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SPDE with generalized drift and fractional-type noise

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Abstract. We consider a stochastic partial differential equation involving a second order differential operator whose drift is discontinuous. The equation is driven by a Gaussian noise which behaves as a Wiener process in space and the time covariance generates a signed measure. This class includes the Brownian motion, fractional Brownian motion and other related processes. We give a necessary and sufficient condition for the existence of the solution and we study the path regularity of this solution.

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1. Introduction

In this paper we consider the stochastic partial differential equation (SPDE)

$$\frac{\partial}{\partial t}u(t,x) = Lu(t,x) + \dot{W}(t,x) \tag{1}$$

where $t \geqslant 0$, $x \in \mathcal{E} = \mathbb{R}^d$ with vanishing initial condition $u(0,x) = 0$ for every $x \in \mathcal{E}$ and $u(t,\cdot)/\partial\mathcal{E} = 0$ for every $t \geqslant 0$.

In (1), L denotes the operator defined by

$$L = C\Delta + 2Cq\delta_S\nabla \tag{2}$$

with Δ the Laplacian on $\mathbb{R}^d$, S is the hyperplane $S = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; x_d = 0\}$, C is a strictly positive constant, q is some constant such that

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$| q | \leq 1$ and δ_S is a generalized function on $\mathbb{R}^d$ the action of which onto basic functions reduces to integration over S, i.e.

$$\int_{\mathbb{R}^d} \varphi(x)\delta_S(x)dx = \int_S \varphi(x)d\sigma \tag{3}$$

where $\varphi : \mathbb{R}^d \to \mathbb{R}$ is any real-valued continuous function of compact support defined on $\mathbb{R}^d$, and the integral on the right is a surface integral.

The notation $\dot{W}$ in (1) refers to the formal derivative of a Gaussian noise W that behaves as a Wiener process with respect to the space variable (it is "white in space") and it is a process that admits a covariance measure structure (in the sense of [3]) in time. In particular, W may behave as a Brownian motion, fractional Brownian motion, bifractional Brownian motion or other related processes with respect to the time variable.

The study of diffusion processes with generalized drift has been initiated by Portenko. A complete exposition of this field can be found in the monograph [10]. They appeared as natural models for extremely irregular motions (for example, the velocity in the liquid) that can be very large at some points of space or at some times. Operators with a singular first-order differential term appeared at the end of the 1970s in [8] and [9] as an interesting way to model some permeable barrier. In these papers, the author constructed a continuous Markov process in $\mathbb{R}^d$ whose infinitesimal generator has a singular drift concentrated on an hyperplane, orthogonal to some unit vector. The operator L given by (2) has been analyzed in many references due to the fact that it generates the so-called skew Brownian motion. We refer to [7] for the fundamental solution of the equation $(\frac{\partial}{\partial t} - L)u = 0$ and to the survey [5] and the references therein for its connection with the skew Brownian motion. Moreover, as noted in [8,9], solving $(\frac{\partial}{\partial t} - L)u = 0$ with L given by (2) is also equivalent in solving a *transmission problem.*

Our purpose is to solve the stochastic counterpart of this equation, namely the SPDE (1). This can be seen as a generalization of the deterministic model to a random perturbation and it also represents a natural extension of the stochastic heat equation driven by Gaussian noises, such as fractional Brownian motion (which has been widely studied recently, see [11] and the references therein). We prove a necessary and sufficient condition for the existence of the mild solution to (1) for a general class of Gaussian noises, including fractional Brownian motion, bifractional Brownian motion and other related process. Our findings extend the results obtained recently and less recently for the case of the heat equation with fractional noise in time. We also study the Hölder continuity of the solution when the noise is fractional in time.

Our paper is organized as the following way: in Sect. 2 we describe the basic elements related to the operator L (2) and the fundamental solution associated to (1), together with the random noise in this SPDE. Section 3 focuses on the necessary and sufficient condition for the existence of a mild solution to (1) while in Sect. 4 we regard some important particular cases. In the last section we study the pathwise regularity of the solution when the noise behaves as a fractional Brownian motion with respect to the time variable.

2. Preliminaries

In this preliminary paragraph we introduce the operator L that appears in (1), we describe the solution to $(\frac{\partial}{\partial t} - L)u = 0$ and we present the random noise that drives the SPDE (1).

2.1. The PDE with constant and generalized drift coefficients

On the Euclidien space $\mathbb{R}^d$, $d \geq 2$, let us consider two domains

$$\mathcal{D}_1 = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; x_d < 0\} \ \ and \ \ \mathcal{D}_2 = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; x_d > 0\},$$

and the hyperplane

$$S = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d; x_d = 0\}.$$

In this paragraph, we will study the second-order parabolic partial differential equation

$$\left(L - \frac{\partial}{\partial t}\right) u \equiv 0, \tag{4}$$

where L is defined by

$$Lu(x,t) = C \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2}(x,t) + 2Cq\delta_S(x)\nabla_x u, \tag{5}$$

C is a strictly positive constant, q is some constant such that $\mid q \mid \leq 1$, $\delta_S(x)$ is a generalized function satisfying (3).

Here, the operator L is generalized in the sense that for all couple of functions u and g, of class $\mathcal{C}^\infty$ and with compact supports,

$$\int_{\mathbb{R}^d} Lu(x)g(x)dx = C \int_{\mathbb{R}^d} \Delta u(x)g(x)dx + 2C \int_S g(x)q(x)\nabla u(x)\delta_S(dx).$$

One fundamental interest in our study is the expression of the fundamental solution associated to the SPDE (1). The following result is due to [7].

Theorem 1. *There exists a unique fundamental solution* $p(x-y,t,s)=p(s,x,t,y)$ *of the partial differential equation* $Lu=\frac{\partial u}{\partial t}$*. It can be explicitly expressed as, for* $x \in \mathcal{E}, 0 \leqslant s \leqslant t$

$$\begin{aligned} p(x-y,t,s) = \frac{1}{(4\pi C(t-s))^{d/2}} \Bigg[& \exp\left(-\frac{\mid x-y \mid^2}{4C(t-s)}\right) \\ & + sign(y) q \exp\left(-\frac{(\mid x-\tilde{x} \mid + \mid y-\tilde{y} \mid)^2 + \mid \tilde{x}-\tilde{y} \mid^2}{4C(t-s)}\right)\Bigg] \end{aligned} \tag{6}$$

where $\tilde{x}$ *denotes the orthogonal projection of* x *on* S*, and* $sign(y)$ $= \begin{cases} +1 \ if \ y \in \mathcal{D}_1 \\ -1 \ if \ y \in \mathcal{D}_2. \end{cases}$

2.2. The random noise

Let R be a covariance function on $[0,T]^2$. It defines naturally a finite additive measure $\mu_R := \mu$ on the algebra $\mathcal{R}$ of finite disjoint rectangles included in the set $[0,T]^2$ by

$$\mu(I) = \Delta_I R$$

where $\Delta_I R$ denotes the rectangular increment of R over the rectangle I given by

$$\Delta_I R = R(b_1,b_2) - R(a_1,b_2) - R(a_2,b_1) + R(a_1,a_2)$$

if $I = [a_1,b_1) \times [a_2,b_2)$.

We will say that the process $(X_t)_{t\in[0,T]}$ generates a covariance measure if μ can be extended to a signed sigma-finite measure on $\mathcal{B}([0,T]^2)$. An example of a class of processes with covariance measure structure is the set of processes whose covariance satisfies

$$\frac{\partial^2 R}{\partial s \partial t} \in L^1([0,T]^2).$$

In this case the measure μ generated by R admits a density with respect to the Lebesgue measure on $[0,T]^2$ given by $\frac{\partial^2 R}{\partial s\partial t}$.

We give few examples of processes with covariance measure structure.

Example **2.** Suppose that X is a fractional Brownian motion B^H with Hurst parameter $H \in (\frac{1}{2}, 1)$. That is, X is a centered Gaussian process with covariance

$$\mathbf{E} X_t X_s = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$$

for every $s,t \in [0,T]$. Then X admits a covariance measure structure on $[0,T]^2$ which has a density given by

$$H(2H-1)|t-s|^{2H-2}.$$

Example **3.** If X is a Wiener process, then X defines a covariance measure μ given by $\mu(du,dv) = \delta_0(u-v)dudv$, where δ_0 is the Dirac measure.

Example **4.** Let us denote by $M^H = \{M_t^H(a,b); t \geq 0\} = \{M_t^H; t \geq 0\}$ the mixed-fractional Brownian motion (mfBm, see [13]) of parameters a, b and H such that $0 < H < 1$, $(a,b) \in \mathbb{R}^2 \backslash \{(0,0)\}$; that is the centered Gaussian process, starting from zero, with covariance

$$R^{H,a,b}(t,s) := R(t,s) = a^2(t \wedge s) + \frac{b^2}{2}\Big(t^{2H} + s^{2H} - \mid t-s \mid^{2H}\Big), \tag{7}$$

where $t \wedge s = \frac{1}{2}\Big(t + s - \mid t-s \mid\Big)$. When $a=0$ and $b=1$, $M^H(0,1) = B^H$ is a fractional Brownian motion. When $a=1$ and $b=0$, $M^H(1,0) = B$ is a Brownian motion. So the mfBm is clearly an extension of the fractional Brownian motion and of the Wiener process. We refer to [13] for further information on this process.

If $(a,b,H) \in \mathbb{R}\times\mathbb{R}^\star\times\left(\dfrac{1}{2},1\right)$ or $(a,b) \in \mathbb{R}^\star\times\{0\}$, $M^H(a,b)$ admits a covariance structure on $[0,T]^2$ which has a density given by

$$a^2\delta_0(u-v) + b^2H(2H-1)|u-v|^{2H-2},$$

where δ_0 is the Dirac measure and $\mathbb{R}^\star$ is the set $\mathbb{R}\backslash\{0\}$.

Example **5.** The bifractional Brownian motion $(B^{H,K}t)_{t\geq 0}$ is a centered Gaussian process, starting from zero, with covariance

$$R^{H,K}(t,s) := R(t,s) = \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - \mid t-s \mid^{2HK}\right) \tag{8}$$

with $H \in (0,1)$ and $K \in (0,1]$.

Clearly $B^{H,1}$ is a fractional Brownian motion with Hurst parameter $H \in (0,1)$. The bifractional Brownian motion admits the following decomposition (see [4]).

Note that the covariance of the bifractional Brownian motion can be written as

$$\begin{aligned} R^{H,K}(t,s) &= \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - t^{2HK} - s^{2HK}\right) \\ &\quad + \frac{1}{2^K}(t^{2HK}+s^{2HK}-|t-s|^{2HK}) \\ &:= R^{H,K}_{(1)}(t,s) + R^{H,K}_{(2)}(t,s). \end{aligned} \tag{9}$$

The function denoted by $R^{H,K}_{(2)}$ is, modulo a constant, the covariance of the fractional Brownian motion with Hurst parameter HK. The function $-R^{H,K}_{(1)}$ is also a covariance function, see [3] or [12]. Actually, we have

$$\frac{\partial^2 R^{H,K}_{(1)}}{\partial t\partial s}(t,s) = \frac{1}{2^K}(2H)^2K(K-1)(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1} \tag{10}$$

and

$$\frac{\partial^2 R^{H,K}_{(2)}}{\partial t\partial s}(t,s) = 2^{-K}2HK(2HK-1)|t-s|^{2HK-2} \tag{11}$$

for every $s,t \in [0,T]$. It has been proved in [3] and [12] that $\frac{\partial^2 R^{H,K}_{(1)}}{\partial t\partial s}$, $\frac{\partial^2 R^{H,K}_{(2)}}{\partial t\partial s}$ are integrable over $[0,T]^2$ when $2HK \geqslant 1$ and thus the bifractional Brownian motion generates a covariance measure if $2HK \geqslant 1$.

Let us now introduce the random noise that drives the parabolic equation (16). On a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$, we consider a zero-mean stochastic process $W = \{W(t,A); t\in[0,T], A\in\mathcal{B}_b(\mathbb{R}^d)\}$ with covariance:

$$\mathbf{E}(W(t,A)W(s,B)) = R(t,s)\lambda(A\cap B) \tag{12}$$

where λ is the Lebesgue measure, and R is the covariance of a stochastic process that generates a covariance measure.

To the Gaussian field W we can associate a Hilbert space that will be called the canonical Hilbert space of W and will be denoted by $\mathcal{H}$. Consider $\mathcal{E}$ the set of linear combinations of elementary functions $\mathbf{1}_{[0,t]}\times A$, $t\in[0,T]$,

$A \in \mathcal{B}_b(\mathbb{R}^d)$, and let $\mathcal{H}$ be the Hilbert space defined as the closure of $\mathcal{E}$ with respect to the inner product

$$< \mathbf{1}_{[0,t]} \times A, \mathbf{1}_{[0,s]} \times B >_{\mathcal{H}} := \mathbf{E}(W(t,A)W(s,B)).$$

We have the following expression of the scalar product in $\mathcal{H}$:

$$< g, h >_{\mathcal{H}} = \int_0^T \int_0^T \mu(du, dv) \int_{\mathbb{R}^d} dy g(y,u)h(y,v) \tag{13}$$

which holds in particular for any $f, g \in \mathcal{H}$ such that

$$\int_0^T \int_0^T |\mu|(du, dv) \int_{\mathbb{R}^d} dy |g(y,u)||h(y,v)| < \infty. \tag{14}$$

It is possible to define Wiener integrals with respect to the process W whose covariance is given by (12). This Wiener integral will act as an isometry between the Hilbert space $\mathcal{H}$ and $L^2(\Omega)$ in the sense that

$$\begin{aligned} &\mathbf{E} \int_0^T \int_{\mathbb{R}^d} \varphi(u,y) W(du,dy) \int_0^T \int_{\mathbb{R}^d} \psi(u,y) W(du,dy) \\ &\quad = \int_0^T \int_0^T \mu(du,dv) \int_{\mathbb{R}^d} dy \varphi(u,y)\psi(v,y) \end{aligned} \tag{15}$$

for any function φ, ψ satisfying (14).

3. Stochastic PDE with generalized drift: existence of the solution

In this paragraph we will consider the stochastic counterpart of PDE (4). Namely, we discuss the stochastic partial differential equation with generalized drift

$$Lu(t,x) = \dot{W}(t,x), \quad t \in [0,T],\ x \in \mathbb{R}^d \tag{16}$$

with vanishing initial condition $u(0,x) = 0$ for every $x \in \mathbb{R}^d$.

The notion of solution to (16) is defined in the mild sense. We call a mild solution to (16) the stochastic process

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} p(t-s, x-y) W(ds, dy), \quad t \in [0,T],\ x \in \mathbb{R}^d \tag{17}$$

where W is the Gaussian noise with covariance given by (12), p denotes the fundamental solution (6) and the integral in (17) is a Wiener integral with respect to the Gaussian noise W. The solution exists when this Wiener integral is well-defined.

Let us now give the necessary condition for the existence of the mild solution (17). Recall that μ is the measure generates by R from (12) and by $|\mu|$ we denote its variation measure. For $z \in \mathbb{R}$ we introduce the complementary error function

$$\mathbf{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{+\infty} e^{-u^2} du.$$

Proposition 6. *If*

$$\int_0^t \int_0^t |\mu|(du, dv)(2t-u-v)^{-d/2} < +\infty \tag{18}$$

for every $t \in [0,T]$, then the mild solution (17) is well-defined for every $t \in [0,T]$ and $x \in \mathbb{R}^d$. Moreover, its covariance is given by

$$\begin{aligned}\mathbf{E}(u(t,x)u(s,x)) = \frac{1}{(4\pi C)^{d/2}} \int_0^t \int_0^s \mu(du, dv)(t+s-u-v)^{-d/2} \\ \times \Bigg[1 + \Big(q^2 - qsign(x_d)\Big) \mathbf{erfc}\left(\frac{|\, x_d \,| \sqrt{t+s-u-v}}{\sqrt{4C(t-u)(s-v)}} \right) \\ + qsign(x_d) \exp\left(- \frac{x_d^2}{C(t-u+s-v)} \right) \Bigg] \end{aligned} \tag{19}$$

for every $x \in \mathbb{R}^d$ and for every $s,t \in [0,T]$.

Proof. Let p be the fundamental solution (6). We will decompose it as follows

$$p(t, x-y) = p_1(t, x-y) + qsign(y) p_2(t, x-y)$$

where

$$p_1(t, x-y) = \frac{1}{(4\pi Ct)^{d/2}} \exp\left(- \frac{|\, x-y \,|^2}{4C(t-s)} \right)$$

and

$$p_2(t, x-y) = \frac{1}{(4\pi Ct)^{d/2}} \exp\left(- \frac{(|\, x-\tilde{x} \,| + |\, y-\tilde{y} \,|)^2 + |\, \tilde{x}-\tilde{y} \,|^2}{4Ct} \right). \tag{20}$$

Note that p_1 above is the fundamental solution of the heat equation.

The solution (17) exists when the function $p(t-\cdot, x-\cdot)$ belongs to the Hilbert space $\mathcal{H}$ from Sect. 2.2 since $\mathbf{E}u(t,x)^2 = \|p(t-\cdot, x-\cdot)\|_{\mathcal{H}}^2$.

Fix $s,t \in [0,T]$ and $x \in \mathbb{R}^d$. We will calculate

$$\mathbf{E}u(t,x)u(s,x) = \langle p(t-\cdot, x-\cdot), p(s-\cdot, x-\cdot)\rangle_{\mathcal{H}}$$

with the scalar product in $\mathcal{H}$ given by (13). So,

$$\begin{aligned}\mathbf{E}(u(t,x)u(s,x)) &= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_1(t-u, x-y) p_1(s-v, x-y) \\ &+ q \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_1(t-u, x-y) sign(y) p_2(s-v, x-y) \\ &+ q \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy sign(y) p_2(t-u, x-y) p_1(s-v, x-y) \\ &+ q^2 \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_2(t-u, x-y) p_2(s-v, x-y) \\ &:= A_1(t,s,x) + qA_2(t,s,x) + qA_3(t,s,x) + q^2 A_4(t,s,x). \end{aligned} \tag{21}$$

Let us start by calculating the first term denoted by $A_1(t,s,x)$. Since p_1 is the Green associated to the heat equation, the computation of A_1 follows closely the computation in Chapter 1 in [11]. We will have

$$\begin{aligned}
A_1(t,s,x) &= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_1(t-u,x-y) p_1(s-v,x-y) \\
&= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} \frac{1}{(4\pi C(t-u))^{d/2}} \exp\left(- \frac{\mid x-y \mid^2}{4C(t-u)} \right) \\
&\quad \times \frac{1}{(4\pi C(s-v))^{d/2}} \exp\left(- \frac{\mid x-y \mid^2}{4C(s-v)} \right) \\
&= \int_0^t \int_0^s \mu(du,dv) \frac{1}{(4\pi C(t-u))^{d/2}} \frac{1}{(4\pi C(s-v))^{d/2}} \times I_1(x,u,v),
\end{aligned}$$

where we used the notation

$$I_1(x,u,v) = \int_{\mathbb{R}^d} dy \exp\left(- \frac{\mid x-y \mid^2}{4C(t-u)} \right) \exp\left(- \frac{\mid x-y \mid^2}{4C(s-v)} \right).$$

By using the change of variables

$$z_i = \frac{x_i - y_i}{\sqrt{2C}} \sqrt{\frac{t+s-u-v}{(t-u)(s-v)}}, \qquad i = 1, \dots, d$$

it is easy to check that

$$\begin{aligned}
I_1(x,u,v) &= \prod_{i=1}^d \int_{\mathbb{R}} \exp\left(- \frac{t+s-u-v}{4C(t-u)(s-v)} \right) (x_i - y_i)^2 dy_i \\
&= \left(\frac{t+s-u-v}{4C\pi(t-u)(s-v)} \right)^{-d/2}.
\end{aligned}$$

So,

$$A_1(t,s,x) = \frac{1}{(4\pi C)^{d/2}} \int_0^t \int_0^s \mu(du,dv)(t+s-u-v)^{-d/2}. \tag{22}$$

The terms denoted by A_2 and A_3 are symmetric. We will calculate A_2. We can write

$$\begin{aligned}
A_2(t,s,x) &= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_1(t-u,x-y) sign(y) p_2(s-v,x-y) \\
&= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} \frac{1}{(4\pi C(t-u))^{d/2}} \exp\left(- \frac{\mid x-y \mid^2}{4C(t-u)} \right) \\
&\quad \times sign(y) \frac{1}{(4\pi C(s-v))^{d/2}} \\
&\quad \times \exp\left(- \frac{(\mid x-\tilde{x} \mid + \mid y-\tilde{y} \mid)^2 + \mid \tilde{x}-\tilde{y} \mid^2}{4C(s-v)} \right) dy
\end{aligned}$$

$$= \int_0^t \int_0^s \mu(du,dv) \frac{1}{(4\pi C(t-u))^{d/2}} \frac{1}{(4\pi C(s-v))^{d/2}} \times I_2(\tilde{x},u,v) \times I_2(x_d,u,v),$$

with

$$I_2(\tilde{x},u,v) = \int_{\mathbb{R}^{d-1}} d\tilde{y} \exp\left(- \frac{|\tilde{x}-\tilde{y}|^2}{4C(t-u)} \right) \exp\left(- \frac{|\tilde{x}-\tilde{y}|^2}{4C(s-v)} \right)$$

and

$$I_2(x_d,u,v) = \int_{\mathbb{R}} \exp\left(- \frac{|x_d-y_d|^2}{4C(t-u)} \right) sign(y_d) \exp\left(- \frac{(|x_d|+|y_d|)^2}{4C(s-v)} \right) dy_d.$$

Exactly as in the calculation of $I_1(\tilde{x},u,v)$ we get:

$$I_2(\tilde{x},u,v) = \left(\frac{t+s-u-v}{4C\pi(t-u)(s-v)} \right)^{-(d-1)/2}.$$

To calculate $I_2(x_d,u,v)$, we start by writing:

$$I_2(x_d,u,v) = \int_0^{+\infty} \exp\left(- \frac{|x_d-y_d|^2}{4C(t-u)} \right) \exp\left(- \frac{(|x_d|+y_d)^2}{4C(s-v)} \right) dy_d$$
$$- \int_{-\infty}^0 \exp\left(- \frac{|x_d-y_d|^2}{4C(t-u)} \right) \exp\left(- \frac{(|x_d|-y_d)^2}{4C(s-v)} \right) dy_d := T_1 - T_2.$$

We discuss separately the cases $x_d \geqslant 0$ and $x_d < 0$.

If $x_d < 0$, then

$$T_1 = \int_0^{+\infty} \exp\left(- \frac{(x_d-y_d)^2}{4C(t-u)} \right) \exp\left(- \frac{(x_d-y_d)^2}{4C(s-v)} \right) dy_d$$
$$= \int_0^{+\infty} \exp\left(- B(x_d-y_d)^2 \right) dy_d$$

where $B = \dfrac{t+s-u-v}{4C(t-u)(s-v)}$. By the change of variable: $u = \sqrt{B}(y_d - x_d)$ we get,

$$T_1 = \frac{1}{\sqrt{B}} \int_{-x_d\sqrt{B}}^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2\sqrt{B}} \mathbf{erfc}(-x_d\sqrt{B}).$$

Let us calculate T_2. We get

$$T_2 = \int_{-\infty}^0 \exp\left(- \frac{(x_d-y_d)^2}{4C(t-u)} \right) \exp\left(- \frac{(x_d+y_d)^2}{4C(s-v)} \right) dy_d$$
$$= \int_0^{+\infty} \exp\left(- \frac{(x_d+y_d)^2}{4C(t-u)} \right) \exp\left(- \frac{(x_d-y_d)^2}{4C(s-v)} \right) dy_d$$
$$= \exp(-x_d^2 B) \int_0^{+\infty} \exp\left(- By_d^2 + C'y_d \right) dy_d$$

where $C' = \dfrac{x_d(t-u+v-s)}{2C(t-u)(s-v)}$. So,

$$T_2 = \exp(-x_d^2 B) \exp\left(\frac{C'^2}{4B}\right) \int_0^{+\infty} \exp\left(-\left(y_d\sqrt{B} - \frac{C'}{2\sqrt{B}}\right)^2\right) dy_d$$

Now, by the change of variable $u = y_d\sqrt{B} - \dfrac{C'}{2\sqrt{B}}$ we get

$$T_2 = \exp(-x_d^2 B) \exp\left(\frac{C'^2}{4B}\right) \frac{1}{\sqrt{B}} \int_{-\frac{C'}{2\sqrt{B}}} e^{-u^2} du = \exp(-x_d^2 B) \times \exp\left(\frac{C'^2}{4B}\right) \frac{\sqrt{\pi}}{2\sqrt{B}} \mathbf{erfc}\left(-\frac{C'}{2\sqrt{B}}\right).$$

We deduce that

$$\begin{aligned} I_2(x_d,u,v) &= \frac{\sqrt{\pi}}{2\sqrt{B}} \left[\mathbf{erfc}(-x_d\sqrt{B}) - \exp(-x_d^2 B) \exp\left(\frac{C'^2}{4B}\right) \mathbf{erfc}\left(-\frac{C'}{2\sqrt{B}}\right)\right] \\ &= \sqrt{\frac{\pi C(t-u)(s-v)}{t+s-u-v}} \left[\mathbf{erfc}\left(-x_d\sqrt{\frac{t+s-u-v}{4C(t-u)(s-v)}}\right)\right. \\ &\quad - \mathbf{erfc}\left(-\frac{x_d(t-u+v-s)}{2\sqrt{C(t-u)(s-v)(t+s-u-v)}}\right) \\ &\quad \times \exp\left(-x_d^2 \frac{t+s-u-v}{4C(t-u)(s-v)}\right) \\ &\quad \left.\times \exp\left(\frac{x_d^2(t-u+v-s)^2}{4C(t-u)(s-v)(t+s-u-v)}\right)\right] \\ &= \sqrt{\frac{\pi C(t-u)(s-v)}{t+s-u-v}} \left[\mathbf{erfc}\left(-x_d\sqrt{\frac{t+s-u-v}{4C(t-u)(s-v)}}\right)\right. \\ &\quad - \mathbf{erfc}\left(-\frac{x_d(t-u+v-s)}{2\sqrt{C(t-u)(s-v)(t+s-u-v)}}\right) \\ &\quad \left.\times \exp\left(-x_d^2 \frac{1}{C(t+s-u-v))}\right)\right]. \end{aligned}$$

If $x_d \geq 0$, then

$$\begin{aligned} T_1 &= \int_0^{+\infty} \exp\left(-\frac{(x_d-y_d)^2}{4C(t-u)}\right) \exp\left(-\frac{(x_d+y_d)^2}{4C(s-v)}\right) dy_d \\ &= \exp(-x_d^2 B) \int_0^{+\infty} \exp\left(-By_d^2 - C'y_d\right) dy_d \end{aligned}$$

where C' and B are the same as those of the case $x_d < 0$.

It is clear that T_1 has the same form as that of T_2 of the case $x_d < 0$. The only difference is that here, we have $-C'$ in place of C'. So,

$$T_1 = \exp(-x_d^2 B) \exp\left(\frac{C'^2}{4B}\right) \frac{1}{\sqrt{B}} \int_{\frac{C'}{2\sqrt{B}}} e^{-u^2} du = \exp(-x_d^2 B) \\ \times \exp\left(\frac{C'^2}{4B}\right) \frac{\sqrt{\pi}}{2\sqrt{B}} \mathbf{erfc}\left(\frac{C'}{2\sqrt{B}}\right).$$

Concerning T_2,

$$T_2 = \int_{-\infty}^{0} \exp\left(-\frac{(x_d - y_d)^2}{4C(t-u)}\right) \exp\left(-\frac{(x_d - y_d)^2}{4C(s-v)}\right) dy_d \\ = \int_{0}^{+\infty} \exp\left(-\frac{(x_d + y_d)^2}{4C(t-u)}\right) \exp\left(-\frac{(x_d + y_d)^2}{4C(s-v)}\right) dy_d,$$

and we see that T_2 has the same form as that of T_1 of the case $x_d < 0$. The only difference is that here, we have x_d in place of $-x_d$. So,

$$T_2 = \frac{1}{\sqrt{B}} \int_{x_d\sqrt{B}}^{+\infty} e^{-u^2} du = \frac{\sqrt{\pi}}{2\sqrt{B}} \mathbf{erfc}(x_d\sqrt{B}).$$

We conclude that

$$\begin{aligned} I_2(x_d, u, v) &= \frac{\sqrt{\pi}}{2\sqrt{B}} \left[\exp(-x_d^2 B) \exp\left(\frac{C'^2}{4B}\right) \mathbf{erfc}\left(\frac{C'}{2\sqrt{B}}\right) - \mathbf{erfc}(x_d\sqrt{B}) \right] \\ &= \sqrt{\frac{\pi C(t-u)(s-v)}{t+s-u-v}} \Bigg[-\mathbf{erfc}\left(x_d\sqrt{\frac{t+s-u-v}{4C(t-u)(s-v)}}\right) \\ &\quad + \mathbf{erfc}\Big(\frac{x_d(t-u+v-s)}{2\sqrt{C(t-u)(s-v)(t+s-u-v)}}\Big) \\ &\quad \times \exp\left(-x_d^2 \frac{t+s-u-v}{4C(t-u)(s-v)}\right) \\ &\quad \times \exp\left(\frac{x_d^2 (t-u+v-s)^2}{4C(t-u)(s-v)(t+s-u-v)}\right) \Bigg] \\ &= \sqrt{\frac{\pi C(t-u)(s-v)}{t+s-u-v}} \Bigg[-\mathbf{erfc}\left(x_d\sqrt{\frac{t+s-u-v}{4C(t-u)(s-v)}}\right) \\ &\quad + \mathbf{erfc}\Big(\frac{x_d(t-u+v-s)}{2\sqrt{C(t-u)(s-v)(t+s-u-v)}}\Big) \\ &\quad \times \exp\left(-x_d^2 \frac{1}{C(t+s-u-v))}\right) \Bigg]. \end{aligned}$$

Hence,

$$
\begin{aligned}
A_2(t,s,x) &= \frac{2^{-1}}{(4C\pi)^{d/2}} \int_0^t \int_0^s \mu(du,dv)(t+s-u-v)^{-d/2} sign(x_d) \\
&\quad \times \Bigg[-\mathbf{erfc}\left(\frac{|\, x_d \,| \sqrt{t+s-u-v}}{\sqrt{4C(t-u)(s-v)}} \right) + \exp\Big(- \frac{x_d^2}{C(t-u+s-v)} \Big) \\
&\quad \times \mathbf{erfc}\left(\frac{|\, x_d \,| \, (t-u+v-s)}{\sqrt{4C(t-u+s-v)(t-u)(s-v)}} \right) \Bigg]
\end{aligned} \tag{23}
$$

Similarly, we obtain the expression of the term $A_3(t,s,x)$

$$
\begin{aligned}
A_3(t,s,x) &= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_2(t-u,x-y) sign(y) p_1(s-v,x-y) \\
&= \frac{2^{-1}}{(4C\pi)^{d/2}} \int_0^t \int_0^s \mu(du,dv)(t+s-u-v)^{-d/2} sign(x_d) \\
&\quad \times \Bigg[-\mathbf{erfc}\left(\frac{|\, x_d \,| \sqrt{t+s-u-v}}{\sqrt{4C(t-u)(s-v)}} \right) + \exp\Big(- \frac{x_d^2}{C(t-u+s-v)} \Big) \\
&\quad \times \mathbf{erfc}\left(\frac{|\, x_d \,| \, (s-v+u-t)}{\sqrt{4C(t-u+s-v)(t-u)(s-v)}} \right) \Bigg]
\end{aligned} \tag{24}
$$

Concerning the last summand $A_4(t,s,x)$,

$$
\begin{aligned}
A_4(t,s,x) &= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} dy p_2(t-u,x-y) p_2(s-v,x-y) \\
&= \int_0^t \int_0^s \mu(du,dv) \int_{\mathbb{R}^d} \frac{1}{(4\pi C(t-u))^{d/2}} \\
&\quad \times \exp\left(- \frac{(|\, x-\tilde{x} \,| + |\, y-\tilde{y} \,|)^2 + |\, \tilde{x}-\tilde{y} \,|^2}{4C(t-u)} \right) \\
&\quad \times \frac{1}{(4\pi C(s-v))^{d/2}} \exp\left(- \frac{(|\, x-\tilde{x} \,| + |\, y-\tilde{y} \,|)^2 + |\, \tilde{x}-\tilde{y} \,|^2}{4C(s-v)} \right) dy \\
&= \int_0^t \int_0^s \mu(du,dv) \frac{1}{(4\pi C(t-u))^{d/2}} \frac{1}{(4\pi C(s-v))^{d/2}} \\
&\quad \times I_4(\tilde{x},u,v) \times I_4(x_d,u,v),
\end{aligned}
$$

with

$$
\begin{aligned}
I_4(\tilde{x},u,v) &= \int_{\mathbb{R}^{d-1}} d\tilde{y} \exp\left(- \frac{|\, \tilde{x}-\tilde{y} \,|^2}{4C(t-u)} \right) \exp\left(- \frac{|\, \tilde{x}-\tilde{y} \,|^2}{4C(s-v)} \right) \\
&= I_2(\tilde{x},u,v) = \Big(\frac{t+s-u-v}{4C\pi(t-u)(s-v)} \Big)^{-(d-1)/2} .+
\end{aligned}
$$

and

$$\begin{aligned} I_4(x_d,u,v) &= \int_{\mathbb{R}} \exp\Big(-\frac{(\mid x_d \mid + \mid y_d \mid)^2}{4C(t-u)}\Big)\exp\Big(-\frac{(\mid x_d \mid + \mid y_d \mid)^2}{4C(s-v)}\Big)dy_d \\ &= 2\int_0^{+\infty} \exp\Bigg(-\frac{(\mid x_d \mid + y_d)^2(s-v+t-u)}{4C(t-u)(s-v)}\Bigg)dy_d \\ &= \frac{\sqrt{4C\pi(s-v)(t-u)}}{\sqrt{s-v+t-u}}\mathbf{erfc}\Bigg(\mid x_d \mid \frac{\sqrt{s-v+t-u}}{\sqrt{4C(s-v)(t-u)}}\Bigg) \end{aligned}$$

Then,

$$\begin{aligned} A_4(t,s,x) = \frac{1}{(4\pi C)^{d/2}} &\int_0^t\int_0^s \mu(du,dv)(t+s-u-v)^{-d/2} \\ &\times \mathbf{erfc}\Bigg(\mid x_d \mid \frac{\sqrt{s-v+t-u}}{\sqrt{4C(s-v)(t-u)}}\Bigg). \end{aligned} \tag{25}$$

By Eqs. (21), (22), (23), (24) and (25) we get the formula of the covariance (19). By taking $s=t$, we have

$$\begin{aligned} \mathbf{E}(u(t,x)^2) = \frac{1}{(4\pi C)^{d/2}} &\int_0^t\int_0^t \mu(du,dv)(2t-u-v)^{-d/2} \\ &\times\Bigg[1+\Big(q^2-q sign(x_d)\Big)\mathbf{erfc}\Bigg(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\Bigg) \\ &+ q sign(x_d)\exp\Bigg(-\frac{x_d^2}{C(2t-u-v)}\Bigg)\Bigg]. \end{aligned} \tag{26}$$

By majorizing the complementary error function by $\sqrt{2}$ and the exponential in (26), by 1, we easily get

$$\mathbf{E}u(t,x)^2 \leqslant c\int_0^t\int_0^t |\mu|(du,dv)(2t-u-v)^{-d/2}$$

and this implies that $u(t,x)$ is well-defined as a random variable in $L^2(\Omega)$ under condition (18). We obtain the conclusion of Proposition 6. $\square$

Remark 7. Relation (19) shows that the covariance of the solution in time depends on the space variable $x \in \mathbb{R}^d$ through its last component. So this solution is not stationary with respect to the space variable (as happens in the case of the heat equation, i.e. $q=0$). This is due to the "part" p_2 (20) of the fundamental solution.

When the covariance R from (12) of the random noise with respect to the time variable generates a positive measure μ, it is possible to obtain a necessary and sufficient condition for the existence of the mild solution (17).

Theorem 8. *Assume that μ is a positive measure. Then the solution $(u(t,x), t \in [0,T], x \in \mathbb{R}^d)$ is well-defined and satisfies*

$$\sup_{t\in[0,T]} \mathbf{E}u(t,x)^2 < \infty \tag{27}$$

if and only if

$$\int_0^t \int_0^t \mu(du, dv)(2t-u-v)^{-\frac{d}{2}} < \infty \tag{28}$$

for every $t \in [0,T]$.

Proof. The implication (28) $\Longrightarrow$ (27) is an immediate consequence of the proof of Proposition 6.

If $-1 < q < 0$, then for every $t \in [0,T]$ and $x \in \mathcal{D}_2$, we have $\mathbf{E}u(t,x)^2 < \infty$. On the other hand, by using the formula (26)

$$\begin{aligned}\infty > \mathbf{E}u(t,x)^2 &= c\int_0^t \int_0^t \mu(du, dv)(2t-u-v)^{-d/2} \\ &\quad \times \left(1 + (q^2-q)\mathbf{erfc}\left(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\right)\right. \\ &\quad \left. + q\exp\left(-\frac{x_d^2}{C(2t-u-v)}\right)\right) \\ &\geqslant c\int_0^t \int_0^t \mu(du, dv)(2t-u-v)^{-d/2}.\end{aligned}$$

Thus, relation (28) is obtained.

If $0 \le q < 1$, then for every $t \in [0,T]$ and $x \in \mathcal{D}_1$, we have $\mathbf{E}u(t,x)^2 < \infty$. On the other hand, by using the formula (26)

$$\begin{aligned}\infty > \mathbf{E}u(t,x)^2 &= c\int_0^t \int_0^t \mu(du, dv)(2t-u-v)^{-d/2} \\ &\quad \times \left(1 + (q^2+q)\mathbf{erfc}\left(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\right)\right. \\ &\quad \left. -q\exp\left(-\frac{x_d^2}{C(2t-u-v)}\right)\right) \\ &= c\int_0^t \int_0^t \mu(du, dv)(2t-u-v)^{-d/2} \\ &\quad \times \left(1 - q\left[(-q-1)\mathbf{erfc}\left(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\right)\right.\right. \\ &\quad \left.\left. + \exp\left(-\frac{x_d^2}{C(2t-u-v)}\right)\right]\right).\end{aligned}$$

Since $-q-1 < 0$ we have

$$\begin{aligned}&(-q-1)\mathbf{erfc}\left(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\right) + \exp\left(-\frac{x_d^2}{C(2t-u-v)}\right) \\ &\quad \le \exp\left(-\frac{x_d^2}{C(2t-u-v)}\right) \le 1.\end{aligned}$$

Then, since $q > 0$,

$$-q\left[(-q-1)\mathbf{erfc}\left(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}}\right) + \exp\left(-\frac{x_d^2}{C(2t-u-v)}\right)\right] \geq -q,$$

and consequently

$$\infty > \mathbf{E}u(t,x)^2 \geqslant c(1-q)\int_0^t\int_0^t \mu(du,dv)(2t-u-v)^{-d/2}.$$

So here also, relation (28) is obtained. □

Remark 9. The condition (28) extends the findings in [2], [1] or [12] for the cases of Brownian, fractional Brownian or bifractional Brownian motion.

4. Examples

We obtained in Theorem 8 an "iff" condition for the existence of the solution to the SPDE with generalized drift (16) by assuming that the covariance measure of the noise with respect to the time variable is positive.

The processes X such that their covariance R satisfies $\frac{\partial^2 R}{\partial s \partial t} \in L^1([0,T]^2)$ and $\frac{\partial^2 R(t,s)}{\partial s \partial t} \geqslant 0$ for every $s,t \in [0,T]$ generate a positive covariance measure μ. The typical example is the fractional Brownian motion with Hurst index bigger than one half.

Example **10.** Suppose that the noise is a fractional Brownian motion with respect to the time variable with $H > \frac{1}{2}$. Then its covariance (see Example 2) $\frac{\partial^2 R(t,s)}{\partial s \partial t} = H(2H-1)|t-s|^{2H-2}$ is positive and integrable. By Theorem 8, the solution exists and satisfies (27) if and only if for every $t \in [0,T]$,

$$\int_0^t\int_0^t |u-v|^{2H-2}(2t-u-v)^{-\frac{d}{2}}dudv < \infty \tag{29}$$

which is equivalent (see [1] or [11])

$$d < 4H.$$

Consequently the mild solution u exists in dimension $1, 2$ or 3. Notice that the same condition (29) appears in the case of the heat equation with white noise in space and fractional noise in time.

Let us discuss now the cases of the Wiener process and of the mixed-fractional Brownian motion. These processes generate positive covariance measures.

Example **11.** The Wiener process also generates a positive measure since $\mu(du,dv) = \delta_0(u-v)dudv$ (see Example 3). In this case (28) becomes

$$\int_0^t (2t-2u)^{-\frac{d}{2}}du < \infty$$

which is equivalent to $d = 1$.

Example **12.** Theorem 8 can be also applied to the mixed-fractional Brownian motion from Example 4. In this case, the solution exists if and only if

$$\int_0^t (t-u)^{-\frac{d}{2}} du + \int_0^t \int_0^t |u-v|^{2H-2}(2t-u-v)^{-\frac{d}{2}} dudv < \infty$$

which is true if and only if $d = 1$.

The covariance measure generated by the bifractional Brownian motion is not necessarily positive, therefore Theorem 8 cannot be applied to it. Nevertheless, from the calculations contained in the proof of Proposition 6, we can deduce an "iff"condition for the existence of the solution.

Example **13.** If the noise W from Sect. 2.2 behaves in time as a bifractional Brownian motion with $2HK > 1$, then the covariance measure μ generated by $R^{H,K}$ exists and it admits a density with respect to the Lebesgue measure on $[0,T]^2$ given by

$$\frac{\partial^2 R^{H,K}}{\partial s \partial t}(s,t) = 2^{-K}(2HK) \Big(2H(K-1)(t^{2H}+s^{2H})^{K-2}(ts)^{2H-1} + (2HK-1) \mid t-s \mid^{2HK-2}\Big), \tag{30}$$

where in the last equation we have used (9), (10) and (11). From (30) we see that μ is not necessarily positive. But using the proof of the above theorem 8 and Proposition 4.2 in [12], we can prove the necessary and sufficient condition for the existence of the solution.

Indeed, by using (30), the formula (26) for squared mean of $u(t,x)$ becomes

$$\begin{aligned}
\mathbf{E}(u(t,x)^2) &= \frac{2^{-K}2HK}{(4\pi C)^{d/2}} \Bigg((2HK-1) \int_0^t \int_0^t dudv |u-v|^{2HK-2}(2t-u-v)^{-d/2} \\
&\quad \times \Bigg[1 + \Big(q^2 - qsign(x_d)\Big) \mathbf{erfc}\Bigg(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}} \Bigg) \\
&\quad + qsign(x_d) \exp\Big(- \frac{x_d^2}{C(2t-u-v)} \Big) \Bigg] \\
&\quad - 2H(1-K) \int_0^t \int_0^t dudv (u^{2H}+v^{2K})^{K-2}(uv)^{2H-1}(2t-u-v)^{-d/2} \\
&\quad \times \Bigg[1 + \Big(q^2 - qsign(x_d)\Big) \mathbf{erfc}\Bigg(\frac{\mid x_d \mid \sqrt{2t-u-v}}{\sqrt{4C(t-u)(t-v)}} \Bigg) \\
&\quad + qsign(x_d) \exp\Big(- \frac{x_d^2}{C(2t-u-v)} \Big) \Bigg] \Bigg) \\
&:= T_1 - T_2.
\end{aligned}$$

The conclusion will follows from the following facts: when $d < 4HK$ we can show that both T_1 and T_2 are finite and thus $\mathbf{E}(u(t,x)^2)$ is finite. When $d \geqslant$

$4HK$, we show that T_1 is infinite and dominates T_2 (which can be finite or infinite). Indeed, the proof of Theorem 8 shows that T_1 is finite if and only if $d < 4HK$ and moreover

$$\begin{aligned} T_1 &\geqslant c_1 \int_0^t \int_0^t |u-v|^{2HK-2}(2t-u-v)^{-d/2} dudv \\ &= c_1 t^{-\frac{d}{2}+2HK} \int_0^1 u^{-\frac{d}{2}+2HK-1} du. \end{aligned} \tag{31}$$

Also, again from Proposition 4.2 in [12], T_2 is finite if $d < 2HK + 2$ and

$$\begin{aligned} T_2 &\leqslant c_2 \int_0^t dudv((uv)^{HK-1}(2t-u-v)^{-d/2} \\ &= c_2 t^{-\frac{d}{2}+2HK} \int_0^1 u^{-\frac{d}{2}+KH}(1-u)^{HK-1}. \end{aligned} \tag{32}$$

Therefore if $d < 4HK < 2HK + 2$, both T_1 and T_2 are finite and obviously $\mathbf{E}(u(t,x)^2) < \infty$ while for $d \geqslant 4HK$, from (31) and (32),

$$\begin{aligned} \mathbf{E}(u(t,x)^2) &= T_1 - T_2 \geqslant t^{-\frac{d}{2}+2HK} \\ &\times \left(c_1 \int_0^1 u^{-\frac{d}{2}+2HK-1} du - c_2 \int_0^1 du u^{-\frac{d}{2}+KH}(1-u)^{HK-1} \right) \end{aligned}$$

and as in [12], Proof of Proposition 4.2, we can show that the right-hand side above is infinite when $d \geqslant 4HK$. Indeed,

$$\begin{aligned} & c_1 \int_0^1 u^{-\frac{d}{2}+2HK-1} du - c_2 \int_0^1 du u^{-\frac{d}{2}+KH}(1-u)^{HK-1} \\ &= \lim_{N\to\infty} \left[c_1 \int_{1/N}^1 u^{-\frac{d-\alpha}{2}+2HK-1} du - c_2 \int_{1/N}^1 u^{-\frac{d-\alpha}{2}+HK}(1-u)^{HK-1} du \right] \\ &\sim \lim_N (c_1 N^{\frac{d-\alpha}{2}-2HK} - c_2 N^{\frac{d-\alpha}{2}-HK-1}) \end{aligned}$$

which converges to infinity when $N \to \infty$ when $d \geqslant 4HK$ (above $\sim$ means that the sides have the same behavior as $N \to \infty$).

Therefore, if the noise is given by a bifractional Brownian motion with $2HK > 1$ in time, the "iff" condition for the existence of the solution is $d < 4HK$.

5. The regularity of the solution

The purpose of this section is to give a sharp estimate for the regularity of the solution to the stochastic partial differential equation (16) with fractional-white noise. Assume that W has the covariance (12) and the time covariance R is he covariance of the fBm, see Example 2.

We have the following result.

Proposition 14. *Assume* (18) *holds and let* u *be the mild solution to* (16). *For any* $s,t \in [T_0,T]$ *and for any* $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, $|q|\leqslant 1$ *such that*

$$q^2 - qsign(x_d) \leqslant 0 \tag{33}$$

we have

$$\mathbf{E}\,|u(t,x)-u(s,x)|^2 \leqslant C_2|t-s|^{2H-\frac{d}{2}}.$$

In particular, under (33), *the process* $t\to u(t,x)$ *is Hölder continuous of order* δ *with* $0<\delta<H-\frac{d}{4}$.

Proof. Let us denote by R_u the covariance of the process u with respect to the time variable for fixed $x\in\mathbb{R}^d$

$$R_u(t,s)=\mathbf{E}u(t,x)u(s,x).$$

for every $s,t\in[T_0,T]$.

By (19) we have

$$\begin{aligned} R_u(t,s) &= \frac{1}{(4\pi C)^{d/2}}\int_0^t\int_0^s H(2H-1)|u-v|^{2H-2}(t+s-u-v)^{-d/2}\\ &\quad\times\left[1+\left(q^2-qsign(x_d)\right)\mathbf{erfc}\left(\frac{|\,x_d\,|\sqrt{t+s-u-v}}{\sqrt{4C(t-u)(s-v)}}\right)\right.\\ &\quad\left.+\,qsign(x_d)\exp\left(-\frac{x_d^2}{C(t-u+s-v)}\right)\right] \end{aligned} \tag{34}$$

for every $x\in\mathbb{R}^d$ and for every $s,t\in[0,T]$. Thus

$$\begin{aligned} R_u(t,s) = \frac{H(2H-1)}{(4\pi C)^{d/2}}\Bigg(& R_{1,u}(t,s)+(q^2-qsign(x_d))R_{2,u}(t,s)\\ &+qsign(x_d)R_{3,u}(t,s)\Bigg) \end{aligned}$$

where

$$R_{1,u}(t,s)=\int_0^t\int_0^s|u-v|^{2H-2}(t+s-u-v)^{-d/2} \tag{35}$$

$$R_{2,u}(t,s)=\int_0^t\int_0^s|u-v|^{2H-2}(t+s-u-v)^{-d/2}\mathbf{erfc}\left(\frac{|\,x_d\,|\sqrt{t+s-u-v}}{\sqrt{4C(t-u)(s-v)}}\right) \tag{36}$$

and

$$R_{3,u}(t,s)=\int_0^t\int_0^s|u-v|^{2H-2}(t+s-u-v)^{-d/2}\exp\left(-\frac{x_d^2}{C(t-u+s-v)}\right) \tag{37}$$

So, we can write

$$\begin{aligned}\mathbf{E}\left|u(t,x)-u(s,x)\right|^2 &= R_u(t,t)-2R_u(t,s)+R_u(s,s)\\ &= \frac{H(2H-1)}{(4\pi C)^{d/2}}\Bigg(\Big(R_{1,u}(t,t)-2R_{1,u}(t,s)+R_{1,u}(s,s)\Big)\\ &\quad +(q^2-qsign(x_d))\Big(R_{2,u}(t,t)-2R_{2,u}(t,s)+R_{2,u}(s,s)\Big)\\ &\quad +qsign(x_d)\Big(R_{3,u}(t,t)-2R_{3,u}(t,s)+R_{3,u}(s,s)\Big)\Bigg)\end{aligned} \tag{38}$$

By the proof of Theorem 2.2 in [11], we know that there exists two strictly positive constants C_1, C_2 such that for any $t,s\in[0,T]$ and for any $x\in\mathbb{R}^d$,

$$C_1|t-s|^{2H-\frac{d}{2}} \leqslant R_{1,u}(t,t)-2R_{1,u}(t,s)+R_{1,u}(s,s) \leqslant C_2|t-s|^{2H-\frac{d}{2}} \tag{39}$$

Let us analyze the part $R_{3,u}$ from (38) of the covariance R_u. Some parts of its estimation are related to the proof of the inequality (39) in [6] but the increments of the exponential function will also be involved. We have

$$\begin{aligned}&R_{3,u}(t,t)-2R_{3,u}(t,s)+R_{3,u}(s,s)\\ &\quad=\int_s^t\int_s^t dudv|u-v|^{2H-2}(2t-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(2t-u-v)}}\\ &\quad\quad+2\int_s^t du\int_0^s dv|u-v|^{2H-2}\\ &\left[(2t-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(2t-u-v)}}-(t+s-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(t+s-u-v)}}\right]\\ &\quad\quad+\int_0^s du\int_0^s dv|u-v|^{2H-2}\left[(2t-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(2t-u-v)}}\right.\\ &\quad\quad\left.-2(t+s-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(t+s-u-v)}}+(2s-u-v)^{-\frac{d}{2}}e^{-\frac{x_d^2}{C(2s-u-v)}}\right]\\ &\quad=: A+B+D.\end{aligned}$$

The first term A is the easiest to handle. Indeed,

$$\begin{aligned}A &\leqslant \int_s^t\int_s^t dudv|u-v|^{2H-2}(2t-u-v)^{-\frac{d}{2}}\\ &\leqslant c|t-s|^{2H-\frac{d}{2}}\end{aligned}$$

where the last bound follows from the successive change of variables $\tilde{u}=t-u,\tilde{v}=t-v$ and $\tilde{u}=\frac{u}{t-s},\tilde{v}=\frac{v}{t-s}$. Let us look to the term denoted by B. We can express it as

$$\begin{aligned} B &= 2\int_s^t du \int_0^s dv(u-v)^{2H-2} \\ &\quad \times \left[(2t-u-v)^{-\frac{d}{2}} - (t+s-u-v)^{-\frac{d}{2}}\right] e^{-\frac{x_d^2}{C(2t-u-v)}} \\ &\quad + 2\int_s^t du \int_0^s dv(u-v)^{2H-2}(t+s-u-v)^{-\frac{d}{2}} \\ &\quad \times \left[e^{-\frac{x_d^2}{C(2t-u-v)}} - e^{-\frac{x_d^2}{C(t+s-u-v)}}\right] \\ &=: B_1 + B_2. \end{aligned}$$

Note that B_1 is negative, so $B \leqslant B_2$. Thus it suffices to estimate B_2. In order to do this, notice that

$$\left| e^{-\frac{x_d^2}{C(2t-u-v)}} - e^{-\frac{x_d^2}{C(t+s-u-v)}} \right| \leqslant c|t-s| \tag{40}$$

for every u, v, x_d. Indeed, the function

$$f(y) = e^{-\frac{x_d^2}{C(a+y)}} - e^{-\frac{x_d^2}{Ca}} \tag{41}$$

defined on $[0,\infty)$ satisfies $f(0) = 0$ and

$$f'(y) = e^{-\frac{x_d^2}{C(a+y)}} \frac{x_d^2}{C(a+y)^2}.$$

It is easy to see that $|f'(y)| \leqslant M$ for every y with some $M > 0$. Using the bound (40), we get

$$\begin{aligned} B_2 &\leqslant c|t-s| \int_s^t du(u-s)^{2H-2} \int_0^s (t+s-u-v)^{-\frac{d}{2}} dv \\ &\leqslant c(t-s) \int_s^t du(u-s)^{2H-2} \left|(t-u+s)^{-\frac{d}{2}+1} - (t-u)^{-\frac{d}{2}+1}\right| \\ &\leqslant c(t-s) \int_s^t du(u-s)^{2H-2}(t-u)^{-\frac{d}{2}+1} \\ &= c(t-s)^{2H-\frac{d}{2}} \end{aligned}$$

since $\int_s^t du(u-s)^{2H-2}(t-u)^{-\frac{d}{2}+1} \leqslant c(t-s)^{2H-\frac{d}{2}}$ by using the successive change of variable $\tilde{u} = u-s$ and $\tilde{u} = \frac{u}{t-s}$, where c denotes a universal constant.

Concerning the summand D, it can be written as

$$D = \int_0^s du \int_0^s dv|u-v|^{2H-2} g(t-s, a_{u,v})$$

where, for fixed u and v, $a_{u,v} = 2s-u-v$ and $g(., a_{u,v})$ denotes the function

$$\begin{aligned} g(., a_{u,v}) : y \longmapsto &(2y+a_{u,v})^{-\frac{d}{2}} e^{-\frac{x_d^2}{C(2y+a_{u,v})}} - 2(y+a_{u,v})^{-\frac{d}{2}} e^{-\frac{x_d^2}{C(y+a_{u,v})}} \\ &+ a_{u,v}^{-\frac{d}{2}} e^{-\frac{x_d^2}{Ca_{u,v}}}. \end{aligned}$$

It is clear that $g(0, a_{u,v}) = 0$, and by an easy calculus we get

$$g'(y, a_{u,v}) = \left[- d(2y + a_{u,v})^{-d/2-1} + \frac{2x_d^2}{C}(2y+a_{u,v})^{-d/2-2} \right] \exp\left(- \frac{x_d^2}{C(2y+a)} \right) + \left[d(y + a_{u,v})^{-d/2-1} - \frac{2x_d^2}{C}(y + a_{u,v})^{-d/2-2} \right] \exp\left(- \frac{x_d^2}{C(y + a)} \right),$$

and

$$\begin{aligned} g''(y, a_{u,v}) = \exp\left(- \frac{x_d^2}{C(2y + a)} \right) &\Bigg[2d(d/2 + 1)(2y + a_{u,v})^{-d/2-2} \\ &- \frac{x_d^2}{C}(4d + 8)(2y + a_{u,v})^{-d/2-3} + \frac{4x_d^4}{C^2}(2y + a_{u,v})^{-d/2-4} \Bigg] \\ &- \exp\left(- \frac{x_d^2}{C(y + a)} \right) \Bigg[d(d/2 + 1)(y + a_{u,v})^{-d/2-2} \\ &- \frac{x_d^2}{C}(2d + 4)(y + a_{u,v})^{-d/2-3} + \frac{2x_d^4}{C^2}(y + a_{u,v})^{-d/2-4} \Bigg]. \end{aligned}$$

Since, for every $k \in \{2, 3, 4\}$,

$$(2y + a_{u,v})^{-d/2-k} \exp\left(- \frac{x_d^2}{C(2y + a_{u,v})} \right) \leq Cte$$

and

$$(y + a_{u,v})^{-d/2-k} \exp\left(- \frac{x_d^2}{C(y + a_{u,v})} \right) \leq Cte,$$

where Cte denotes a positive real constant, independent of u, v, s and t, we easily get $g'(0, a_{u,v}) = 0$ and $\mid g''(y, a_{u,v}) \mid \leq M$, where M is a positive constant, independent of u, v, s and t. Thus

$$\mid g(t - s, a_{u,v}) \mid \leq M(t - s)^2,$$

and consequently

$$\begin{aligned} qD \leq \mid q \mid \times \mid D \mid &\leq c(t - s)^2 \int_0^s du \int_0^s dv \mid u - v \mid^{2H-2} \\ &\leq c(t - s)^2 \\ &\leq c(t - s)^{2H-d/2}, \end{aligned}$$

where the last inequality is due to the fact that $2-2H+d/2 > 0$ and $(t-s) \leq T$.

Therefore,

$$R_{3,u}(t, t) - 2R_{3,u}(t, s) + R_{3,u}(s, s) \leqslant c(t - s)^{2H-\frac{d}{2}}. \tag{42}$$

The proof of Proposition 6 shows that $R_{2,u}(t,s)$ is the covariance of the process

$$u_2(t,x) = \int_0^t \int_{\mathbb{R}^d} p_2(t-s, x-y) W(ds, dy)$$

up to a multiplicative constant, with

$$R_{2,u}(t,t) - 2R_{2,u}(t,s) + R_{2,u}(s,s) = E|u_2(t,x) - u_2(s,x)|^2 \geqslant 0.$$

Thus, if under the condition (33) we have

$$\left(q^2 - qsign(x_d)\right)\left(R_{2,u}(t,t) - 2R_{2,u}(t,s) + R_{2,u}(s,s)\right) \leqslant 0. \tag{43}$$

□

Remark 15. We notice that under Condition (33), the process u solution to (1) keeps the same Hölder regularity in time as the solution to the heat equation with fractional -white noise (see [6,11]). This means that the part of L given by the second summand in the right-hand side of (2) does not perturb too much the paths of the process.

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