



Null controllability in large time of a parabolic equation involving the Grushin operator with an inverse-square potential

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Abstract. We prove the null controllability in large time of the following linear parabolic equation involving the Grushin operator with an inverse-square potential

$$u_t - \Delta_x u - |x|^{2s} \Delta_y u - \frac{\mu}{|x|^2} u = v 1_\omega$$

in a bounded domain $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ ($N_1 \geq 3, N_2 \geq 1$) intersecting the surface $\{x = 0\}$ under an additive control supported in an open subset $\omega = \omega_1 \times \Omega_2$ of Ω .

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1. Introduction and statement of the main result

Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$, $N_1 \geq 3, N_2 \geq 1$, be a bounded domain with $0_{\mathbb{R}^{N_1}} \in \Omega_1$ and $\partial\Omega_1$ is smooth enough. We study the null controllability of the following problem

$$\begin{cases} u_t - \Delta_x u - |x|^{2s} \Delta_y u - \frac{\mu}{|x|^2} u = v 1_\omega & \text{for } (x, y, t) \in \Omega \times (0, T), \\ u = 0 & \text{for } (x, y, t) \in \partial\Omega \times (0, T), \\ u(0) = u^0 & \text{for } (x, y) \in \Omega, \end{cases} \quad (1.1)$$

where 1_ω denotes the characteristic function of the open subset ω of Ω , and $\mu \leq \mu^* = \mu^*(N_1)$ with $\mu^*(N_1) = (N_1 - 2)^2/4$ is the best constant in the following Hardy inequality for the Grushin operator (see [1, Theorem 3.3]):

$$\int_{\Omega} (|\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2) dx dy \geq \mu^* \int_{\Omega} \frac{u^2}{|x|^2} dx dy. \quad (1.2)$$

Here, because we are considering the case of internal singularity, the assumption $N_1 \geq 3$ is made to ensure that the best constant $\mu^* = \mu^*(N_1)$ in the above Hardy inequality is strictly positive.

We say that problem (1.1) is **null controllable in time T** if for every $u_0 \in L^2(\Omega)$ given, there exists a control $v \in L^2(\Omega \times (0, T))$ such that the solution $u(x, y, t)$ of (1.1) satisfies $u(\cdot, \cdot, T) = 0$.

The aim of this paper is to prove the following result.

Theorem 1.1. *Let $\omega = \omega_1 \times \Omega_2$ be an open subset of Ω such that $0_{\mathbb{R}^{N_1}} \notin \bar{\omega}_1$. If $\mu < \mu^*$ and $s = 1$, then there exists a time $T^* > 0$ such that problem (1.1) is null controllable in any time $T > T^*$.*

We first review some existing controllability results related to degenerate/singular parabolic equations. The controllability for degenerate parabolic equations in dimension one has been studied widely in recent years by many authors (see e.g., [2, 7–11, 18, 19]). The null controllability of parabolic equations involving the Grushin operator has been studied first in dimension two [4], and then in some multi-dimensional domains [3, 5]. On the other hand, the controllability results of parabolic equations with an inverse-square potential were obtained in [14, 19] for the case of internal singularity, and in [13] for the case of boundary singularity. Recently, in the dimension two, the approximate controllability of the parabolic Grushin operator with a singular potential has been studied in [17] thanks to the unique continuation of the corresponding operator. Moreover, in [12], the authors also proved the null controllability in large time of the parabolic Grushin operator involving a singular potential when $s = 1$ and spatial domain is $(0, 1) \times (0, 1)$, that is, with the boundary degeneracy and singularity. As mentioned in [12, 17], the null controllability problem is completely open when the degeneracy of the diffusion coefficient and singularity of the potential occur at the interior of the domain. This paper is an attempt to partly solve this open question by proving the null controllability for this operator when $s = 1$ in the multi-dimensional domains intersecting the surface $\{x = 0\}$.

We now explain the method used in the paper. By the Hilbert Uniqueness Method (HUM) introduced by J.-L. Lions, it is well-known that the null controllability of problem (1.1) is equivalent to the observability of the adjoint problem. To get the observability of the adjoint problem, the classical method is to construct a global Carleman inequality for the solutions to the adjoint system of (1.1). However, up to now, there is no existing way for constructing such a Carleman inequality for the parabolic problem involving the Grushin operator. Here to prove the main result, we exploit the ideas introduced in [4] for proving the null controllability problem of the parabolic problem involving the Grushin operator in dimension two. More precisely, thanks to the Fourier decomposition for the solution of the equation, the observability of the adjoint problem can be reduced to the uniform observability with respect to Fourier frequencies, and the later is proved by using a suitable Carleman inequality

and a dissipation speed of the Fourier components. This approach, however, requires that the control domain must have the form $\omega = \omega_1 \times \Omega_2$. It is noticed that if we apply the Carleman inequality in [14] directly, we only obtain an observability constant as $\exp(C(T)\gamma_n^{2/3})$, which is not enough for our purpose while we need an observability constant as $\exp(C\sqrt{\gamma_n}T)$. To construct the new necessary Carleman inequality, a basic tool of the proof, we follow the general lines of the approach in [14] and consider the potential $\gamma_n|x|^2$ in the principal part of the operator to follow precisely the dependence on γ_n . Because of the results in [4], the null controllability for the parabolic Grushin operator with singular potential is only expected to hold in the case $0 < s \leq 1$. However, in the present paper (and also [12]) we only can prove the null controllability in large time in the case $s = 1$. The reason is that although we can prove the dissipation speed for any $s > 0$ (see the remark after Proposition 2.2), we are only able to prove the Carleman inequality in the case $s = 1$ (see Theorem 2.5). This is also the situation for the case of boundary degeneracy and singularity in [12]. It is also noticed that the assumption $\mu < \mu^*$ is needed to ensure that the constant C_* in Proposition 2.2 is positive due to the classical Hardy inequality, while the smoothness of Ω_1 is needed to construct the Carleman inequality.

In order to study the problem (1.1), we use the function space $S_{\mu,0}^1(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{S_{\mu,0}^1(\Omega)} = \left(\int_{\Omega} \left(|\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2 - \frac{\mu}{|x|^2} u^2 \right) dx dy \right)^{1/2}.$$

By the Hardy inequality (1.2), we know that $S_{\mu,0}^1(\Omega)$, $\mu \leq \mu^*$, is a Banach space endowed with the above norm, and when $\mu < \mu^*$,

$$\left(1 - \frac{\max\{0, \mu\}}{\mu^*} \right) \|u\|_{S_0^1(\Omega)}^2 \leq \|u\|_{S_{\mu,0}^1(\Omega)}^2 \leq \left(1 - \frac{\min\{0, \mu\}}{\mu^*} \right) \|u\|_{S_0^1(\Omega)}^2,$$

where $S_0^1(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_{S_0^1(\Omega)} = \left(\int_{\Omega} \left(|\nabla_x u|^2 + |x|^{2s} |\nabla_y u|^2 \right) dx dy \right)^{1/2}.$$

This means that when $\mu < \mu^*$, the two spaces $S_{\mu,0}^1(\Omega)$ and $S_0^1(\Omega)$ are equal. Therefore, the embedding $S_{\mu,0}^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact if $\mu < \mu^*$ (see e.g., [15] for more details). However, in the critical case $\mu = \mu^*$, the space $S_{\mu^*,0}^1(\Omega)$ is strictly larger than $S_0^1(\Omega)$. As in the case without potential [4, Sect. 2.1], using the Galerkin approximation method or the standard theory of semigroups, one can prove that for any $u_0 \in L^2(\Omega)$ and $v \in L^2(0, T; L^2(\Omega))$ given, problem (1.1) has a unique weak solution u satisfying

$$u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; S_{\mu,0}^1(\Omega)).$$

The rest of the paper is organized as follows. In Sect. 2, due to the Hilbert Uniqueness Method, we prove Theorem 1.1 by showing that the adjoint system is observable. The proof relies on uniform observability estimates with respect to Fourier frequencies and this is proved using a new Carleman estimate and a dissipation speed of the Fourier components. For clarity of the presentation,

the long and technical proof of our new Carleman inequality is given later in the last section.

2. Proof of the main result

2.1. Fourier decomposition and dissipation speed

Let $(\gamma_n)_{n \in \mathbb{N}^*}$ be the nondecreasing sequence of eigenvalues of the operator $-\Delta_y$ in $H^2(\Omega_2) \cap H_0^1(\Omega_2)$ and the associated eigenfunctions $(\varphi_n(y))_{n \in \mathbb{N}^*}$, that is,

$$\begin{cases} -\Delta_y \varphi_n(y) = \gamma_n \varphi_n(y), & y \in \Omega_2, \\ \varphi_n(y) = 0, & y \in \partial\Omega_2. \end{cases}$$

For any weak solution $u(x, y, t)$ of (1.1) and any control $v(x, y, t)$, we set

$$u_n(x, t) = \int_{\Omega_2} u(x, y, t) \varphi_n(y) dy, \quad v_n(x, t) = \int_{\Omega_2} v(x, y, t) \varphi_n(y) dy, \quad (2.1)$$

then by substituting (2.1) into (1.1), we obtain the following

Proposition 2.1. *Let $u_0 \in L^2(\Omega)$ be given and let u be the corresponding unique weak solution of (1.1) with $\mu \leq \mu^*$ and $s = 1$. Then, for every $n \in \mathbb{N}^*$, the function $u_n(x, t)$ is the unique weak solution of the problem*

$$\begin{cases} \frac{\partial u_n}{\partial t} - \Delta_x u_n + \gamma_n |x|^2 u_n - \frac{\mu}{|x|^2} u_n = v_n 1_{\omega_1}(x) & \text{in } \Omega_1 \times (0, T), \\ u_n = 0 & \text{on } \partial\Omega_1 \times (0, T), \\ u_n(x, 0) = u_{0,n}(x) & \text{in } \Omega_1, \end{cases} \quad (2.2)$$

where $u_{0,n}(x) = \int_{\Omega_2} u_0(x, y) \varphi_n(y) dy$.

Proof. The proof is very similar to the one of Proposition 2 in [12], see also the original proof in [4, Prop. 2], so we omit it here. \square

We know that the smallest eigenvalue of $-\Delta\varphi(x) + \gamma_n |x|^2 \varphi(x) - \frac{\mu}{|x|^2} \varphi(x)$ in $H^2(\Omega_1) \cap H_0^1(\Omega_1)$ is given by

$$\lambda_{n,\mu} := \min \left\{ \frac{\int_{\Omega_1} \left(|\nabla \varphi(x)|^2 + \left(\gamma_n |x|^2 - \frac{\mu}{|x|^2} \right) |\varphi|^2 \right) dx}{\int_{\Omega_1} |\varphi|^2 dx} \mid \varphi \in H_0^1(\Omega_1), \varphi \neq 0 \right\}.$$

Proposition 2.2. *For any $N_1 \geq 3$ and $\mu < \mu^*$, there exist $C_* = C_*(\mu) > 0$ and $C^* = C^*(\mu) > 0$ such that*

$$C_* \gamma_n^{\frac{1}{2}} \leq \lambda_{n,\mu} \leq C^* \gamma_n^{\frac{1}{2}} \quad \forall n \in \mathbb{N}^*. \quad (2.3)$$

Proof. The proof is an adaptation of [12, Prop. 4]. We first prove the lower bound. By the change of variable $\varphi(x) = \gamma_n^{\frac{N_1}{8}} \phi(\gamma_n^{1/4} x) = \gamma_n^{\frac{N_1}{8}} \phi(y)$, we get

$$\begin{aligned} \lambda_{n,\mu} &= \inf_{\substack{\phi \in C_0^\infty(\Omega_1) \\ \|\varphi\|_{L^2(\Omega_1)}=1}} \left\{ \int_{\Omega_1} \left(|\nabla \varphi(x)|^2 + \left(\gamma_n |x|^2 - \frac{\mu}{|x|^2} \right) |\varphi(x)|^2 \right) dx \right\} \\ &= \gamma_n^{1/2} \inf_{\substack{\phi \in C_0^\infty(\gamma_n^{1/4} \Omega_1) \\ \|\varphi\|_{L^2(\gamma_n^{1/4} \Omega_1)}=1}} \left\{ \int_{\gamma_n \Omega_1} \left(|\nabla \phi(y)|^2 + \left(|y|^2 - \frac{\mu}{|y|^2} \right) |\phi(y)|^2 \right) dy \right\} \\ &\geq C_* \gamma_n^{1/2}, \end{aligned}$$

where

$$C_* := \inf_{\substack{\phi \in C_0^\infty(\mathbb{R}^{N_1}) \\ \|\varphi\|_{L^2(\mathbb{R}^{N_1})}=1}} \left\{ \int_{\mathbb{R}^{N_1}} \left(|\nabla \phi(y)|^2 + \left(|y|^2 - \frac{\mu}{|y|^2} \right) |\phi(y)|^2 \right) dy \right\}$$

is positive via the classical Hardy inequality (see e.g., [6]).

We now prove the upper bound for $\lambda_{n,\mu}$ by choosing suitable test functions. For every $k > 1$ large enough such that $\bar{B}_{2/k}(0) \subset \Omega_1$, we consider the function

$$\varphi_k(x) = \begin{cases} k|x| & \text{if } |x| \leq 1/k, \\ 2 - k|x| & \text{if } 1/k \leq |x| \leq 2/k \\ 0 & \text{if } |x| \geq 2/k. \end{cases}$$

One can see that φ_k belongs to $H_0^1(\Omega_1)$ for each $k > 1$. We have

$$\begin{aligned} \int_{\Omega_1} |\varphi_k(x)|^2 dx &= k^2 \int_{|x| \leq 1/k} |x|^2 dx + \int_{1/k \leq |x| \leq 2/k} (2 - k|x|)^2 dx \\ &= 4 \int_{1/k \leq |x| \leq 2/k} dx - 4k \int_{1/k \leq |x| \leq 2/k} |x| dx + k^2 \int_{|x| \leq 2/k} |x|^2 dx. \end{aligned}$$

Using the change of variables in spherical coordinates, we have

$$4 \int_{1/k \leq |x| \leq 2/k} dx = \frac{4(2^{N_1} - 1)C_{N_1}}{k^{N_1}},$$

where

$$C_{N_1} = \begin{cases} \frac{\pi^{N_1/2}}{\left(\frac{N_1}{2}\right)!} & \text{for even } N_1, \\ \frac{2^{\frac{N_1+1}{2}} \pi^{\frac{N_1-1}{2}}}{N_1!!} & \text{for odd } N_1, \end{cases}$$

and

$$\begin{aligned} -4k \int_{1/k \leq |x| \leq 2/k} |x| dx &= \frac{1}{k^{N_1}} \frac{-4(2^{N_1+1} - 1)}{N_1 + 1} C_{N_1}, \\ k^2 \int_{|x| \leq 2/k} |x|^2 dx &= \frac{1}{k^{N_1}} \frac{2^{N_1+2}}{N_1 + 2} C_{N_1}, \end{aligned}$$

where

$$\mathcal{C}_{N_1} := \pi \int_{(0,\pi)^{N_1-2}} \sin^{N_1-2} \phi_1 \cdots \sin \phi_{N_1-2} d\phi_1 d\phi_2 \cdots d\phi_{N_1-2}.$$

Hence

$$\int_{\Omega_1} |\varphi_k(x)|^2 dx = \mathcal{C}_{1,N_1} \frac{1}{k^{N_1}},$$

where

$$\mathcal{C}_{1,N_1} := 4(2^{N_1} - 1)C_{N_1} - \left(\frac{4(2^{N_1+1} - 1)}{N_1 + 1} - \frac{2^{N_1+3}}{N_1 + 2} \right) C_{N_1} > 0.$$

Similarly, we have

$$\int_{\Omega_1} |\nabla \varphi_k(x)|^2 dx = 2^{N_1} C_{N_1} \frac{k^2}{k^{N_1}}$$

and

$$\int_{\Omega_1} |x|^2 |\varphi_k(x)|^2 dx = \mathcal{C}_{2,N_1} \frac{1}{k^{N_1+2}},$$

where

$$\mathcal{C}_{2,N_1} := 4 \left(\frac{2^{N_1+2} - 1}{N_1 + 2} - \frac{2^{N_1+3} - 1}{N_1 + 3} + \frac{2^{N_1+2}}{N_1 + 4} \right) C_{N_1} > 0,$$

and

$$\int_{\Omega_1} \frac{1}{|x|^2} |\varphi_k(x)|^2 dx = \mathcal{C}_{3,N_1} \frac{k^2}{k^{N_1}},$$

where

$$\mathcal{C}_{3,N_1} := 2^{N_1} C_{N_1} + 4C_{N_1} \left(\frac{2^{N_1-2} - 1}{N_1 - 2} - \frac{(2^{N_1-1} - 1)}{N_1 - 1} \right).$$

Thus,

$$\lambda_{n,\mu} \leq h_{n,\mu}(k) := \frac{2^{N_1} C_{N_1} - \mu \mathcal{C}_{3,N_1}}{\mathcal{C}_{1,N_1}} k^2 + \gamma_n \frac{\mathcal{C}_{2,N_1}}{\mathcal{C}_{1,N_1}} k^{-2}$$

for all $k > 1$.

We note that $2^{N_1} C_{N_1} - \mu \mathcal{C}_{3,N_1} > 0$ for $\mu < \mu^*(N_1)$. Then since $h_{n,\mu}$ attains its minimum at $\bar{k} = \bar{\mathcal{C}}(\mu) \gamma_n^{1/4}$, we have

$$\lambda_{n,\mu} \leq h_{n,\mu}(\bar{k}) = C^* \gamma_n^{1/2},$$

where

$$C^* = \frac{(\mathcal{C}_{2,N_1})^{1/2}}{\mathcal{C}_{1,N_1}} 2 \left(2^{N_1} C_{N_1} - \mu \mathcal{C}_{3,N_1} \right)^{1/2}.$$

□

Remark 2.3. By a similar method, one can prove the dissipation speed result for any $s > 0$. More precisely, we have

$$C_* \gamma_n^{\frac{1}{1+s}} \leq \lambda_{n,s,\mu} \leq C^* \gamma_n^{\frac{1}{1+s}} \quad \forall n \in \mathbb{N}^*,$$

where $\lambda_{n,s,\mu}$ is the smallest eigenvalue of $-\Delta\varphi(x) + \gamma_n|x|^{2s}\varphi(x) - \frac{\mu}{|x|^2}\varphi(x)$ in $H^2(\Omega_1) \cap H_0^1(\Omega_1)$.

2.2. Uniform observability of the adjoint system

By duality, the null controllability of problem (1.1) is equivalent to an observability inequality for the adjoint problem of problem (1.1):

$$\begin{cases} w_t + \Delta_x w + |x|^2 \Delta_y w + \frac{\mu}{|x|^2} w = 0 & (x, y, t) \in \Omega \times (0, T), \\ w = 0 & (x, y, t) \in \partial\Omega \times (0, T), \\ w(x, y, T) = w_T(x, y) & (x, y) \in \Omega. \end{cases} \quad (2.4)$$

We say that the adjoint problem (2.4) is observable in ω in time T if there exists $C > 0$ such that for every $w_T \in L^2(\Omega)$, the solution w of (2.4) satisfies

$$\|w(\cdot, \cdot, 0)\|_{L^2(\Omega)}^2 \leq C \iint_{\omega \times (0, T)} |w(x, y, t)|^2 dx dy dt.$$

Using (2.1), we get adjoint problem of (2.2) as follows

$$\begin{cases} \partial_t w_n + \Delta_x w_n - \gamma_n |x|^2 w_n + \frac{\mu}{|x|^2} w_n = 0, & (x, t) \in \Omega_1 \times (0, T), \\ w_n = 0, & (x, t) \in \partial\Omega_1 \times (0, T), \\ w_n(x, T) = w_{T,n}(x), & x \in \Omega_1, \end{cases} \quad (2.5)$$

where $w_n(x, t) = \int_{\Omega_2} w(x, y, t) \varphi_n(y) dy$ and $w_{T,n}(x) = \int_{\Omega_2} w_T(x, y) \varphi_n(y) dy$.

Therefore, by the Bessel–Parseval equality, in order to prove the observability of problem 2.4 it is sufficient to prove that the adjoint problem (2.5) is observable in ω_1 uniformly with respect to $n \in \mathbb{N}^*$, that is, for $\omega_1 \subset \Omega_1$, there exists $C > 0$ (independent of n) such that for any $n \in \mathbb{N}^*$ and any $w_{T,n} \in L^2(\Omega_1)$, the solution w_n of (2.5) satisfies

$$\|w_n(\cdot, 0)\|_{L^2(\Omega_1)}^2 \leq C \iint_{\omega_1 \times (0, T)} |w_n(x, t)|^2 dx dt.$$

We now prove the following uniform observability result, which will imply Theorem 1.1 due to the above reasons.

Theorem 2.4. *Let $\omega_1 \subset \Omega_1$ such that $0_{\mathbb{R}^{N_1}} \notin \bar{\omega}_1$ and $\mu < \mu^*$. Then there exists $T^* > 0$ such that for every $T > T^*$, problem (2.5) is observable in ω_1 uniformly with respect to $n \in \mathbb{N}^*$.*

Proof. The proof relies on a new Carleman estimate for the solutions of (2.5) (Theorem 2.5 below) and the dissipation speed (see (2.3)).

In order to construct our Carleman inequality, we consider the following weight function as in [14]:

$$\sigma(x, t) = \frac{\left(e^{2\lambda \sup \psi} - \frac{1}{2}|x|^2 - e^{\lambda \psi(x)}\right)}{(t(T-t))^3} := \frac{\beta(x)}{(t(T-t))^3},$$

where λ is a positive parameter aimed at being large, and ψ is a smooth function such that

$$\begin{cases} \psi(x) = \ln(|x|/\delta), & x \in B_\delta(0), \\ \psi(x) = 0, & x \in \partial\Omega_1, \\ \psi(x) > 0, & x \in \Omega_1 \setminus \overline{B_\delta(0)}, \end{cases}$$

and there exist an open set $\tilde{\omega}_1$ satisfying $\tilde{\omega}_1 \subset \omega_1$ and $m_* > 0$ such that

$$|\nabla \psi(x)| \geq m_*, \quad x \in \tilde{\omega}_1. \quad (2.6)$$

Here the fixed number $0 < \delta \leq 1$ is chosen such that the ball $\overline{B_\delta(0)} \subset \Omega_1$ and $\overline{B_\delta(0)} \cap \bar{\omega}_1 = \emptyset$.

As explained in [14], we can choose ψ such that the weight function β is at least of class C^4 when λ is large enough and this is enough for our purpose. We refer the reader to [14] for more discussions on choosing of the weight function.

We have the following Carleman inequality whose long and technical proof is postponed in the Sect. 3. \square

Theorem 2.5. *Let $\omega_1 \subset \Omega_1$ such that $0_{\mathbb{R}^{N_1}} \notin \bar{\omega}_1$. If $\mu < \mu^*$, then there is a positive constant λ_0 such that for $\lambda \geq \lambda_0$, there exist $\mathcal{K}_1 = \mathcal{K}_1(\lambda, \beta)$ and $\mathcal{K}_2 = \mathcal{K}_2(\lambda, \beta)$ such that for any $w \in C([0, T]; L^2(\Omega_1)) \cap L^2(0, T; H_0^1(\Omega_1))$, the following inequality holds*

$$\begin{aligned} \mathcal{K}_1 & \left[\iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{M}{(t(T-t))^3} |\nabla w|^2 dx dt + \iint_{\Omega_1 \times (0, T)} e^{-2M\sigma} \right. \\ & \quad \times \frac{M}{(t(T-t))^3} \frac{|w|^2}{|x|} dx dt + \iint_{B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{M^3 |x|^2}{(t(T-t))^9} |w|^2 dx dt \\ & \quad \left. + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{M^3}{(t(T-t))^9} |w|^2 dx dt \right] \\ & \leq \iint_{\omega_1 \times (0, T)} e^{-2M\sigma} \frac{M^3}{(t(T-t))^9} |w|^2 dx dt + \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n, \mu} w|^2 dx dt. \end{aligned} \quad (2.7)$$

Here

$$M = M(\lambda, T, \gamma_n, \beta) = \mathcal{K}_2 \max\{T^3 + T^4 + T^5 + T^6; \sqrt{\gamma_n} T^6\},$$

and

$$G_{n, \mu} w = w_t + \Delta w - \gamma_n |x|^2 w + \frac{\mu}{|x|^2} w.$$

We now continue the proof of Theorem 2.4. The following arguments are inspired from the proofs of Propositions 5 and 6 in [5].

Applying the Carleman inequality (2.7) for solution $w_n(x, t)$ of (2.5), we have

$$\mathcal{K}_1 \iint_{\Omega_1 \times (0, T)} \frac{e^{-2M\sigma} M}{(t(T-t))^3} \frac{|w_n|^2}{|x|} dx dt \leq \iint_{\omega_1 \times (0, T)} e^{-2M\sigma} \frac{M^3}{(t(T-t))^9} |w_n(x, t)|^2 dx dt. \quad (2.8)$$

Noting that

$$\frac{Me^{-2M\sigma(x, t)}}{(t(T-t))^3} \frac{1}{|x|} \geq \frac{64M}{R_{\Omega_1} T^6} e^{-\frac{128\beta^*}{27T^6} M} \quad \forall (x, t) \in \Omega_1 \times (T/4, 3T/4),$$

where $R_{\Omega_1} := \sup_{x \in \overline{\Omega_1}} |x|$ and

$$e^{-2M\sigma(x, t)} \frac{M^3}{(t(T-t))^9} \leq K_3 M^3 \quad \forall (x, t) \in \omega_1 \times (0, T).$$

Here $K_3 = K_3(\beta) := \max\{e^{-2\beta_* \zeta} \zeta^3 : \zeta \in \mathbb{R}_+\}$ with $\beta^* = \max\{\beta(x) : x \in \Omega_1\}$ and $\beta_* = \min\{\beta(x) : x \in \omega_1\}$.

Hence (2.8) implies that

$$\int_{T/4}^{3T/4} \int_{\Omega_1} |w_n(x, t)|^2 dx dt \leq \mathcal{K}_3 T^6 M^2 e^{\frac{128\beta^*}{27} \frac{M}{T^6}} \iint_{\omega_1 \times (0, T)} |w_n(x, t)|^2 dx dt, \quad (2.9)$$

where $\mathcal{K}_3 := K_3 R_{\Omega_1} / (64K_1)$.

Now, multiplying (2.5) by $-w_n$, then integrating over Ω_1 , we obtain

$$\begin{aligned} & -\frac{1}{2} \frac{d}{dt} \int_{\Omega_1} |w_n(x, t)|^2 dx \\ & + \int_{\Omega_1} \left[|\nabla w_n(x, t)|^2 + \left(\gamma_n |x|^2 - \frac{\mu}{|x|^2} \right) |w_n(x, t)|^2 \right] dx = 0. \end{aligned} \quad (2.10)$$

Using (2.3), we get from (2.10) that

$$\frac{d}{dt} \left(e^{-2C_* \sqrt{\gamma_n} t} \int_{\Omega_1} |w_n(x, t)|^2 dx \right) \geq 0 \quad \forall t \geq 0.$$

Hence

$$\int_{\Omega_1} |w_n(x, 0)|^2 dx \leq e^{-2C_* \sqrt{\gamma_n} t} \int_{\Omega_1} |w_n(x, t)|^2 dx \quad \forall t \geq 0.$$

Integrating from $T/4$ to $3T/4$, we obtain

$$\int_{\Omega_1} |w_n(x, 0)|^2 dx \leq \frac{2}{T} e^{-\frac{C_*}{2} \sqrt{\gamma_n} T} \int_{T/4}^{3T/4} \int_{\Omega_1} |w_n(x, t)|^2 dx dt. \quad (2.11)$$

Substituting (2.9) into (2.11) to get

$$\begin{aligned} \int_{\Omega_1} |w_n(x, 0)|^2 dx & \leq 2\mathcal{K}_3 T^5 M^2 \exp \left(\frac{128\beta^*}{27} \frac{M}{T^6} - \frac{C_*}{2} \sqrt{\gamma_n} T \right) \\ & \times \iint_{\omega_1 \times (0, T)} |w_n(x, t)|^2 dx dt. \end{aligned}$$

We consider two cases:

- If $\sqrt{\gamma_n} < 1 + \frac{1}{T} + \frac{1}{T^2} + \frac{1}{T^3}$, then $M = \mathcal{K}_2(T^3 + T^4 + T^5 + T^6)$, and thus

$$\begin{aligned} \int_{\Omega_1} |w_n(x, 0)|^2 dx &\leq 2\mathcal{K}_3\mathcal{K}_2^2 T^{11} (1 + T + T^2 + T^3)^2 \\ &\times \exp\left(\frac{128\beta^*}{27}\mathcal{K}_2\left(1 + \frac{1}{T} + \frac{1}{T^2} + \frac{1}{T^3}\right)\right) \iint_{\omega_1 \times (0, T)} |w_n(x, t)|^2 dx dt. \end{aligned}$$

- If $\sqrt{\gamma_n} \geq 1 + \frac{1}{T} + \frac{1}{T^2} + \frac{1}{T^3}$, then $M = \mathcal{K}_2\sqrt{\gamma_n}T^6$, and thus

$$\frac{128\mathcal{C}}{27} \frac{M}{T^6} - \frac{C_*}{2} \sqrt{\gamma_n} T = \left(\frac{128\beta^*}{27}\mathcal{K}_2 - \frac{C_*}{2}T\right) \sqrt{\gamma_n}.$$

So, if $T > T^* = \frac{256\beta^*\mathcal{K}_2}{27C_*}$, then

$$\begin{aligned} \int_{\Omega_1} |w_n(x, 0)|^2 dx &\leq 2\mathcal{K}_3\mathcal{K}_2^2 T^{17} \gamma_n \exp\left[\left(\frac{128\beta^*}{27}\mathcal{K}_2 - \frac{C_*}{2}T\right) \sqrt{\gamma_n}\right] \\ &\leq 2\mathcal{K}_3\mathcal{K}_2^2 T^{17} \mathcal{K}_4 \iint_{\omega_1 \times (0, T)} |w_n(x, t)|^2 dx dt, \end{aligned}$$

$$\text{where } \mathcal{K}_4 = \max\left\{\zeta \exp\left[\left(\frac{128\beta^*}{27}\mathcal{K}_2 - \frac{C_*}{2}T\right) \sqrt{\zeta}\right] : \zeta \in \mathbb{R}_+\right\}.$$

The proof of Theorem 2.4 is complete. \square

3. Proof of the Carleman inequality

3.1. Some properties of weight functions

Let us define a smooth positive radial function $\eta(x) = \eta(|x|)$, $0 \leq \eta(x) \leq \frac{1}{N_1}$, such that

$$\eta(x) = 0, \quad |x| \leq \frac{\delta}{2}, \quad \text{and} \quad \eta(x) = \frac{1}{N_1}, \quad |x| \geq \frac{3\delta}{4}.$$

We have the following properties of the weight function β .

Proposition 3.1. *Denote by \mathcal{O} the open set $\Omega_1 \setminus (\overline{B_\delta(0)} \cup \widetilde{\omega}_1)$. We have*

(1) *On $\partial\Omega_1$, we have*

$$\frac{\partial\beta}{\partial\nu} \geq 0 \quad \text{for} \quad \lambda > \frac{R_{\Omega_1}}{m_*}, \quad (3.1)$$

where ν denotes the outward normal vector and m_* is the constant in (2.6).

(2) There is a number $\lambda_0 > 0$ such that for $\lambda \geq \lambda_0$, there exist two positive constants C_1 and C_2 such that

$$-\eta(x)\Delta\beta(x)|Z|^2 - 2D^2\beta(x)(Z, Z) \geq C_1|Z|^2, \quad \forall Z \in \mathbb{R}^{N_1}, x \in \Omega_1 \setminus \tilde{\omega}_1, \quad (3.2)$$

$$\eta(x)\Delta\beta(x)|\nabla\beta(x)|^2 - 2D^2\beta(x)(\nabla\beta(x), \nabla\beta(x)) \geq C_2, \quad \forall x \in \mathcal{O}, \quad (3.3)$$

$$\eta(x)\Delta\beta(x)|\nabla\beta(x)|^2 - 2D^2\beta(x)(\nabla\beta(x), \nabla\beta(x)) \geq |x|^2, \quad \forall x \in B_\delta(0), \quad (3.4)$$

where the constant C_1 can be taken such that

$$C_1 \geq \eta N_1 + 2 \quad \text{as } \lambda \text{ large enough.} \quad (3.5)$$

It is noticed that from the proof below, we can give the precise value of λ_0 ; but this does not play any role, so we omit it.

Proof. The proof is quite elementary and similar to that of Proposition 13 in [5], but we give it for the completeness.

(1) We can see that on $\partial\Omega_1$,

$$\frac{\partial\beta}{\partial\nu} = -x \cdot \nu - \lambda \nabla\psi \cdot \nu \geq -R_{\Omega_1} + \lambda |\nabla\psi| \geq -R_{\Omega_1} + \lambda m_*.$$

Hence, we get (3.1).

(2) From the definition of ψ outside the ball $B_\delta(0)$, we get a positive constant m^* such that

$$|\nabla\psi(x)|, |\Delta\psi(x)|, |D^2\psi(x)| \leq m^* \quad \text{for } x \in \tilde{\omega}_1 \cup \Omega_1 \setminus \overline{B_\delta(0)}.$$

Proof of (3.2). We see from the definition of $\beta(x)$ in $\Omega_1 \setminus \overline{B_\delta(0)}$ that

$$\begin{aligned} \nabla\beta(x) &= -\lambda \nabla\psi e^{\lambda\psi} - x, \\ D^2\beta(x) &= -(\lambda^2 \nabla\psi(x) \otimes \nabla\psi(x) + \lambda D^2\psi(x)) e^{\lambda\psi} - I_{N_1}, \\ \Delta\beta(x) &= -(\lambda^2 |\nabla\psi|^2 + \lambda \Delta\psi) e^{\lambda\psi} - N_1, \end{aligned} \quad (3.6)$$

where I_{N_1} denotes the unit matrix order N_1 and $a \otimes b = (a_i b_j)_{N_1 \times N_1}$ for $a = (a_1, \dots, a_{N_1}), b = (b_1, \dots, b_{N_1})$.

So, for any $x \in \mathcal{O}$, we have

$$\begin{aligned} & -\eta\Delta\beta(x)|Z|^2 - 2D^2\beta(x)(Z, Z) \\ &= \eta(\lambda^2 |\nabla\psi|^2 + \lambda \Delta\psi) e^{\lambda\psi} |Z|^2 + \eta N_1 |Z|^2 \\ &\quad + 2(\lambda^2 (\nabla\psi \cdot Z)^2 + \lambda D^2\psi(Z, Z)) e^{\lambda\psi} + 2|Z|^2 \\ &= \lambda^2 (\eta |\nabla\psi|^2 |Z|^2 + 2(\nabla\psi \cdot Z)^2) e^{\lambda\psi} \\ &\quad + \lambda (\eta \Delta\psi |Z|^2 + 2D^2\psi(Z, Z)) e^{\lambda\psi} + (\eta N_1 + 2) |Z|^2 \\ &\geq (\eta m_*^2 \lambda^2 - (\eta + 2) m^* \lambda + \eta N_1 + 2) |Z|^2 \\ &\geq C_1 |Z|^2, \end{aligned} \quad (3.7)$$

where C_1 can be chosen to satisfy (3.5) if λ large enough, for instance, when λ so that

$$\eta m_*^2 \lambda^2 - (\eta + 2) m^* \lambda \geq 0.$$

Now, in the ball $B_\delta(0)$, we have that $\beta(x) = e^{2\lambda\|\psi\|_\infty} - \frac{1}{2}|x|^2 - |x|^\lambda$. Hence, for $\lambda > 4$, we have

$$\begin{cases} \nabla\beta(x) &= -x(1 + \lambda|x|^{\lambda-2}), \\ D^2\beta(x) &= -(I_{N_1} + \lambda I_{N_1}|x|^{\lambda-2} + \lambda(\lambda-2)(x \otimes x)|x|^{\lambda-4}), \\ \Delta\beta(x) &= -N_1 - (\lambda N_1 + \lambda(\lambda-2))|x|^{\lambda-2}. \end{cases} \quad (3.8)$$

So,

$$\begin{aligned} & -\eta\Delta\beta|Z|^2 - 2D^2\beta(Z, Z) \\ &= [\eta N_1 + 2 + \{2\lambda + \eta(\lambda N_1 + \lambda(\lambda-2))\}|x|^{\lambda-2}]|Z|^2 \\ & \quad + 2\lambda(\lambda-2)(x \cdot Z)^2|x|^{\lambda-4} \geq (\eta N_1 + 2)|Z|^2. \end{aligned}$$

Combining this with (3.7) we obtain (3.2) with $C_1 \geq \eta N_1 + 2$.

Proof of (3.3). After some computations, we get from (3.6) that

$$\begin{aligned} & \eta\Delta\beta|\nabla\beta|^2 - 2D^2\beta(\nabla\beta, \nabla\beta) \\ &= \lambda^4(2-\eta)|\nabla\psi|^4 e^{3\lambda\psi} \\ & \quad + \lambda^3 \left[-\eta(\Delta\psi|\nabla\psi|^2 e^{3\lambda\psi} + 2x \cdot \nabla\psi|\nabla\psi|^2 e^{2\lambda\psi}) \right. \\ & \quad \left. + 4|\nabla\psi|^2 \nabla\psi \cdot x e^{2\lambda\psi} + 2D^2\psi(\nabla\psi, \nabla\psi) e^{3\lambda\psi} \right] \\ & \quad + \lambda^2 \left[-\eta(|x|^2|\nabla\psi|^2 e^{\lambda\psi} + 2x \cdot \nabla\psi\Delta\psi e^{2\lambda\psi} + N_1|\nabla\psi|^2 e^{2\lambda\psi}) \right. \\ & \quad \left. + 2|\nabla\psi|^2 e^{2\lambda\psi} + 4D^2\psi(\nabla\psi, x) e^{2\lambda\psi} + 2(\nabla\psi \cdot x)^2 e^{\lambda\psi} \right] \\ & \quad + \lambda \left[-\eta(|x|^2\Delta\psi e^{\lambda\psi} + 2N_1x \cdot \nabla\psi e^{\lambda\psi}) + (2D^2\psi(x, x) + 4x \cdot \nabla\psi) e^{\lambda\psi} \right] \\ & \quad + (2 - \eta N_1 e^{\lambda\psi})|x|^2 \\ &\geq \lambda^4(2-\eta)m_*^4 e^{3\lambda\psi} - \lambda^3(\eta+2)(1+2R_{\Omega_1})(m^*)^3 e^{3\lambda\psi} \\ & \quad - \lambda^2(\eta(R_{\Omega_1}^2 + 2R_{\Omega_1} + N_1)(m^*)^2 + 4R_{\Omega_1}m^*) e^{3\lambda\psi} \\ & \quad - \lambda(\eta(R_{\Omega_1}^2 + 2N_1R_{\Omega_1}) + 2(R_{\Omega_1}^2 + 2R_{\Omega_1}))m^* e^{3\lambda\psi} - (2-\eta N_1)R_{\Omega_1}^2 e^{3\lambda\psi} \\ &\geq C_2 > 0, \text{ when } \lambda \text{ is large enough.} \end{aligned}$$

Proof of (3.4). In the ball $B_\delta(0)$, using (3.8) we have

$$\begin{aligned} & \eta\Delta\beta|\nabla\beta|^2 - 2D^2\beta(\nabla\beta, \nabla\beta) \\ &= (2 - \eta N_1)|\nabla\beta|^2 + \lambda[(\lambda(2-\eta) - 2 + \eta(2-N_1))|x|^\lambda (1 + \lambda|x|^{\lambda-2})^2 \\ & \geq (2 - \eta N_1)|\nabla\beta|^2 \geq |x|^2 \quad \text{as } \lambda > 2. \end{aligned}$$

□

3.2. Proof of Theorem 2.5

We will follow the general lines of the proof in [14] and consider the potential $\gamma_n|x|^2$ in the principal part of the operator to follow precisely the dependence on γ_n .

To prove our Carleman inequality, we will use the following improved Hardy inequality.

Lemma 3.2. *For any bounded domain Ω_1 of \mathbb{R}^{N_1} ($N_1 \geq 3$), there exists a positive constant $C_0 > 0$ such that*

$$\int_{\Omega_1} |\nabla z|^2 dx - \mu^*(N_1) \int_{\Omega_1} \frac{|z|^2}{|x|^2} dx \geq C_0 \int_{\Omega_1} \frac{|z|^2}{|x|} dx, \quad \forall z \in H_0^1(\Omega_1). \quad (3.9)$$

Proof. Applying Corollary 3 in [16, Section 2.1.6] with $m = N_1, n = 0, p = 2, 2 < q \leq (2N_1)/(N_1 - 2), \gamma = -1 + N_1(2^{-1} - q^{-1})$, we have

$$\int_{\Omega_1} |\nabla z|^2 dx - \mu^*(N_1) \int_{\Omega_1} \frac{|z|^2}{|x|^2} dx \geq \tilde{c} \| |x|^\gamma z \|_{L^q(\Omega_1)}^2, \quad \forall z \in H_0^1(\Omega_1).$$

Choosing $q = 2N_1/(N_1 - 1)$ so that $\gamma = -1/2$ and noting that $L^q(\Omega_1) \hookrightarrow L^2(\Omega_1)$ since $q > 2$, we get

$$\| |x|^\gamma z \|_{L^q(\Omega_1)}^2 \geq \tilde{C} \| |x|^\gamma z \|_{L^2(\Omega_1)}^2 = \tilde{C} \int_{\Omega_1} \frac{|z|^2}{|x|} dx.$$

Combining these two inequalities, we get the desired inequality (3.9). \square

Now, we prove Theorem 2.5. By a density argument, we can assume that

$$w \in H^1([0, T]; L^2(\Omega_1)) \cap L^2(0, T; H^2(\Omega_1) \cap H_0^1(\Omega_1)).$$

Let

$$z(x, t) = \exp(-M\sigma(x, t))w(x, t),$$

where $M = M(\lambda, T, \gamma_n, \beta) > 0$ will be chosen later on. We can see that

$$z(T) = z(0) = 0 \text{ in } H_0^1(\Omega_1)$$

due to the assumptions on σ . One has the identity

$$e^{-M\sigma} G_{n,\mu} w = G_1 z + G_2 z + G_3 z, \quad (3.10)$$

where

$$G_1 z = \Delta z + (M\sigma_t + M^2 |\nabla \sigma|^2) z + \frac{\mu}{|x|^2} z - \gamma_n |x|^2 z,$$

$$G_2 z = z_t + 2M \nabla \sigma \cdot \nabla z + M \Delta \sigma z (1 + \eta),$$

$$G_3 z = -M \eta z \Delta \sigma.$$

From (3.10), we deduce that

$$\begin{aligned} & \iint_{\Omega_1 \times (0, T)} G_1 z G_2 z \, dx dt - \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |G_3 z|^2 \, dx dt \\ & \leq \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n,\mu} w|^2 \, dx dt. \end{aligned} \quad (3.11)$$

Step 1. Computation of $\iint_{\Omega_1 \times (0, T)} G_1 z G_2 z \, dx dt - \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |G_3 z|^2 \, dx dt$.

Term concerning Δz : Integrating by parts and using the fact that

$$z_t(., t) = 0 \quad \text{on } \partial\Omega_1; \quad z(., 0) = z(., T) = 0 \quad \text{in } \Omega_1,$$

we have

$$\iint_{\Omega_1 \times (0,T)} \Delta z z_t \, dx dt = - \iint_{\Omega_1 \times (0,T)} \nabla z \cdot \nabla z_t \, dx dt = \frac{1}{2} \int_0^T \frac{d}{dt} \int_{\Omega_1} |\nabla z|^2 \, dx dt = 0. \quad (3.12)$$

Using Green's formula and noticing that $\nabla z = \frac{\partial z}{\partial \nu} \nu$ on $\partial\Omega_1$, we get

$$\begin{aligned} 2M \iint_{\Omega_1 \times (0,T)} \Delta z \nabla \sigma \cdot \nabla z \, dx dt &= 2M \iint_{\partial\Omega_1 \times (0,T)} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{\partial \sigma}{\partial \nu} \, ds dt \\ &\quad + M \iint_{\Omega_1 \times (0,T)} (|\nabla z|^2 \Delta \sigma - 2D^2 \sigma (\nabla z, \nabla z)) \, dx dt. \end{aligned} \quad (3.13)$$

Also by Green's formula, we have

$$\begin{aligned} M \iint_{\Omega_1 \times (0,T)} \Delta z \Delta \sigma z (1 + \eta) \, dx dt &= -M \iint_{\Omega_1 \times (0,T)} \nabla z \cdot \nabla (z \Delta \sigma (1 + \eta)) \, dx dt \\ &= M \iint_{\Omega_1 \times (0,T)} \left(\frac{\Delta^2 \sigma}{2} |z|^2 (1 + \eta) - \Delta \sigma |\nabla z|^2 (1 + \eta) \right) \, dx dt \\ &\quad + M \iint_{\Omega_1 \times (0,T)} |z|^2 (\nabla \eta \cdot \nabla \Delta \sigma + \frac{1}{2} \Delta \sigma \Delta \eta) \, dx dt. \end{aligned} \quad (3.14)$$

Term concerning $(M\sigma_t + M^2|\nabla\sigma|^2)zz_t$: Integrating by parts, we have

$$\iint_{\Omega_1 \times (0,T)} (M\sigma_t + M^2|\nabla\sigma|^2) z z_t \, dx dt = -\frac{1}{2} \iint_{\Omega_1 \times (0,T)} (M\sigma_t + M^2|\nabla\sigma|^2)_t |z|^2 \, dx dt. \quad (3.15)$$

Using Green's formula, then

$$\begin{aligned} 2M \iint_{\Omega_1 \times (0,T)} (M\sigma_t + M^2|\nabla\sigma|^2) z \nabla \sigma \cdot \nabla z \, dx dt \\ = -M \iint_{\Omega_1 \times (0,T)} \operatorname{div} [(M\sigma_t + M^2|\nabla\sigma|^2) \nabla \sigma] |z|^2 \, dx dt. \end{aligned} \quad (3.16)$$

And the last term is

$$M \iint_{\Omega_1 \times (0,T)} (M\sigma_t + M^2|\nabla\sigma|^2) \Delta \sigma (1 + \eta) |z|^2 \, dx dt. \quad (3.17)$$

Term concerning $-\gamma_n |x|^2 z$: integrating by parts and using the fact that $z(T) = z(0) = 0$, we get

$$\begin{aligned} -\gamma_n \iint_{\Omega_1 \times (0,T)} |x|^2 z (z_t + 2M \nabla \sigma \cdot \nabla z + M z \Delta \sigma (1 + \eta)) \, dx dt \\ = M \gamma_n \iint_{\Omega_1 \times (0,T)} (\operatorname{div}(|x|^2 \nabla \sigma) - |x|^2 \Delta \sigma (1 + \eta)) |z|^2 \, dx dt. \end{aligned} \quad (3.18)$$

Term concerning $\frac{\mu}{|x|^2} z$: integrating by parts, we get

$$\iint_{\Omega_1 \times (0,T)} \frac{\mu}{|x|^2} z z_t \, dx dt = 0, \quad (3.19)$$

and

$$2\mu M \iint_{\Omega_1 \times (0,T)} \frac{z}{|x|^2} \nabla \sigma \cdot \nabla z \, dx dt = -\mu M \iint_{\Omega_1 \times (0,T)} \frac{|z|^2}{|x|^2} \Delta \sigma \, dx dt + 2\mu M \iint_{\Omega_1 \times (0,T)} \frac{|z|^2}{|x|^3} \partial_r \sigma \, dx dt, \quad (3.20)$$

where $\partial_r \sigma = \frac{x}{|x|} \cdot \nabla \sigma$. And the final term is

$$\iint_{\Omega_1 \times (0,T)} \mu \frac{z}{|x|^2} M \Delta \sigma z (1 + \eta) \, dx dt = \mu M \iint_{\Omega_1 \times (0,T)} \frac{|z|^2}{|x|^2} \Delta \sigma (1 + \eta) \, dx dt. \quad (3.21)$$

Combining from (3.12) to (3.21), then (3.11) becomes

$$\begin{aligned} & 2M \iint_{\partial\Omega_1 \times (0,T)} \left(\frac{\partial z}{\partial \nu} \right)^2 \frac{\partial \sigma}{\partial \nu} \, ds dt - M \iint_{\Omega_1 \times (0,T)} (2D^2 \sigma (\nabla z, \nabla z) + \eta \Delta \sigma |\nabla z|^2) \, dx dt \\ & + \mu M \iint_{\Omega_1 \times (0,T)} \frac{|z|^2}{|x|^2} \eta \Delta \sigma \, dx dt + 2\mu M \iint_{\Omega_1 \times (0,T)} \frac{|z|^2}{|x|^3} \partial_r \sigma \, dx dt \\ & + \iint_{\Omega_1 \times (0,T)} \{f + M\gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - |x|^2 \Delta \sigma (1 + \eta)]\} |z|^2 \, dx dt \\ & \leq \frac{1}{2} \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 \, dx dt, \end{aligned} \quad (3.22)$$

where

$$\begin{aligned} f = & \frac{M}{2} \Delta^2 \sigma (1 + \eta) - \frac{1}{2} (M\sigma_t + M^2 |\nabla \sigma|^2)_t - M \operatorname{div} [(M\sigma_t + M^2 |\nabla \sigma|^2) \nabla \sigma] \\ & - \frac{1}{2} M^2 \eta^2 (\Delta \sigma)^2 + M (M\sigma_t + M^2 |\nabla \sigma|^2) \Delta \sigma (1 + \eta) \\ & + M \left(\nabla \eta \cdot \nabla \Delta \sigma + \frac{1}{2} \Delta \sigma \Delta \eta \right). \end{aligned}$$

Step 2. Estimation of the terms in (3.22). We can see that

$$\begin{aligned} f = & \frac{1}{(t(T-t))^9} \left(M^3 [\eta |\nabla \beta|^2 \Delta \beta - 2D^2 \beta (\nabla \beta, \nabla \beta)] \right. \\ & + M^2 \left[(T-2t)(t(T-t))^2 (6|\nabla \beta|^2 - 3\eta \beta \Delta \beta) - \frac{\eta^2}{2} (t(T-t))^3 (\Delta \beta)^2 \right] \\ & + M \left[\left(\frac{1+\eta}{2} \Delta^2 \beta + \nabla \eta \cdot \nabla \Delta \beta + \frac{1}{2} \Delta \beta \Delta \eta \right) (t(T-t))^6 \right. \\ & \left. \left. - 3(2T^2 - 7Tt + 7t^2)(t(T-t))^4 \beta \right] \right). \end{aligned}$$

Hence, using (3.2)–(3.3) and the C^4 –regularity of β on $\overline{\Omega}_1$, the definition of η , we get positive constants $C_3 = C_3(\beta)$, $C_4 = C_4(\beta)$, $c = c(\beta)$ such that

$$-\eta \Delta \beta |\nabla z|^2 - 2D^2 \sigma (\nabla z, \nabla z) \geq C_1 M |\nabla z|^2, \quad \forall (x, t) \in \overline{\Omega}_1 \setminus \tilde{\omega}_1 \times (0, T), \quad (3.23)$$

$$|-\eta \Delta \beta |\nabla z|^2 - 2D^2 \sigma (\nabla z, \nabla z)| \leq C_3 M |\nabla z|^2, \quad \forall (x, t) \in \overline{\omega}_1 \times (0, T), \quad (3.24)$$

and

$$f \geq \frac{1}{(t(T-t))^9} [C_2 M^3 - c(T^5 + T^6)M^2 - c(T^5 + T^6)^2 M], \quad \forall (x, t) \in \overline{\mathcal{O}} \times (0, T),$$

$$|f| \leq \frac{1}{(t(T-t))^9} [C_4 M^3 + c(T^5 + T^6)M^2 + c(T^5 + T^6)^2 M], \quad \forall (x, t) \in \overline{\omega_1} \times (0, T).$$

Thus, there exist positive constants $m_1 = m_1(\beta)$, $C'_2 = C'_2(\beta)$, $C'_4 = C'_4(\beta)$ such that for $M \geq M_1(T, \beta) := m_1(\beta)(T^5 + T^6)$, we have

$$f \geq \frac{C'_2 M^3}{(t(T-t))^9}, \quad \forall (x, t) \in \overline{\mathcal{O}} \times (0, T),$$

$$|f| \leq \frac{C'_4 M^3}{(t(T-t))^9}, \quad \forall (x, t) \in \overline{\omega_1} \times (0, T). \quad (3.25)$$

Now, we consider f in $B_\delta(0)$. In $B_\delta(0)$ we have $\beta(x) = e^{2\lambda\|\psi\|_\infty} - \frac{|x|^2}{2} - |x|^\lambda$. Thus, by (3.8) with note that $\eta(x)$ vanishes in $B_{\delta/2}(0)$, there exists $\bar{c}(\lambda, \delta)$ such that for $\lambda > 4$,

$$(T-2t)(t(T-t))^2 (9|\nabla\beta|^2 3\eta\beta\Delta\beta) - \eta^2(t(T-t))^3 (\Delta\beta)^2$$

$$\geq -\bar{c}(T^5 + T^6)|x|^2, \quad \forall (x, t) \in B_\delta(0) \times (0, T). \quad (3.26)$$

Using (3.4) and (3.26), we obtain

$$f - f_1 \geq \frac{|x|^2}{(t(T-t))^9} [M^3 - \bar{c}(T^5 + T^6)M^2], \quad \forall (x, t) \in B_\delta(0) \times (0, T), \quad (3.27)$$

where

$$f_1 := \frac{M}{(t(T-t))^5} \left[\left(\frac{1+\eta}{2} \Delta^2 \beta + \nabla\eta \cdot \nabla\Delta\beta + \frac{1}{2} \Delta\beta\Delta\eta \right) (t(T-t))^2 \right. \\ \left. - 3(2T^2 - 7Tt + 7t^2)\beta \right].$$

From (3.27), there exist a positive constant $\bar{m}_1 = \bar{m}_1(\lambda, \delta)$ such that for $M \geq \bar{M}_1(T, \lambda) := \bar{m}_1(T^5 + T^6)$, we have

$$f - f_1 \geq \frac{M^3|x|^2}{2(t(T-t))^9} \quad \forall (x, t) \in B_\delta(0) \times (0, T). \quad (3.28)$$

Using the definition of β and η in $B_\delta(0)$, we get the positive constant $\tilde{c}(\lambda, \eta)$ such that

$$f_1 \geq -\frac{M}{(t(T-t))^5} \tilde{c}(T^2 + T^4), \quad \forall (x, t) \in B_\delta(0) \times (0, T). \quad (3.29)$$

By (3.1), (3.23), (3.24), (3.25), (3.28) and (3.29) then for $M \geq M_2(T, \beta, \lambda)$, we get from (3.22) that

$$\begin{aligned}
& \iint_{\Omega_1 \setminus \tilde{\omega}_1 \times (0, T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dx dt + \iint_{\Omega_1 \times (0, T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dx dt \\
& - \iint_{B_\delta(0) \times (0, T)} \frac{\tilde{c}(T^2 + T^4) M}{(t(T-t))^5} |z|^2 dx dt + \iint_{B_\delta(0) \times (0, T)} \frac{|x|^2 M^3}{2(t(T-t))^9} |z|^2 dx dt \\
& + \iint_{\mathcal{O} \times (0, T)} \frac{C'_2 M^3}{(t(T-t))^9} |z|^2 dx dt + \iint_{\Omega_1 \setminus \tilde{\omega}_1 \times (0, T)} M \gamma_n \\
& \times [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^2 \Delta \sigma] |z|^2 dx dt \\
& \leq \iint_{\tilde{\omega}_1 \times (0, T)} \left(\frac{C'_4 M^3}{(t(T-t))^9} - M \gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^{2s} \Delta \sigma] \right) |z|^2 dx dt \\
& + \iint_{\tilde{\omega}_1 \times (0, T)} \frac{C_2 M}{(t(T-t))^3} |\nabla z|^2 dx dt + \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n, \mu} w|^2 dx dt,
\end{aligned} \tag{3.30}$$

where $M_2 = M_2(T, \beta, \lambda) := \tilde{m}_1(\beta, \lambda)(T^5 + T^6)$ with $\tilde{m}_1(\beta, \lambda) = \max\{m_1, \overline{m}_1\}$.

Step 2.1. Estimation of the degeneracy. There exists constant $C_5 = C_5(\beta) > 0$ such that

$$\begin{aligned}
& |M \gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^2 \Delta \sigma]| \leq \frac{C_5 M \gamma_n}{(t(T-t))^3}, \\
& \forall (x, t) \in (\Omega_1 \setminus B_\delta(0)) \times (0, T).
\end{aligned} \tag{3.31}$$

Let $M_3 = M_3(T, \gamma_n, \beta)$ be defined by $M_3 = M_3(T, \gamma_n, \beta) := \sqrt{(2C_5)/C'_2}(T/2)^6 \sqrt{\gamma_n}$. So, for $M \geq M_3$, we deduce from (3.31) that

$$\begin{aligned}
& |M \gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^2 \Delta \sigma]| \leq \frac{C'_2 M^3}{2(t(T-t))^9}, \\
& \forall (x, t) \in (\Omega_1 \setminus B_\delta(0)) \times (0, T).
\end{aligned} \tag{3.32}$$

On the other hand, in the ball $B_\delta(0)$, we can see that there exists $C_6(\lambda)$ such that

$$|M \gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^2 \Delta \sigma]| \leq \frac{C_6 |x|^2 M \gamma_n}{(t(T-t))^3}, \quad \forall (x, t) \in B_\delta(0) \times (0, T). \tag{3.33}$$

Let $\overline{M}_3 = \overline{M}_3(\lambda, T, \gamma_n)$ be defined by $\overline{M}_3 = \overline{M}_3(\lambda, T, \gamma_n) := 2\sqrt{C_6}(T/2)^6 \sqrt{\gamma_n}$. So, for $M \geq \overline{M}_3$, we deduce from (3.33) that

$$|M \gamma_n [\operatorname{div}(|x|^2 \nabla \sigma) - (1 + \eta) |x|^2 \Delta \sigma]| \leq \frac{M^3 |x|^2 \gamma_n}{4(t(T-t))^9}, \quad \forall (x, t) \in B_\delta(0) \times (0, T). \tag{3.34}$$

Taking $M = M(\lambda, T, \gamma_n, \beta) := K_2 \max\{T^5 + T^6, \sqrt{\gamma_n} T^6\}$ with

$$K_2 := \max \left\{ \tilde{m}_1, \frac{\sqrt{2C_5/C'_2}}{64}, \frac{\sqrt{C_6}}{32} \right\},$$

then $M \geq M_2, M_3$ and \overline{M}_3 . So, from (3.32) and (3.34), then (3.30) becomes

$$\begin{aligned}
& \iint_{\Omega_1 \setminus \tilde{\omega}_1 \times (0,T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dxdt + \iint_{\Omega_1 \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dxdt \\
& - \iint_{B_\delta(0) \times (0,T)} \frac{\tilde{c}(T^2 + T^4)M}{(t(T-t))^5} |z|^2 dxdt \\
& + \iint_{B_\delta(0) \times (0,T)} \frac{M^3 |x|^2}{4(t(T-t))^9} |z|^2 dxdt + \iint_{\mathcal{O} \times (0,T)} \frac{C'_2 M^3}{2(t(T-t))^9} |z|^2 dxdt \\
& \leq \iint_{\tilde{\omega}_1 \times (0,T)} \frac{C_7 M^3}{(t(T-t))^9} |z|^2 dxdt + \iint_{\tilde{\omega}_1 \times (0,T)} \frac{C_2 M}{(t(T-t))^3} |\nabla z|^2 dxdt \\
& + \frac{1}{2} \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 dxdt, \tag{3.35}
\end{aligned}$$

where $C_7 = C_7(\beta, \lambda) := C'_4 + C'_2/2$.

We add the same quantity to both sides of (3.35) to obtain

$$\begin{aligned}
& \iint_{\Omega_1 \times (0,T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dxdt + \iint_{\Omega_1 \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dxdt \\
& - \iint_{B_\delta(0) \times (0,T)} \frac{\tilde{c}(T^2 + T^4)M}{(t(T-t))^5} |z|^2 dxdt \\
& + \iint_{B_\delta(0) \times (0,T)} \frac{M^3 |x|^2}{4(t(T-t))^9} |z|^2 dxdt + \iint_{\mathcal{O} \times (0,T)} \frac{C'_2 M^3}{2(t(T-t))^9} |z|^2 dxdt \\
& \leq \iint_{\tilde{\omega}_1 \times (0,T)} \frac{C_7 M^3}{(t(T-t))^9} |z|^2 dxdt + \iint_{\tilde{\omega}_1 \times (0,T)} \frac{C_8 M}{(t(T-t))^3} |\nabla z|^2 dxdt \\
& + \frac{1}{2} \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 dxdt, \tag{3.36}
\end{aligned}$$

where $C_8 = C_1 + C_2$.

Step 2.2. Estimation of the singular potential: we consider

$$\begin{aligned}
& \iint_{\Omega_1 \times (0,T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dxdt + \iint_{\Omega_1 \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dxdt \\
& = I_{in} + I_{out},
\end{aligned}$$

where

$$\begin{aligned}
I_{in} &:= \iint_{B_\delta(0) \times (0,T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dxdt \\
& + \iint_{B_\delta(0) \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dxdt
\end{aligned}$$

and

$$\begin{aligned}
I_{out} &:= \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dxdt \\
& + \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dxdt.
\end{aligned}$$

We first consider I_{in} . Recall that in the ball $B_\delta(0)$, $\beta(x) = e^{2\lambda\|\psi\|_\infty} - \frac{1}{2}|x|^2 - |x|^\lambda$ and

$$\begin{cases} \nabla\beta(x) &= -x(1 + \lambda|x|^{\lambda-2}), \\ \Delta\beta(x) &= -N_1 - (\lambda N_1 + \lambda(\lambda - 2))|x|^{\lambda-2}. \end{cases}$$

Hence

$$\begin{aligned} & \iint_{B_\delta(0) \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dx dt \\ &= -\mu \iint_{B_\delta(0) \times (0,T)} (\eta N_1 + 2) \frac{M}{(t(T-t))^3} \frac{|z|^2}{|x|^2} dx dt \\ &\quad - \mu \iint_{B_\delta(0) \times (0,T)} [\eta \lambda (\lambda + N_1 - 2) + 2\lambda] \frac{M}{(t(T-t))^3} |x|^{\lambda-4} |z|^2 dx dt \\ &= -2\mu \iint_{B_\delta(0) \times (0,T)} \frac{M}{(t(T-t))^3} \frac{|z|^2}{|x|^2} dx dt \\ &\quad - \mu \iint_{B_\delta(0) \times (0,T)} \left([\eta \lambda (\lambda + N_1 - 2) + 2\lambda] |x|^{\lambda-4} + \eta N_1 \frac{1}{|x|^2} \right) \\ &\quad \times \frac{M}{(t(T-t))^3} |z|^2 dx dt. \end{aligned} \quad (3.37)$$

We note that since η vanishes in $B_{\delta/2}(0)$, then when $\lambda > 4$, there exists $C_9(\lambda, \delta) > 0$ such that

$$\mu[\eta \lambda (\lambda + N_1 - 2) + 2\lambda] |x|^{\lambda-4} + \mu \eta N_1 \frac{1}{|x|^2} \leq C_9, \quad \forall x \in B_\delta(0).$$

Therefore, from (3.37) with note that $C_1 \geq \eta N_1 + 2 \geq 2$ and (3.5), then

$$\begin{aligned} I_{in} &\geq 2 \iint_{B_\delta(0) \times (0,T)} \frac{M}{(t(T-t))^3} \left(|\nabla z|^2 - \mu \frac{|z|^2}{|x|^2} \right) dx dt \\ &\quad - C_9 \iint_{B_\delta(0) \times (0,T)} \frac{M}{(t(T-t))^3} |z|^2 dx dt. \end{aligned} \quad (3.38)$$

Now we consider I_{out} . For $x \in \Omega_1 \setminus B_\delta(0)$ then

$$\begin{aligned} & \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dx dt \\ &= -\mu \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} \frac{(\eta N_1 + 2)M}{(t(T-t))^3} \frac{|z|^2}{|x|^2} dx dt \\ &\quad - 2\mu \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} \left(\frac{\eta(\lambda \Delta \psi + \lambda^2 |\nabla \psi|^2)}{|x|^2} + \lambda \nabla \psi \cdot x \frac{1}{|x|^4} \right) \\ &\quad \times e^{\lambda \psi} \frac{M}{(t(T-t))^3} |z|^2 dx dt. \end{aligned}$$

Hence, there exists a positive constant $C_{11}(\beta, \mu, \delta)$ such that

$$\begin{aligned} \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dx dt &\geq -\mu \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \\ &\times \frac{(\eta N_1 + 2)M}{(t(T-t))^3} \frac{|z|^2}{|x|^2} dx dt - C_{10} \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{M}{(t(T-t))^3} |z|^2 dx dt. \end{aligned} \quad (3.39)$$

So, using (3.5) once again, then we have from (3.39) that

$$\begin{aligned} I_{out} &\geq 2 \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{M}{(t(T-t))^3} \left(|\nabla z|^2 - \mu \frac{|z|^2}{|x|^2} \right) dx dt - \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \\ &\times \frac{C_{11}M}{(t(T-t))^3} |z|^2 dx dt + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{\eta N_1 M}{(t(T-t))^3} |\nabla z|^2 dx dt. \end{aligned} \quad (3.40)$$

Combining (3.38), (3.40) and using the improved Hardy inequality (3.9), we have

$$\begin{aligned} &\iint_{\Omega_1 \times (0, T)} \frac{C_1 M}{(t(T-t))^3} |\nabla z|^2 dx dt + \iint_{\Omega_1 \times (0, T)} \left[\mu M \eta \Delta \sigma \frac{|z|^2}{|x|^2} + 2\mu M \frac{|z|^2}{|x|^3} \partial_r \sigma \right] dx dt \\ &\geq \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{M}{(t(T-t))^3} |\nabla z|^2 dx dt + \iint_{\Omega_1 \times (0, T)} \frac{2C_0 M}{(t(T-t))^3} \frac{|z|^2}{|x|} dx dt \\ &\quad - \iint_{\Omega_1 \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dx dt \end{aligned} \quad (3.41)$$

with $C_{11} = \max\{C_9; C_{10}\}$.

From (3.36) and (3.41), we deduce

$$\begin{aligned} &\iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{M}{(t(T-t))^3} |\nabla z|^2 dx dt + \iint_{\Omega_1 \times (0, T)} \frac{2C_0 M}{(t(T-t))^3} \frac{|z|^2}{|x|} dx dt \\ &\quad - \iint_{B_\delta(0) \times (0, T)} \frac{\tilde{c}(T^2 + T^4)M}{(t(T-t))^5} |z|^2 dx dt - \iint_{\Omega_1 \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dx dt \\ &\quad + \iint_{B_\delta(0) \times (0, T)} \frac{M^3 |x|^2}{4(t(T-t))^9} |z|^2 dx dt + \iint_{\mathcal{O} \times (0, T)} \frac{C'_2 M^3}{2(t(T-t))^9} |z|^2 dx dt \\ &\leq \iint_{\tilde{\omega}_1 \times (0, T)} \frac{C_7 M^3}{(t(T-t))^9} |z|^2 dx dt + \iint_{\tilde{\omega}_1 \times (0, T)} \frac{C_8 M}{(t(T-t))^3} |\nabla z|^2 dx dt \\ &\quad + \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n,\mu} w|^2 dx dt. \end{aligned} \quad (3.42)$$

Now, by the inequality $ab \leq \frac{1}{3}a^3 + \frac{2}{3}b^{3/2}$ ($a, b > 0$), we have

$$\begin{aligned}
& \iint_{B_\delta(0) \times (0, T)} \frac{\tilde{c}(T^2 + T^4)M}{(t(T-t))^5} |z|^2 dxdt + \iint_{\Omega_1 \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dxdt \\
& \leq \iint_{B_\delta(0) \times (0, T)} \frac{(\tilde{c} + C_{11}/16)(T^2 + T^4)M}{(t(T-t))^5} |z|^2 dxdt \\
& \quad + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dxdt \\
& = M \iint_{B_\delta(0) \times (0, T)} \left(\frac{(\tilde{c} + C_{11}/16)(T^2 + T^4)|x|^{2/3}|z|^{2/3}}{C_0^{2/3}(t(T-t))^3} \right) \\
& \quad \times \left(\frac{C_0^{2/3}|z|^{4/3}}{(t(T-t))^2|x|^{2/3}} \right) dxdt + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dxdt \\
& \leq \iint_{B_\delta(0) \times (0, T)} \frac{2C_0M}{3(t(T-t))^3} \frac{|z|^2}{|x|} dxdt \\
& \quad + \iint_{B_\delta(0) \times (0, T)} \frac{(\tilde{c} + C_{11}/16)^3 (T^2 + T^4)^3 M |x|^2}{3C_0^2(t(T-t))^9} |z|^2 dxdt \\
& \quad + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{C_{11}M}{(t(T-t))^3} |z|^2 dxdt. \tag{3.43}
\end{aligned}$$

From now on, if we take

$$M = M(\lambda, T, \gamma_n, \beta) := \mathcal{K}_2 \max\{T^3 + T^4 + T^5 + T^6; \sqrt{\gamma_n}T^6\},$$

where

$$\mathcal{K}_2 = \mathcal{K}_2(\beta, \delta) := \max \left\{ \sqrt{8(\tilde{c} + C_{11}/16)^3/(3C_0^2)}; \sqrt{\frac{C_{11}}{C_2'}}/(32); K_2 \right\},$$

then

$$\frac{(\tilde{c} + C_{11}/16)^3 (T^2 + T^4)^3 M}{3C_0^2} \leq \frac{M^3}{8}.$$

Thus, from (3.43), (3.42) becomes

$$\begin{aligned}
& \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{M}{(t(T-t))^3} |\nabla z|^2 dxdt + \iint_{\Omega_1 \times (0, T)} \frac{4C_0M}{3(t(T-t))^3} \frac{|z|^2}{|x|} dxdt \\
& \quad + \iint_{B_\delta(0) \times (0, T)} \frac{M^3|x|^2}{8(t(T-t))^9} |z|^2 dxdt + \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} \frac{C_2'M^3}{4(t(T-t))^9} |z|^2 dxdt \\
& \leq \iint_{\tilde{\omega}_1 \times (0, T)} \frac{C_{12}M^3}{(t(T-t))^9} |z|^2 dxdt + \iint_{\tilde{\omega}_1 \times (0, T)} \frac{C_8M}{(t(T-t))^3} |\nabla z|^2 dxdt \\
& \quad + \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n,\mu} w|^2 dxdt, \tag{3.44}
\end{aligned}$$

where $C_{12} = C_7 + C_2'/2$.

Since $z = e^{-M\sigma}w$, by the Cauchy inequality, we have

$$\begin{aligned}
 & \frac{M}{(t(T-t))^3} |\nabla z|^2 + \frac{C'_2 M^3}{4(t(T-t))^9} |z|^2 \\
 &= e^{-2M\sigma} \left(\frac{M}{(t(T-t))^3} |\nabla w - M(\nabla\sigma)w|^2 + \frac{C'_2 M^3}{4C_0(t(T-t))^3} |w|^2 \right) \\
 &\geq e^{-2M\sigma} \left(\frac{C_{13}M}{(t(T-t))^3} |\nabla w|^2 + \frac{C'_2 M^3}{8(t(T-t))^9} |w|^2 \right), \\
 &\quad \forall (x, t) \in \Omega_1 \setminus B_\delta(0) \times (0, T),
 \end{aligned} \tag{3.45}$$

where $C_{13} = C_{13}(\beta) = C'_2/(4\|\nabla\beta\|_\infty^2 + C'_2)$, and

$$\begin{aligned}
 & \frac{C_8 M}{(t(T-t))^3} |\nabla z|^2 = e^{-2M\sigma} \frac{C_8 M}{(t(T-t))^3} |\nabla w - M(\nabla\sigma)w|^2 \\
 &\leq e^{-2M\sigma} \left(\frac{2C_8 M}{(t(T-t))^3} |\nabla w|^2 + \frac{2C_8 \|\nabla\beta\|_\infty^2 M^3}{(t(T-t))^9} |w|^2 \right), \\
 &\quad \forall (x, t) \in \tilde{\omega}_1 \times (0, T).
 \end{aligned} \tag{3.46}$$

Hence, from (3.45) and (3.46) then (3.44) becomes

$$\begin{aligned}
 & \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{C_{13}M}{(t(T-t))^3} |\nabla w|^2 dx dt \\
 &+ \iint_{\Omega_1 \times (0, T)} e^{-2M\sigma} \frac{4C_0 M}{3(t(T-t))^3} \frac{|w|^2}{|x|} dx dt \\
 &+ \iint_{B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{M^3 |x|^2}{8(t(T-t))^9} |w|^2 dx dt \\
 &+ \iint_{\Omega_1 \setminus B_\delta(0) \times (0, T)} e^{-2M\sigma} \frac{C'_2 M^3}{8(t(T-t))^9} |w|^2 dx dt \\
 &\leq \iint_{\tilde{\omega}_1 \times (0, T)} e^{-2M\sigma} \frac{(2C_8 \|\nabla\beta\|_\infty^2 + C_{12})M^3}{(t(T-t))^9} |w|^2 dx dt \\
 &+ \iint_{\tilde{\omega}_1 \times (0, T)} e^{-2M\sigma} \frac{C_{14}M}{(t(T-t))^3} |\nabla w|^2 dx dt \\
 &+ \frac{1}{2} \iint_{\Omega_1 \times (0, T)} |e^{-M\sigma} G_{n,\mu} w|^2 dx dt.
 \end{aligned} \tag{3.47}$$

Here $C_{14} = 2C_8$.

Step 3. Estimation of $\iint_{\tilde{\omega}_1 \times (0, T)} e^{-2M\sigma} \frac{C_{14}M}{(t(T-t))^3} |\nabla w|^2 dx dt$. This step is to prove that the term $\iint_{\tilde{\omega}_1 \times (0, T)} e^{-2M\sigma} \frac{C_{14}M}{(t(T-t))^3} |\nabla w|^2 dx dt$ can be absorbed by the other terms. This proof is similar to the proof of Cacciopoli's type inequality (see [20]). To end this, we consider the function $\xi : \bar{\Omega}_1 \rightarrow \mathbb{R}$

such that

$$\begin{cases} 0 \leq \xi(x) \leq 1, & x \in \overline{\Omega_1}, \\ \xi(x) = 1, & x \in \tilde{\omega}_1, \\ \xi(x) = 0, & x \notin \omega_1. \end{cases}$$

By the definitions of σ and ξ , we integrate by parts to obtain

$$\begin{aligned} & - \iint_{\Omega_1 \times (0,T)} G_{n,\mu} w \frac{w \xi e^{-2M\sigma}}{(t(T-t))^3} dx dt \\ &= \iint_{\Omega_1 \times (0,T)} \left(-w_t - \Delta w - \frac{\mu}{|x|^2} w + \gamma_n |x|^2 w \right) \frac{w \xi e^{-2M\sigma}}{(t(T-t))^3} dx dt \\ &= \iint_{\Omega_1 \times (0,T)} \frac{\xi e^{-2M\sigma}}{(t(T-t))^3} |\nabla w|^2 dx dt + \gamma_n \iint_{\Omega_1 \times (0,T)} |x|^2 \frac{\xi e^{-2M\sigma}}{(t(T-t))^3} |w|^2 dx dt \\ & \quad - \iint_{\Omega_1 \times (0,T)} \frac{\xi |w|^2 e^{-2M\sigma}}{2(t(T-t))^3} [\Delta \xi - 4M \nabla \xi \cdot \nabla \sigma + \xi (4M^2 |\nabla \sigma|^2 - 2M \Delta \sigma)] dx dt \\ & \quad - \iint_{\Omega_1 \times (0,T)} \xi \frac{|w|^2 e^{-2M\sigma}}{(t(T-t))^3} \left[M \sigma_t - 3 \frac{T-2t}{2(t(T-t))^4} + \frac{\mu}{|x|^2} \right] dx dt. \end{aligned}$$

Hence

$$\begin{aligned} & \iint_{\Omega_1 \times (0,T)} \frac{\xi C_{14} M e^{-2M\sigma}}{(t(T-t))^3} |\nabla w|^2 dx dt \leq - \iint_{\Omega_1 \times (0,T)} G_{n,\mu} w \frac{C_{14} M w \xi e^{-2M\sigma}}{(t(T-t))^3} dx dt \\ & \quad + \iint_{\Omega_1 \times (0,T)} \frac{C_{14} M |w|^2 e^{-2M\sigma}}{2(t(T-t))^3} [\Delta \xi - 4M \nabla \xi \cdot \nabla \sigma + \rho (4M^2 |\nabla \sigma|^2 - 2M \Delta \sigma)] dx dt \\ & \quad + \iint_{\Omega_1 \times (0,T)} \frac{C_{14} M \xi |w|^2 e^{-2M\sigma}}{(t(T-t))^3} \left[\frac{\mu}{|x|^2} - 2M \sigma_t - 3 \frac{T-2t}{(t(T-t))^4} \right] dx dt \\ & \leq \frac{1}{2} \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 dx dt + \iint_{\omega_1 \times (0,T)} \frac{C_{15} M^3 e^{-2M\sigma}}{(t(T-t))^9} |w|^2 dx dt \end{aligned}$$

for some positive constant $C_{15} = C_{15}(\beta, \xi)$. Here, we have used the fact that $0_{\mathbb{R}^{N_1}} \notin \tilde{\omega}_1$ and $\text{supp}(\xi), \text{supp}(\Delta \xi), \text{supp}(\nabla \xi) \subset \omega_1$. So,

$$\begin{aligned} & \iint_{\tilde{\omega}_1 \times (0,T)} e^{-2M\sigma} \frac{C_{14} M}{(t(T-t))^3} |\nabla w|^2 dx dt \leq \iint_{\Omega_1 \times (0,T)} \frac{\xi C_{14} M e^{-2M\sigma}}{(t(T-t))^9} |\nabla w|^2 dx dt \\ & \leq \frac{1}{2} \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 dx dt + \iint_{\omega_1 \times (0,T)} \frac{C_{15} M^3 e^{-2M\sigma}}{(t(T-t))^9} |w|^2 dx dt. \end{aligned} \tag{3.48}$$

Combining (3.48) with (3.47), we get

$$\begin{aligned}
& \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} e^{-2M\sigma} \frac{C_{13}M}{(t(T-t))^3} |\nabla w|^2 dxdt \\
& + \iint_{\Omega_1 \times (0,T)} e^{-2M\sigma} \frac{4C_0M}{3(t(T-t))^3} \frac{|w|^2}{|x|} dxdt \\
& + \iint_{B_\delta(0) \times (0,T)} e^{-2M\sigma} \frac{M^3|x|^2}{8(t(T-t))^9} |w|^2 dxdt \\
& + \iint_{\Omega_1 \setminus B_\delta(0) \times (0,T)} e^{-2M\sigma} \frac{C'_2M^3}{8(t(T-t))^9} |w|^2 dxdt \\
& \leq \iint_{\omega_1 \times (0,T)} e^{-2M\sigma} \frac{C_{16}M^3}{(t(T-t))^9} |w|^2 dxdt + \iint_{\Omega_1 \times (0,T)} |e^{-M\sigma} G_{n,\mu} w|^2 dxdt
\end{aligned}$$

with $C_{16} = C_{16}(\beta) = 2C_8 \|\nabla \beta\|_\infty^2 + C_{12} + C_{15}$. Then the Carleman inequality (2.7) holds with

$$\mathcal{K}_1 = \mathcal{K}_1(\beta) := \frac{\min\{C_{13}, 4C_0/3, 1/8, C'_2/8\}}{\max\{1, C_{16}\}}.$$

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