2 = \varepsilon$.

Proof. Let $i \in \{1, 2\}$. By the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. Then, by [4, Lemma 1] and the continuous embedding $H_V^1 \hookrightarrow L^2_{\mathrm{loc}}(\mathbb{R}^N)$, there exists a constant $c^{(i)}_{R_1,R_2} > 0$ such that for all $u \in H^1_{V,\mathrm{r}}$ we have

$$\int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_i}\, dx \leq c^{(i)}_{R_1,R_2} \|u\|^{q_i}.$$

Therefore, by (24) and the definitions of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, we obtain

$$\begin{aligned}
&\left|\int_{\mathbb{R}^N} G(|x|, u)\, dx\right| \\
&\leq \int_{\mathbb{R}^N} |G(|x|, u) - g(|x|, 0)\, u|\, dx + \left|\int_{\mathbb{R}^N} g(|x|, 0)\, u\, dx\right| \\
&\leq \tilde{M} \int_{\mathbb{R}^N} K(|x|) \min\left\{|u|^{q_1}, |u|^{q_2}\right\} dx + L_0 \|u\| \\
&\leq \tilde{M} \left(\int_{B_{R_1}} K(|x|)\, |u|^{q_1}\, dx + \int_{B^c_{R_2}} K(|x|)\, |u|^{q_2}\, dx \right. \\
&\quad \left. + \int_{B_{R_2} \setminus B_{R_1}} K(|x|)\, |u|^{q_i}\, dx\right) + L_0 \|u\| \\
&\leq \tilde{M} \left(\|u\|^{q_1} \mathcal{S}_0(q_1, R_1) + \|u\|^{q_2} \mathcal{S}_\infty(q_2, R_2) + c^{(i)}_{R_1,R_2} \|u\|^{q_i}\right) + L_0 \|u\| \\
&= c_1 \|u\|^{q_1} + c_2 \|u\|^{q_2} + L_0 \|u\|,
\end{aligned} \tag{26}$$

with obvious definition of the constants c_1 and c_2, independent of u. This yields (25). If $\left(\mathcal{S}''_{q_1,q_2}\right)$ also holds, then $\forall \varepsilon > 0$ we can fix $R_{1,\varepsilon} < R_{2,\varepsilon}$ such that

$\tilde{M}\mathcal{S}_0(q_1, R_{1,\varepsilon}) < \varepsilon$ and $\tilde{M}\mathcal{S}_\infty(q_2, R_{2,\varepsilon}) < \varepsilon$, so that inequality (26) becomes

$$\left|\int_{\mathbb{R}^N} G(|x|, u)\, dx\right| \leq \varepsilon \|u\|^{q_1} + \varepsilon \|u\|^{q_2} + c^{(i)}_{R_{1,\varepsilon},R_{2,\varepsilon}} \|u\|^{q_i} + L_0 \|u\|.$$

The result then ensues by taking $i = 2$ and $c_2(\varepsilon) = \varepsilon + c^{(2)}_{R_{1,\varepsilon},R_{2,\varepsilon}}$, or $i = 1$ and $c_1(\varepsilon) = \varepsilon + c^{(1)}_{R_{1,\varepsilon},R_{2,\varepsilon}}$. □

Henceforth, we will assume that the hypotheses of Theorem 12 also include the following condition:

$$g(r, t) = 0 \quad \text{for all } r > 0 \text{ and } t < 0. \tag{27}$$

This can be done without restriction, since the theorem concerns *nonnegative* critical points and all its assumptions still hold true if we replace $g(r, t)$ with $g(r, t)\,\chi_{\mathbb{R}_+}(t)$ ($\chi_{\mathbb{R}_+}$ is the characteristic function of $\mathbb{R}_+$).

Lemma 22. *Under the assumptions of each of Theorems 12 [including (27)] and 14, the functional $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ satisfies the Palais-Smale condition.*

Proof. By (27) and (g_5) respectively, under the assumptions of each of Theorems 12 and 14 we have that either g satisfies (g_1) for all $t \in \mathbb{R}$, or $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and g satisfies

$$\theta G(r, t) \leq g(r, t)\, t \quad \text{for almost every } r > 0 \text{ and all } |t| \geq t_0. \tag{28}$$

Let $\{u_n\}$ be a sequence in $H^1_{V,\mathrm{r}}$ such that $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in the dual space of $H^1_{V,\mathrm{r}}$. Hence

$$\frac{1}{2}\|u_n\|^2 - \int_{\mathbb{R}^N} G(|x|, u_n)\, dx = O(1)$$

and

$$\|u_n\|^2 - \int_{\mathbb{R}^N} g(|x|, u_n)\, u_n dx = o(1)\|u_n\|.$$

If g satisfies (g_1), then we get

$$\begin{aligned}\frac{1}{2}\|u_n\|^2 + O(1) &= \int_{\mathbb{R}^N} G(|x|, u_n)\, dx \leq \frac{1}{\theta}\int_{\mathbb{R}^N} g(|x|, u_n)\, u_n dx \\ &= \frac{1}{\theta}\|u_n\|^2 + o(1)\|u_n\|,\end{aligned}$$

which implies that $\{\|u_n\|\}$ is bounded since $\theta > 2$. If $K(|\cdot|) \in L^1(\mathbb{R}^N)$ and g satisfies (28), then we slightly modify the argument: we have

$$\begin{aligned}&\int_{\{|u_n|\geq t_0\}} g(|x|, u_n)\, u_n dx \\ &\quad= \int_{\mathbb{R}^N} g(|x|, u_n)\, u_n dx - \int_{\{|u_n|<t_0\}} g(|x|, u_n)\, u_n dx \\ &\quad\leq \int_{\mathbb{R}^N} g(|x|, u_n)\, u_n dx + \int_{\{|u_n|<t_0\}} |g(|x|, u_n)\, u_n|\, dx\end{aligned}$$

where (thanks to (h_1) and (f_{q_1,q_2}))

$$\begin{aligned}\int_{\{|u_n|<t_0\}} |g(|x|, u_n) u_n| \, dx &\le \int_{\{|u_n|<t_0\}} K(|x|) |f(u_n)| |u_n| \, dx \\ &\le M \int_{\{|u_n|<t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\} \, dx \\ &\le M \min\{t_0^{q_1}, t_0^{q_2}\} \int_{\{|u_n|<t_0\}} K(|x|) \, dx \\ &\le M \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)},\end{aligned}$$

so that, by (24), we obtain

$$\begin{aligned}\frac{1}{2}\|u_n\|^2 + O(1) &= \int_{\mathbb{R}^N} G(|x|, u_n) \, dx \\ &= \int_{\{|u_n|<t_0\}} G(|x|, u_n) \, dx + \int_{\{|u_n|\ge t_0\}} G(|x|, u_n) \, dx \\ &\le \tilde{M} \int_{\{|u_n|<t_0\}} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\} \, dx \\ &\quad + \frac{1}{\theta} \int_{\{|u_n|\ge t_0\}} g(|x|, u_n) u_n dx \\ &\le \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta} \int_{\mathbb{R}^N} g(|x|, u_n) u_n dx \\ &\quad + \frac{M}{\theta} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} \\ &= \left(\tilde{M} + \frac{M}{\theta}\right) \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} + \frac{1}{\theta}\|u_n\|^2 \\ &\quad + o(1)\|u_n\|.\end{aligned}$$

This yields again that $\{\|u_n\|\}$ is bounded. Now, since the embedding $H^1_{V,\mathrm{r}} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is compact by assumption $\left(\mathcal{S}''_{q_1,q_2}\right)$ (see [4, Theorem 1]) and the operator $u \mapsto \int_{\mathbb{R}^N} G(|x|, u) \, dx$ is of class C^1 on $L^{q_1}_K + L^{q_2}_K$ (see the proof of Proposition 10 above), it is a standard exercise to conclude that $\{u_n\}$ has a strongly convergent subsequence in $H^1_{V,\mathrm{r}}$. □

Proof of Theorem 12 We want to apply the Mountain-Pass Theorem [1]. To this end, from (25) of Lemma 21 we deduce that, since $L_0 = 0$ and $q_1, q_2 > 2$, there exists $\rho > 0$ such that

$$\inf_{u \in H^1_{V,\mathrm{r}},\, \|u\|=\rho} I(u) > 0 = I(0). \tag{29}$$

Therefore, taking into account Proposition 10 and Lemma 22, we need only to check that $\exists \bar{u} \in H^1_{V,\mathrm{r}}$ such that $\|\bar{u}\| > \rho$ and $I(\bar{u}) < 0$. In order to check this, from assumption (g_3) (which holds in any case, according to Remark 13.1), we infer that

$$G(r,t) \ge \frac{G(r,t_0)}{t_0^\theta} t^\theta \quad \text{for almost every } r > 0 \text{ and all } t \ge t_0.$$

Then, by assumption (h_0), we fix a nonnegative function $u_0 \in C_c^\infty(B_{r_2} \setminus \overline{B}_{r_1}) \cap H^1_{V,\mathrm{r}}$ such that the set $\{x \in \mathbb{R}^N : u_0(x) \geq t_0\}$ has positive Lebesgue measure. We now distinguish the case of assumptions (g_1) and (g_2) from the case of $K(|\cdot|) \in L^1(\mathbb{R}^N)$. In the first one, (g_1) and (g_2) ensure that $G \geq 0$ and $G(\cdot, t_0) > 0$ almost everywhere, so that for every $\lambda > 1$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} G(|x|, \lambda u_0)\, dx \\
&\quad \geq \int_{\{\lambda u_0 \geq t_0\}} G(|x|, \lambda u_0)\, dx \geq \frac{\lambda^\theta}{t_0^\theta} \int_{\{\lambda u_0 \geq t_0\}} G(|x|, t_0)\, u_0^\theta dx \\
&\quad \geq \frac{\lambda^\theta}{t_0^\theta} \int_{\{u_0 \geq t_0\}} G(|x|, t_0)\, u_0^\theta dx \geq \lambda^\theta \int_{\{u_0 \geq t_0\}} G(|x|, t_0)\, dx > 0.
\end{aligned}$$

Since $\theta > 2$, this gives

$$\lim_{\lambda \to +\infty} I(\lambda u_0) \leq \lim_{\lambda \to +\infty} \left(\frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta \int_{\{u_0 \geq t_0\}} G(|x|, t_0)\, dx \right) = -\infty.$$

If $K(|\cdot|) \in L^1(\mathbb{R}^N)$, assumption (g_3) still gives $G(\cdot, t_0) > 0$ almost everywhere and from (24) we infer that

$$G(r,t) \geq -\tilde{M} K(r) \min\{t_0^{q_1}, t_0^{q_2}\} \quad \text{for almost every } r > 0 \text{ and all } 0 \leq t \leq t_0.$$

Therefore, arguing as before about the integral over $\{\lambda u_0 \geq t_0\}$, for every $\lambda > 1$ we obtain

$$\begin{aligned}
\int_{\mathbb{R}^N} G(|x|, \lambda u_0)\, dx &= \int_{\{\lambda u_0 < t_0\}} G(|x|, \lambda u_0)\, dx + \int_{\{\lambda u_0 \geq t_0\}} G(|x|, \lambda u_0)\, dx \\
&\geq -\tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \int_{\{\lambda u_0 < t_0\}} K(|x|)\, dx \\
&\quad + \lambda^\theta \int_{\{u_0 \geq t_0\}} G(|x|, t_0)\, dx,
\end{aligned}$$

which implies

$$\begin{aligned}
\lim_{\lambda \to +\infty} I(\lambda u_0) \leq \lim_{\lambda \to +\infty} &\left(\frac{\lambda^2}{2} \|u_0\|^2 + \tilde{M} \min\{t_0^{q_1}, t_0^{q_2}\} \|K\|_{L^1(\mathbb{R}^N)} \right. \\
&\left. + -\lambda^\theta \int_{\{u_0 \geq t_0\}} G(|x|, t_0)\, dx \right) = -\infty.
\end{aligned}$$

So, in any case, we can take $\bar{u} = \lambda u_0$ with λ sufficiently large and the Mountain-Pass Theorem provides the existence of a nonzero critical point $u \in H^1_{V,\mathrm{r}}$ for I. Since (27) implies $I'(u)\, u_- = -\|u_-\|^2$ (where $u_- \in H^1_{V,\mathrm{r}}$ is the negative part of u), one concludes that $u_- = 0$, i.e., u is nonnegative. □

Proof of Theorem 14 By the oddness assumption (g_5), for all $u \in H^1_{V,\mathrm{r}}$ one has $I(u) = I(-u)$ and thus we can apply the Symmetric Mountain-Pass Theorem (see e.g. [24, Chapter 1]). To this end, we deduce (29) as in the proof of Theorem 12 and therefore, thanks to Proposition 10 and Lemma 22, we need only to show that I satisfies the following geometrical condition: for any finite dimensional subspace $Y \neq \{0\}$ of $H^1_{V,\mathrm{r}}$ there exists $R > 0$ such that $I(u) \leq 0$ for all $u \in Y$ with $\|u\| \geq R$. In fact, it is sufficient to prove that any diverging sequence in Y admits a subsequence on which I is nonpositive. So, let $\{u_n\} \subseteq Y$ be such that $\|u_n\| \to +\infty$. Since all norms are equivalent on Y, by (22) one has

$$\|u_n\|_{L^p_K(\Lambda_{u_n})} + \|u_n\|_{L^q_K(\Lambda^c_{u_n})} \geq \|u_n\|_{L^{q_1}_K + L^{q_2}_K} \geq m_1 \|u_n\| \to +\infty \tag{30}$$

for some constant $m_1 > 0$, where $p := \min\{q_1, q_2\}$ and $q := \max\{q_1, q_2\}$. Hence, up to a subsequence, at least one of the sequences $\{\|u_n\|_{L^p_K(\Lambda_{u_n})}\}$, $\{\|u_n\|_{L^q_K(\Lambda^c_{u_n})}\}$ diverges. We now use assumptions (g_4) and (g_5) to deduce that

$$G(r,t) \geq mK(r) \min\{|t|^{q_1}, |t|^{q_2}\} \quad \text{for almost every } r > 0 \text{ and all } t \in \mathbb{R},$$

which implies

$$\begin{aligned}\int_{\mathbb{R}^N} G(|x|, u_n)\, dx &\geq m \int_{\mathbb{R}^N} K(|x|) \min\{|u_n|^{q_1}, |u_n|^{q_2}\}\, dx \\ &= m \int_{\Lambda_{u_n}} K(|x|) |u_n|^p\, dx + m \int_{\Lambda^c_{u_n}} K(|x|) |u_n|^q\, dx.\end{aligned}$$

Hence, using inequalities (30), there exists a constant $m_2 > 0$ such that

$$\begin{aligned}I(u_n) \leq\ & m_2 \left(\|u_n\|^2_{L^p_K(\Lambda_{u_n})} + \|u_n\|^2_{L^q_K(\Lambda^c_{u_n})} \right) + \\ & - m \left(\|u_n\|^p_{L^p_K(\Lambda_{u_n})} + \|u_n\|^q_{L^q_K(\Lambda^c_{u_n})} \right),\end{aligned}$$

so that $I(u_n) \to -\infty$ since $p, q > 2$. The Symmetric Mountain-Pass Theorem thus implies the existence of an unbounded sequence of critical values for I and this completes the proof. $\square$

Lemma 23. *Under the assumptions of each of Theorems 16 and 19, the functional $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ is bounded from below and coercive. In particular, if g satisfies (g_6), then*

$$\inf_{v \in H^1_{V,\mathrm{r}}} I(v) < 0. \tag{31}$$

Proof. The fact that I is bounded below and coercive on $H^1_{V,\mathrm{r}}$ is a consequence of Lemma 21. Indeed, the result readily follows from (25) if $q_1, q_2 \in (1,2)$, while, if $\max\{q_1, q_2\} = 2 > \min\{q_1, q_2\} > 1$, we fix $\varepsilon < 1/2$ and use the second part of the lemma in order to get

$$I(u) \geq \left(\frac{1}{2} - \varepsilon\right) \|u\|^2 - c(\varepsilon) \|u\|^{\min\{q_1,q_2\}} - L_0 \|u\| \qquad \text{for all } u \in H^1_{V,\mathrm{r}},$$

which yields again the conclusion. In order to prove (31), we use assumption (h_0) to fix a function $u_0 \in C^\infty_c(B_{r_2} \setminus \overline{B}_{r_1}) \cap H^1_{V,\mathrm{r}}$ such that $0 \leq u_0 \leq t_0$, $u_0 \neq 0$. Then, by assumption (g_6), for every $0 < \lambda < 1$ we get that $\lambda u_0 \in H^1_{V,\mathrm{r}}$ satisfies

$$I(\lambda u_0) = \frac{1}{2} \|\lambda u_0\|^2 - \int_{\mathbb{R}^N} G(|x|, \lambda u_0)\, dx \leq \frac{\lambda^2}{2} \|u_0\|^2 - \lambda^\theta m \int_{\mathbb{R}^N} K(|x|)\, u_0^\theta dx.$$

Since $\theta < 2$, this implies $I(\lambda u_0) < 0$ for λ sufficiently small and therefore (31) ensues. $\square$

Proof of Theorem 16 Let

$$\mu := \inf_{v \in H^1_{V,\mathrm{r}}} I(v)$$

and take any minimizing sequence $\{v_n\}$ for μ. From Lemma 23 we have that the functional $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ is bounded from below and coercive, so that $\mu \in \mathbb{R}$ and $\{v_n\}$ is bounded in $H^1_{V,\mathrm{r}}$. Thanks to assumption $\left(\mathcal{S}''_{q_1,q_2}\right)$, the embedding $H^1_{V,\mathrm{r}} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is compact (see [4, Theorem 1]) and thus we can assume that there exists $u \in H^1_{V,\mathrm{r}}$ such that, up to a subsequence, one has:

$$\begin{array}{ll} v_n \rightharpoonup u & \text{in } H^1_{V,\mathrm{r}}, \\ v_n \to u & \text{in } L^{q_1}_K + L^{q_2}_K. \end{array}$$

Then, thanks to (h_3) and the continuity of the functional (23) on $L^{q_1}_K + L^{q_2}_K$ (see the proof of Proposition 10 above), u satisfies

$$\begin{aligned} \int_{\mathbb{R}^N} G(|x|, v_n)\, dx &= \int_{\mathbb{R}^N} \left(G(|x|, v_n) - g(|x|, 0)\, v_n\right) dx + \int_{\mathbb{R}^N} g(|x|, 0)\, v_n dx \\ &\to \int_{\mathbb{R}^N} G(|x|, u)\, dx. \end{aligned}$$

By the weak lower semi-continuity of the norm, this implies

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(|x|, u)\, dx \le \lim_{n\to\infty} \left(\frac{1}{2}\|v_n\|^2 - \int_{\mathbb{R}^N} G(|x|, v_n)\, dx\right) \\ &= \mu \end{aligned}$$

and thus we conclude $I(u) = \mu$. It remains to show that $u \neq 0$. If g satisfies (g_6), then we have $\mu < 0$ by Lemma 23 and therefore it must be $u \neq 0$, since $I(0) = 0$. If (g_7) holds, assume by contradiction that $u = 0$. Since u is a critical point of $I \in C^1(H^1_{V,\mathrm{r}}; \mathbb{R})$, from (19) we get

$$\int_{\mathbb{R}^N} g(|x|, 0)\, h\, dx = 0, \quad \forall h \in C^\infty_{c,\mathrm{rad}}(B_{r_2} \setminus \overline{B}_{r_1}) \subset H^1_{V,\mathrm{r}}.$$

This implies $g(\cdot, 0) = 0$ almost everywhere in (r_1, r_2), which is a contradiction. $\square$

Proof of Corollary 17 Setting

$$\widetilde{g}(r,t) := \begin{cases} g(r,t) & \text{if } t \ge 0 \\ 2g(r,0) - g(r,|t|) & \text{if } t < 0 \end{cases}$$

and

$$\widetilde{G}(r,t) := \int_0^t \widetilde{g}(r,s)\, ds = \begin{cases} G(r,t) & \text{if } t \ge 0 \\ 2g(r,0)\, t + G(r,|t|) & \text{if } t < 0, \end{cases}$$

it is easy to check that the function $\widetilde{g}$ still satisfies all the assumptions of Theorem 16. We just observe that $\widetilde{g}$ satisfies (g_6) or (g_7) if so does g, and that for almost every $r > 0$ and all $t \in \mathbb{R}$ one has

$$|\widetilde{g}(r,t) - \widetilde{g}(r,0)| = |g(r,|t|) - g(r,0)| \le K(r)\, |f(|t|)|$$

with $|f(|t|)| \leq M \min\left\{|t|^{q_1-1}, |t|^{q_2-1}\right\}$. Then, by Theorem 16, there exists $\widetilde{u} \neq 0$ such that

$$\widetilde{I}(\widetilde{u}) = \min_{u \in H^1_{V,\mathrm{r}}} \widetilde{I}(u), \quad \text{where} \quad \widetilde{I}(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} \widetilde{G}(|x|, u)\, dx.$$

For every $u \in H^1_{V,\mathrm{r}}$ one has

$$\begin{aligned}
\widetilde{I}(u) &= \frac{1}{2}\|u\|^2 - \int_{\{u \geq 0\}} G(|x|, u)\, dx - 2\int_{\{u<0\}} g(|x|, 0)\, u\, dx + \\
&\quad - \int_{\{u<0\}} G(|x|, |u|)\, dx \\
&= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^N} G(|x|, |u|)\, dx + 2\int_{\mathbb{R}^N} g(|x|, 0)\, u_-\, dx \\
&= I(|u|) + 2\int_{\mathbb{R}^N} g(|x|, 0)\, u_-\, dx, \qquad (32)
\end{aligned}$$

which implies that $\widetilde{u}$ satisfies (21), as one readily checks that

$$\inf_{u \in H^1_{V,\mathrm{r}}} \left(I(|u|) + 2\int_{\mathbb{R}^N} g(|x|, 0)\, u_-\, dx \right) = \inf_{u \in H^1_{V,\mathrm{r}},\, u \geq 0} I(u).$$

Moreover, since $G(r, |t|) = \widetilde{G}(r, |t|)$ and $g(\cdot, 0) \geq 0$, (32) gives

$$\widetilde{I}(u) = \widetilde{I}(|u|) + 2\int_{\mathbb{R}^N} g(|x|, 0)\, u_-\, dx \geq \widetilde{I}(|u|)$$

and hence $|\widetilde{u}| \in H^1_{V,\mathrm{r}}$ is still a minimizer for $\widetilde{I}$, so that we can assume $\widetilde{u} \geq 0$. Finally, $\widetilde{u}$ is a critical point for I since $\widetilde{u}$ is a critical point of $\widetilde{I}$ and $\widetilde{g}(r, t) = g(r, t)$ for avery $t \geq 0$. □

In proving Theorem 19, we will use a well known abstract result from [16,21]. We recall it here in a version given in [30].

Theorem 24. ([30, Lemma 2.4]) *Let X be a real Banach space and let $J \in C^1(X; \mathbb{R})$. Assume that J satisfies the Palais-Smale condition, is bounded from below, even and such that $J(0) = 0$. Assume furthermore that $\forall k \in \mathbb{N} \setminus \{0\}$ there exist $\rho_k > 0$ and a k-dimensional subspace X_k of X such that*

$$\sup_{u \in X_k,\, \|u\|_X = \rho_k} J(u) < 0. \qquad (33)$$

Then J has a sequence of critical values $c_k < 0$ such that $\lim_{k \to \infty} c_k = 0$.

Proof of Theorem 19 Since $I : H^1_{V,\mathrm{r}} \to \mathbb{R}$ satisfies $I(0) = 0$ and is of class C^1 by Proposition 10, even by assumption (g_5) and bounded below by Lemma 23, for applying Theorem 24 (with $X = H^1_{V,\mathrm{r}}$ and $J = I$) we need only to show that I satisfies the Palais-Smale condition and the geometric condition (33). By coercivity (Lemma 23), every Palais-Smale sequence for I is bounded in $H^1_{V,\mathrm{r}}$ and one obtains the existence of a strongly convergent subsequence as in the proof of Lemma 22. In order to check (33), we first deduce from (g_5) and (g_6) that

$$G(r,t) \geq mK(r)\,|t|^{\theta} \quad \text{for almost every } r>0 \text{ and all } |t| \leq t_0. \tag{34}$$

Then, for any $k \in \mathbb{N} \backslash \{0\}$, we take k linearly independent functions $\phi_1, \ldots, \phi_k \in C^{\infty}_{c,\mathrm{rad}}(B_{r_2} \setminus \overline{B}_{r_1})$ such that $0 \leq \phi_i \leq t_0$ for every $i = 1, \ldots, k$ and set

$$X_k := \operatorname{span}\{\phi_1, \ldots, \phi_k\} \quad \text{and} \quad \|\lambda_1\phi_1 + \cdots + \lambda_k\phi_k\|_{X_k} := \max_{1\leq i\leq k} |\lambda_i|.$$

This defines a subspace of $H^1_{V,\mathrm{r}}$ by assumption (h_0) and all norms are equivalent on X_k, so that there exist $m_k, l_k > 0$ such that for all $u \in X_k$ one has

$$\|u\|_{X_k} \leq m_k \|u\| \quad \text{and} \quad \|u\|^{\theta}_{L^{\theta}_K(\mathbb{R}^N)} \geq l_k \|u\|^{\theta}. \tag{35}$$

Fix $\rho_k > 0$ small enough that $km_k\rho_k < 1$ and $\rho_k^2/2 - ml_k\rho_k^{\theta} < 0$ (which is possible since $\theta < 2$) and take any $u = \lambda_1\phi_1 + \cdots + \lambda_k\phi_k \in X_k$ such that $\|u\| = \rho_k$. Then by (35) we have

$$|\lambda_i| \leq \|u\|_{X_k} \leq m_k\rho_k < \frac{1}{k} \qquad \text{for every } i = 1, \ldots, k$$

and therefore

$$|u(x)| \leq \sum_{i=1}^{k} |\lambda_i|\,\phi_i(x) \leq t_0 \sum_{i=1}^{k} |\lambda_i| < t_0 \qquad \text{for all } x \in \mathbb{R}^N.$$

By (34) and (35), this implies

$$\int_{\mathbb{R}^N} G(|x|, u)\,dx \geq m \int_{\mathbb{R}^N} K(|x|)\,|u|^{\theta}\,dx \geq ml_k \|u\|^{\theta}$$

and hence we get

$$I(u) \leq \frac{1}{2}\|u\|^2 - ml_k\|u\|^{\theta} = \frac{1}{2}\rho_k^2 - ml_k\rho_k^{\theta} < 0.$$

This proves (33) and the conclusion thus follows from Theorem 24. □

5. Proof of the existence results for problem (P_Q)

Let $\Omega \subseteq \mathbb{R}^N$, $N \geq 3$, be a spherically symmetric domain (bounded or unbounded). Let V, K, f be as in $(\mathbf{V}), (\mathbf{K}), (\mathbf{f})$ and let $Q \in L^2(\Omega_{\mathrm{r}}, r^{N+1}dr)$. If $\Omega \neq \mathbb{R}^N$, extend the definition of V, K, Q by setting

$$V(r) := +\infty \quad \text{and} \quad K(r) := Q(r) := 0 \quad \text{for every } r \in \mathbb{R}_+ \setminus \Omega_{\mathrm{r}}.$$

Define a Carathéodory function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ by setting

$$g(r,t) := K(r)\,f(t) + Q(r).$$

The next lemma shows that we need only to study problem (P_Q) on $\mathbb{R}^N$, to which the case with $\Omega \neq \mathbb{R}^N$ reduces. Recall the definitions (1) and (15) of the spaces $H^1_{0,V}(\Omega)$ and $H^1_V(\mathbb{R}^N)$.

Lemma 25. *If $\Omega \neq \mathbb{R}^N$, then, up to restriction to Ω and null extension on $\mathbb{R}^N \setminus \Omega$, we have that $H_V^1(\mathbb{R}^N) = H_{0,V}^1(\Omega)$ and any weak solution to problem (P_Q) on $\mathbb{R}^N$ is a weak solution to (P_Q) on Ω.*

Note that the result is obvious if $\Omega = \mathbb{R}^N$ (since $D^{1,2}(\mathbb{R}^N) = D_0^{1,2}(\mathbb{R}^N)$).

Proof. Let $u \in H_V^1(\mathbb{R}^N)$. By the Lebesgue integration theory of functions with real extended values (see e.g. [22]), $\int_{\mathbb{R}^N} V(|x|)\, u^2 dx < \infty$ implies $u = 0$ almost everywhere on $\mathbb{R}^N \setminus \Omega$ (where $V(|x|) = +\infty$) and

$$\int_{\mathbb{R}^N} V(|x|)\, u^2 dx = \int_\Omega V(|x|)\, u^2 dx. \tag{36}$$

Hence $u \in D^{1,2}(\mathbb{R}^N)$ implies $u \in D_0^{1,2}(\Omega)$ and therefore $u \in H_{0,V}^1(\Omega)$. Conversely, if $u \in H_{0,V}^1(\Omega)$, then $u \in D^{1,2}(\mathbb{R}^N)$ and one has $\int_\Omega V(|x|)\, u^2 dx = \int_{\mathbb{R}^N} V(|x|)\, u^2 \chi_\Omega dx$ with $u^2\chi_\Omega = u^2$ almost everywhere (recall from the statement of the lemma that we extend $u = 0$ on $\mathbb{R}^N \setminus \Omega$), so that (36) holds and therefore $u \in H_V^1(\mathbb{R}^N)$. This proves that $H_V^1(\mathbb{R}^N) = H_{0,V}^1(\Omega)$ and the last part of the lemma readily follows, since all the integrals involved in the definition of weak solutions to (P_Q) on $\mathbb{R}^N$ are computed on functions that vanish almost everywhere on $\mathbb{R}^N \setminus \Omega$. □

The proof of our existence results for problem (P_Q) relies on the application of the general results of Sect. 3, whose assumptions (h_0)-(h_3) are satisfied. Indeed, since $f(0) = 0$, one has $g(\cdot, 0) = Q$ and therefore g trivially satifies assumption (h_1). Moreover, the potentials V, K satisfy assumptions $(h_0), (h_2)$ thanks to $(\mathbf{V}), (\mathbf{K})$. Finally, assumption (h_3) holds because $Q \in L^2(\mathbb{R}_+, r^{N+1}dr)$ means $Q(|\cdot|) \in L^2(\mathbb{R}^N, |x|^2 dx)$ and thus implies

$$\begin{aligned}\left|\int_{\mathbb{R}^N} Q(|x|)\, u\, dx\right| &\leq \left(\int_{\mathbb{R}^N} |x|^2 Q(|x|)^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} dx\right)^{\frac{1}{2}} \\ &\leq \frac{2}{N-2}\left(\int_{\mathbb{R}^N} |x|^2 Q(|x|)^2 dx\right)^{\frac{1}{2}} \|u\|\end{aligned}$$

for all $u \in H_V^1 \hookrightarrow D^{1,2}(\mathbb{R}^N)$, by Hölder and Hardy inequalities.

In order to apply the general results, we will need a compactness lemma from [4], which gives sufficient conditions in order that $\left(\mathcal{S}''_{q_1,q_2}\right)$ and $(\mathcal{R}_{q_1,q_2})$ hold. For stating this lemma, we introduce some functions, whose graphs are partially sketched in Figs. 1, 2, 3, 4, 5, 6, 7 and 8 with a view to easing the application of the lemma itself. For $\alpha \in \mathbb{R}$, $\beta \in [0,1]$ and $\gamma \geq 2$, we define

$$\underline{\alpha}_0(\beta,\gamma) := \begin{cases} \max\{\alpha_2(\beta), \alpha_3(\beta,\gamma)\} & \text{if } 2 \leq \gamma < N \\ \alpha_1(\beta,\gamma) & \text{if } N \leq \gamma \leq 2N-2 \\ -\infty & \text{if } \gamma > 2N-2, \end{cases}$$

$$\underline{q}_0(\alpha,\beta,\gamma) := \begin{cases} \max\{1, 2\beta\} & \text{if } 2 \leq \gamma \leq N \\ \max\{1, 2\beta, q_*(\alpha,\beta,\gamma)\} & \text{if } N < \gamma \leq 2N-2 \\ \max\{1, 2\beta, q_*(\alpha,\beta,\gamma), q_{**}(\alpha,\beta,\gamma)\} & \text{if } \gamma > 2N-2 \end{cases}$$

and

$$\overline{q}_0(\alpha,\beta,\gamma) := \begin{cases} \min\{q_*(\alpha,\beta,\gamma), q_{**}(\alpha,\beta,\gamma)\} & \text{if } 2 \le \gamma < N \\ q_{**}(\alpha,\beta,\gamma) & \text{if } N \le \gamma < 2N-2 \\ +\infty & \text{if } \gamma \ge 2N-2 \end{cases}$$

(recall the definitions (8)–(10) and (5) of $\alpha_1, \alpha_2, \alpha_3$ and q_*, q_{**}). For $\alpha \in \mathbb{R}$, $\beta \in [0,1]$ and $\gamma \le 2$, we define the function

$$\underline{q}_\infty(\alpha,\beta,\gamma) := \max\{1, 2\beta, q_*(\alpha,\beta,\gamma), q_{**}(\alpha,\beta,\gamma)\}.$$

Lemma 26. ([4, Theorems 2, 3, 4, 5]) (i) *Suppose that Ω contains a neighbourhood of the origin and assume that $(\mathbf{VK}_0)$ holds with $\alpha_0 > \underline{\alpha}_0$, where $\underline{\alpha}_0 = \underline{\alpha}_0(\beta_0, 2)$. Then*

$$\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = 0 \quad \textit{for every } \underline{q}_0 < q_1 < \overline{q}_0,$$

where $\underline{q}_0 = \underline{q}_0(\alpha_0,\beta_0,2)$ and $\overline{q}_0 = \overline{q}_0(\alpha_0,\beta_0,2)$. If V also satisfies $(\mathbf{V}_0)$, then the same result holds true with $\underline{\alpha}_0 = \underline{\alpha}_0(\beta_0,\gamma_0)$, $\underline{q}_0 = \underline{q}_0(\alpha_0,\beta_0,\gamma_0)$ and $\overline{q}_0 = \overline{q}_0(\alpha_0,\beta_0,\gamma_0)$.

(ii) *Suppose that Ω contains a neighbourhood of infinity and assume $(\mathbf{VK}_\infty)$. Then*

$$\lim_{R\to +\infty} \mathcal{R}_\infty(q_2, R) = 0 \quad \textit{for every } q_2 > \underline{q}_\infty,$$

where $\underline{q}_\infty = \underline{q}_\infty(\alpha_\infty,\beta_\infty,2)$. If V also satisfies $(\mathbf{V}_\infty)$, then one can take $\underline{q}_\infty = \underline{q}_\infty(\alpha_\infty,\beta_\infty,\gamma_\infty)$.

Proof of Theorem 1 The theorem concerns problem (P_0), so that we have $g(\cdot,0) = Q = 0$. We want to apply Theorem 12. To this aim, since $(\mathbf{F}_1)$ implies (g_1)-(g_2) and $(\mathbf{F}_2)$ implies (g_3), we need only to check that $\exists q_1, q_2 > 2$ such that $\left(\mathcal{S}''_{q_1,q_2}\right)$ and (f_{q_1,q_2}) hold. This follows from the other assumptions of the theorem, since $(\mathbf{f}_1)$ and $f(0) = 0$ imply (f_{q_1,q_2}) for $t \ge 0$ (which is actually enough by Remark 13. 2), and inequalities (6) imply $q_1, q_2 > 2$ and

$$\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = \lim_{R\to +\infty} \mathcal{R}_\infty(q_2, R) = 0$$

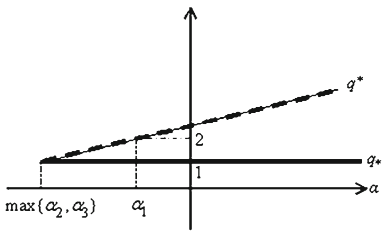

FIGURE 1. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*) and $\overline{q}_0(\cdot,\beta,\gamma)$ (*dashed*) for $\alpha > \underline{\alpha}_0$, with fixed $\beta \in [0,1]$, $\gamma = 2$

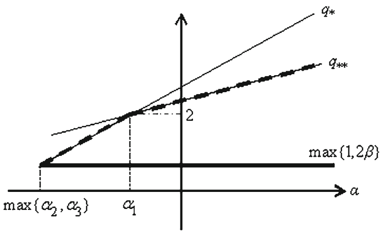

FIGURE 2. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*) and $\overline{q}_0(\cdot,\beta,\gamma)$ (*dashed*) for $\alpha > \underline{\alpha}_0$, with fixed $\beta \in [0,1]$, $2 \leq \gamma < N$

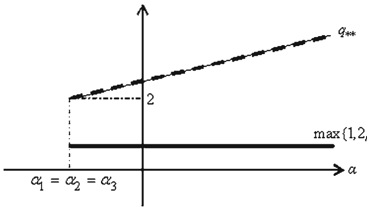

FIGURE 3. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*) and $\overline{q}_0(\cdot,\beta,\gamma)$ (*dashed*) for $\alpha > \underline{\alpha}_0$, with fixed $\beta \in [0,1]$, $\gamma = N$

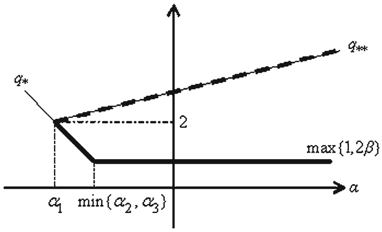

FIGURE 4. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*) and $\overline{q}_0(\cdot,\beta,\gamma)$ (*dashed*) for $\alpha > \underline{\alpha}_0$, with fixed $\beta \in [0,1]$, $N < \gamma < 2N-2$

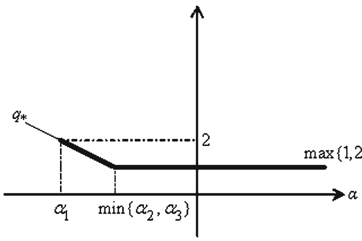

FIGURE 5. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*) for $\alpha > \underline{\alpha}_0$, with fixed $\beta \in [0,1]$, $\gamma = 2N-2$

by Lemma 26 (cf. Figs. 1–8). Note that also Proposition 11 applies, since $(\mathcal{R}_{q_1,q_2})$ holds. Hence the proof is complete, as the result follows from Theorem 12 and Proposition 11. $\square$

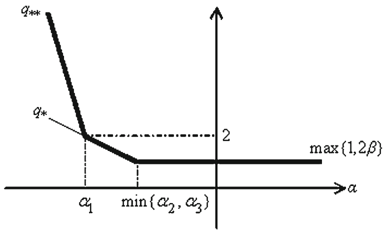

FIGURE 6. $\underline{q}_0(\cdot,\beta,\gamma)$ (*solid*), with fixed $\beta\in[0,1]$, $\gamma>2N-2$

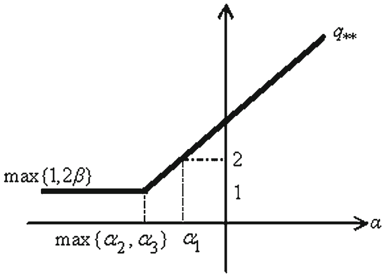

FIGURE 7. $\underline{q}_\infty(\cdot,\beta,\gamma)$ (*solid*), with fixed $\beta\in[0,1]$, $\gamma=2$

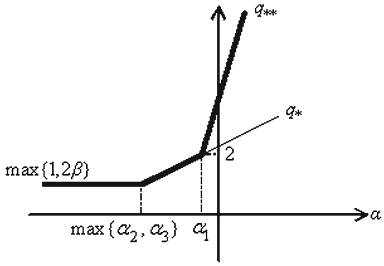

FIGURE 8. $\underline{q}_\infty(\cdot,\beta,\gamma)$ (*solid*), with fixed $\beta\in[0,1]$, $\gamma<2$

Proof of Theorems 3 and 4 The proof runs exactly as the proof of Theorem 1 and then ends by applying Lemma 25. The only difference is that, under the assumptions of Theorem 3, we use inequalities (7) to deduce $q>2$ and

$$\lim_{R\to 0^+}\mathcal{R}_0\left(q,R\right)=0$$

by part (i) of Lemma 26 (cf. Figs. 1–6), and then we observe that $\mathcal{R}_\infty\left(q,R\right)=0$ for every $R>0$ large enough, since the integrand $K\left(\left|\cdot\right|\right)\left|u\right|^{q-1}\left|h\right|$ vanishes almost everywhere outside Ω. Similarly, under the assumptions of Theorem 4, we deduce

$$\lim_{R\to +\infty}\mathcal{R}_\infty\left(q_2,R\right)=0$$

by part (ii) of Lemma 26 (cf. Figs. 7, 8), and then we observe that $\mathcal{R}_0(q,R) = 0$ for every $R > 0$ small enough. Hence, in both cases, conditions $\left(\mathcal{S}''_{q_1,q_2}\right)$, $(\mathcal{R}_{q_1,q_2})$ and (f_{q_1,q_2}) hold with $q_1 = q_2 = q$ and we can apply Theorem 12 and Proposition 11. □

Proof of Theorems 5 and 6 The proof is analogous to the one of Theorem 1 and reduces to the application of Corollary 17. We have $g(\cdot,0) = Q \geq 0$ by assumption. This implies $G(r,t) = K(r)F(t) + Q(r)t \geq K(r)F(t)$ for almost every $r > 0$ and all $t \geq 0$, so that (g_6) or (g_7) holds true, respectively according as f satisfies $(\mathbf{F}_3)$ or Q does not vanish almost everywhere in (r_1,r_2). Thus, in order to apply Corollary 17, we need only to check that the other assumptions of the theorems yield the existence of $q_1,q_2 \in (1,2)$ such that $\left(\mathcal{S}''_{q_1,q_2}\right)$ and (f_{q_1,q_2}) hold. On the one hand, inequalities (11) and (13) imply $q,q_1,q_2 \in (1,2)$ and

$$\lim_{R\to 0^+} \mathcal{R}_0(q,R) = \lim_{R\to+\infty} \mathcal{R}_\infty(q,R) = \lim_{R\to 0^+} \mathcal{R}_0(q_1,R) = \lim_{R\to+\infty} \mathcal{R}_\infty(q_2,R) = 0$$

by Lemma 26 (cf. Figs. 1–8). This also allows us to apply Proposition 11, since $(\mathcal{R}_{q_1,q_2})$ holds. On the other hand, $(\mathbf{f}_1)$ and $(\mathbf{f}_2)$ respectively imply (f_{q_1,q_2}) and $(f_{q,q})$ for $t \geq 0$, which is actually enough by Remark 18.2. Hence the result follows from Corollary 17 and Proposition 11, so that the proof is complete. □

The proof of Theorem 8 is very similar to the ones of Theorems 1, 5 and 6, so we leave it to the interested reader (apply Theorems 14 and 19 instead of Theorem 12 and Corollary 17).

6. Appendix

This Appendix is devoted to the derivation of some radial estimates, which has been announced and used in [4, Lemmas 3 and 4] to prove the compactness results yielding Lemma 26 above. Such estimates has been also proved in [29, Lemmas 4 and 5], but under slightly different assumptions from the ones used here (and in [4]).

Assume $N \geq 3$ and denote by σ_N the $(N-1)$-dimensional measure of the unit sphere of $\mathbb{R}^N$.

Lemma 27. *Let u be any radial map of $D^{1,2}(\mathbb{R}^N)$ and let $\tilde{u} : (0,+\infty) \to \mathbb{R}$ be such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Then $\tilde{u} \in W^{1,1}(\mathcal{I})$ for any bounded open interval $\mathcal{I} \subset (0,+\infty)$ such that* $\inf \mathcal{I} > 0$.

Proof. Denote $\lambda := \inf \mathcal{I} > 0$ and set $\mathcal{I}_N := \left\{x \in \mathbb{R}^N : |x| \in \mathcal{I}\right\}$. Since $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we readily have

$$\int_{\mathcal{I}} |\tilde{u}(r)|\, dr \leq \frac{1}{\lambda^{N-1}} \int_{\mathcal{I}} r^{N-1} |\tilde{u}(r)|\, dr = \frac{1}{\lambda^{N-1}\sigma_N} \int_{\mathcal{I}_N} |u(x)|\, dx < \infty\,.$$

Now we exploit the density of $C_c^\infty(\mathbb{R}^N)$ radial mappings in the space of $D^{1,2}(\mathbb{R}^N)$ radial mappings (which follows from standard convolution and regularization arguments) in order to infer that, as u is radial, $\nabla u(x) \cdot x$ only depends on $|x|$. Thus there exists $\phi : (0, +\infty) \to \mathbb{R}$ such that $\nabla u(x) \cdot x = \phi(|x|)$ for almost every $x \in \mathbb{R}^N$ and one has

$$\int_{\mathcal{I}} \frac{|\phi(r)|}{r} dr = \int_{\mathcal{I}} r^{N-1} \frac{|\phi(r)|}{r^N} dr = \frac{1}{\sigma_N} \int_{\mathcal{I}_N} \frac{|\nabla u(x) \cdot x|}{|x|^N} dx$$
$$\leq \frac{1}{\lambda^{N-1}\sigma_N} \int_{\mathcal{I}_N} |\nabla u(x)|\, dx < \infty .$$

Then, letting $\varphi \in C_c^\infty(\mathcal{I})$ and setting $\psi(x) := \varphi(|x|)$, we get

$$\begin{aligned}\int_{\mathcal{I}} \tilde{u}(r) \varphi'(r)\, dr &= \frac{1}{\sigma_N} \int_{\mathcal{I}_N} \frac{\tilde{u}(|x|)}{|x|^{N-1}} \varphi'(|x|)\, dx \\ &= \frac{1}{\sigma_N} \int_{\mathcal{I}_N} \frac{u(x)}{|x|^{N-1}} \sum_{i=1}^N \frac{\partial \psi}{\partial x_i}(x) \frac{x_i}{|x|} dx \\ &= -\frac{1}{\sigma_N} \sum_{i=1}^N \int_{\mathcal{I}_N} \frac{\partial u}{\partial x_i}(x) \frac{x_i}{|x|^N} \psi(x)\, dx \\ &= -\frac{1}{\sigma_N} \int_{\mathcal{I}_N} \frac{\nabla u(x) \cdot x}{|x|^N} \psi(x)\, dx \\ &= -\frac{1}{\sigma_N} \int_{\mathcal{I}_N} \frac{\phi(|x|)}{|x|^N} \varphi(|x|)\, dx = -\int_{\mathcal{I}} \frac{\phi(r)}{r} \varphi(r)\, dr\end{aligned}$$

because $\psi \in C_c^\infty(\mathcal{I}_N)$. □

Let $V : \mathbb{R}_+ \to [0, +\infty]$ be a measurable function satisfying (h_0) and define $H^1_{V,\mathrm{r}}$ as in (16).

Proposition 28. *Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\lambda_\infty := \operatorname*{ess\,inf}_{r > R_2} r^{\gamma_\infty} V(r) > 0 \quad \textit{for some } \gamma_\infty \leq 2.$$

Then every $u \in H^1_{V,\mathrm{r}}$ satisfies

$$|u(x)| \leq c_\infty \lambda_\infty^{-\frac{1}{4}} \|u\| \, |x|^{-\frac{2(N-1)-\gamma_\infty}{4}} \qquad \textit{almost everywhere in } B^c_{R_2},$$

where $c_\infty = \sqrt{2/\sigma_N}$.

Proof. Let $u \in H^1_{V,\mathrm{r}}$ and let $\tilde{u} : (0, +\infty) \to \mathbb{R}$ be continuous and such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Set

$$v(r) := r^{N-1-\gamma_\infty/2} \tilde{u}(r)^2 \quad \text{for all } r > 0.$$

If $\lambda := \liminf_{r \to +\infty} v(r) > 0$, then for every r large enough one has

$$r^{N-1-\gamma_\infty} \tilde{u}(r)^2 \geq \frac{\lambda}{2r^{\gamma_\infty/2}},$$

whence, since $\gamma_\infty \leq 2$, one gets the contradiction

$$\begin{aligned}\int_{B_{R_2}^c} V(|x|)\, u^2 dx &\geq \lambda_\infty \int_{B_{R_2}^c} \frac{u^2}{|x|^{\gamma_\infty}} dx = \lambda_\infty \sigma_N \int_{R_2}^{+\infty} \frac{\tilde{u}(r)^2}{r^{\gamma_\infty}} r^{N-1} dr \\ &\geq \lambda_\infty \sigma_N \int_{R_2}^{+\infty} \frac{\lambda}{2r^{\gamma_\infty/2}} dr = +\infty.\end{aligned}$$

Therefore it must be $\lambda = 0$ and thus there exists $r_n \to +\infty$ such that $v(r_n) \to 0$. By Lemma 27, we have $v \in W^{1,1}((r, r_n))$ for every $R_2 < r < r_n < +\infty$ and hence

$$v(r_n) - v(r) = \int_r^{r_n} v'(s)\, ds\,.$$

Moreover, for almost every $s \in (r, r_n)$ one has

$$\begin{aligned} v'(s) &= \left(N - 1 - \frac{\gamma_\infty}{2}\right) s^{N-2-\frac{\gamma_\infty}{2}} \tilde{u}(s)^2 + 2s^{N-1-\frac{\gamma_\infty}{2}} \tilde{u}(s)\, \tilde{u}'(s) \\ &\geq 2s^{N-1-\frac{\gamma_\infty}{2}} \tilde{u}(s)\, \tilde{u}'(s) \geq -2s^{N-1-\frac{\gamma_\infty}{2}} |\tilde{u}(s)|\, |\tilde{u}'(s)| \end{aligned}$$

(note that $N - 1 - \frac{\gamma_\infty}{2} \geq N - 2 > 0$). Hence

$$v(r_n) - v(r) = \int_r^{r_n} v'(s)\, ds \geq -2 \int_r^{r_n} s^{N-1-\frac{\gamma_\infty}{2}} |\tilde{u}(s)|\, |\tilde{u}'(s)|\, ds$$

and therefore

$$\begin{aligned} v(r) - v(r_n) &\leq 2 \int_r^{r_n} s^{\frac{N-1}{2}} \frac{|\tilde{u}(s)|}{s^{\gamma_\infty/2}} s^{\frac{N-1}{2}} |\tilde{u}'(s)|\, ds \\ &\leq 2 \left(\int_r^{r_n} s^{N-1} \frac{\tilde{u}(s)^2}{s^{\gamma_\infty}} ds \right)^{\frac{1}{2}} \left(\int_r^{r_n} s^{N-1} \tilde{u}'(s)^2\, ds \right)^{\frac{1}{2}} \\ &\leq 2 \left(\int_{R_2}^{+\infty} s^{N-1} \frac{\tilde{u}(s)^2}{s^{\gamma_\infty}} ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} \tilde{u}'(s)^2\, s^{N-1} ds \right)^{\frac{1}{2}} \\ &\leq 2 \left(\frac{1}{\lambda_\infty} \int_{R_2}^{+\infty} V(s)\, \tilde{u}(s)^2\, s^{N-1} ds \right)^{\frac{1}{2}} \left(\frac{1}{\sigma_N} \int_{\mathbb{R}^N} |\nabla u|^2\, dx \right)^{\frac{1}{2}} \\ &\leq \frac{2}{\sigma_N} \left(\frac{1}{\lambda_\infty} \int_{\mathbb{R}^N} V(|x|)\, u^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2\, dx \right)^{\frac{1}{2}}. \end{aligned}$$

This implies $v(r) \leq 2\lambda_\infty^{-1/2} \|u\|^2 / \sigma_N$, whence the result readily ensues. $\square$

Proposition 29. *Assume that there exists $R > 0$ such that $V(r) < +\infty$ almost everywhere on $(0, R)$ and*

$$\lambda_0 := \operatorname*{ess\,inf}_{r \in (0,R)} r^{\gamma_0} V(r) > 0 \quad \textit{for some } \gamma_0 \geq 2.$$

Then every $u \in H^1_{V,\mathrm{r}}$ *satisfies*

$$|u(x)| \leq c_0 \left(\frac{1}{\sqrt{\lambda_0}} + \frac{R^{\frac{\gamma_0 - 2}{2}}}{\lambda_0} \right)^{\frac{1}{2}} \|u\| \, |x|^{-\frac{2N-2-\gamma_0}{4}} \qquad \textit{almost everywhere in } B_R,$$

where $c_0 = \sqrt{\max\{2, N-2\} / \sigma_N}$.

Proof. Let $u \in H^1_{V,\mathrm{r}}$ and $\tilde{u} : (0, +\infty) \to \mathbb{R}$ continuous and such that $u(x) = \tilde{u}(|x|)$ for almost every $x \in \mathbb{R}^N$. Set

$$v(r) := r^{N-1-\gamma_0/2} \tilde{u}(r)^2 \quad \text{for all } r > 0.$$

If $\lambda := \liminf_{r \to 0^+} v(r) > 0$, then for every r small enough one gets

$$r^{N-1-\gamma_0} \tilde{u}(r)^2 \geq \frac{\lambda}{2 r^{\gamma_0/2}}$$

and thus, since $\gamma_0 \geq 2$, one deduces the contradiction

$$\begin{aligned} \int_{B_R} V(|x|)\, u^2 dx \geq \lambda_0 \int_{B_R} \frac{u^2}{|x|^{\gamma_0}} dx = \lambda_0 \sigma_N \int_0^R \frac{\tilde{u}(r)^2}{r^{\gamma_0}} r^{N-1} dr \\ \geq \lambda_0 \sigma_N \int_0^R \frac{\lambda}{2 r^{\gamma_0/2}} dr = +\infty. \end{aligned}$$

So we have $\lambda = 0$ and thus there exists $r_n \to 0^+$ such that $v(r_n) \to 0$. By Lemma 27, we have $v \in W^{1,1}((r_n, r))$ for every $0 < r_n < r < R$ and hence

$$v(r) - v(r_n) = \int_{r_n}^r v'(s)\, ds\,.$$

Moreover, for almost every $s \in (r_n, r)$ one has

$$v'(s) = \left(N - 1 - \frac{\gamma_0}{2} \right) s^{N-2-\frac{\gamma_0}{2}} \tilde{u}(s)^2 + 2 s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s) \tilde{u}'(s)\,.$$

Notice that

$$\begin{aligned} \int_{r_n}^r s^{N-2-\frac{\gamma_0}{2}} \tilde{u}(s)^2 ds = \int_{r_n}^r s^{\frac{\gamma_0-2}{2}} \frac{\tilde{u}(s)^2}{s^{\gamma_0}} s^{N-1} ds \leq r^{\frac{\gamma_0-2}{2}} \int_0^R \frac{\tilde{u}(s)^2}{s^{\gamma_0}} s^{N-1} ds \\ \leq r^{\frac{\gamma_0-2}{2}} \frac{1}{\lambda_0} \int_0^R V(s)\, \tilde{u}(s)^2 s^{N-1} ds \\ \leq R^{\frac{\gamma_0-2}{2}} \frac{1}{\lambda_0} \frac{1}{\sigma_N} \|u\|^2 \end{aligned}$$

and

$$\begin{aligned} & \int_{r_n}^r s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s)\, \tilde{u}'(s)\, ds \\ & \leq \int_{r_n}^r s^{N-1-\frac{\gamma_0}{2}} |\tilde{u}(s)| \, |\tilde{u}'(s)|\, ds \\ & \leq \left(\int_{r_n}^r s^{N-1} \frac{\tilde{u}(s)^2}{s^{\gamma_0}} ds \right)^{\frac{1}{2}} \left(\int_{r_n}^r s^{N-1} \tilde{u}'(s)^2 ds \right)^{\frac{1}{2}} \end{aligned}$$

$$\leq \left(\int_0^R s^{N-1} \frac{\tilde{u}(s)^2}{s^{\gamma_0}} ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} s^{N-1} \tilde{u}'(s)^2 ds \right)^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{\lambda_0} \int_0^R V(s) \tilde{u}(s)^2 s^{N-1} ds \right)^{\frac{1}{2}} \left(\int_0^{+\infty} \tilde{u}'(s)^2 s^{N-1} ds \right)^{\frac{1}{2}}$$

$$\leq \frac{1}{\sigma_N} \left(\frac{1}{\lambda_0} \int_{\mathbb{R}^N} V(|x|) u^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\nabla u|^2 dx \right)^{\frac{1}{2}} = \frac{1}{\sigma_N} \frac{1}{\lambda_0^{1/2}} \|u\|^2 .$$

If $2 \leq \gamma_0 < 2N-2$ (i.e., $N-1-\frac{\gamma_0}{2} > 0$), then

$$\begin{aligned} v(r) - v(r_n) &= \int_{r_n}^r v'(s) ds \leq \left(N-1-\frac{\gamma_0}{2} \right) \int_{r_n}^r s^{N-2-\frac{\gamma_0}{2}} \tilde{u}(s)^2 ds + \\ &\quad + 2 \int_{r_n}^r s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s) \tilde{u}'(s) ds \\ &\leq \left(N-1-\frac{\gamma_0}{2} \right) R^{\frac{\gamma_0-2}{2}} \frac{1}{\lambda_0} \frac{1}{\sigma_N} \|u\|^2 + \frac{2}{\sigma_N} \frac{1}{\lambda_0^{1/2}} \|u\|^2 \\ &\leq (N-2) R^{\frac{\gamma_0-2}{2}} \frac{1}{\lambda_0} \frac{1}{\sigma_N} \|u\|^2 + \frac{2}{\sigma_N} \frac{1}{\lambda_0^{1/2}} \|u\|^2 . \end{aligned}$$

If $\gamma_0 \geq 2N-2$ (i.e., $N-1-\frac{\gamma_0}{2} \leq 0$), then

$$\begin{aligned} v'(s) &= \left(N-1-\frac{\gamma_0}{2} \right) s^{N-2-\frac{\gamma_0}{2}} \tilde{u}(s)^2 + 2 s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s) \tilde{u}'(s) \\ &\leq 2 s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s) \tilde{u}'(s) \end{aligned}$$

and therefore

$$v(r) - v(r_n) = \int_{r_n}^r v'(s) ds \leq 2 \int_{r_n}^r s^{N-1-\frac{\gamma_0}{2}} \tilde{u}(s) \tilde{u}'(s) ds \leq \frac{2}{\sigma_N} \frac{1}{\lambda_0^{1/2}} \|u\|^2 .$$

So, in any case, we have

$$v(r) - v(r_n) \leq \frac{1}{\sigma_N} \left(\frac{2}{\lambda_0^{1/2}} + (N-2) \frac{R^{\frac{\gamma_0-2}{2}}}{\lambda_0} \right) \|u\|^2 ,$$

which implies

$$v(r) \leq \frac{1}{\sigma_N} \left(\frac{2}{\lambda_0^{1/2}} + (N-2) \frac{R^{\frac{\gamma_0-2}{2}}}{\lambda_0} \right) \|u\|^2 .$$

Hence the result readily follows. □

Remark 30. *By Hardy inequality, the radial subspace of $D^{1,2}(\mathbb{R}^N)$ equals $H^1_{1/|x|^2,\mathrm{r}}$, with equivalent norms. Therefore, taking $\gamma_0 = \gamma_\infty = 2$ and $R = R_2$ in Propositions 28 and 29, we get $\lambda_0 = \lambda_\infty = 1$ and find the renowned Radial Lemma known as Ni's inequality: there exists $c_N > 0$ such that every radial function in $D^{1,2}(\mathbb{R}^N)$ satisfies $|u(x)| \leq c_N \|\nabla u\|_{L^2(\mathbb{R}^N)} |x|^{-(N-2)/2}$ almost everywhere in $\mathbb{R}^N$.*

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