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Two solutions for a singular elliptic problem indefinite in sign

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Abstract. In this paper we deal with a singular elliptic problem involving a nonlinearity which is indefinite in sign. We prove the existence of two non negative solutions, one of them being positive. The approach relies on suitable truncation methods and variational arguments.

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1. Introduction

In the present paper we deal with a semilinear problem involving a singular term of the following type:

$$\begin{cases} -\Delta u = \left(\lambda u^{s-1} - u^{r-1}\right)\chi_{\{u>0\}}, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases} \qquad (P_\lambda)$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \in \mathbb{N})$ with smooth boundary $\partial\Omega$, $0 < r \leq 1 < s < 2$, λ is a positive parameter and $\chi_{\{u>0\}}$ the characteristic function corresponding to the set $\{u > 0\}$.

A *weak* solution of (P_λ) is a function $u \in W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \nabla \varphi - \int_\Omega \left(\lambda u^{s-1} - u^{r-1}\right)\chi_{\{u>0\}}\varphi = 0$$

for every $\varphi \in C_c^1(\Omega)$.

A *classical* solution of (P_λ) is a function $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ such that (P_λ) is satisfied pointwisely.

Problems of (P_λ) type have been studied for instance in [3–5].

In [3], Dávila and Montenegro prove the existence, for every $\lambda > 0$, of a non negative, classical solution obtained as the limit of a sequence of solutions of perturbed problems.

The existence of a positive solution has been proved in [4] by Hernàndez, Mancebo and Vega in

Theorem A. ([4], Theorem 3.13) *There exists* $\Lambda > 0$ *such that, for all* $\lambda \in]\Lambda, +\infty[$*, problem* (P_λ) *has a classical solution belonging to* $\mathrm{int}(C_0^1(\overline{\Omega})_+)$*. If* $\lambda \in]0, \Lambda[$ *problem* (P_λ) *has no positive solution.*

The existence part of the above result comes from the implicit function theorem. More precisely, if $\mathcal{S}$ is the set of λ's such that (P_λ) has a positive solution, then, via sub-supersolution techniques, it is shown that $\mathcal{S}$ is an interval with $\inf \mathcal{S} > 0$.

The problem of finding multiple solutions for (P_λ) has been carried on in [5], where Montenegro and Silva have established the following result:

Theorem B. ([5], Theorem 1.1) *There exists* $\lambda_0 > 0$ *such that for each* $\lambda \in]\lambda_0, +\infty[$*, problem* (P_λ) *has two distinct nontrivial nonnegative weak solutions.*

Following the approach of [3], in [5] the authors consider a sequence of ε-problems where the singularity is replaced by the "non singular" term $u^q/(u+\varepsilon)^{q+r-1}$ for some $0 < q < s-1$. For big values of the parameter, the existence of two critical points, a global minimum and a mountain pass critical point is ensured for the associated perturbed energy functional. As $\varepsilon \to 0$, by employing careful estimates on the gradient of the solutions of the perturbed problems, two nontrivial nonnegative solutions of (P_λ) are provided.

In the above result nothing is said about the positivity of the solutions, although the authors conjecture that one of them is positive. Indeed, we can observe that this occurs if $\lambda > \max\{\Lambda, \lambda_0\}$, where Λ and λ_0 are as in Theorems A and B, respectively. Our aim, in this paper, is to show that the existence of a positive solution for problem (P_λ) ensures the existence of a second solution, that is to say $\Lambda \geq \lambda_0$. Moreover, we also prove that the positive solution is stable, i.e. it corresponds to a local minimum point of the associated energy functional.

Our result reads as follows:

Theorem 1. *There exists* $\Lambda > 0$ *such that problem* (P_λ) *has no positive solution for* $\lambda < \Lambda$ *and two distinct nontrivial nonnegative weak solutions for* $\lambda > \Lambda$*. One of them belongs to* $int(C_0^1(\overline{\Omega})_+)$ *and corresponds to a local minimum point of the energy functional associated to problem* (P_λ)*.*

We point out that the energy functional associated to (P_λ) is not Gâteaux differentiable in the Sobolev space $W_0^{1,2}(\Omega)$ and the classical critical point theory does not apply. In our proof truncation techniques and variational arguments are employed. More precisely, the first positive solution is obtained as global minimum of a suitable truncated functional, while the existence of the second solution relies on an appropriate application of the Mountain Pass Theorem for a regularized functional in the same spirit as in [5]. We point out that our approach, in the proof of the existence of the positive solution differs from that of [5]: indeed, we introduce a suitable truncation involving the positive solution given by Theorem A. The truncated functional is of class C^1 and its global minimum turns out to be a local minimum of the energy functional and it belongs to $\mathrm{int}(C_0^1(\overline{\Omega})_+)$.

We believe that the idea of the proof can be also carried on in the study of the case $0 < r < s \leq 1$.

2. Preliminaries

We endow the Sobolev space $W_0^{1,2}(\Omega)$ with the classical norm

$$\|u\| = \left(\int_\Omega |\nabla u|^2\right)^{\frac{1}{2}}.$$

As usual, for every $u \in W_0^{1,2}(\Omega)$, put $u_+ = \max\{0, u\}$ and $u_- = \max\{0, -u\}$.

Recall also that

$$C_0^1(\overline{\Omega}) = \left\{u \in C^1(\overline{\Omega}), \quad u = 0 \text{ on } \partial\Omega\right\}$$

is an ordered Banach space with positive cone

$$C_+ = \left\{u \in C_0^1(\overline{\Omega}),\ u \geq 0 \text{ in } \Omega\right\}$$

which has a nonempty interior given by

$$\mathrm{int}C_+ = \left\{u \in C_0^1(\overline{\Omega}) : u > 0 \text{ in } \Omega,\ \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega\right\},$$

where $\nu(\cdot)$ is the outer unit normal to $\partial\Omega$. It is also well known that if $u \in \mathrm{int}C_+$, there exists $c > 0$ such that $u(x) \geq cd(x)$ for every $x \in \Omega$, being $d(x) = d(x, \partial\Omega)$.

Define

$$\Psi_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{s}\int_\Omega u_+^s dx + \frac{1}{r}\int_\Omega u_+^r dx \ \text{ for all } \ u \in W_0^{1,2}(\Omega).$$

The functional Ψ_λ is well defined and continuous on $W_0^{1,2}(\Omega)$, but not differentiable because of the presence of the singular term. However, we can prove the following result.

Lemma 1. *For every $\lambda > 0$, Ψ_λ is Gâteaux differentiable at each point of* $\mathrm{int}C_+$ *and for every $u \in \mathrm{int}C_+$, one has*

$$\Psi'_\lambda(u)\varphi = \int_\Omega \nabla u \nabla\varphi dx - \lambda\int_\Omega u^{s-1}\varphi dx + \int_\Omega u^{r-1}\varphi dx$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

Proof. It is well known that the functionals

$$\psi_1(u) = \frac{1}{2}\|u\|^2$$

and

$$\psi_2(u) = \frac{\lambda}{s}\int_\Omega u_+^s dx$$

are Gâteaux differentiable in $W_0^{1,2}(\Omega)$ with derivative

$$\psi_1'(u)\varphi = \int_\Omega \nabla u \nabla \varphi dx,$$
$$\psi_2'(u)\varphi = \lambda \int_\Omega u_+^{s-1} \varphi dx,$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

Define $\psi_3 : W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$\psi_3(u) = \frac{1}{r} \int_\Omega u_+^r dx.$$

Let us prove that if $u \in \mathrm{int}C_+$, then

$$\lim_{t\to 0} \frac{\psi_3(u+t\varphi) - \psi_3(u)}{t} = \int_\Omega u^{r-1} \varphi dx$$

for every $\varphi \in W_0^{1,2}(\Omega)$. Assume first that $\varphi \geq 0$ in Ω and fix a sequence $t_n \to 0$, $t_n \in]0, 1[$.

Since $u \in \mathrm{int}C_+$, there exists $c > 0$ such that $u(x) \geq cd(x)$ for every $x \in \Omega$. Moreover, the function $x \to d(x)^{r-1}$ belongs to $L^p(\Omega)$ for every $p < \frac{N}{1-r}$. Choose p in such a way that $\frac{2N}{N+2} < p < \frac{N}{1-r}$. Since $\varphi \in W_0^{1,2}(\Omega)$, then, $\varphi \in L^{p'}(\Omega)$.

One has

$$\begin{aligned}\frac{\psi_3(u+t_n\varphi) - \psi_3(\varphi)}{t_n} &= \int_\Omega \frac{(u+t_n\varphi)^r - u^r}{t_n} dx \\ &= r \int_\Omega (u+\tau_n\varphi)^{r-1} \varphi dx\end{aligned}$$

for a sequence $\tau_n \to 0^+$. Since

$$(u+\tau_n\varphi)^{r-1}\varphi \to u^{r-1}\varphi \qquad \text{a.e. in } \Omega$$

and

$$(u(x)+\tau_n\varphi(x))^{r-1}\varphi(x) \leq u(x)^{r-1}\varphi(x) \leq c^{r-1}d(x)^{r-1}\varphi(x) \in L^1(\Omega),$$

from Lebesgue theorem, we deduce that

$$\lim_{n\to\infty} \frac{\psi_3(u+t_n\varphi) - \psi_3(\varphi)}{t_n} = \int_\Omega u^{r-1}\varphi dx.$$

For a generic $\varphi \in W_0^{1,2}(\Omega)$, by the equality $\varphi = \varphi_+ - \varphi_-$, it follows that

$$\psi_3'(u)\varphi = \int_\Omega u^{r-1}\varphi dx.$$

It is also easily seen that $\psi_3'(u)$ is linear and continuous on $W_0^{1,2}(\Omega)$. The proof is concluded. □

For the next property of Ψ_λ, we need an auxiliary result which we state in a general form. Denote by $\mathcal{A}$ the class of all Carathèodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that there exist $C > 0$ and $q > 0$:

$$|f(x,t)| \leq C(1+|t|^{q-1}) \quad \text{for all } t \in \mathbb{R}, \text{almost all } x \in \Omega$$

with $q \in [1, \frac{2N}{N-2}]$ if $N > 2$, $q \in [1, +\infty[$ if $N = 2$. If $N = 1$, $\mathcal{A}$ is the class of all Carathèodory functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ such that for all $s \in]0, +\infty[$

$$\sup_{|t| \leq s} |f(\cdot, t)| \in L^1(\Omega).$$

For each $f \in \mathcal{A}$ and $\mu > 0$, put

$$I_{\mu,f}(u) = \mu \|u\|^2 - \int_\Omega \left(\int_0^{u(x)} f(x,t)dt \right) dx, \quad \text{for all} \quad u \in W_0^{1,2}(\Omega).$$

Proposition 1. *Let $f \in \mathcal{A}$ and assume that there exists $\delta > 0$ such that*

$$f(x,t)t \leq 0 \textit{ for each} t \in [-\delta, \delta] \textit{and almost all} x \in \Omega.$$

Then, for every positive μ, 0 is a local minimum for $I_{\mu,f}$.

Proof. Let $\mu > 0$. Notice first that, in view of the assumption on f, $I_{\mu,f}$ is well defined in $W_0^{1,2}(\Omega)$. Assume, by contradiction, that 0 is not a local minimum for $I_{\mu,f}$. Then, by Theorem 1 of [2], 0 is not a local minimum for $I_{\mu,f}$ in the $C^1(\overline{\Omega})$ topology. So, there exists a sequence $\{u_n\}$ in $C^1(\overline{\Omega})$ with $\|u_n\|_{C^1(\overline{\Omega})} \to 0$ such that $I_{\mu,f}(u_n) < 0$ for all $n \in \mathbb{N}$. Nevertheless, $\max_{\overline{\Omega}} |u_n| \leq \delta$ for n big enough. Therefore, $I_{\mu,f}(u_n) \geq 0$ for n big enough, a contradiction. □

Lemma 2. *There exists $\delta_0 > 0$ such that* $\inf_{\|u\|=\delta} \Psi_\lambda(u) > 0 = \Psi_\lambda(0) = 0$ *for all $\delta \in]0, \delta_0]$.*

Proof. Assume, by contradiction, that there exists a sequence $\{\delta_n\}$ in $]0, +\infty[$ such that $\delta_n \to 0$ and

$$\inf_{\|u\|=\delta_n} \Psi_\lambda(u) \leq 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{1}$$

Fix $\tilde{r} \in]1, s[$ and define

$$g(t) = \frac{t_+^{\tilde{r}}}{1 + t_+^{\tilde{r}}} \quad \text{for all} \quad t \in \mathbb{R}.$$

It is easy to check that $g(t) \leq t_+^r$ for all $t \in \mathbb{R}$ and g is of class C^1. Therefore, if we define $f : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$f(x,t) = \lambda t_+^{s-1} - \frac{1}{\tilde{r}} g'(t) = \lambda t_+^{s-1} - \frac{t_+^{\tilde{r}-1}}{(1 + t_+^{\tilde{r}})^2} \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

we have $f \in \mathcal{A}$ and $I_{\frac{1}{2},f}(u) \leq \Psi_\lambda(u)$ for all $u \in W_0^{1,2}(\Omega)$. So,

$$I_{\frac{1}{4},f}(u) = I_{\frac{1}{2},f}(u) - \frac{1}{4}\|u\|^2 \leq \Psi_\lambda(u) - \frac{1}{4}\|u\|^2 \quad \text{for all } u \in W_0^{1,2}(\Omega).$$

Taking into account (1), we have

$$\inf_{\|u\|=\delta_n} I_{\frac{1}{4},f}(u) < 0 \quad \text{for all} \quad n \in \mathbb{N}. \tag{2}$$

On the other hand, f satisfies the assumptions of Proposition 1. Therefore, 0 is a local minimum for $I_{\frac{1}{4},f}$. This is in contradiction with (2). □

3. Proof of Theorem 1

The proof of Theorem 1 will be divided in two steps: in the first one we will prove that the positive solution given by Theorem A can be constructed in such way that it turns out a local minimum point for the energy functional. In the second step, we will use this information to find a second non negative solution.

Step 1. *Variational characterization of the solution given by Theorem A.*

Let Λ as in Theorem A, $\lambda \in]\Lambda, +\infty[$ and $\lambda' \in]\Lambda, \lambda[$. From Theorem A there exists $u_{\lambda'} \in \text{int}C_+$, classical solution of $(P_{\lambda'})$.

Consider the continuous function

$$f(x,t) = \begin{cases} \lambda u_{\lambda'}(x)^{s-1} - u_{\lambda'}(x)^{r-1} & \text{if} \quad x \in \Omega \ \text{ and } \ t \le u_{\lambda'}(x) \\ \lambda t^{s-1} - t^{r-1} & \text{if} \quad x \in \Omega \ \text{ and } \ t > u_{\lambda'}(x) \end{cases}$$

which satisfies the following estimate:

$$|f(x,t)| \le \lambda \|u_{\lambda'}\|_\infty^{s-1} + \lambda |t|^{s-1} + cd(x)^{r-1} \tag{3}$$

for a.e. $x \in \Omega$ and every $t \in \mathbb{R}$ (being c a positive constant). Define

$$\tilde{\Psi}_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\int_0^{u(x)} f(x,t)dt \right) dx \quad \text{for all} \quad u \in W_0^{1,2}(\Omega).$$

$\tilde{\Psi}_\lambda$ is coercive and sequentially weakly lower semicontinuous as it follows from (3), so it admits a global minimum u_λ. Moreover, since $u_{\lambda'} \in \text{int}C_+$, it is easy to prove that $\tilde{\Psi}_\lambda$ is differentiable in $W_0^{1,2}(\Omega)$ (similar arguments as in Lemma 1). Therefore, one has $\tilde{\Psi}'_\lambda(u_\lambda) = 0$. This means that u_λ is a weak solution of the problem

$$\begin{cases} -\Delta u = f(x,u) & \text{in} \ \ \Omega, \\ u = 0 & \text{on} \ \ \partial\Omega, \end{cases}$$

i.e.

$$\int_\Omega \nabla u_\lambda \nabla \varphi dx = \int_\Omega f(x, u_\lambda) \varphi dx \tag{4}$$

for every $\varphi \in W_0^{1,2}(\Omega)$. Also, from Theorems 8.2 and 8.2' of [1], $u_\lambda \in C^1(\overline{\Omega})$. Let us observe that $u_\lambda \ge u_{\lambda'}$ in Ω. Indeed if it was

$$A \stackrel{\text{def}}{=} \{x \in \Omega : u_\lambda(x) < u_{\lambda'}(x)\} \neq \emptyset,$$

then, we should have

$$\begin{cases} -\Delta(u_\lambda - u_{\lambda'}) = (\lambda - \lambda')\, u_{\lambda'}^{s-1} \ge 0 & \text{in} \ \ A, \\ u_\lambda - u_{\lambda'} = 0 & \text{on} \ \ \partial A, \end{cases}$$

and, by the maximum principle, $u_\lambda(x) \ge u_{\lambda'}(x)$ in A, a contradiction. The same argument shows that $u_\lambda \neq u_{\lambda'}$ (otherwise $u_{\lambda'} = 0$). Hence, from (4) we infer that u_λ verifies

$$\int_\Omega \nabla u_\lambda \nabla \varphi dx = \int_\Omega \left(\lambda u_\lambda^{s-1} - u_\lambda^{r-1} \right) \varphi dx \tag{5}$$

for every $\varphi \in W_0^{1,2}(\Omega)$, which implies that u_λ is a weak solution of problem (P_λ) according to the definition given in the Introduction.

We also have

$$-\Delta(u_\lambda - u_{\lambda'}) = \lambda u_\lambda^{s-1} - \lambda' u_{\lambda'}^{s-1} - u_\lambda^{r-1} + u_{\lambda'}^{r-1} \geq (\lambda - \lambda')\, u_{\lambda'}^{s-1} \geq 0 \quad \text{in} \quad \Omega.$$

Therefore, by the strong maximum principle we infer that $u_\lambda - u_{\lambda'} \in \mathrm{int}C_+$. Hence, $u_\lambda \in \mathrm{int}C_+$.

u_λ is a local minimum in the $C^1(\overline{\Omega})$-topology for Ψ_λ. Note that $\mathrm{int}C_+$ is open in $C^1(\overline{\Omega})$ and $u_\lambda \in u_{\lambda'} + \mathrm{int}C_+$ is in particular a global minimum for $\tilde{\Psi}_\lambda$ in the $C^1(\overline{\Omega})$-topology. Now, observe that, for each $u \in u_{\lambda'} + \mathrm{int}C_+$, one has $\tilde{\Psi}_\lambda(u) \geq \tilde{\Psi}_\lambda(u_\lambda)$, and

$$\begin{aligned}
\Psi_\lambda(u) = \frac{1}{2}\|u\|^2 &- \int_\Omega \int_0^{u_{\lambda'}(x)} \left(\lambda t_+^{s-1} - t_+^{r-1}\right) dtdx \\
&- \int_\Omega \int_{u_{\lambda'}(x)}^{u(x)} \left(\lambda t_+^{s-1} - t_+^{r-1}\right) dtdx \\
= \frac{1}{2}\|u\|^2 &- \int_\Omega \int_0^{u_{\lambda'}(x)} \left(\lambda t_+^{s-1} - t_+^{r-1}\right) dtdx \\
&- \int_\Omega \int_0^{u(x)} f(x,t)dtdx + \int_\Omega \int_0^{u_{\lambda'}(x)} f(x,t)dtdx \\
= \tilde{\Psi}_\lambda(u) &+ \text{const.}
\end{aligned}$$

Then, we infer that u_λ is a local minimum in the $C^1(\overline{\Omega})$-topology for Ψ_λ as well. Consequently, 0 is a local minimum point in the $C^1(\overline{\Omega})$-topology for the functional

$$\begin{aligned}
\tilde{\Phi}_\lambda(u) &:= \Psi_\lambda(u + u_\lambda) - \Psi_\lambda(u_\lambda) \\
&= \frac{1}{2}\|u\|^2 + \int_\Omega \nabla u_\lambda \nabla u dx \\
&\quad - \int_\Omega \left(\frac{\lambda}{s}(u+u_\lambda)_+^s - \frac{\lambda}{s}u_\lambda^s - \frac{1}{r}(u+u_\lambda)_+^r + \frac{1}{r}u_\lambda^r\right) dx \\
(\text{by } (5)) &= \frac{1}{2}\|u\|^2 - \int_\Omega \left[\lambda\left(\frac{1}{s}(u+u_\lambda)_+^s - \frac{1}{s}u_\lambda^s - u_\lambda^{s-1}u\right)\right. \\
&\quad \left. - \left(\frac{1}{r}(u+u_\lambda)_+^r - \frac{1}{r}u_\lambda^r - u_\lambda^{r-1}u\right)\right] dx. \qquad (6)
\end{aligned}$$

The auxiliary functional Φ_λ. Fix $\varepsilon_0 \in]0,1[$ and let $G : \Omega \times \mathbb{R} \to \mathbb{R}$ defined as follows:

$$\begin{aligned}
G(x,t) &= \frac{\varepsilon_0^r - 1 - r(\varepsilon_0 - 1)}{r} u_\lambda(x)^r - \frac{r}{2}\varepsilon_0^{r-2} u_\lambda(x)^{r-2}\left[t - (\varepsilon_0 - 1)u_\lambda(x)\right]^2 \\
&\quad + \left(\varepsilon_0^{r-1} - 1\right) u_\lambda(x)^{r-1}\left[t - (\varepsilon_0 - 1)u_\lambda(x)\right] \quad \text{for all } x \in \Omega \text{ and } t < -(1-\varepsilon_0)u_\lambda(x); \\
G(x,t) &= \frac{1}{r}(t + u_\lambda(x))_+^r - \frac{1}{r}u_\lambda(x)^r - u_\lambda(x)^{r-1}t \quad \text{for all } x \in \Omega \text{ and } t \\
&\quad \geq -(1-\varepsilon_0)u_\lambda(x);
\end{aligned}$$

We can see that $G \in C^0(\Omega \times \mathbb{R})$ and $G(x,\cdot) \in C^1(\mathbb{R})$ for all $x \in \Omega$. Moreover, one has

(i) $G(x,t) \leq \frac{1}{r}(t+u_\lambda(x))_+^r - \frac{1}{r}u_\lambda(x)^r - u_\lambda(x)^{r-1}t$ for all $(x,t) \in \Omega \times \mathbb{R}$
(ii) $G(x,0) = 0$ for all $x \in \Omega$.

Let us define also $H : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$H(x,t) = \lambda\left[\frac{1}{s}(t+u_\lambda(x))_+^s - \frac{1}{s}u_\lambda(x)^s - u_\lambda(x)^{s-1}t\right].$$

Put also $G'(x,t) = g(x,t)$ and $H'(x,t) = h(x,t)$ for every $(x,t) \in \Omega \times \mathbb{R}$. Finally let

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u(x))dx + \int_\Omega G(x,u(x))dx$$

for all $u \in W_0^{1,2}(\Omega)$.

The functional Φ_λ is well defined because of the generalized Hardy Sobolev inequality and is differentiable in $W_0^{1,2}(\Omega)$ with derivative at $u \in W_0^{1,2}(\Omega)$ given by

$$\Phi_\lambda'(u)\varphi = \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega (h(x,u) - g(x,u))\varphi dx$$

for every $\varphi \in W_0^{1,2}(\Omega)$.

0 *is a local minimum of* Φ_λ. Indeed, since $(1-\varepsilon_0)u_\lambda \in \mathrm{int}C_+$, we can find a neighborhood V of 0 in $C^1(\overline{\Omega})$ such that $(1-\varepsilon_0)u_\lambda + V \subset \mathrm{int}C_+$ and $\tilde{\Phi}_\lambda(u) \geq \tilde{\Phi}_\lambda(0) = 0$ for all $u \in V$. Consequently, noticing that $\Phi_\lambda = \tilde{\Phi}_\lambda$ in V, 0 is a local minimum in the $C^1(\overline{\Omega})$-topology for Φ_λ.

Let us prove that 0 is a local minimum for Φ_λ in the topology of $W_0^{1,2}(\Omega)$. The proof follows with some modification that of Theorem 1 of [2]. Indeed, assume by contradiction that 0 is not a local minimum for Φ_λ in the topology of $W_0^{1,2}(\Omega)$. Then, we can find a sequence $\{\delta_n\}$ of positive numbers and a sequence $\{u_n\}$ in $W_0^{1,2}(\Omega)$ such that for all $n \in \mathbb{N}$,

(j) $\delta_n \to 0$,
(jj) $\|u_n\| \leq \delta_n$,
(jjj) $\Phi_\lambda(u_n) = \inf_{\|u\| \leq \delta_n} \Phi_\lambda(u) < 0 = \Phi_\lambda(0)$.

By the Lagrange Multipliers Theorem, for each $n \in \mathbb{N}$, there exists $\mu_n \leq 0$ such that

$$\Phi_\lambda'(u_n) = \mu_n u_n.$$

In other words,

$$\begin{cases} -(1-\mu_n)\Delta u_n = \lambda\left[(u_\lambda + u_n)_+^{s-1} - u_\lambda^{s-1}\right] - g(x,u_n) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

At this point, let $p > N$ such that $(1-r)p < N$. Observe that, from the definition of G, we can find a positive constant C_1 such that

$$\begin{aligned} &|\lambda\left[(u_\lambda(x) + u_n(x))_+^{s-1} - u_\lambda(x)^{s-1}\right] - g(x,u_n(x))| \\ &\quad \leq C_1\left(|u_n(x)|^{s-1} + |u_\lambda(x)|^{s-1} + |u_\lambda(x)|^{r-1} + 1\right), \end{aligned}$$

for all $x \in \Omega$. By standard regularity results, one has $u_n \in C^1(\overline{\Omega})$. Moreover, since $u_\lambda \in \text{int}C_+$ and $(1-r)p < N$ the right hand side in the above inequality is p-summable. So, is the left hand side. It follows, from Theorem 8.2 and Theorem 8.2' of [1], that $u_n \in W^{2,p}(\Omega)$ and for some constant C_3,

$$\|u_n\|_{W^{2,p}(\Omega)} \leq C_3 \left[\left(\int_\Omega |u_n|^p dx \right)^{\frac{1}{p}} + 1 \right].$$

Let $\alpha \in]0,1[$ such that $p\alpha < c_p$, where c_p is the embedding constant of $W_0^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$. From the above inequality and in view of the embedding $W^{2,p}(\Omega) \hookrightarrow C^{1+\beta}(\overline{\Omega})$ for some $\beta \in]0,1[$, there exists a positive constant C_4 such that

$$\begin{aligned} \|u_n\|_{C^{1+\beta}(\overline{\Omega})} &\leq C_4 \left[\|u_n\|^{1-\alpha}_{C^{1+\beta}(\overline{\Omega})} \cdot \left(\left(\int_\Omega |u_n|^{\alpha p} dx \right)^{\frac{1}{p}} + 1 \right) \right] \\ &\leq C_4 \left[\|u_n\|^{1-\alpha}_{C^{1+\beta}(\overline{\Omega})} \cdot (\|u_n\|^\alpha + 1) \right], \end{aligned}$$

for all $n \in \mathbb{N}$. In view of (j) and (jj), it follows that the sequence $\{u_n\}$ is bounded in $C^{1+\beta}(\overline{\Omega})$. Since $u_n \to 0$ strongly in $W_0^{1,2}(\Omega)$, up to a subsequence, by the Ascoli–Arzelà Theorem, $u_n \to 0$ in $C^1(\overline{\Omega})$. This is a contradiction with (jjj) and the fact that 0 is a local minimum for Φ_λ in the $C^1(\overline{\Omega})$ topology. Therefore 0 is a local minimum for Φ_λ in the $W_0^{1,2}(\Omega)$-topology.

At this point, observe that, by construction (see (i) and (ii)), one has $\Phi_\lambda(u) \leq \tilde{\Phi}_\lambda(u)$, for all $u \in W_0^{1,2}(\Omega)$, and $\Phi_\lambda(0) = \tilde{\Phi}_\lambda(0) = 0$. From this, we easily infer that u_λ is a local minimum point for Ψ_λ in the $W_0^{1,2}(\Omega)$-topology.

Step 2. *Existence of a second solution.*

From the previous step, there exists $\delta_0 > 0$, which can also be assumed strictly less than $\|u_\lambda\|$, such that

$$0 = \Phi_\lambda(0) \leq \inf_{\|u\|=\delta} \Phi_\lambda(u)$$

for all $\delta \in]0, \delta_0[$. Now, let us distinguish the following cases:

(I) $\inf_{\|u\|=\delta} \Phi_\lambda(u) = 0$ for all $\delta \in]0, \delta_0[$;

(II) $\inf_{\|u\|=\tilde{\delta}} \Phi_\lambda(u) > 0$ for some $\tilde{\delta} \in]0, \delta_0[$.

Suppose that case (I) occurs. Let $\{\delta_n\}$ be a sequence in $]0, \delta_0[$ such that $\delta_n \to 0$ and

$$\inf_{\|u\|=\delta_n} \Phi_\lambda(u) = 0$$

for all $n \in \mathbb{N}$. Since the functional

$$I(u) := \int_\Omega (G(x,u) - H(x,u))\, dx, \quad u \in W_0^{1,2}(\Omega),$$

is sequentially weakly continuous in $W_0^{1,2}(\Omega)$ and, in particular, weakly continuous on every bounded set of $W_0^{1,2}(\Omega)$ (by the Eberlein–Smulian Theorem), one has:

$$0 = \inf_{\|u\|=\delta_n} \Phi_\lambda(u) = \frac{1}{2}\delta_n^2 + \inf_{\|u\|=\delta_n} \int_\Omega (G(x,u) - H(x,u))dx$$
$$= \frac{1}{2}\delta_n^2 + \inf_{\|u\|\le\delta_n} \int_\Omega (G(x,u) - H(x,u))dx. \qquad (7)$$

Therefore, there exists $u_n \in W_0^{1,2}(\Omega)$ such that $\|u_n\| \le \delta_n$ and $I(u_n) = \inf_{\|u\|\le\delta_n} I(u)$. If it was $\|u_n\| < \delta_n$, then u_n should be a local minimum for I and so $I'(u_n) = 0$. In other words,

$$\int_\Omega (g(x,u_n) - h(x,u_n))\, v dx = 0 \ \text{ for all } \ v \in W_0^{1,2}(\Omega). \qquad (8)$$

By the Hahn–Banach theorem, the functional $I'(u_n)$ can be extended to $L^2(\Omega)$. Since $W_0^{1,2}(\Omega)$ is dense in $L^2(\Omega)$, from (8) we have $I'(u_n) = 0$ in $L^2(\Omega)$. This implies

$$g(x,u_n(x)) - h(x,u_n(x)) = 0, \quad \text{for almost every } \ x \in \Omega. \qquad (9)$$

Since the function $g(x,\cdot) - h(x,\cdot)$ is strictly decreasing in $\mathbb{R}$ and $g(x,0) - h(x,0) = 0$, for all $x \in \Omega$, we infer, from (9), that $u_n(x) = 0$ a.e. in Ω. Then, from (7) we should have $\delta_n = 0$ which is an absurd. Consequently, $\|u_n\| = \delta_n$, and so $\Phi_\lambda(u_n) = \inf_{\|u\|=\delta_n} \Phi_\lambda(u) = 0 = \Phi_\lambda(0)$. Since $\delta_n \in]0,\delta_0[$, this means that u_n is a nonzero local minimum for Φ_λ for all $n \in \mathbb{N}$. Clearly, we have

$$\begin{cases} -\Delta u_n = h(x,u_n) - g(x,u_n) & \text{in } \ \Omega, \\ u_n = 0 & \text{on } \ \partial\Omega. \end{cases}$$

Using again Theorem 8.2 and Theorem 8.2' of [1], we infer that $u_n \to 0$ in $C^1(\overline{\Omega})$. Therefore, for $n \in \mathbb{N}$ large enough, we have $u_n(x) \ge -(1-\varepsilon_0)u_\lambda(x)$ for all $x \in \Omega$. This implies that u_n is a nonzero weak solution of the problem

$$\begin{cases} -\Delta u = \lambda(u+u_\lambda)_+^{s-1} - \lambda u_\lambda^{s-1} - (u+u_\lambda)_+^{r-1} + u_\lambda^{r-1} & \text{in } \ \Omega, \\ u = 0 & \text{on } \ \partial\Omega. \end{cases}$$

Recalling that u_λ is a weak solution of problem (P_λ), we infer that $u_n + u_\lambda$ is a nonzero weak solution for (P_λ) distinct from u_λ. So, in this case, the conclusion is achieved.

Now, suppose that case (II) occurs. Let $\tilde{\Phi}_\lambda$ be the functional defined in (6). Recalling that

$$G(x,t) \le \frac{1}{r}(t+u_\lambda(x))_+^r - \frac{1}{r}u_\lambda(x)^r - u_\lambda(x)^{r-1}t$$

for all $(x,t) \in \Omega \times \mathbb{R}$, we have

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)dx + \int_\Omega G(x,u)dx$$
$$\le \frac{1}{2}\|u\|^2 - \int_\Omega H(x,u)dx + \int_\Omega \left(\frac{1}{r}(u+u_\lambda)_+^r - \frac{1}{r}u_\lambda^r - u_\lambda^{r-1}u\right) dx$$
$$= \tilde{\Phi}_\lambda(u)$$

for all $u \in W_0^{1,2}(\Omega)$. Consequently,

$$\inf_{\|u\|=\tilde{\delta}} (\Psi_\lambda(u+u_\lambda) - \Psi_\lambda(u_\lambda)) = \inf_{\|u\|=\tilde{\delta}} \tilde{\Phi}_\lambda(u) \geq \inf_{\|u\|=\tilde{\delta}} \Phi_\lambda(u) > 0,$$

that is

$$\inf_{\|u\|=\tilde{\delta}} \Psi_\lambda(u+u_\lambda) > \Psi_\lambda(u_\lambda) \tag{10}$$

Moreover, since $\tilde{\delta} < \|u_\lambda\|$, in view of Lemma 2, we can find $\delta_1 \in \mathbb{R}$ with $0 < \delta_1 < \|u_\lambda\| - \tilde{\delta}$ such that

$$\inf_{\|u\|=\delta_1} \Psi_\lambda(u) > \Psi_\lambda(0) = 0. \tag{11}$$

Introduce now the continuous function

$$f_\varepsilon(t) = \begin{cases} \dfrac{t^{q-1}}{(t+\varepsilon)^{q-r}} & \text{if } t \geq 0, \\ 0 & \text{if } t \leq 0 \end{cases}$$

with $1 < q < s$. Let $F_\varepsilon(t) = \int_0^t f_\varepsilon(s)ds$. Define also the functions $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ by

$$\varphi_1(\varepsilon) = \frac{1}{2}\|u_\lambda\|^2 - \int_\Omega \left[\frac{\lambda}{s} u_\lambda^s - F_\varepsilon(u_\lambda)\right] dx,$$

$$\varphi_2(\varepsilon) = \inf_{\|u\|=\tilde{\delta}} \left(\frac{1}{2}\|u+u_\lambda\|^2 - \int_\Omega \left[\frac{\lambda}{s}(u+u_\lambda)_+^s - F_\varepsilon((u+u_\lambda)\right] dx\right).$$

Clearly, φ_1 is continuous in $\mathbb{R}$. Let us to show that φ_2 is continuous in $\mathbb{R}$ as well. For all $u \in W_0^{1,2}(\Omega)$ and $\varepsilon \in \mathbb{R}$, put

$$J(\varepsilon, u) = \int_\Omega \nabla u_\lambda \nabla u dx - \int_\Omega \left[\frac{\lambda}{s}(u+u_\lambda)_+^s - F_\varepsilon(u+u_\lambda)\right] dx,$$

It is easy to check that J is sequentially continuous in $\mathbb{R} \times W_0^{1,2}(\Omega)$ whenever $W_0^{1,2}(\Omega)$ is equipped with the weak topology. Since the closed ball $B_{\tilde{\delta}}(0)$ is weakly compact and $J(\varepsilon, \cdot)$ is weakly continuous on $B_{\tilde{\delta}}(0)$ (from the Eberlein–Smulian Theorem), one has

$$\inf_{\|u\|=\tilde{\delta}} J(\varepsilon, u) = \inf_{\|u\|\leq\tilde{\delta}} J(\varepsilon, u),$$

and the continuity of the marginal function

$$\varepsilon \in \mathbb{R} \to \inf_{\|u\|\leq\tilde{\delta}} J(\varepsilon, u).$$

From this and noticing that

$$\varphi_2(\varepsilon) = \frac{1}{2}\tilde{\delta}^2 + \frac{1}{2}\|u_\lambda\|^2 + \inf_{\|u\|=\tilde{\delta}} J(\varepsilon, u),$$

we obtain the continuity of φ_2.

Now, observe that

$$F_0(t) = \frac{1}{r} t_+^r \text{ for all } t \in \mathbb{R},$$

so

$$\varphi_1(0) = \frac{1}{2}\|u_\lambda\|^2 - \int_\Omega \left[\frac{\lambda}{s} u_\lambda^s - \frac{1}{r} u_\lambda^r\right] dx = \Psi_\lambda(u_\lambda),$$

$$\begin{aligned}\varphi_2(0) &= \inf_{\|u\|=\tilde{\delta}} \left(\frac{1}{2}\|u+u_\lambda\|^2 - \int_\Omega \left[\frac{\lambda}{s}(u+u_\lambda)_+^s - \frac{1}{r}(u+u_\lambda)_+^r\right] dx\right)\\ &= \inf_{\|u\|=\tilde{\delta}} \Psi_\lambda(u+u_\lambda).\end{aligned}$$

From (10) one has $\varphi_2(0) - \varphi_1(0) > 0$. Then, if we put $\eta_1 = \frac{\varphi_2(0)-\varphi_1(0)}{2}$, we can find $\varepsilon_1 \in]0,1[$ such that

$$\begin{aligned}&\inf_{\|u\|=\tilde{\delta}} \left[\frac{1}{2}\|u+u_\lambda\|^2 - \int_\Omega \left(\frac{\lambda}{s}(u+u_\lambda)_+^s - F_\varepsilon(u+u_\lambda)\right) dx\right]\\ &\quad - \left[\frac{1}{2}\|u_\lambda\|^2 - \int_\Omega \left(\frac{\lambda}{s} u_\lambda^s - F_\varepsilon(u_\lambda)\right) dx\right] = \varphi_2(\varepsilon) - \varphi_1(\varepsilon) > \eta_1 > 0 \qquad (12)\end{aligned}$$

for all $\varepsilon \in [0, \varepsilon_1]$. Following the same argument, thanks to (11), we can also find $\eta_2 > 0$ and $\varepsilon_2 \in]0,1[$ such that

$$\inf_{\|u\|=\delta_1} \left[\frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{\lambda}{s} u_+^s - F_\varepsilon(u)\right) dx\right] > \eta_2 > 0, \qquad (13)$$

for all $\varepsilon \in [0, \varepsilon_2]$.

Let $\tilde{\varepsilon} = \min\{\varepsilon_1, \varepsilon_2\} < 1$ and put

$$\Psi_{\varepsilon,\lambda}(u) = \frac{1}{2}\|u\|^2 - \int_\Omega \left(\frac{\lambda}{s} u_+^s - F_\varepsilon(u)\right) dx$$

for all $u \in W_0^{1,2}(\Omega)$ and $\varepsilon \in [0, \tilde{\varepsilon}]$. Notice that $\Psi_{\varepsilon,\lambda}$ is a C^1 functional and its critical points are non negative functions in $W_0^{1,2}(\Omega)$. Now, let $\varepsilon \in]0, \tilde{\varepsilon}[$. Inequalities (12) and (13) say that $\Psi_{\varepsilon,\lambda}$ has two local minima $u_{1,\varepsilon}, u_{2,\varepsilon} \in W_0^{1,2}(\Omega)$ such that $\|u_{1,\varepsilon}\| < \delta_1$ and $\|u_\lambda - u_{2,\varepsilon}\| < \tilde{\delta}$. Recalling that $\delta_1 < \|u_\lambda\| - \tilde{\delta}$, we have that $u_{1,\varepsilon} \neq u_{2,\varepsilon}$. Since $\Psi_{\varepsilon,\lambda}$ satisfies the Palais–Smale condition (see [7, Example 38.25]), using a standard Mountain Pass Theorem (see [6]) we find a critical point $v_\varepsilon \in W_0^{1,2}(\Omega)$ for $\Psi_{\varepsilon,\lambda}$ which, thanks to (12) and (13), satisfies:

$$\Psi_{\varepsilon,\lambda}(v_\varepsilon) \geq \eta_1 + \Psi_{\varepsilon,\lambda}(u_\lambda) \quad \text{and} \quad \Psi_{\varepsilon,\lambda}(v_\varepsilon) \geq \eta_2. \qquad (14)$$

Fix now a sequence $\{\varepsilon_n\} \subset]0, \tilde{\varepsilon}[$. From above, there exists a sequence of non negative functions $\{v_n\} \subset W_0^{1,2}(\Omega)$ (with $v_n = v_{\varepsilon_n}$), satisfying the following conditions:

(I) $\Psi'_{\varepsilon_n,\lambda}(v_n) = 0$,
(II) $\Psi_{\varepsilon_n,\lambda}(v_n) \geq \eta_1 + \Psi_{\varepsilon_n,\lambda}(u_\lambda)$,
(III) $\Psi_{\varepsilon_n,\lambda}(v_n) \geq \eta_2$.

From (l), we deduce that the sequence $\{v_n\}$ is bounded in $W_0^{1,2}(\Omega)$. Indeed,

$$\begin{aligned} 0 = \Psi'_{\varepsilon_n,\lambda}(v_n)(v_n) &= \|v_n\|^2 - \lambda \int_\Omega v_n^s dx + \int_\Omega f_{\varepsilon_n}(v_n) v_n dx \\ &\geq \|v_n\|^2 - \lambda \|v_n\|^s_{L^s(\Omega)} \geq \|v_n\|^2 - \lambda c_s \|v_n\|^s. \end{aligned}$$

(c_s being the embedding constant of $W_0^{1,2}(\Omega)$ into $L^s(\Omega)$). Since $s \in]1,2[$, v_n is bounded in $W_0^{1,2}(\Omega)$. Therefore, there exists $v_\lambda \in W_0^{1,2}(\Omega)$ such that $\{v_n\}$ weakly converges to v_λ in $W_0^{1,2}(\Omega)$. Clearly, $v_\lambda \geq 0$ a.e. in Ω.

Let us prove that $v_\lambda \neq u_\lambda$.

From (l) and (ll) we get that

$$\begin{aligned} 2\left(\eta_1 + \Psi_{\varepsilon_n,\lambda}(u_\lambda)\right) &\leq 2\Psi_{\varepsilon_n,\lambda}(v_n) - \Psi'_{\varepsilon_n,\lambda}(v_n)(v_n) \\ &= \lambda \left(1 - \frac{2}{s}\right) \int_\Omega v_n^s + \int_\Omega \left[2F_{\varepsilon_n}(v_n) - f_{\varepsilon_n}(v_n) v_n\right] dx \end{aligned} \tag{15}$$

It is easily seen that

$$\int_\Omega F_{\varepsilon_n}(u_\lambda) dx \to \frac{1}{r} \int_\Omega u_\lambda^r dx.$$

Indeed from Beppo Levi Theorem, it follows that

$$F_{\varepsilon_n}(u_\lambda(x)) \to \frac{1}{r} u_\lambda(x)^r \qquad \text{a.e. } x \in \Omega$$

and also

$$|F_{\varepsilon_n}(u_\lambda)| \leq \frac{1}{r} u_\lambda^r \in L^1(\Omega),$$

which clearly implies our claim. Therefore,

$$\Psi_{\varepsilon_n,\lambda}(u_\lambda) \to \Psi(u_\lambda). \tag{16}$$

Also, from the weak convergence $v_n \rightharpoonup v_\lambda$, it follows that $v_n \to v_\lambda$ a.e. $x \in \Omega$ and $v_n \to v_\lambda$ strongly in $L^s(\Omega)$ and in $L^r(\Omega)$. So, in particular,

$$\int_\Omega v_n^s dx \to \int_\Omega v_\lambda^s dx. \tag{17}$$

Moreover, there exists $h \in L^r(\Omega)$ such that

$$v_n(x) \leq h(x) \quad \text{a.e. in } \Omega \text{ and } n \in \mathbb{N}.$$

Therefore,

$$0 \leq f_{\varepsilon_n}(v_n) v_n = \frac{v_n^q}{(v_n + \varepsilon_n)^{q-r}} \to v_\lambda^r \qquad \text{a.e. in } \Omega$$

and

$$0 \leq f_{\varepsilon_n}(v_n) v_n \leq h^r \in L^1(\Omega),$$

and from Lebesgue theorem

$$\int_\Omega f_{\varepsilon_n}(v_n) v_n dx \to \int_\Omega v_\lambda^r dx. \tag{18}$$

In a similar way it is easy to prove that

$$F_{\varepsilon_n}(v_n) \to \frac{1}{r} v_\lambda^r$$

and

$$|F_{\varepsilon_n}(v_n)| \leq \frac{1}{r} h^r \in L^1(\Omega),$$

so from Lebesgue theorem

$$\int_\Omega F_{\varepsilon_n}(v_n)dx \to \frac{1}{r}\int_\Omega v_\lambda^r dx. \tag{19}$$

Bearing in mind (16)–(19) we get

$$2\Psi_{\varepsilon_n}(v_n) - \Psi'_{\varepsilon_n}(v_n)(v_n) \to \lambda\left(1-\frac{2}{s}\right)\int_\Omega v_\lambda^s dx + \left(1-\frac{2}{r}\right)\int_\Omega v_\lambda^r dx \tag{20}$$

So, taking the limit in (15) we get

$$2\left(\eta_1 + \Psi(u_\lambda)\right) \leq \lambda\left(1-\frac{2}{s}\right)\int_\Omega v_\lambda^s dx - \left(1-\frac{2}{r}\right)\int_\Omega v_\lambda^r dx$$

If $v_\lambda = u_\lambda$, then, from the above inequality

$$\begin{aligned} 2\left(\eta_1 + \Psi(u_\lambda)\right) &\leq \lambda\left(1-\frac{2}{s}\right)\int_\Omega u_\lambda^s dx - \left(1-\frac{2}{r}\right)\int_\Omega u_\lambda^r dx \\ &= 2\Psi(u_\lambda) - \Psi'(u_\lambda)(u_\lambda) \\ &= 2\Psi(u_\lambda), \end{aligned}$$

a contradiction. In a similar way, using (l) and (lll), we get that

$$2\eta_2 \leq 2\Psi_{\varepsilon_n,\lambda}(v_n) - \Psi'_{\varepsilon_n,\lambda}(v_n)(v_n)$$

and by (20)

$$0 < 2\eta_2 \leq \lambda\left(1-\frac{2}{s}\right)\int_\Omega v_\lambda^s dx - \left(1-\frac{2}{r}\right)\int_\Omega v_\lambda^r dx$$

which implies $v_\lambda \neq 0$.

It remains to show that v_λ is a solution of (P_λ). This follows as in ([5, Proof of Theorem 1.1]). The proof is concluded.

□

Remark 1. Comparing our Theorem 1 with Theorem B in the Introduction, some natural questions arise:

(Q_1) Is it true that $\Lambda = \lambda_0$?

(Q_2) From Theorem A we know that for $\lambda < \Lambda$ problem (P_λ) has no positive solutions. What about the existence of sign changing solutions?

(Q_3) What about the positivity of the second solution v_λ?

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